Anderson localization for 2D discrete Schrödinger operator with random vector potential

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Abstract

We prove the Anderson localization near the bottom of the spectrum for two dimensional discrete Schrödinger operators with a class of random vector potentials and no scalar potentials. Main lemmas are the Lifshitz tail and the Wegner estimate on the integrated density of states. Then, the Anderson localization, i.e., the pure point spectrum with exponentially decreasing eigenfunctions, is proved by the standard multiscale argument.

1 Introduction

We consider a magnetic Schrödinger operator on $\mathbb{Z}^2$ defined as follows: let

$$\mathcal{E} = \left\{(x, y) \,\middle|\, x, y \in \mathbb{Z}^2, |x - y| = 1\right\}$$

be the set of the directed edges on $\mathbb{Z}^2$, and let

$$A \;:\; \mathcal{E} \to \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$$

be a vector potential such that

$$A((x, y)) = -A((y, x)) \quad \text{for } (x, y) \in \mathcal{E}.$$

Then, our Hamiltonian is defined by

$$H(A)u(x) = \sum_{|x-y|=1} \left(u(x) - e^{iA((x,y))}u(y)\right), \quad x \in \mathbb{Z}^2,$$

for $u \in \ell^2(\mathbb{Z}^2)$. It is easy to show that $H(A)$ is a bounded self-adjoint operator on $\ell^2(\mathbb{Z}^2)$ and

$$0 \le H(A) \le 8$$

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for any vector potential A. The magnetic field induced by A is defined as follows: let

$$\mathcal{F} = \big\{\{x_1, x_1+1\} \times \{x_2, x_2+1\} \subset \mathbb{Z}^2 \,\big|\, (x_1,x_2) \in \mathbb{Z}^2\big\}$$

be the set of unit squares in $\mathbb{Z}^2$. For $f \in \mathcal{F}$, the boundary ∂f is defined by

$$\partial f_x = \big\{(x, x+e_1), (x+e_1, x+e_1+e_2), (x+e_1+e_2, x+e_2), (x+e_2, x)\big\} \subset \mathcal{E}$$

where $f_x = \{x_1, x_1+1\} \times \{x_2, x_2+1\}$ and $e_1 = (1,0), e_2 = (0,1) \in \mathbb{Z}^2$. The magnetic field $B = dA$ is then given by

$$B(f) = \sum_{e \in \partial f} A(e), \quad B \;:\; \mathcal{F} \to \mathbb{T}.$$

It is well-known that the spectral properties of $H(A)$ depend only on B, and are independent of the choice of A such that $B = dA$.

We suppose A has the following form. Let $f_x \in \mathcal{F}$ be defined as above. For $0 < b < \pi$, we set

$$B_0^b(f_x) = \begin{cases} b & \text{if } x_1+x_2 \text{ is even,} \\ -b & \text{if } x_1+x_2 \text{ is odd,} \end{cases}$$

and we fix A_0^b so that $dA_0^b = B_0^b$. For example, we can set

$$A_0^b((x, x+e_1)) = \begin{cases} b/2 & \text{if } x_1+x_2 \text{ is even,} \\ -b/2 & \text{if } x_1+x_2 \text{ is odd,} \end{cases}$$

and $A_0^b((x, x+e_2)) = 0$ for all $x \in \mathbb{Z}^2$. Let

$$a^\omega(e) \;:\; \mathcal{E} \to [-1, 1]$$

be independent identically distributed (i.i.d.) random variables on a probability space $(\Omega, \mathfrak{B}, \mu)$. Let $\lambda > 0$ and we set

$$A^\omega(e) = A_0^b(e) + \lambda a^\omega(e) \in \mathbb{T} \quad \text{for } e \in \mathcal{E}.$$

We denote the common distribution of $a^\omega(e)$ by ν. We note

$$\sigma(H(A_0^b)) = [4(1-\cos(b/4)), 4(1+\cos(b/4))]$$

(see [15], Example 1). Moreover, if

$$\nu([-1, -1+\varepsilon]) > 0, \quad \nu([1-\varepsilon, 1]) > 0 \quad \text{for } \varepsilon > 0, \tag{1.1}$$

then we also have

$$\sigma(H(A_\omega)) = [4(1-\cos((b-4\lambda)/4)), 4(1+\cos((b-4\lambda)/4))]$$

almost surely, provided $b - 4\lambda \geq 0$.

Assumption A. ν has a bounded density function $g(t)$, and ν satisfies (1.1). Moreover, $g(t)$ is Lipschitz continuous on $[-1, 1]$.

Now we can state our main result:

Theorem 1.1. *There exists* $\lambda_0 > 0$ *such that if* $0 < \lambda < \lambda_0$*, then the Anderson localization holds near the bottom of the spectrum. Namely, there exist*

$$E_1 > E_0 := 4(1 - \cos((b - 4\lambda)/4))$$

such that $H(A^\omega)$ *has dense pure point spectrum on* $[E_0, E_1]$*, and each eigenfunction decays exponentially as* $|x| \to \infty$*.*

Remark 1.1. In the small λ regime, following [13], one can give a lower bound for $E_1 = E_1(\lambda)$ as well as a lower bound on the rate of exponential decay of the eigenfunctions associated to eigenvalues in $[E_0(\lambda), E_1(\lambda)]$ (here $E_0(\lambda) = E_0 = 4(1 - \cos((b - 4\lambda)/4))$. One finds that, for any $0 < \varepsilon < 3/4$, for λ sufficiently small, one has

$$E_1(\lambda) \geq E_0(0) - \lambda^{1+\varepsilon}.$$

Here $\mathbb{E}(\cdot)$ denotes the expectation with respect to ω.

For $0 < \varepsilon < 3/4$ fixed, there exists $a > 0$, such that, for λ sufficiently small, the rate of exponential decay of an eigenfunction associated to the eigenvalue $E \in [E_0(\lambda), E_0(0) - \lambda^{1+\varepsilon}]$ is lower bounded by $a\sqrt{|E - E_0(0)|}$. See remark 2.1 for more details.
One can compare the exponent 3/4 obtained here with the one obtained in [13] (note that $E_0(0) = \mathbb{E}(W_B(0)) + O(\lambda^2)$; hence, in the present case, the interval $[E_0(\lambda), E_0(0) - \lambda^{1+\varepsilon}]$ is the counterpart of the interval $I_{\gamma,\eta} := [0, \gamma(\overline{\omega} - \gamma^\eta)]$ used in [13]). It is much larger than what was obtained in the case of a potential perturbation (indeed, in [13], the exponent η introduced above has to satisfy $\eta < 1/6$ in dimension 2). This is due to the fact that the averaged flux of the magnetic field over large areas vanishes (which in turn is due to the special form of our random perturbation; indeed, we chose the magnetic potential to be random i.i.d (see equation (2.1))).

The constant λ_0 depends only on b, and the condition on λ_0 is given in Theorem 3.2. Theorem 1.1 is proved by the standard multiscale argument (see, e.g., [6], [4], [18] and references therein), combined with the Lifshitz tail (Theorem 1.2) and the Wegner estimate (Theorem 1.3). In order to state these results explicitly, we introduce the integrated density of states (IDS). For $L > 0$, we set

$$\Lambda_L = [-L, L]^2 \cap \mathbb{Z}^2 \subset \mathbb{Z}^2$$

be the finite lattice of size $|\Lambda_L| = (2L + 1)^2$. Let $H_{\Lambda_L}(A)$ be the magnetic Schrödinger operator on Λ_L. We give the precise definition in Section 2. For $E \in \mathbb{R}$, the integrated density of states is defined by

$$k(E) = \lim_{L\to\infty} \frac{1}{|\Lambda_L|} \sharp\{\text{eigenvalues of } H_{\Lambda_L}(A^\omega) \leq E\}.$$

See Appendix C of [15] for the proof of the existence of $k(E)$ for discrete magnetic Schrödinger operators. Note that $k(E)$ is a nonrandom quantity, i.e., it is independent of the choice of the sample ω, a.s.

Theorem 1.2. *Under the above assumptions,*

$$\lim_{E \downarrow E_0} \log(-\log k(E))/\log(E - E_0) \le -1.$$

This result implies, roughly speaking,

$$k(E) \lesssim e^{-(E-E_0)^{-1}} \quad \text{as } E \downarrow E_0,$$

and hence the IDS is very thin near the bottom of the spectrum. We note that the equality holds with an additional assumption on ν (cf. Theorem 2.3). On the other hand, the Wegner estimate implies that the distribution of the eigenvalues of $H_{\Lambda_L}(A)$ admits a density in a low energy region. In particular, it follows from the Wegner estimate that the IDS is Lipschitz continuous.

Theorem 1.3. *There exists $\lambda_1 > 0$ such that if $0 < \lambda < \lambda_1$ then there are $E_2 > E_0$ and $C > 0$ such that*

$$\mathbb{P}\big(\mathrm{dist}(\sigma(H_{\Lambda_L}(A^\omega)), E) < \varepsilon\big) \le C\varepsilon|\Lambda_L|$$

for $E \in [E_0, E_2]$ and $\varepsilon > 0$.

Once Theorems 1.2 and 1.3 are proved, Theorem 1.1 follows by the multiscale argument, and we omit the detail. We note that the Combes-Thomas estimate and the decomposition of resolvents in the multiscale argument work for magnetic Schrödinger operators with essentially no modifications.

Whereas a large amount of work has been done on the spectral properties of Schrödinger operators with random potentials (see, e.g., [2], [5], [18] and references therein), only a few results have been obtained on Schrödinger operators with random magnetic fields. Ueki ([20]) proved the Lifshitz tail for a class of Gaussian random magnetic field, and Nakamura ([15], [16]) proved it for the 2D discrete case, and the continuous case, respectively. Hislop and Klopp proved the Wegner estimate near the bottom edge of the spectrum for continuous case ([8]). They suggested the Anderson localization (combined with the result of [16]), but it was not clear if there exists an interval in the spectrum that satisfies both conditions. In a recent paper [21], Ueki proved the Anderson localization for Schrödinger operator with a random potential and a correlated random magnetic field.

There have been active discussion about the spectrum of random magnetic Schrödinger operators in physics literature, most of them are mainly numerical computations (see, e.g., [7], [14], [9], [17]). There seems to be no agreement if there exists continuous spectrum in the middle of the spectrum, but it appears that the localization near the spectrum edges is widely believed, which is the subject of this paper.

We prove Theorem 1.2 under more general assumptions in Section 2, and Theorem 1.3 is proved in Section 3.

2 The Lifshitz tail

We first define $H_{\Lambda_L}(A)$ on $\ell^2(\Lambda_L)$. We note

$$\begin{aligned}\langle u|Hu\rangle &= \frac{1}{2}\sum_{e\in\mathcal{E}}\big|u(i(e))-e^{iA(e)}u(t(e))\big|^2\\ &= \frac{1}{2}\sum_{f\in\mathcal{F}}\sum_{e\in\partial f}\big|u(i(e))-e^{iA(e)}u(t(e))\big|^2,\end{aligned}$$

where $i(e)$ and $t(e)$ denote the initial point and the terminal point of e, respectively. Namely,

$$i(e)=x,\quad t(e)=y\quad \text{for } e=(x,y).$$

Then, we set

$$\langle u|H_{\Lambda_L}u\rangle = \frac{1}{2}\sum_{\substack{f\in\mathcal{F},\\ f\subset\Lambda_L}}\sum_{e\in\partial f}\big|u(i(e))-e^{iA(e)}u(t(e))\big|^2 + \sum_{x\in\partial\Lambda_L}|u(x)|^2,$$

where

$$\partial\Lambda_L = \big\{x\in\Lambda_L \;\big|\; |x_i| = L \text{ for } i=1 \text{ or } 2\big\}.$$

The boundary term $\sum_{\partial\Lambda_L}|u(x)|^2$ does not affect the IDS since it is an operator of rank $4(L+1) << |\Lambda_L|$. One may consider our Hamiltonian H_{Λ_L} is analogous to the Dirichlet Hamiltonian, though they are slightly different.

In order to prove Theorem 1.2, we follow the argument of [15], but we need a more precise local energy estimate since $\inf\sigma(H(A^\omega))>0$. We fix

$$0<\alpha<1-1/\sqrt{2},$$

and define

$$\beta(t)=\min(1-\cos(t/4),\alpha)$$

for $t\in\mathbb{T}\cong[-\pi,\pi)$. We set

$$W_B(x)=\sum_{x\in\partial f}\beta(B(f)),\quad x\in\mathbb{Z}^2.$$

Then, we have

Theorem 2.1. *For $u\in\ell^2(\mathbb{Z}^2)$,*

$$\langle u|Hu\rangle \ge \big\langle u\big|W_B u\big\rangle + \gamma\big\langle |u|\big|H_0|u|\big\rangle$$

where

$$\gamma=\frac{1}{4}\Big(1-\frac{1}{\sqrt{2}}-\alpha\Big)>0,$$

and H_0 is the free discrete Schrödinger operator on $\mathbb{Z}^2$.

Proof. We mimic the argument of Theorem 2 of [15]. We consider a Hamiltonian H_f on $\ell^2(f) \cong \mathbb{C}^4$ defined by

$$\big\langle u_f \big| H_f u_f \big\rangle = \frac{1}{2} \sum_{e \in \partial f} \big| u_f(e) - e^{iA(e)} u_f(e) \big|^2$$

for $u_f \in \ell^2(f)$. We may write

$$f = \big\{ y_0, y_1, y_2, y_3 \big\}, \quad e_j = (y_j, y_{j+1}),$$

where $y_4 = y_0$, and

$$\big\langle u_f \big| H_f u_f \big\rangle = \frac{1}{2} \sum_{j=0}^{3} \big| u_f(y_j) - e^{iA(e_j)} u_f(y_{j+1}) \big|^2.$$

By a gauge transform, we may suppose $A(e_j) = B/4$ where $B = B(f)$. Namely, there exists $\{g_j\}_{j=0}^3$ with $|g_j| = 1$ such that if we set

$$\tilde{u}_f(y_j) = g_j u_f(y_j)$$

then

$$\big\langle u_f \big| H_f u_f \big\rangle = \frac{1}{2} \sum_{j=0}^{3} \big| \tilde{u}_f(y_j) - e^{iB/4} \tilde{u}_f(y_{j+1}) \big|^2.$$

We define $\tilde{H}_f$ on $\ell^2(f)$ so that

$$\big\langle u_f \big| H_f u_f \big\rangle = \big\langle \tilde{u}_f \big| \tilde{H}_f \tilde{u}_f \big\rangle.$$

It is easy to show

$$\sigma(H_f) = \sigma(\tilde{H}_f) = \big\{ \lambda_j \ \big| \ j = 0, 1, 2, 3 \big\}$$

where

$$\lambda_j = 1 - \cos((B + 2\pi j)/4),$$

and the eigenvectors of $\tilde{H}_f$ is given by

$$\mathbf{v}_j = \frac{1}{2}(1, e^{i\pi j/2}, e^{2i\pi j/2}, e^{3i\pi j/2}), \quad j = 0, 1, 2, 3.$$

Let Π_j be the orthogonal projection to the eigenspace with the eigenvalue λ_j. Then, we have

$$\begin{aligned}
\langle u_f | H_f u_f \rangle &= \sum_{j=0}^{3} \lambda_j \|\Pi_j \tilde{u}_f\|^2 \\
&\geq \beta(B) \|\Pi_0 \tilde{u}_f\|^2 + \sum_{j=1}^{3} \lambda_j \|\Pi_j \tilde{u}_f\|^2 \\
&= \beta(B) \|\tilde{u}_f\|^2 + \sum_{j=1}^{3} (\lambda_j - \beta(B)) \|\Pi_j \tilde{u}_f\|^2 \\
&\geq \beta(B) \|u_f\|^2 + 4\gamma \|(1 - \Pi_0) \tilde{u}_f\|^2
\end{aligned}$$

by the definitions of $\beta(t)$ and γ, and the fact that $\lambda_j \geq 1 - 1/\sqrt{2}$ for $j = 1, 2, 3$. We then estimate the last term in the right hand side. For $v \in \ell^2(f)$, we have

$$\begin{aligned}\|(1 - \Pi_0)v\|^2 &= \sum_{j=0}^{3} |v(y_j) - \bar{v}|^2 \geq \frac{1}{4}\sum_{j=0}^{3} |v(y_j) - v(y_{j+1})|^2 \\ &\geq \frac{1}{4}\sum_{j=0}^{3} \big||v(y_j)| - |v(y_{j+1})|\big|^2\end{aligned}$$

where $\bar{v} = \frac{1}{4}\sum_{j=0}^{3} v(y_j)$ is the average of v. Hence we have

$$\begin{aligned}\|(1 - \Pi_0)\tilde{u}_f\|^2 &\geq \frac{1}{4}\sum_{j=0}^{3} \big||\tilde{u}_f(y_j)| - |\tilde{u}_f(y_{j+1})|\big|^2 \\ &= \frac{1}{4}\sum_{j=0}^{3} \big||u_f(y_j)| - |u_f(y_{j+1})|\big|^2 = \frac{1}{4}\big\langle |u_f|\big|H_{0,f}|u_f|\big\rangle\end{aligned}$$

where $H_{0,f}$ is the free Schrödinger operator on $\ell^2(f)$. Combining these, we learn

$$\langle u_f | H_f u_f \rangle \geq \beta(B(f))\|u_f\|^2 + \gamma\big\langle |u_f|\big|H_{0,f}|u_f|\big\rangle.$$

If we set $u_f = u\big|_f$ and sum up this inequality over $f \in \mathcal{F}$, then we obtain

$$\begin{aligned}\langle u | H u \rangle &= \sum_{f\in\mathcal{F}} \beta(B(f))\|u_f\|^2 + \gamma\sum_{f\in\mathcal{F}} \big\langle |u_f|\big|H_{0,f}|u_f|\big\rangle \\ &= \big\langle u\big|W_B u\big\rangle + \gamma\big\langle |u|\big|H_0|u|\big\rangle.\end{aligned}$$

□

We can also prove the following estimate for H_{Λ_L} in exactly the same way as above. Note that we use the boundary term in the definition of H_{Λ_L} near the boundary.

Theorem 2.2. *For $u \in \ell^2(\Lambda_L)$,*

$$\langle u | H_{\Lambda_L} u \rangle \geq \big\langle u\big|W_{B,\Lambda_L} u\big\rangle + \gamma\big\langle |u|\big|H_{0,\Lambda_L}|u|\big\rangle,$$

where

$$W_{B,\Lambda_L}(x) = \sum_{x\in f\subset\Lambda_L} \beta(B(f)),$$

and H_{0,Λ_L} is the free Schrödinger operator on $\ell^2(\Lambda_L)$ defined by

$$\langle u | H_{0,\Lambda_L} u \rangle = \frac{1}{2}\sum_{f\in\mathcal{F}}\sum_{e\in\partial f} |u(i(e)) - u(t(e))|^2.$$

Given this estimate, we can prove the following generalization of Theorem 1.2, using the argument of [15] and the large deviation argument of [10, 19]. We omit the details.

Theorem 2.3. *Suppose* $\{B(f)|f \in \mathcal{F}\}$ *are metrically transitive random variables with finite correlation length, i.e., there exists* $R > 0$ *such that* $\{B(f)|f \in F_1\}$ *and* $\{B(f)|f \in F_2\}$ *are independent if* $\mathrm{dist}(F_1, F_2) \geq R$. *Let* μ *be the common distribution of* $B(f)$, *and suppose*

$$\mathrm{supp}\ \mu \subset [-b_+, -b_-] \cup [b_-, b_+]$$

with $0 < b_- < b_+ \leq \pi$. *Then,*

$$\lim_{E \downarrow E_0} \log(-\log k(E))/\log(E - E_0) \leq -1,$$

where $E_0 = \inf \sigma(H^\omega) = 4(1 - \cos(b_-/4))$. *Moreover, if in addition,*

$$\mu([b_-, b_- + \varepsilon]) \geq C\varepsilon^a, \quad \mu([-b_- - \varepsilon, -b_-]) \geq C\varepsilon^a \quad \textit{for } \varepsilon > 0,$$

with some $C, a > 0$, *then*

$$\lim_{E \downarrow E_0} \log(-\log k(E))/\log(E - E_0) = -1.$$

Remark 2.1. To study the small λ regime, rather than Dirichlet approximations, one uses periodic approximations to the random operator as in [12, 13]. It essentially means that we consider periodic realization of our magnetic operators (with very large periods) with quasi-periodic boundary conditions and study the eigenvalue distributions for these realizations. One can prove an analogue of Theorem 2.2 for these periodic realization of our magnetic operators with quasi-periodic boundary conditions. Using the analysis done in [13], one sees that to estimate the density of states for $H(A_\omega)$ in $[E_0, E_0 + c\lambda]$, one has to estimate the probability that for $\Lambda = \Lambda_L$ and $L = L_\lambda = \lambda^{-\rho}$ ($\rho > 0$ is large but fixed), there exists a function $u \in \ell^2(\Lambda)$, $\|u\| = 1$, so that

$$\begin{aligned}\langle u|W_{B,\Lambda_L} u\rangle + \gamma\langle |u||H^\theta_{0,\Lambda_L}|u|\rangle &= \langle v|W_{B,\Lambda_L} v\rangle + \gamma\langle v|H^\theta_{0,\Lambda_L} v\rangle \\ &\leq E_0 + c\lambda \text{ where } v = |u|.\end{aligned}$$

Here H^θ_{0,Λ_L} is the discrete Laplacian with quasi-periodic boundary conditions. As $W_{B,\Lambda_L} - E_0 \geq 0$, one gets $\langle |u||H^\theta_{0,\Lambda_L}|u|\rangle \leq c\lambda$. So $|u|$ has to be flat on cubes of size roughly $\lambda^{-1/2}$. The positivity of the Laplace operator then implies that $\langle v|(W_{B,\Lambda_L} - E_0)v\rangle \leq c\lambda$. Using the small λ expansion for W_{B,Λ_L}, one roughly has to estimate (see [13] for details) the probability that

$$\begin{aligned}&\lambda \sin(b/4) \sum_{f \in \Lambda_{\lambda^{-1/2}}} B^\omega(f) + C\lambda^2 \sum_{f \in \Lambda_{\lambda^{-1/2}}} (B^\omega(f))^2 \\ &= \lambda \sin(b/4) \sum_{e \in \partial\Lambda_{\lambda^{-1/2}}} a^\omega(e) + C\lambda^2 \sum_{f \in \Lambda_{\lambda^{-1/2}}} (B^\omega(f))^2 \leq c.\end{aligned} \tag{2.1}$$

Standard large deviation arguments tell us that this probability stays exponentially small with λ as long as $c < -\lambda^{3/4-\delta}$ (here $\delta > 0$ is fixed and arbitrary).

3 The Wegner estimate

In this section, we prove Theorem 1.3 using an idea similar to [8]. Let $A^\omega(e)$ be the vector potential as in Section 1. Namely,

$$A^\omega(e) = A_0^b(e) + \lambda a^\omega(e),$$

where $\{a^\omega(e) | e \in \mathcal{E}\}$ are i.i.d. random variables with the density function $g(t)$. We set

$$F_0 = \inf \sigma(H(A_0^b)) = 4(1 - \cos(b/4)).$$

We always assume

$$b - 4\lambda \geq 0, \quad \text{i.e., } 0 < \lambda \leq b/4$$

so that

$$E_0 = \inf \sigma(H(A^\omega)) = 4(1 - \cos((b - 4\lambda)/4).$$

Our Hamiltonian $H(A^\omega)$ may be considered as a function of $\{a^\omega(e)\}$, and we compute the partial derivative of $H_{\Lambda_L}(A^\omega)$ with respect to each $a^\omega(e)$. Our main lemma in this section is the following:

Lemma 3.1. *Let ψ be a normalized eigenfunction of $H_{\Lambda_L}(A^\omega)$ with an eigenvalue $E < F_0$. Then,*

$$\frac{1}{2} \sum_{e \in \mathcal{E}'_L} a^\omega(e) \left\langle \psi \middle| \frac{\partial H_{\Lambda_L}(A^\omega)}{\partial a^\omega(e)} \psi \right\rangle \geq F_0 - E - 6\lambda^2,$$

where

$$\mathcal{E}'_L = \big\{ e \in \mathcal{E} \bigm| e \subset \Lambda_L \big\}.$$

Proof. We decompose $H_{\Lambda_L}(A^\omega)$ as follows:

$$\begin{aligned} H_{\Lambda_L}(A^\omega)u(x) &= \sum_{|x-y|=1} \big(u(x) - e^{iA_0^b((x,y))}u(y)\big) + \chi_{\partial\Lambda_L}(x)u(x) \\ &\quad + \sum_{|x-y|=1} \big(1 - e^{i\lambda a^\omega((x,y))}\big) e^{iA_0^b((x,y))} u(y) \\ &= H_{\Lambda_L}(A_0^b)u(x) + V^\omega u(x), \end{aligned}$$

with $u \in \ell^2(\Lambda_L)$. By direct computations, we have

$$
\begin{aligned}
&\frac{1}{2}\sum_{e\in\mathcal{E}'_L}\left\langle u \middle| a^\omega(e)\frac{\partial V^\omega}{\partial a^\omega(e)}\, u\right\rangle \\
&= -\frac{i}{2}\Big\{\sum_e \lambda a^\omega(e) e^{i(\lambda a^\omega(e)+A_0^b(e))}\overline{u(i(e))}u(t(e)) \\
&\qquad + \sum_e \lambda a^\omega(e) e^{-i(\lambda a^\omega(e)+A_0^b(e))}\overline{u(t(e))}u(i(e))\Big\} \\
&= -i\sum_e \lambda a^\omega(e) e^{i(\lambda a^\omega(e)+A_0^b(e))}\overline{u(i(e))}u(t(e)) \\
&= -\sum_e \big(e^{i\lambda a^\omega(e)}-1\big)e^{iA_0^b(e)}\overline{u(i(e))}u(t(e)) \\
&\qquad + \sum_e r(e)\overline{u(i(e))}u(t(e)) \\
&= -\langle u|V^\omega u\rangle + \langle u|Ru\rangle
\end{aligned}
\tag{3.1}
$$

where

$$r(e) := \big((e^{i\lambda a^\omega(e)}-1) - \lambda a^\omega(e)e^{i\lambda a^\omega(e)}\big)e^{iA_0^b(e)}.$$

It is easy to observe by Taylor's theorem that

$$|r(e)| \le \frac{3}{2}\lambda^2|a^\omega(e)|^2 \le \frac{3}{2}\lambda^2,$$

and hence

$$\|R\| \le 4\sup_e |r(e)| \le 6\lambda^2. \tag{3.2}$$

On the other hand, we have

$$
\begin{aligned}
E = \langle\psi|H_{\Lambda_L}(A^\omega)\psi\rangle &= \langle\psi|H_{\Lambda_L}(A_0^b)\psi\rangle + \langle\psi|V^\omega\psi\rangle \\
&\ge F_0 + \langle\psi|V^\omega\psi\rangle
\end{aligned}
$$

and hence

$$\langle\psi|V^\omega\psi\rangle \le -(F_0 - E). \tag{3.3}$$

Combining (3.1) with (3.2) and (3.3), we obtain

$$
\begin{aligned}
\frac{1}{2}\sum_{e\in\mathcal{E}'_L}\left\langle \psi \middle| a^\omega(e)\frac{\partial H_{\Lambda_L}(A^\omega)}{\partial a^\omega(e)}\, \psi\right\rangle &= -\langle\psi|V^\omega\psi\rangle + \langle\psi|R\psi\rangle \\
&\ge (F_0 - E) - 6\lambda^2.
\end{aligned}
$$

□

Given Lemma 3.1, we can prove the Wegner estimate using the idea of Hislop and Klopp [8].

Theorem 3.2. *Suppose*

$$E_0 < F_0 - 6\lambda^2,$$

and let $E_2 \in (E_0, F_0 - 6\lambda^2)$. *Then, there exists* $C > 0$ *such that*

$$\mathbb{P}\big(\mathrm{dist}(\sigma(H_{\Lambda_L}(A^\omega))), E\big) \leq C\varepsilon|\Lambda_L|$$

for $\varepsilon > 0, L > 0$ *and* $E \in [E_0, E_2]$.

Remark 3.1. The assumption is satisfied if λ is sufficiently small since

$$E_0 = F_0 - 4(\sin b/4)\lambda + O(\lambda^2) \quad \text{for } \lambda \sim 0.$$

Proof. The proof is analogous to [8] (see also [3]), but we prove it for the completeness[1]. We fix $E \in [E_0, E_2]$, and for $\varepsilon > 0$ we set $\eta \in C_0^\infty(\mathbb{R})$ so that

$$\eta(t) = \begin{cases} 1, & (|t-E| \leq \varepsilon), \\ 0, & (|t-E| \geq 2\varepsilon), \end{cases}$$

and $0 \leq \eta(t) \leq 1$ for all $t \in \mathbb{R}$. By Chebyshev's inequality, we have

$$\mathbb{P}(\mathrm{dist}(\sigma(H_{\Lambda_L}(A^\omega))), E) \leq \mathbb{E}(\mathrm{Tr}(\eta(H_{\Lambda_L}(A^\omega)))). \tag{3.4}$$

Let $\{E_j^\omega | j = 1, 2, \dots\}$ be the eigenvalues of $H_{\Lambda_L}(A^\omega)$, and let $\{\psi_j^\omega\}$ be the corresponding eigenfunctions. Then, by a standard computation of the analytic perturbation theory, we have

$$\frac{\partial E_j^\omega}{\partial a^\omega(e)} = \left\langle \psi_j \middle| \frac{\partial H_{\Lambda_L}(A^\omega)}{\partial a^\omega(e)} \psi_j \right\rangle.$$

Let

$$\xi(t) = \int_t^\infty \eta(s)ds \in C^\infty(\mathbb{R}).$$

Then, we learn

$$\frac{\partial}{\partial a^\omega(e)} \mathrm{Tr}(\xi(H_{\Lambda_L}(A^\omega))) = \frac{\partial}{\partial a^\omega(e)} \sum_j \xi(E_j^\omega) = -\sum_j \frac{\partial E_j^\omega}{\partial a^\omega(e)} \eta(E_j^\omega),$$

where the sum is taken over j such that $E_j^\omega \in [E - 2\varepsilon, E + 2\varepsilon]$. Combining these with Lemma 3.1, we obtain

$$\begin{aligned} &\sum_{e \in \mathcal{E}_L'} a^\omega(e) \frac{\partial}{\partial a^\omega(e)} \mathrm{Tr}(\xi(H_{\Lambda_L}(A^\omega))) \\ &\quad = \sum_{e \in \mathcal{E}_L'} \sum_j a^\omega(e) \left\langle \psi_j \middle| \frac{\partial H_{\Lambda_L}(A^\omega)}{\partial a^\omega(e)} \psi_j \right\rangle \eta(E_j^\omega) \\ &\quad \geq \sum_j (F_0 - E_j^\omega - 6\lambda^2) \eta(E_j^\omega) \\ &\quad \geq (F_0 - E - 2\varepsilon - 6\lambda^2) \mathrm{Tr}(\eta(H_{\Lambda_L}(A^\omega))). \end{aligned} \tag{3.5}$$

[1] In fact, the proof is much simpler in our situation.

Now we compute the expectation of the left hand side of (3.5):

$$\mathbb{E}\Big(\sum_{e\in\mathcal{E}'_L} a^\omega(e)\frac{\partial}{\partial a^\omega(e)}\mathrm{Tr}(\xi(H_{\Lambda_L}(A^\omega)))\Big)$$
$$=\sum_{e\in\mathcal{E}'_L}\int\cdots\int a^\omega(e)\frac{\partial}{\partial a^\omega(e)}\mathrm{Tr}(\xi(H_{\Lambda_L}(A^\omega)))\prod_{e'\in\mathcal{E}'_L} g(a^\omega(e'))da^\omega(e').$$

We denote $K_e^t = H_{\Lambda_L}(A_e^t)$ with

$$A_e^t(e') = \begin{cases} A^\omega(e'), & (e'\neq e),\\ t\quad , & (e'=e).\end{cases}$$

By an integration by parts, we have

$$\begin{aligned}
&\int a^\omega(e)\frac{\partial}{\partial a^\omega(e)}\mathrm{Tr}(\xi(H_{\Lambda_L}(A^\omega)))da^\omega(e)\\
&=\int \frac{\partial}{\partial t}\Big(\mathrm{Tr}(\xi(K_e^t)-\xi(K_e^0))\Big)tg(t)dt\\
&=g(1)\mathrm{Tr}(\xi(K_e^1)-\xi(K_e^0)) + g(-1)\mathrm{Tr}(\xi(K_e^{-1})-\xi(K_e^0))\\
&\quad+\int \mathrm{Tr}(\xi(K_e^t)-\xi(K_e^0))(g(t)+tg'(t))dt\\
&\leq (3\sup|g|+\sup|g'|)\sup_{-1\leq t\leq 1}|\mathrm{Tr}(\xi(K_e^t)-\xi(K_e^0)|. \qquad (3.6)
\end{aligned}$$

Since $K_e^t - K_e^0$ is an operator of rank 2, we have

$$\begin{aligned}
|\mathrm{Tr}(\xi(K_e^t)-\xi(K_e^0))| &= -\int \xi'(s)|\Xi(s;K_e^t,K_e^0))|ds\\
&\leq 2\int \eta(s)ds \leq 8\varepsilon, \qquad (3.7)
\end{aligned}$$

where $\Xi(s;A,B)$ denotes the spectral shift function for the pair of operators A and B. Note that $\Xi(s;A,B)$ is uniformly bounded by the rank of $A-B$ (see. e.g., [1]). Thus (3.6) and (3.7) imply

$$\mathbb{E}\Big(\sum_{e\in\mathcal{E}'_L} a^\omega(e)\frac{\partial}{\partial a^\omega(e)}\mathrm{Tr}(\xi(H_{\Lambda_L}(A^\omega)))\Big)$$
$$\leq \sum_{e\in\mathcal{E}'_L}\int\cdots\int C_1\varepsilon\prod_{e'\neq e} g(a^\omega(e'))da^\omega(e') \leq C_1\varepsilon|\Lambda_L|,$$

where $C_1 = 8(3\sup|g|+\sup|g'|)$. It follows from (3.4), (3.5) and this estimate:

$$\mathbb{P}(\mathrm{dist}(\sigma(H_{\Lambda_L}(A^\omega))),E)\leq C_2\varepsilon|\Lambda_L|$$

with $C_2 = C_1(F_0 - E - 2\varepsilon - 6\lambda^2)^{-1}$. Theorem 3.2 now follows if $\varepsilon < (F_0 - E_2 - 6\lambda^2)/4$. □

References

[1] Birman, M. Sh., Yafaev, D. R.: The spectral shift function. The papers of M. G. Krein and their further development. St. Petersburg Math. J. **4** (1993), 833–870.

[2] Carmona, R., Lacroix, J.: Spectral Theory of Random Schrödinger Operators. Birkhauser, Boston, MA, 1990.

[3] Combes, J. M., Hislop, P. D., Nakamura, S.: The L^p-theory of the spectral shift function, the Wegner estimate, and the integrated density of states for some random operators. Comm. Math. Phys. **218** (2001), 113–130.

[4] von Dreifus, H., Klein, A.: A new proof of localization in the Anderson tight binding model. Comm. Math. Phys. **124** (1989), 285–299.

[5] Pastur, L., Figotin, A.: Spectra of Random and Almost-Periodic Operators. Springer-Verlag, Berlin, 1992.

[6] Fröhlich, J., Spencer, T.: Absence of diffusion in the Anderson tight binding model for large disorder or low energy. Comm. Math. Phys. **88** (1983), 151–184.

[7] Halperin, B. I., Lee, P. A., Read, N.: Theory of half-filled Landau level. Phys. Rev. B **47** (1993), 7312–7343.

[8] Hislop, P. D., Klopp, F.: The integrated density of states for some random operators with nonsign definite potentials. Preprint 2001 (`mp_arc` 01-139), To appear in J. Functional Analysis.

[9] Kawarabayashi, T., Otsuki, T.: Diffusion of electrons in random magnetic fields. Phys. Rev. B **51** (1995), 10897–10904.

[10] Kirsch, W., Martinelli, F.: Large deviations and Lifshitz singularity of the integrated density of states of random Hamiltonians. Comm. Math. Phys. **89** (1983), 27–40.

[11] Klopp, F.: Internal Lifshits tails for random perturbations of periodic Schrödinger operators. *Duke Math. J.*, 98(2):335–396, 1999.

[12] Klopp, F.: Lifshitz tails for random perturbations of periodic Schrödinger operators. *Proc. Ind. Acad. Sciences*, 2002.

[13] Klopp, F.: Weak disorder localization and Lifshitz tails. Technical report, Université Paris-Nord, 2001.

[14] Lee, D., K., Chalker, J. T.: Unified model of two localization problems: Electron states in spin-degenerate Landau levels and in a random magnetic field. Phys. Rev. Lett. **72** (1994), 1510–1513

[15] Nakamura, S.: Lifshitz tail for 2D discrete Schrödinger operator with random magnetic field. Ann. Henri Poincaré **1** (2000), 823–835.

[16] Nakamura, S.: Lifshitz tail for Schrödinger operator with random magnetic field. Comm. Math. Phys. **214** (2000), 565–572.

[17] Sheng, D. N., Weng, Z. Y.: Delocalization of electrons in a random magnetic field. Phys. Rev. Lett. **75** (1995), 2388–2391.

[18] Stollmann, P.: Caught by Disorder. Bound States in Random Media. Birkhauser, Boston, MA, 2001.

[19] Simon, B.: Lifshitz tails for the Anderson model. *Journal of Statistical Physics*, 38:65–76, 1985.

[20] Ueki, N.: Simple examples of Lifschitz tails in Gaussian random magnetic fields. Ann. Henri Poincaré **1** (2000), 473–498.

[21] Ueki, N.: Seminar talk at RIMS, 2001.