PLANE-LIKE MINIMIZERS IN PERIODIC MEDIA

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Abstract. We show that given an elliptic integrand $\mathcal{J}$ in $\mathbb{R}^d$ which is periodic under integer translations, given any plane in $\mathbb{R}^d$, there is at least one minimizer of $\mathcal{J}$ which remains at a bounded distance from this plane. This distance can be bounded uniformly on the planes. We also show that, when folded back to $\mathbb{R}^d/\mathbb{Z}^d$ the minimizers we construct give rise to a lamination. One particular case of these results is minimal surfaces for metrics invariant under integer translations.

The same results hold for other functionals that involve volume terms (small and average zero). In such a case the minimizers satisfy the prescribed mean curvature equation. A further generalization allows to formulate and prove similar results in other manifolds than the torus provided that their fundamental group and universal cover satify some hypothesis.

1. Introduction

The main goal of this paper is to consider minimizers of periodic variational problems (elliptic integrals plus a small volume term) on sets of finite perimeter in $\mathbb{R}^d$ (and other manifolds). These problems include as a particular case the problem of finding hypersurfaces of codimension one of minimal area. (Indeed, for the case of minimal area, several of our lemmas simplify considerably and can be found directly in the references we give.)

We show, under very general circumstances, that there are plane-like (i.e. that stay at a finite distance of a plane, this distance is bounded a priori by properties of the metric and independently of the plane) minimizers along every direction. These minimizers also enjoy other geometric properties.

More precisely, a particular case of our results (see Theorem 4.1 formulated later) is:

Theorem 1.1. *Let g be a C^2 strictly positive metric in $\mathbb{R}^d$ invariant under integer translations. Then, we can find a number M depending only on the oscillation properties of the metric such that, for every $d-1$ dimensional hyperplane Π we can find a minimal surface Σ such that $d(\Sigma, \Pi) \le M$.*

We will also establish certain other properties of the minimizer. Notably, if we consider the problem as a problem in $\mathbb{T}^d$ using the quotient under integer translations, the hypersurfaces Σ produced in Theorem 1.1 give rise to a lamination by minimal surfaces. We also show that the surfaces produced

The work of both authors was supported by NSF grants.

are quasiperiodic (i.e. approximated in an appropriate sense, which we will detail, by periodic surfaces.)

The method we present can accommodate functionals that contain volume terms provided that they are small enough and of average zero over the unit cell. This means that the mean curvature of the surface is prescribed as a function of the point. (See Section 11.1.)

We also obtain similar results (see Section 11.2) in manifolds M in place of $\mathbb{R}^d$, when M is the universal cover of a manifold with a residually finite fundamental group (this is a mild hypothesis and is satisfied automatically by manifolds with metrics with negative curvature as well as many other manifolds.) Laminations and, specially, foliations by minimal surfaces have been studied for a long time. (Recent references referring to previous work are [Gro91b] and [Gro91a].) Many results are known on the question on when a foliation can be made a foliation by minimal leaves for an appropriately chosen metric.

The problem of constructing quasiperiodic minimal surfaces for periodic metrics was proposed in [Mos86] which proved similar results for minimizers of a similar – but different – class of those that we consider. In that paper, the problem of generalizing the results to manifolds which are universal covers of manifolds with a non-abelian fundamental group was posed.

Even if we do not deal with currents, only with boundaries of sets, the related problem of minimizing currents has been studied. For $d = 3$ [Ban90a] studied the minimizing currents on the torus. The characterization of minimizers of currents in $\mathbb{T}^d$ was announced in [Ban96]. This reference also poses the question of characterizing the minimizers in manifolds of negative curvature. We also call attention to [Aue97] which studies the problem for non-intersecting currents.

In [Mos86] it was also shown that its results on variational problems could be considered as extensions of results in Aubry-Mather theory and that, indeed, they implied the results of Aubry and Mather (see [Mat82]) on existence of quasi-periodic orbits. (We refer to [MF94] for a recent survey on developments in Aubry-Mather theory beyond the existence of quasi-periodic orbits.)

The same class of functionals as in [Mos86] has been considered in [Ban89]. In [dlL00], one can find another proof of the results of [Mos86] and extensions to ΨDE's and to other manifolds.

We also note that, for metrics which are close to flat metric in an smooth enough topology and for planes that satisfy Diophantine properties, in [Mos88], extending results of [Koz83] it was shown that the laminations induced on the torus are indeed foliations by smooth minimal surfaces and have smooth holonomy maps. In [Ban87] one can find examples showing that these foliations may have gaps. In [Gro91b] we find the result that in small deformations along a family of metrics of constant negative curvature, the foliations persist.

The results we prove here also have a close relation to the classical results of [Mor24], [Hed32] on geodesic in manifolds of dimension 2. Note that geodesics are surfaces of codimension 1 in manifolds of dimension 2. Hence, our results, give an independent proof of the the results on existence on asymptotically linear geodesics. For a modern survey relating the classical theory of minimizing geodesics to Aubry-Mather theory, we refer to [Ban88] and [Ban90b].

Methods of geometric measure theory have been used in [Ban99] to study minimal one-dimensional currents in higher dimensions. One dimensional currents can be considered a generalization of geodesics. It has been known since the work of Hedlund that for $\mathbb{T}^d$ $d \geq 3$ there are no minimizing geodesics in most of the homotopy classes. Similar phenomena happen for objects of codimension greater that one. In this paper, we show that if we consider objects of codimension 1, it is possible to obtain minimizers that are asymptotic to a plane in every direction.

The method used in this paper is based on the direct methods of the calculus of variations. We will work in the class of sets of finite perimeter also known as Caccioppoli sets and consider functionals based on elliptic integrals. (A summary of the results of the theory of Cacciopoli sets is collected in Appendix 12.) An important case of elliptic integrands is the area integral, hence we obtain minimal surfaces as a particular case of our results. Later, we will also show how to incorporate volume terms.

First we show that the integrands reach their minimum when we consider sets restricted to lie on bounded sets and to contain another one. (This uses lower semicontinuity of the functional as well as some compactness and coerciveness.) Moreover, due to some subadditivity properties, it is possible to define an *infimal minimizer* which is contained in all the other minimizers. These infimal minimizers satisfy several monotonicity properties which allow us to construct them also in sets that are the limit of increasing compact sets.

Another crucial result is that minimizers satisfy density estimates which imply that in all balls centered in points in the boundary we can find other balls of comparable size which are included in the set or in its complement.

In the case that the plane has a rational normal vector, by applying the previous results to the set obtained identifying points under integer translations that preserve the plane, we can construct infimal minimizer sets which are periodic.

The periodic minimizers thus constructed satisfy a geometric property (quite analogous to the property called Birkhoff in Aubry-Mather theory). This property, roughly, says that the boundary of the infimal minimizer can not cross its integer translates (this would contradict that the infimal minimizer is contained in all the other minimizers).

Using the density estimates and the Birkhoff property, we can show that the boundary of the infimal minimizer is contained in a band whose width is independent of the direction of the plane.

Using some mild regularity of the minimizers (e.g. that the BV norm of the intersection with unit cubes is uniformly bounded) and the fact that they are contained in bands of uniform width, we can pass to the limit of a sequence of planes and, therefore, consider planes with a general normal vector, establishing in this way the desired result, Theorem 4.1.

Some of these steps are somewhat easier for the case that the functional considered is the area, but we hope that the slight complication is worth the extra generality obtained.

We devote a short section to the most elementary results about the study of the average value of the functional as a function of the direction of the minimizer. This is analogous to quantities studied in Aubry-Mather theory and in the theory of homogenization.

In a final section, we discuss some generalizations to other functionals including terms with volume (so that the stationary points of the functional, rather than than being minimal hypersurfaces satisfy the prescribed mean curvature equations) as well as to manifolds other than $\mathbb{R}^d$.

We have collected in an appendix some standard results in the theory of Caccioppoli sets. This will serve to set our notation and a quick reference for some of the results.

Another appendix collects the proof of some lemmas which we use and which are, nevertheless, quite similar to results in the literature.

2. Elliptic integrands.

In this paper, we will be concerned with functionals defined on a certain class $\mathcal{A}$ of closed Caccioppoli sets. We refer to Appendix 12 for the definition of Caccioppoli sets, their boundary elements and other standard notations.

The functionals we will be considering are of the form:

$$\mathcal{J}(E) = \int F(x,\nu)\, d|\omega_E| \tag{1}$$

where ν denotes the inner unit normal to the set E (defined in (45)) $d\omega_E$ is the boundary measure (defined in (44)) and where $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^+$ will be a non-negative function that satisfies certain conditions H1-H5 formulated below.

Given a closed set B, we will define:

$$\mathcal{J}_B = \int_B F(x,\nu)\, d|\omega_E|$$

the restriction of the functional to a set.

Notice that to define the functional, it suffices to have F defined only when ν ranges over the unit sphere. Nevertheless, it is more convenient to consider F defined for all values of $\nu \in \mathbb{R}^d$ so that it satisfies certain convexity and homogeneity properties.

We will refer to functionals of the form (1) as boundary functionals. Later, we will also called them elliptic integrals when they satisfy some extra properties.

In this paper, we will be concerned with boundary functionals whose defining function F satisfies the following properties:

H1 F is homogeneous of degree 1 in its second variable ν. That is $F(x, a\nu) = aF(x, \nu)$ for all $a \in \mathbb{R}^+$, $x \in \mathbb{R}^d$, $\nu \in \mathbb{R}^d$.
H2 F is convex in ν
H2' For some $c > 0$, $F - c|\nu|$ is convex in ν
H3 F is continuous in x, ν.
H4 There are $0 < \lambda \le \Lambda$ such that

$$0 < \lambda \le F(x,\nu) \le \Lambda \qquad \forall x \in \mathbb{R}^d, |\nu| = 1 \tag{2}$$

Later on, we will add another property that involves periodicity.

Remark 2.1. Functionals of the form (1) and satisfying hypothesis H1-H2 defined on currents (more general objects than boundaries of Caccioppoli sets since they include objects of higher codimension and less regularity) have been considered in geometric measure theory. See e.g. [Fed69] Ch. 5.

Indeed, many of the properties that we will use for Cacciopoli have analogues for currents. In particular in [Fed69] Ch. 5 one can find compactness, lower semicontinuity and existence of minimizers. A very useful dictionary between properties of Cacciopoli sets and properties of currents can be found in [Giu73].

We have found the theory of Caccioppoli sets well suited to our purposes since the formulation of monotonicity under inclusion and comparisons, which play an important role in our argument are quite clear using sets. They also allow us to avoid topological issues such as orientability, etc. □

Remark 2.2. Clearly, property H2' implies property H2. It is shown in [Fed69] p. 517 that H2' in the case we are considering is equivalent to the property of ellipticity, which can be defined for currents of any dimension.

Most of this paper will use property H2, but the regularity theory requires property H2'. We will refer to functionals satisfying H2 or H2' as elliptic integrands. In this paper, we will use mainly H2, so this will not lead to confusion. In the literature, one finds the name *semielliptic* applied to functionals that satisfy H2 and *elliptic* is reserved for H2'. □

An important example of a functional of the form (1) and satisfying all the assumptions is the perimeter (or area) itself which corresponds to taking $F(x, \nu) = |\nu|$. This is valid even if we take any Riemannian metric in $\mathbb{R}^d$ provided that in Definitions 12.1 and 12.9 we take as the divergence the one corresponding to the metric we are considering.

The area with respect to an arbitrary metric can also be incorporated in a functional of the form (1) satisfying hypotheses H1-H4 in two different ways.

One is to observe that if we define the divergence with respect to the metric (and the volume form induced by the metric) and, therefore the ω_E perimeter, we can still consider $F(x, \nu) = |\nu|$ to obtain the area formula.

Another, more elementary way is to observe that the area functional with respect to a Riemannian metric can be written using the standard perimeter and taking $F(x, \nu) = B(x)|G(x)\nu|$ where $B(x)$ is an scalar function and $G(x)$ is a positive definite linear function. Of course, in a general manifold, there may not be a standard perimeter.

In the following, we will consider a formulation that encompasses both. That is, we will assume that there is an underlying metric with respect to which we define divergence – and therefore, perimeter – and also a functional $\mathcal{J}$ of the form (1) with F satisfying H1-H4. This formulation is well suited to the discussions of problems in a general manifold.

Remark 2.3. The functionals considered in [Mos86] and in [Ban89] do not fall in the class of functionals considered here. They are integrals of the form

$$\int_{\mathbb{R}^{d-1}} G(x, u(x), Du(x))\, dx.$$

We can see that if we consider them as functionals on the surface $(x, u(x))$, H2 and the lower bound in H4 are not satisfied, so that not all the results of this paper apply directly – some do –. Nevertheless, we can treat them with the methods of this paper.

With the same techniques (see also [CC95b]) we can construct minimizers of periodic Landau-Ginzburg functionals

$$F(x, \nabla u) + W(x, u)$$

with plane-like level surfaces. (Here $F(x, \nabla u)$ is uniformly elliptic and W is a double –well potential.)

We plan to come back to these issues in future work. □

The main goal of this paper is to consider situations in which functionals of the form (1) satisfy some periodicity properties. So, we will assume:

- H5 The functional is periodic: This amounts to
  - H5.1 The metric on $\mathbb{R}^d$ used to define the volume, divergence (and, hence, perimeter, bounded variation etc.) is a C^2 strictly positive metric periodic under translations in $\mathbb{Z}^d$.
  - H5.2 F is periodic in $\mathbb{R}^d$. That is:

$$F(x + e, \nu) = F(x, \nu) \qquad \forall e \in \mathbb{Z}^d \tag{3}$$

Of course, when considering problems in $\mathbb{R}^d$, all the statements can be referred to the standard metric.

3. Minimizers.

In this section we will precise different notions of minimizers that we will be considering. The definitions are quite standard in the calculus of variations, but we recall them since we need to use them more or less at the same time.

Definition 3.1. *Given closed sets $S_- \subset S_+$, we will denote by $\mathcal{A}_{S_+,S_-}$ (but omit the subindices S_-, S_+ if they are clear from the context) the class of closed Caccioppoli sets E such that*

$$\mathcal{A}_{S_+,S_-} = \{E \text{ Caccioppoli } | S_- \subset E \subset S_+\} \tag{4}$$

We will refer to the condition (4) as constraints.

We say that a Caccioppoli set E_0 is an absolute minimizer of the functional $\mathcal{J}$ in a class $\mathcal{A}$ when $E_0 \in \mathcal{A}$ and

$$\mathcal{J}(E_0) = \inf_{E \in \mathcal{A}} \mathcal{J}(E).$$

We say that a Caccioppoli set E_0 is a an unconstrained minimizer in the class $\mathcal{A}_{S_+,S_-}$ if it is a minimizer and we can find closed sets

$$R_- \subset \text{Int}(S_-) \subset S_+ \subset \text{Int}(R_+)$$

such that.

$$\mathcal{J}(E_0) = \inf_{E \in \mathcal{A}_{S_+,S_-}} \mathcal{J}(E). = \inf_{E \in \mathcal{A}_{R_+,R_-}} \mathcal{J}(E).$$

We say that a closed Caccioppoli set $E \subset \mathbb{R}^d$ is a Class A minimizer when, for all compact sets B, we have that all L such that $L \cup (\mathbb{R}^d - B) = E - \cup(\mathbb{R}^d - B)$ satisfy

$$J_B(E) \le J_B(L)$$

In our applications the sets R_-, R_+, S_-, S_+ will be half spaces of the form $\{x \in \mathbb{R}^d \mid x \cdot \gamma \le \lambda\}$. Most of the time, the sets we will consider will also satisfy some periodicity conditions. Hence, we will be dealing with subsets of $\mathbb{T}^{d-1} \times \mathbb{R}$ which contain $\mathbb{T}^{d-1} \times [\infty, \lambda]$ and which are contained in $\mathbb{T}^{d-1} \times [\infty, \lambda']$.

It will be useful to keep these examples in mind as a motivation for several of our constructions, even if we mention them in greater generality.

We emphasize that in the definition above, there is no need to have S_-, S_+ compact. Nevertheless, we will sometimes require that $S_+ - S_-$ is compact.

Note also that we are not requiring that the sets are subsets of $\mathbb{R}^d$, unless specifically stated later. The examples above show it is quite useful to consider these arguments in a manifold such as the torus.

Remark 3.2. The definition of unconstrained minimizer is what is needed for the study of regularity. (If one makes sufficiently small local modifications, the functional cannot decrease.)

The definition of Class A requires that you cannot make compact (but arbitrarily large) modifications that decrease the functional. This is a global property. It was introduced in the case of geodesics in [Mor24]. We have taken the name from there, even if it appears under other names in other places. □

4. Statement of results

The main result of this paper is the following:

Theorem 4.1. *Let $\mathcal{J}$ be a functional of the form* (1) *satisfying H1, H2, H3, H4, H5. Assume $\mathcal{J}(E) < \infty$ for some $E \in \mathcal{A}$.*

Then, for every $\omega \in \mathbb{R}^d$ we can find a class A minimizing set E_ω in such a way that:

A) *For some M which is independent of ω and depends only on λ, Λ in* (2) *and on d, we have:*
$$\partial E_\omega \subset \{x \in \mathbb{R}^d \quad | \quad |x \cdot \omega| \le M|\omega|\}.$$
B) *E_ω is a class A minimizer for $\mathcal{J}$.*
C) *∂E_ω is quasiperiodic.*
D) *$\partial E_\omega / \mathbb{Z}^d$ is a lamination of $\mathbb{T}^d \equiv \mathbb{R}^d/\mathbb{Z}^d$.*

The meaning of quasiperiodic in C) above is that the sets can be approximated in the local BV sense (See Definition 12.9.) by sets periodic under integer translations.

A particular case of the result, using the functional $F(x, \nu) = |\nu|$ as explained above, is the existence of minimal surfaces for general periodic Riemann metrics in $\mathbb{R}^d$. In that case, the proofs of many of the technical results can be simplified.

In case that $\mathcal{J}$ satisfies H2', there is a rich regularity theory that allows us to conclude that the minimizer is a more regular surface.

For this local part of the argument, we can use the regularity results of standard geometric measure theory.

The regularity results for these problems consist in showing that: a) In a neighborhood of any point of the reduced boundary, (see Definition 12.6,) the map is a graph of an smooth function and that b) the Hausdorff dimension of the singularities (the boundary minus the reduced boundary) is not too big.

Once one proves that the functions mentioned in a) are Lipschitz, it is possible to show that it satisfies the minimal surface equation and, then use the recent developments on fully non-linear equations to obtain further regularity.

Combining [SSA77], [SS82] which establish Lipschitz regularity (see also [DS00]) and upper bounds on the dimension of the singular set [CW93b], [CW93a], which develop a PDE theory starting from Lipschitz, one obtains:

Theorem 4.2. *In the conditions of Theorem 4.1. If the functional, moreover satisfies that H2' and is C^2, then, the $d - 3$ Hausdorff measure of the singular set $\partial E - \partial^* E$ is 0.*

If F is sufficiently C^3 close to being a metric, then $\partial E - \partial^* E$ has finite $d - 8 + \delta$ Hausdorff measure.

If F is C^r $r \geq 3$, then, in the neighborhood of any point of $\partial^* E$, ∂E is a C^{r-1} hypersurface.

A simplified version of the geometric measure theory arguments for minimal surfaces can be found in [Giu84], [CC93], [CC95a].

We refer to Section 11 for results that extend the Theorem 4.1 but are proved by similar methods. We will show how one can incorporate volume terms in the minimization problem and also prove results for other manifolds which are not $\mathbb{R}^d$. In Section 11.1, we will mention other regularity theories for minimizers.

5. Existence of minimizers.

In this section, we will show that, under very general circumstances, functionals of the type we are considering reach the minimum. In subsequent sections, we will establish regularity and geometric properties of minimizers.

The key result to prove the existence of minimizers is a lower semi-continuity of the functional in a topology that makes minimizing sequences precompact. In the case that $\mathcal{J}$ is the area, this argument can be found in [Giu84] p. 17.

Lemma 5.1. *Let $\mathcal{J}$ be a functional of the form (1) and satisfying H1 H2 H3.*

Let $\{E_j\}_{j \in \mathbb{N}}$ be a sequence of Caccioppoli sets $E_j \xrightarrow{L^1} E$. $\mathcal{O}$ an open domain with Lipschitz boundary.

Then,

$$\mathcal{J}_{\mathcal{O}}(E) \leq \liminf \mathcal{J}_{\mathcal{O}}(E_j) \tag{5}$$

This result is an extension of the lower semi-continuity of the perimeter for Caccioppoli sets.

It is quite similar to arguments in the literature. (See e.g. [Fed69] Theorem 5.1.5) We present a proof in Section 13.1.

As an easy corollary of Lemma 5.1 we obtain

Lemma 5.2. *Let $\mathcal{A}$ be a class of Caccioppoli sets such that it is closed under L^1 limits (i.e. $E_j \in \mathcal{A}, E_j \xrightarrow{L^1} E \Rightarrow E \in \mathcal{A}$). Let $\mathcal{J}$ a functional as in (1) satisfying H1,H2,H3,H4.*

Then, there is a set $E_0 \in \mathcal{A}$ such that $\mathcal{J}(E_0) = \inf_{E \in \mathcal{A}} \mathcal{J}(E)$.

An example of classes that are closed under L^1 limits which play a role in our applications is $\mathcal{A} = \{E | S_1 \subset E \subset S_2\}$ with S_1, S_2 closed sets of non-empty interior, with $S_+ - S_-$ compact.

Proof. The proof is quite standard of arguments in the calculus of variations.

Choose a sequence E_j such that $\lim \mathcal{J}(E_j) = \inf_{E \in \mathcal{A}} \mathcal{J}(E)$.

We note that H4 implies that $\mathrm{Per}(E) \le (\lambda)^{-1}\mathcal{J}(E)$. Hence, the minimizing sequence, has uniformly bounded perimeter. By the compactness properties in Lemma 12.10 we can pass to a subsequence such that $E_j \xrightarrow{L^1} E_0$ for some E_0.

By the closedness of $\mathcal{A}$ under L^1 limits, we have $E_0 \in \mathcal{A}$ and, by Lemma 5.1, we have that $\mathcal{J}(E_0) \le \inf_{E\in\mathcal{A}} \mathcal{J}(E)$.

□

We call attention to the fact that the previous proof does not use that $S_-$is compact. We use only that $S_+ - S_-$ is compact.

One elementary property of minimizers is:

Proposition 5.3. *If E is a minimizer in a class as above*
If S_-,S_+ connected, then E is connected, $S_+ - \mathrm{Int}(E)$ is connected.

Proof. One component E_1 of E has to contain S_-. This component will be an admissible set. If there was another component of E, then, we would have $\mathcal{J}(E_1) < \mathcal{J}(E)$ in contradiction with E being a minimizer.

The same argument works to show that $S_+ - \mathrm{Int}(E)$ is connected. □

6. The infimal minimizer.

The goal of this section is to show that among all the minimizers we have constructed in the previous section there is a particularly interesting one which is contained in all the others. As we will see later, this infimal minimizer enjoys interesting geometric properties.

The main technical result of this section is:

Proposition 6.1. *Let E, L be finite perimeter sets. Then*

$$\mathcal{J}(E) + \mathcal{J}(L) \ge \mathcal{J}(E\cap L) + \mathcal{J}(E\cup L) \tag{6}$$

We postpone the proof of Proposition 6.1 till Subsection 13.2 but we develop some consequences here.

As an easy corollary of Proposition 6.1, we have:

Proposition 6.2. *Assume that, with the notations in* (4)

- *E is a minimizer in the class $\mathcal{A}_{S_+,S_-}$,*
- *L is a minimizer in the class $\mathcal{A}_{T_+,T_-}$,*
- *$T_- \subset S_-$, $T_+ \subset S_+$.*

Then

- *$E\cup L$ is a minimizer in the class $\mathcal{A}_{S_+,S_-}$*
- *$E\cap L$ is a minimizer in the class $\mathcal{A}_{T_+,T_-}$*

In particular, if E and L are minimizers in a class $\mathcal{A}_{S_+,S_-}$ then, we have that $E\cup L, E\cap L$ are minimizers in the same class.

Proof. Note that $E \cup L \in \mathcal{A}_{T_+,T_-}$, $E \cap L \in \mathcal{A}_{S_+,S_-}$, Hence, since E, L are minimizers $\mathcal{J}(E) \le \mathcal{J}(E \cap L)$, $\mathcal{J}(L) \le \mathcal{J}(E \cup L)$. Together with (6), we obtain $\mathcal{J}(E) = \mathcal{J}(E \cap L)$, $\mathcal{J}(L) = \mathcal{J}(E \cup L)$. □

We want to obtain two consequences of this result.

Lemma 6.3. *Let $\mathcal{A}$ be a class of sets closed under countable intersection. Then the sets*

$$
\begin{aligned}
E_{*,\mathcal{A}} &= \bigcap_{E \text{ minimizer}} E \\
E^{*,\mathcal{A}} &= \bigcup_{E \text{ minimizer}} E
\end{aligned}
\tag{7}
$$

are minimizers in the class $\mathcal{A}$.

We will refer to E_* as the infimal minimizer and to E^* as the maximal minimizer. If the class $\mathcal{A}$ is of the form in (4), we will denote the sets as E_{*,S_+,S_-} and, if there is no risk of confusion because the class can be inferred from the context, we will omit it from the notation.

Note that E_* (resp. E^*) can be characterized as a minimizer that is contained (resp. contains) all the minimizers. They are clearly unique once the class is specified.

Remark 6.4. One important consequence of the uniqueness of the infimal minimizer is that it has all the symmetries of the functional and the class (i.e. there is no symmetry breaking.) This can be seen because if the functional and the class are invariant under a certain transformation $\mathcal{T}$, $\mathcal{T}E_{*,\mathcal{A}}$ is a minimizer and should contain $E_{*,\mathcal{A}}$.

Hence, when we consider a problems invariant under integer translations in $\mathbb{R}^d$, subject to periodicity conditions, the infimal minimizer will have the period of the functional even if we consider it as defined in sets whose period is a multiple of the original period.

This will be used in the proof of Theorem 4.1 (See Proposition 8.9.) and also have several other consequences (See Proposition 10.2.) □

Proof. Since on the set of minimizers one can bound the perimeter uniformly, the set of minimizers is pre-compact in $L^1(\Gamma)$, (actually it is compact since Lemma 5.1 shows it is L^1 closed) which is separable and, therefore, it suffices to take a countable intersection. Note that if $\tilde{E}_n = \bigcap_n^N E_n$, $\chi_{\tilde{E}_n} = \Pi_n^N \chi_{E_n}$ is a decreasing sequence and it converges in L^1.

By (6), we have that $\mathcal{J}(E_1 \cap E_2) \le \mathcal{J}(E_1) + \mathcal{J}(E_2) - \mathcal{J}(E_1 \cup E_2)$. If $J(E_1) = \mathcal{J}(E_2) = J^*$, noting that $E_1 \cup E_2$ is an admissible set and that, therefore, $\mathcal{J}(E_1 \cup E_2) \ge J^* \equiv \inf_{E \in \mathcal{A}} \mathcal{J}(E)$, we have that $\mathcal{J}(E_1 \cup E_2) = J^*$. Therefore, any finite intersection of minimizers is also a minimizer.

Applying again Lemma 5.1, we obtain the the intersection is a minimizer. □

Another consequence is that the infimal and maximal minimizers in a class have easy monotonicity properties with respect to the constraints.

Lemma 6.5. *Let $\mathcal{A}_{S_+,S_-}$, $\mathcal{A}_{T_+,T_-}$, denote classes of Caccioppoli sets as in* (4).

If $T_- \subset S_-$, $T_+ \subset S_+$, then the infimal and maximal minimizers constructed in Lemma 6.3 satisfy:

$$\begin{aligned} E_{*,S_+,S_-} &\supset E_{*,T_+,T_-} \\ E^{*,S_+,S_-} &\supset E^{*,T_+,T_-} \end{aligned} \tag{8}$$

Proof. Note that by Proposition 6.2, $E_{*,S_+,S_-} \cap E_{*,T_+,T_-}$ is a minimizer in $\mathcal{A}_{T_+,T_-}$. Hence, it contains E_{*,T_+,T_-}. The other result is proved in the same way. □

7. Density properties of minimizers.

In this section we establish some properties of minimizers of elliptic integrals We show that the minimizers have to be rather dense near points of their boundary.

Heuristically, these estimates show that the minimizers do not have thin needles, which is reasonable since needles are penalized by elliptic functionals. Similar arguments are basic for the classical proofs of regularity of minimizing currents (See e.g. [Fed69] p. 523.)

In our case, the main consequence that we will draw from density estimates is that there are reasonably big sets both inside the set and in the complement, which we will later use to establish other properties.

We denote by $B_r(x_0)$ the ball centered in x with radius r.

Lemma 7.1. *Let $\mathcal{A}$ be a class of Caccioppoli sets.*

Let E_0 be a minimizer of a functional $\mathcal{J}$ of the form (1) *among the sets in the class $\mathcal{A}$. Assume that $x \in \partial^* E_0$ and that*

$$\begin{aligned} &E_0 - B_\rho(x) \in \mathcal{A}, 0 \le \rho \le r \\ &(\text{resp.} E_0 \cup B_\rho(x) \in \mathcal{A}, 0 \le \rho \le r) \end{aligned} \tag{9}$$

Then there exists a constant C, depending only on λ, Λ defined in (2) *and in d such that:*

$$\begin{aligned} &|E_0 \cap B_r(x)| \ge C r^d \\ &(\text{resp.}) |(\mathbb{R}^d - E_0) \cap B_r(x)| \ge C r^d) \end{aligned} \tag{10}$$

Moreover,

$$|\omega_{E_0}| [(\partial E_0) \cap B_r(x)] \le C r^{d-1}. \tag{11}$$

Remark 7.2. Condition (9) roughly says that the point x is away from the conditions that define the class $\mathcal{A}$. As an example that will be relevant later if $\mathcal{A} = \{E | S_- \subset E \subset S_+\}$, then all the points $x \in (S_+ - S_-)$ such that $d(x, S_-), d(x, \mathbb{R}^d - S_+) \ge r$ satisfy assumption (9). □

Proof .

We will denote by C all constants that depend only on λ, Λ– defined in (2) – and on d.

Since E_0 is a minimizer in a class that satisfies (9), we have

$$\mathcal{J}(E_0) \le \mathcal{J}(E_0 - B_r(x)).$$

That is,

$$\int_{(\partial E_0)\cap B_r(x)} F(x,\nu) d|\omega_E| \le \int_{E_0 \cap \partial(B_r(x))} F(x,\nu) d|\omega_{B_r(x)}| \tag{12}$$

By assumption H.4, (12) yields:

$$\begin{aligned} |\omega_{E_0}|[(\partial E_0)\cap B_r(x)] &\le (\Lambda/\lambda)\cdot|\omega_{B_r(x)}|(E_0\cap\partial(B_r(x))) \\ &\le (\Lambda/\lambda)\cdot|\omega_{B_r(x)}|(\partial(B_r(x))) \\ &= Cr^{d-1} \end{aligned} \tag{13}$$

The last bound establishes (11).

By the isoperimetric inequality, using the first inequality in (13), we have:

(14)

$$\begin{aligned} |E_0 \cap B_r(x)| &\le C_d \operatorname{Per}(E_0\cap B_r(x))^{d/d-1} \\ &\le C_d \left[|\omega_{E_0}|((\partial E_0)\cap B_r(x)) + |\omega_{B_r(x)}|(E_0\cap\partial(B_r(x))\right]^{d/(d-1)} \\ &\le C_d(1+\Lambda/\lambda)^{d/(d-1)}\,|\omega_{B_r(x)}|(E_0\cap\partial(B_r(x)))^{d/(d-1)} \end{aligned}$$

We denote by $V(r) = |E_0 \cap B_r(x)|$, We see that V is clearly a non-decreasing function and, therefore differentiable almost everywhere. Moreover, by the coarea formula for Bounded Variation functions (see [EG92] p. 185), we have

$$V'(r) = |\omega_{B_r(x)}|(\partial E_0 \cap \partial(B_r(x))). \tag{15}$$

We also note that $V(r)$ is non-negative. Indeed, by Definition 12.6 we see that $V(r) > 0$ for $r > 0$.

Hence for $r > 0$, (14) reads:

$$V'(r)V^{-\frac{d-1}{d}}(r) \ge C \tag{16}$$

where $C = C_d(1+\Lambda/\lambda)^{d/(d-1)}$.

Integrating (16) leads to (10).

The result for the complement is proved in the same way.

□

Now, we prove that given a ball centered at a point of the reduced boundary and which is contained inside the constraints, the minimizer has to exclude some balls inside it.

Lemma 7.3. *There exists a $0 < \delta$ depending only on λ, Λ, d such that the following happens.*

Let $\mathcal{A}$ be a class of sets, E_0 be a minimizer of $\mathcal{J}$ as in (1).

Let $x_0 \in \partial^* E_0$ and assume that the class $\mathcal{A}$ is such that, for some $r > 0$, $1/2 > \delta_0 > 0$, we have:

(17)
$$E \cup B_\rho(x) \quad \text{and} \quad E - B_\rho(x) \in \mathcal{A} \quad \text{whenever} \quad E \in \mathcal{A}, x \in B_r(x_0), \rho \le \delta_0 r$$

Then, we can find $x_1, x_2 \in B_r(x)$ such that

$$B_{\delta r}(x_1) \subset E_0 \cap B_{\delta r}(x_0) \tag{18}$$

$$B_{\delta r}(x_2) \subset (\mathbb{R}^d - E_0) \cap B_{\delta r}(x_0) \tag{19}$$

The meaning of condition (17) is very similar to that of (9). See Remark 7.2. Roughly, we want that the ball $B_r(x_0)$ is away from the constraints.

Proof. Given the hypothesis (17), we can apply Lemma 7.1 for all points $x \in \partial^* E_0$.

For all $x \in \partial^* E_0$ we have from the isoperimetric inequality (since both $E \cap B_\rho$ and $(\mathbb{R}^d - E) \cap B_\rho$ have positive density)

$$|\omega_E|[\partial E \cap B_{\delta r}(x)] \ge C(\delta r)^{d-1}. \tag{20}$$

Choose $\delta_0 > \delta > 0$.

By Vitali's theorem, we can extract a countable sequence $\{y_k\}_{k\in\mathbb{N}} \subset \partial^* E_0$ of centers in such a way that $\{B_{(1/5)\delta r}(y_k)\}_{k\in\mathbb{N}}$ is a disjoint collection of balls and that $\partial^* E_0 \subset \cup_{k\in\mathbb{N}} B_{\delta r}(y_k)$.

By Lemma 7.1, we have

$$|\omega_{E_0}|[\partial^* E_0 \cap B_{(1/5)\delta r}(y_k)] \ge C(\delta r)^{d-1} \tag{21}$$

Since the perimeter of E_0 is finite, we deduce that the collection of balls is finite. Indeed, taking into account that by (11) we have that $|\omega_{E_0}| \le C r^{d-1}$ we have that the number of the balls can be bounded by $C\delta^{-(d-1)}$.

Therefore, we have shown that $C\delta^{-d-1}$ balls $B_{\delta r}(y_k)$ cover $\partial E_0 \cap B_r(x)$.

On the other hand, we can cover $B_r(x)$ with a collection $\mathcal{B}$ of $C\delta^{-d}$ balls of radius δr in such a way that a point does not belong to more than $2d$ of them.

Since one of the balls $B_{\delta r}(y_k)$ cannot intersect more than $4d$ of these balls, we obtain that out of the collection $\mathcal{B}$ only $C\delta^{-(d-1)}$ intersect $\partial^* E_0$.

Since $|E_0 \cap B_r(x)| \ge C r^d$, we obtain that at least one of the balls in $\mathcal{B}$ has to be contained in $E_0 - \partial^* E_0$.

Since we can assume that $\partial^* E_0$ is dense in ∂E_0 (See Lemma :12.8), the desired result (18) is established.

To prove (19) a similar argument works using $\mathbb{R}^d - E_0$ in place of E_0.

This finishes the proof of Lemma 7.3. □

8. Periodic minimizers.

So far, none of our results have involved the assumption on periodicity for the functional and the underlying metric. In this section, we will start using this assumption.

The first goal will be to construct unconstrained minimizers when ω is a rational vector. Nevertheless, we will prove enough uniform bounds that will allow us to pass to the limit of irrational minimizes.

In this section, we will consider $\omega \in (1/N)\mathbb{Z}^d$ with $N \in \mathbb{N}$ fixed.

Given a number L we denote by $\Pi^{\omega,L}$ the plane

$$\Pi^{\omega,L} \equiv \{x \in \mathbb{R}^d \,|\, x \cdot \omega = L\}.$$

Given an interval $M = [M_-, M_+]$, we denote by $\Gamma^{\omega,M}$ the slab

$$\Gamma^{\omega,M} = \Gamma^{\omega,[M_-,M_+]} = \{x \in \mathbb{R}^d \mid x \cdot \omega \in M\} \tag{22}$$

We will consider the class $\mathcal{A}$ defined by

$$\mathcal{A}_{M_+} = \{E \quad | \, \Gamma^{\omega,[-\infty,0]} \subset E \subset \Gamma^{\omega,[-\infty,M^+]}, \quad \mathcal{T}_k E = E \quad \forall k \in N\mathbb{Z}^d, \omega{\cdot}k = 0\}$$

It will be quite important that, when ω is rational, we can recover compactness by considering an identification of the space.

Since the sets in the class $\mathcal{A}$ are periodic under integer translations we can identify them with sets in the manifold

$$\Gamma^{\omega,[-\infty,M^+]}/\approx \tag{23}$$

where $\approx$ is the equivalence relation defined by

$$x \approx y \iff x = y + k \quad \text{for some} k \in \mathbb{Z}^d, \omega \cdot k = 0. \tag{24}$$

The manifold (23) clearly is $[-\infty, M] \times \mathbb{T}^{d-1}$.

Pictorially, the identification $\approx$ has the effect of *gluing* vertical walls (with some translation applied to parallel walls) except in the direction of ω. When we supplement these periodic directions with constraints on the other direction, we obtain compact sets and all the theory developed so far applies.

The set $(S_+ - S_-)/\approx$ is just $\mathbb{T}^{d-1} \times [0.M_+]$ and, hence compact.

Hence, we can apply the arguments developed before and produce an infimal minimizer, which we will denote by E_*^{ω,M_+}.

It will turn out that the set E^ω claimed in Theorem 4.1 will be E_*^{ω,M_+} for sufficiently large M_+. This will become apparent after we develop enough properties of these sets.

In Lemmas 7.1 and 7.3 we have established local (or semilocal) properties Now, we prove a geometric property that controls the global behavior of the set. This property will also play an important role in the proof of the result for irrational frequencies.

Proposition 8.1. *Let $k \in \mathbb{Z}^d$.*
If $k \cdot \omega \le 0$, we have $\mathcal{T}_k E_*^{\omega,M_+} \subset E_*^{\omega,M_+}$.
If $k \cdot \omega \ge 0$, we have $\mathcal{T}_k E_*^{\omega,M_+} \subset E_*^{\omega,M_+}$.

Remark 8.2. This property is a generalization of the so called Birkhoff property in Aubry-Mather theory. (See [Mos86] for a more detailed comparison of similar properties for graphs and the Birkhoff property in dynamical systems.)

□

Proof.
Note that $\mathcal{T}_k E_*^{\omega,M_+}$ should be the infimal minimizer in the class $\mathcal{T}_k \mathcal{A}_{M_+}$.
Consider first the case $\omega \cdot k \le 0$. Then, we have

$$\left(\mathcal{T}_k \Gamma^{\omega,[-\infty,l]}/\approx\right) = \left(\Gamma^{\omega,[-\infty,l+k\cdot\omega]}/\approx\right) \subset \left(\Gamma^{\omega,[-\infty,l]}/\approx\right)$$

where $\approx$ was defined in (24).

Hence, applying the above with $l = 0, M_+$ we see that we can verify the hypothesis of Proposition 6.2 and, therefore that $\mathcal{T}_k E^{\omega,M_+} \subset E^{\omega,M_+}$. Which is what we wanted to show in this case.

The proof for the case $\omega \cdot k \ge 0$ is identical.

□

Theorem 4.1 for rational ω follows easily from

Lemma 8.3. *In the conditions of Theorem 4.1. Let ω be rational.*
Let $L = 3\sqrt{d}/\delta + 1$ where δ was introduced in Lemma 7.3. Then for all $a > 0$.

$$E_*^{\omega,L|\omega|+a} \subset \{x | x\cdot(\omega/|\omega|) \le L\} \tag{25}$$

In particular,

$$E_*^{\omega,L|\omega|+a} = E_*^{\omega,L|\omega|}. \tag{26}$$

That is, all the infimal minimizers are contained in a slab of width L which is independent of ω and only depends on the properties of the minimizer. In particular, when we take the upper constraint high enough it ceases to have any effect.

Out of that, it will be easy to show that $E_*^{\omega,L|\omega|}$ is unconstrained and that it is a class A minimizer.

We will prove Lemma 8.3 and that the set is a class A minimizer in the rest of this Section.

We start by noting that:

Proposition 8.4. *When $M_+ > \sqrt{d}|\omega|$ There is no slab $\Gamma^{\omega,[a,a+\sqrt{d}|\omega|]}$ for $a > 0$ contained in E_*^{ω,M_+}.*

Proof. Otherwise, we could find an integer k such that $0 < \omega\cdot k < \sqrt{d}$ and then consider the set

$$\tilde{E} \equiv \mathcal{T}_{-k}(E_*^{\omega,m_+} \cap \Gamma^{\omega,[a,a+k\cdot\omega]}) \cup (E_*^{\omega,M_+} \cap \Gamma^{\omega,[0,a]}) \tag{27}$$

(Pictorially, the set $\tilde{E}$ can be described as eliminating the slab $\Gamma^{\omega,[a,a+k\cdot\omega]}$, hence dividing $\tilde{E}$ into two pieces. Then, performing an integer translation of one of the pieces so that it fits with the other one. We see that if there is an slab of width $\sqrt{d}$ it has to contain an integer and it can be cut out.)

Since $\partial\tilde{E} - \Pi_0^\omega = \partial E_*^{\omega,M_+} - \Pi_0^\omega$, we see that

$$\mathcal{J}(\tilde{E}) = \mathcal{J}(E_*^\omega)$$

and that

$$\mathrm{dist}(\partial\tilde{E}, \Pi_0^\omega) = \mathrm{dist}(\partial E_*^\omega, \Pi_0^\omega) - k\cdot\omega < \mathrm{dist}(\partial E_*^\omega, \Pi_0^\omega).$$

These two properties are a contradiction with the fact that E_*^ω is a minimizer that contains all the other minimizers. □

An easy consequence of the Birkhoff property (introduced in Proposition 8.1) is:

Proposition 8.5. *If $\mathbb{R}^d - E_*^\omega$ contains a cube of side 2, then it contains an slab of width 1 that intersects the cube.*

Note that by the Birkhoff property, we have that if $\mathcal{C} \subset \mathbb{R}^d - E_*^\omega$ then

$$\bigcup_{k\in\mathbb{Z}^d, \omega\cdot k\le 0} \mathcal{T}_k\mathcal{C} \subset \mathbb{R}^d - E_*^\omega$$

The set in the right always contains an slab of width 1.

□

Propositions 8.4, 8.5 together with Lemma 7.3 will allow us to conclude very quickly the proof of Proposition 8.3.

By Proposition 8.4 we conclude that we have a point $x_0 \in \partial^* E_*^{\omega,M_+}$ which is at a distance smaller than $\sqrt{d}$ from Π_0^ω – the lower constraint.

Suppose that E intersects the plane $x\cdot\omega/|\omega| = 2\sqrt{d}/\delta$. Then, there are points $x_0 \in \partial E_*^\omega$ as close as we wish to that plane.

We can apply Lemma 7.3 for a ball of radius $2\sqrt{d}/\delta$ centered in x_0. It would follow that a cube of size 2 is contained in $\mathbb{R}^d - E^\omega \cap \Gamma^{-\infty,3\sqrt{d}/|\omega|}$

From Proposition 8.5 and the fact that E_*^ω is connected, we obtain Lemma 8.3.

□

Remark 8.6. There is an alternative to the last part of the argument in the proof above which does not use the fact that E_*^ω is connected.

Even if the fact that E_*^ω is connected is interesting in its own right, arguments for the proof of Lemma 8.3 which do not use that fact are useful. Indeed, in Section 11.1 we will consider a situation where connectedness does not work.

Once that we have that there is slab $\Gamma^{\omega,[a,b]}$ of width $\sqrt{d}$ contained in the complement of E_*^ω, we can consider the two sets $E_1 = E_*^\omega \cap \Gamma^{\omega,[-\infty,b]}$ $E_2 = E_*^\omega \cap \Gamma^{\omega,[b,\infty]}$. Note that $\mathcal{J}(E) = \mathcal{J}(E_1) + \mathcal{J}(E_2)$.

We can choose $k \in \mathbb{Z}^d$ in such a way that $k\cdot\omega < 0$ and $\mathcal{T}_k E_2 \subset \Gamma^{\omega,(a,\infty]}$.

We have $\mathcal{J}(E_2) = \mathcal{J}(\mathcal{T}_k(E_2))$ and, hence, $E_1 \cup \mathcal{T}_k E_2$ is a minimizer.

Then, $E_1 \cup (E_2 \cap \mathcal{T}_k E_2)$ will be a minimizer by Proposition 6.2.

Since $\inf_{x \in \mathcal{T}_k E_2} x \cdot \omega < \inf_{x \in E_2} x \cdot \omega$ we reach a contradiction with the fact that E_*^ω was contained in all the minimizers. □

Remark 8.7. We call attention that this proof only requires the validity of Lemma 7.3 for radii up to L, which is a number that depends only on the variational problem and on the manifold.

Similarly, note that the meaning of the $\sqrt{d}$ appearing in the expression of the L is just the diameter of the fundamental domain of $\mathbb{R}^d/\mathbb{T}^d$, $\mathbb{R}^d$ being the manifold that we are considering and $\mathbb{T}^d$ being the symmetry group of the variational problem.

Both parts of this remark will play a role in Section 11. □

Now, we turn our attention to proving the the set thus produced is a class A minimizer.

Since we have already shown that the upper limit is irrelevant, we will suppress it from the notation and denote the set we have produced as E_*^ω.

Proposition 8.8. *The set E_*^ω is an unconstrained minimizer.*

Proof.

Proposition 8.3 establishes that the constraint that the set has to be contained in a semiplane $\Gamma^{\omega,(-\infty,M_+]}$ irrelevant for large enough M_+.

Hence, we only have to worry about showing that the constraint requiring that the set is contained in $\Gamma^{\omega,(-\infty,0]}$ is irrelevant.

Given any positive a, we can arrange to find $k \in \mathbb{Z}^d$ such that $\Gamma^{\omega,(-\infty,a]} \subset \mathcal{T}_k E_*^\omega$.

Since the functional and the class are invariant under integer translations, $\mathcal{T}_k \tilde{E}$ is a minimizer. It is clearly unconstrained. This shows that E is unconstrained.

□

Proposition 8.9. *The set E_*^ω is a class A minimizer.*

Proof. Given a compact perturbation, we can consider E_*^ω as periodic with a period larger than the diameter of the perturbation and as a minimizer among the class of sets contained in an slab larger than the size of the perturbation.

By the uniqueness of the infimal minimizer, (see Remark 6.4) E_*^ω will still be the infimal minimizer among the sets with this large period.

Hence, $\mathcal{J}$ will be smaller when restricted to the larger period. □

If we put together Lemma 8.3 and Proposition 8.9, we have established Theorem 4.1 for the case of $\omega \in \mathbb{Q}^d$.

Note that, we have also established some geometric properties such as Proposition 8.1.

Remark 8.10. There is an alternative argument for the proof of proposition 8.8 which avoids the use of balls of size bigger than $3\sqrt{d}$ and which is clearer in the geometric situations considered in Section 11.2.

We want to show that if M_+ is large enough, then E_*^{ω,M_+} is unconstrained. We assume without loss of generality than N is a multiple of 3.

By the density properties, we note that if there is a cube $\mathcal{C}$ of side 2 that contains a point in the boundary of E_*^{ω,M_+}, then a cube $\tilde{\mathcal{C}}$ of side 3 with the same center will satisfy

$$\mathcal{J}_{\tilde{\mathcal{C}}}(E_*^{\omega,M_+}) \geq \alpha > 0 \tag{28}$$

We also note that, comparing the functional evaluated in the minimum with that evaluated in a plane, we have:

$$\mathcal{J}_{\tilde{\mathcal{C}}}(E_*^{\omega,M_+}) \leq \mathcal{J}_{\tilde{\mathcal{C}}}(\Gamma^{\omega,(-\infty,0]}) \leq AB_\omega \tag{29}$$

where A is a number that depends only on the metric and B_ω is the number of cubes of side 3 that intersect the plane $k \cdot \omega = 0$.

By comparing the inequalities, (28) and (29), we obtain that E_*^{ω,M_+} does not intesect more than $(A/\alpha)B_\omega$ cubes of size 2 so that their dilations of size 3 do not intersect.

On the other hand, we observe that the set $\Gamma^{\omega,[0,M_+]}$ contains M_+B_ω disjoint cubes $\{\tilde{C}_i\}$ of size 3.

Hence, we obtain that there is at least one $\mathcal{C}_i \subset \Gamma^{\omega,[0,M_+]}$ cube of side 2 (with the same center as one of the cubes $\tilde{C}_i$ above.)

Using the Birkhoff property, we can conclude that there is a slab that avoids the boundary and the argument proceeds as before. □

9. The limit of irrational frequencies.

The fact that the minimizers we have constructed are contained in strips of uniform width independent of the frequency makes it possible to construct minimizers along any plane.

Given a frequency $\omega \in \mathbb{R}^d$, we can construct a sequence $\{\omega^n\}_{n\in\mathbb{N}} \subset \mathbb{Q}^d$ and such that $\omega_n \to \omega$.

Let $E_*^{\omega_n}$ denote the minimizers we have constructed in the previous section.

Note that, by (11) and the bounds in (2), we have that for any ball B of radius 1 we have an upper bound

$$\mathrm{Per}(E_*^{\omega_n} \cap B) \leq C$$

with a C which depends only on the functional and is independent on the minimizer.

Hence, given any ball $B_R(0)$ using Lemma 12.10, we can extract a subsequence so that $E_*^{\omega_n} \cap B_R(0)$ converges in L^1 to a limit E_*^ω. By applying a diagonal trick, we can obtain that $E_*^{\omega_n} \xrightarrow{L^1_{\mathrm{loc}}} E_*^\omega$.

Note that, in particular, we have:

$$E^\omega_* \subset \{x \in \mathbb{R}^d \mid 0 \le x \cdot (\omega/|\omega|) \le L\}$$

As in the rational case, we define the set $E^\omega = E^\omega_* \cup \Gamma^{\omega,(-\infty,0]}$

Now, we want to argue that E^ω is a class A minimizer. The following argument is, indeed, quite standard in the theory of class A minimizers it already appears in [Mor24].

Given any set L, and any ball $B_r(x)$ we have: $\mathcal{J}_{B_r(x)}(E^{\omega_n}) \le \mathcal{J}_{B_r(x)}(L)$. Since $E^{\omega_n} \cap B_r(x) \xrightarrow{L^1} E^\omega \cap B_r(x)$, applying Lemma 5.1 we obtain that $\mathcal{J}_{B_r(x)}(E^\omega) \le \mathcal{J}_{B_r(x)}(L)$.

This shows that the resulting set is a Class A minimizer and, hence, concludes the proof of Theorem 4.1. □

10. The averaged functional

In analogy with the *average action* of Aubry-Mather theory (See [Mat90]) or with the stable norm of geodesics on surfaces or of geometric measure theory, (See also [Sen95] for a similar concept for the functionals in [Mos86].) we introduce:

$$\bar{\mathcal{J}}(\omega) = \lim_{L\to\infty} (2L)^{d-1} \mathcal{J}_{[-L,L]^d}(E^\omega), \quad |\omega| = 1 \tag{30}$$

and extend by homogeneity of degree 1.

$$\bar{\mathcal{J}}(\omega) = |\omega| \bar{\mathcal{J}}(\omega)$$

In this paper, we will consider just the most elementary properties.

Proposition 10.1. *The limit in* (30) *exits.*

Proof. It is a very standard subadditivity argument.

For large enough $L \in \mathbb{N}$, we have

$$\mathcal{J}_{[-2L,2L]^d}(E^\omega) \le 2^{d-1} \mathcal{J}_{[-L,L]^d}(E^\omega) + CL^{d-2}$$

This is because if we take 2^{d-1} translates of $E^\omega \cap [-L, L]^d$ (recall that these sets are contained in slabs, which we assume have width much smaller than L.), we can form a compact perturbation of $E^\omega \cap [-2L, 2L]^d$. The boundary of this compact perturbation will consist on the boundaries of the translates of E^ω and, perhaps some components in the faces of the translated cubes intersected with the slab where all the set is contained. Since we can bound the area of the intersection of the faces with the slab by CL^{d-2} we see that the desired inequality is a consequence of the fact that the set E^ω is Class A and, hence, the functional on it, is smaller than that of a compact perturbation.

We also have

$$2^{d-1} \mathcal{J}_{[-L,L]^d}(E^\omega) \le \mathcal{J}_{[-2L,2L]^d}(E^\omega) + CL^{d-2}$$

The reason is that, we can consider translations of E^ω restricted to $[-L,L]^d$ as compact perturbations of E^ω, the functional again has to decrease.

Therefore, $(2^n L)^{-(d-1)} \mathcal{J}_{[-2^n L, 2^n L]^d}$ converges.

Using that $\mathcal{J}_{[-L,L]^d}(E^\omega)$ is increasing with L, we obtain that the limit exits along any sequence. □

Of course, the limit is reached after a finite number of duplications for rational ω. Notice that the argument above shows that it is reached uniformly for all ω of modulus 1.

Lemma 10.2. *The function $\bar{\mathcal{J}}$ is convex.*

Proof. It suffices to show that

$$\bar{\mathcal{J}}((\omega_1+\omega_2)/2) \le (1/2)\bar{\mathcal{J}}(\omega_1) + (1/2)\bar{\mathcal{J}}(\omega_2) \ .$$

which, given the homogeneity is the same

$$\bar{\mathcal{J}}(\omega_1+\omega_2) \le \bar{\mathcal{J}}(\omega_1) + \bar{\mathcal{J}}(\omega_2) \tag{31}$$

In case that ω_1, ω_2 are parallel, the proof is trivial, so we will assume they are not.

We can assume without loss of generality and just for the sake of simplifying the typography that performing a rotation we have: $\omega_1 = (\alpha_1, \beta_1, \dots, 0)$, $\omega_2 = (\alpha_2, \beta_2, 0, \dots, 0)$ Denote by $\omega_3 = (\omega_1+\omega_2)/2$. Denote by γ the smallest angle between $\omega_1, \omega_2, \omega_3$. We can always assume that $\alpha_1 + \alpha_2 = 0$, $\beta_1 + \beta_2 > 0$ so that ω_3 is the y axis.

The idea of the proof is that we basically construct a *triangle* with the planes that contain ω_i (or a large enough multiple) and cut them with planes in the perpendicular direction. The boundaries of the sets E^{ω_i} are within a bounded distance (independent of the multiple.) of these planes. We can construct a compact perturbation of E_3 by joining the boundaries of E_1 and E_2. Given the fact that these sets are contained in a uniform slab, we see that, up to an small error, the functional will be the sum of those corresponding to E_1 and E_2.

We will give some more details in the case when $\alpha_1 > 0, \beta_1 < 0, \beta_2 \le 0$. $\alpha_1 < |\beta_1|$. Similar arguments will work for all the other cases,

Given $\varepsilon > 0$ we choose L_0 so that the of $|\bar{\mathcal{J}}(\omega_i) - L^{d-1}\mathcal{J}_{[-L,L]^d} E^{(\omega_i)}| \le \varepsilon/6$ for all $L > L_0$. We will also choose L_0 so small that $3M(|\beta_1| + |\beta_2|)/\gamma L_0 \le \varepsilon/2$ where M is the constant appearing in Theorem 4.1.

We consider the set

$$E \equiv (E^{\omega_3} \cup E^{\omega_1}) \cap \mathcal{T}_k E_2^\omega$$

where k is the vector with integer components closest to $L(\beta_1, -\alpha_1)$.

given the fact that the boundaries of the sets are contained in strips of width not more than M and with slopes $-\beta_i/\alpha_i$, we have:

For $x < -M/\gamma$, the set E agrees with E^{ω_3}. For $M/\gamma < x < L|\beta_1| - M/\gamma$ it agrees with E^{ω_1}. For $L|\beta_1| + M/\gamma < x < L|\beta_1| + |\beta_2| - M/\gamma$ we see that E agrees with E^{ω_2}. Finally, for $L|\beta_1| + |\beta_2| + M/\gamma < x$, E agrees with E^{ω_3}.

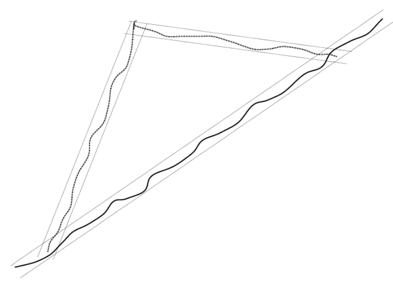

FIGURE 1. Illustration of the argument to show that $\bar{\mathcal{J}}$ is convex

We note that E agrees with E^{ω_3} outside a compact set. Hence, because E^{ω_3} is a Class A minimizer, we have:

$$\mathcal{J}_{[0,(|\beta_1|+|\beta_2|)L]^d}(E) \le \mathcal{J}_{[0,(|\beta_1|+|\beta_2|)L]^d}(E^{\omega_3})$$

We also have that

$$\mathcal{J}_{[0,(|\beta_1|+|\beta_2|)L]^d}(E) = \mathcal{J}_{[0,|\beta_1|L]^d}(E^{\omega_1}) + \mathcal{J}_{[0,|\beta_2|L]^d}(E^{\omega_2}) + R$$

where $R \le 3ML(|\beta_1| + |\beta_2|)/\gamma$.

Using the way that we had to choose ε, we see that:

$$\bar{\mathcal{J}}(\omega_1) + \bar{\mathcal{J}}(\omega_2) + \varepsilon \le \bar{\mathcal{J}}(\omega_1 + \omega_2)$$

Since ε was arbitrary, we obtain the desired result in this case.

The other cases are proved with the same idea. The only thing that changes is the explicit description of the regions where the set agrees with one of the elementary ones.

□

From this, it is not difficult to obtain:

Proposition 10.3. *The function $\bar{\mathcal{J}}$ is Lipschitz.*

Proof. By (31) we have

$$\begin{aligned} \bar{\mathcal{J}}(\omega + \Delta) &\le \bar{\mathcal{J}}(\omega) + \bar{\mathcal{J}}(\Delta) \\ \bar{\mathcal{J}}(\omega) &\le \bar{\mathcal{J}}(\omega + \Delta) + \bar{\mathcal{J}}(-\Delta) \end{aligned} \tag{32}$$

So that the desired result follows from $|\bar{\mathcal{J}}(\Delta)| \le C|\Delta|$ or, equivalently that when, $|\omega| = 1$, we have: $\bar{\mathcal{J}}(\omega) \le C$.

This follows from the fact that, if we take as a test set a semi-plane Π^ω_- , the perimeter when restricted to a cube of size L is $(2L)^{d-1}$. By hypothesis $H4$, we have that $\mathcal{J}_{[-L,L]^d}(\Pi^\omega_-) \le \Lambda \operatorname{Per}(\Pi^\omega_-)$. Since $\mathcal{J}_{[-L,L]^d}(E^\omega_*) \le \mathcal{J}_{[-L,L]^d}(\Pi^\omega_-)$ we obtain the desired result. □

The papers [Mat90], [Sen95] give characterizations of the situations of the points of differentiability of the average action defined in their context. In analogy with these results, we conjecture:

Conjecture 10.4. *The function $\bar{\mathcal{J}}(\omega)$ is differentiable at a point ω_0 if and only if there is a foliation of minimizers in $\mathbb{T}^d$.*

11. Some generalizations of the above results

11.1. **Volume terms in the functional.** The same methods we have used can be used to study functionals of the form

$$\mathcal{V}(E) = \mathcal{J}(E) - \mathcal{H}(E); \quad \mathcal{H}(E) = \int_E h(x)\,dx \tag{33}$$

where $h : \mathbb{R}^d \to \mathbb{R}$

The critical points of (33) satisfy that the mean curvature of the surface passing through the point x is given by the function $h(x)$. Hence the equilibrium equation is often called the prescribed mean curvature problem.

The same strategy of proof we used for the proof of Theorem 4.1 works to prove an analogous result for the functionals $\mathcal{V}$, provided that:

H6) The volume term in the functional satisfies the following properties:
- H6.1) h is continuous
- H6.2) $h(x+k) = h(x) \forall k \in \mathbb{Z}^d$.
- H6.3) $\int_{\mathbb{T}^d} h(x)\,dx = 0$
- H6.4) $||h||_{C^0}$ is sufficiently small depending on the properties of the metric and of F.

Remark 11.1. We note that some version of hypothesis H6.4 is to be expected. If h had a region when it was large and positive – say constant – there could be solutions to the mean curvature equations that are spheres. This would make it impossible to have minimizers of the type that we have discussed.

Note that, since the set is, in the large a plane, the effective curvature at large scales is zero. It is quite remarkable that the surface arranges itself so that it picks the zero average prescribed by H6.3. □

We just discuss the easy modifications to the arguments presented so far, which are needed to incorporate functionals of the form (33)

Note that the functional $\mathcal{H}(E)$ satisfies that if $E_j \xrightarrow{L^1} E$, then, $\lim \mathcal{H}(E_j) = \mathcal{H}(E)$. Since we also have that $\mathcal{H}(E)$ bounded implies bounds on the perimeter, the argument for the existence of minimizers is identical (This remark can also be found in [Giu84] p. 19)

We also have that $\mathcal{H}(E)+\mathcal{H}(L)=\mathcal{H}(E\cup L)+\mathcal{H}(E\cap L)$. Therefore, the argument for the existence or infimal minimizers works also.

Using that $\int h(x)\,dx = 0$, we have that $\mathcal{H}(\mathcal{T}_k E) = \mathcal{H}(E)$ for all sets E and for all $k \in \mathbb{Z}^d$.

The only part of the argument that requires some modification is the density estimates (Lemma 7.1). The version of this Lemma for functionals of the form (33) that will will establish below shows that, provided that $||h||_{C^0}$ is small enough, we can have Lemma 7.1 for all the radius up to an arbitrarily large value.

Lemma 11.2. *Let $\mathcal{A}$ be a class of Caccioppoli sets.*

Let E_0 be a minimizer of a functional $\mathcal{V}$ of the form (33) *among the sets in the class $\mathcal{A}$. Assume that $x \in \partial^* E_0$ and that*

$$E_0 \cap B_\rho(x) \in \mathcal{A}, 0 \le \rho \le r \tag{34}$$

Then there exists a constant C, depending only on λ, Λ defined in (2) *and in d such that:*

$$\begin{aligned} &|E_0 \cap B_r(x)| \ge C r^d \\ &|(\mathbb{R}^d - E_0) \cap B_r(x)| \ge C r^d. \end{aligned} \tag{35}$$

provided that $r \le r_0$ where r_0 depends on the metric and on $||h||_{C^0}$.

We can take $r_0 \ge C||h||_{C^0}^{-1/d}$.

Proof. Because E_0 is a minimizer, comparing with $E_0 - B_r(x_0)$ we have:

$$\int_{(\partial E_0)\cap B_r(x)} F(x,\nu)d|\omega_E| \le \int_{E_0 \cap \partial(B_r(x))} F(x,\nu) d|\omega_{B_r(x)}| - \int_{B_r(x_0)} g(x)dx \tag{36}$$

Hence, proceeding as in the proof of Lemma 7.1 and using the obvious bound $-\int_{B_r(x_0)} g(x)dx \le C||h||_{C^0} r^d$, we have that $V(r) = |E \cup B_r(x_0)|$ satisfies:

$$V(r) \le C V'(r)^{d/(d-1)} + C||h||_{C^0} r^d \tag{37}$$

which leads to:

$$V(r) + C||h||_{C^0} r^d \ge C r^d \tag{38}$$

□

Once we have Lemma 11.2, we can prove in the same way Lemma 7.3 with the extra assumption that the original ball has $r \le r_0$. As observed in Remark 8.7, this is all that is needed for the proof of an analogue of Theorem 4.1 in this case. We also note that the Birkhoff property goes through because it only depends on monotonicity properties of the infimal minimizers with respect to the constraints and on the invariance under translations.

This allows us to prove that there is a plane at a distance from the lower constraint that only depends on the properties of the functional.

Using the alternative argument in Remark 8.6, we can conclude the proof of an analogue of 8.3.

Regularity results for functionals including volume terms (or even more general terms) have been obtained in [Alm76] which obtained in our case that, outside of singularities the minimizers are $C^{1+1/2}$. A simplification of the proof – with slightly weaker regularity is in [Bom82]. Yet another approach to obtaining Lipschitz regularity is in [DS96], [DS98] [DS00].

Again we remark that once one obtains the the boundary is Lipschitz one can obtain further regularity using PDE theory. (e.g [CW93b] [CW93a].)

Also, we remark that the same estimates for the dimension of the singular set hold. The singular set is studied by blow up arguments and the volume terms disappear.

Note that if E is a minimizer for $\mathcal{V}$, then, for all L,

$$\mathcal{J}(E) \leq \mathcal{J}(L) + \mathcal{H}(L) - \mathcal{H}(E).$$

Therefore, we can bound

$$\begin{aligned} \mathcal{H}(L) - \mathcal{H}(E) &\leq C|E - L| \leq C \operatorname{Per}(E\Delta L)^{d/(d-1)} \\ &\leq C\mathcal{J}(E\Delta L)^{d/(d-1)}, \end{aligned} \tag{39}$$

(where by $E\Delta L$ we mean the difference of the two sets $E\Delta L = (E - L) \cup (L - E)$).

In particular, we have for a minimizer of $\mathcal{V}$

$$\mathcal{J}(E) \leq \mathcal{J}(L) + 1/2\mathcal{J}(E\Delta L),$$

which is the hypothesis used by [DS98] or, we have that when $E\Delta L \subset B_r(x)$

$$\mathcal{J}(E) \leq \mathcal{J}(L)(1 + Cr^{\alpha})\, \mathcal{J}(E\Delta L))$$

which is the form of the hypothesis used in [Alm76], [Bom82].

11.2. **Generalization to other manifolds.** In this section, we discuss briefly how the same arguments that we presented here can be used to obtain very similar results for other manifolds that $\mathbb{R}^d$ and $\mathbb{T}^d$. In particular manifolds with a non-abelian fundamental group, which is a question asked in [Mos86].

The main task is to find a generalization of periodicity to other manifolds in such a way that the ingredients of the proof presented here go through. Such a generalization was constructed in [CdlL98] and also used in [dlL00] to obtain results for the variational problems considered in [Mos86] and [Ban89] in manifolds with non-abelian group.

We recall that:

Definition 11.3. *Given a (discrete, finitely generated) group G, a cocycle is a function $\varphi : G \to \mathbb{R}$ such that for any $g, g' \in G$, we have $\varphi(gg') = \varphi(g) + \varphi(g)$.*

We say that a cocycle φ is rational when we can find a discrete index subgroup H such that $\varphi(H) \subset \mathbb{Z}$.

We say that G is residually finite when given any $g \in G$, we can find a finite index H such that $g \notin G$.

In the example $G = \mathbb{Z}^d$, which indeed is residually finite, the cocycles are functions of the form $\varphi_\omega(g) = \omega \cdot g$.

The following easy result can be found in [CdlL98]

Proposition 11.4. *If G is residually finite, rational cocycles are dense in the set of all cocycles.*

It is also well known that the fundamental group of many manifolds (e.g. all manifolds which admit a metric of negative curvature are residually finite. See [CZ93].)

Given a compact, boundaryless Riemannian manifold L, consider $\tilde{L}$ its universal cover. On $\tilde{L}$ we can define an action of the fundamental group $\Pi_1(L)$ by deck transformations (which we will denote by right multiplication). Obviously $L = \tilde{L}/\Pi_1(L)$.

Given a cocycle φ on $\Pi_1(L)$ we can find a harmonic function – which we will denote by $\tilde{\varphi}$ – in such a way that

$$\tilde{\varphi}(xg) = \tilde{\varphi}(x) + \varphi(g) \tag{40}$$

This can be achieved by minimizing the Dirichlet integral among the functions satisfying (40) for a set of generators of $\Pi_1(L)$

Note that if $\varphi_n \to \varphi$ on the generators of $\Pi_1(L)$, then, $\tilde{\varphi}_n \to \tilde{\varphi}$ uniformly on compact sets.

We want to show that if we assume that if:

- $\Pi_1(L)$ is residually finite.
- The balls on the universal cover centered in any point and of diameter Cd where d is the diameter of the fundamental domain and C is a fixed number (which may depend on λ, Λ in (2)) are well defined and there is an isoperimetric inequality for sets in it.

then, we can have a generalization of Theorem 4.1 in which the role of $\mathbb{R}^d$ is taken by $\tilde{L}$, the role of integer translations is taken by the $\Pi_1(L)$ action and the role of the planes is played by the level sets of a cocycle.

The above assumptions are satisfied by the universal cover of a manifold of negative curvature.

We will present two arguments for the proof of this result. For the second one (analogue to that in Remark 8.10), we just need to take $C = 3$.

First note that Lemma 7.3 is local. Even if it assumes that $B_r(x) \approx Cr^d$, $\mathrm{Per}(B_r(x)) \approx Cr^{d-1}$ – both of which could be false in a hyperbolic manifold for large r – only needs it for an r of the order of the diameter of the manifold Λ/λ (or of order 3).

If φ is a rational cocycle on $\Pi_1(L)$ and H is a subgroup of finite index of $Pi_1(L)$ such that $\Pi_1(H) \subset \mathbb{Z}$, we denote by $\tilde{H} = \{h \in H \,|\, \varphi(h) = 0\}$. Note that $\tilde{H}$ is a normal subgroup and that:

$$L = (\tilde{L}/\tilde{H})/(H/\tilde{H}) \tag{41}$$

Since L is compact, this means that if we define $\Gamma^{\varphi,[0,M]} = \{x \in \tilde{L}\,|0 \le \varphi(x) \le M\}$ then $\Gamma^{\varphi,[0,M]}/\tilde{H}$ is compact.

Again we can define $E_*^{\varphi,M}$ infimal minimizer in the class of sets

$$\mathcal{A} = \{E|hE = E\,\forall h \in \tilde{H}, \Gamma^{\varphi,[0,\sigma]} \subset E \subset \Gamma^{\varphi,[0,M]}\} \tag{42}$$

Then, we can define the set $E_*^{\varphi} = \cup_M E_*^{\varphi,M}$ This set is Birkhoff in the sense that $gE_*^{\varphi} \subset E_*^{\varphi}$ or $gE_*^{\varphi} \supset E_*^{\varphi}$ depending on whether $\varphi(g) \ge 0$ or $\varphi(g) \le 0$.

The same argument that we used before can be used to show that $E_*^{\varphi,M} = E_*^{\varphi,M}$ for some M independent of the cocycle.

We argue that at a finite distance independent of φ, either we can find a set that does not contain any point in the boundary of $E_*^{\varphi,M}$ or, appealing to Lemma 7.3 we can find a ball which covers the fundamental domain contained in the complement of the minimizer. In the first case, we are done. In the second, we can use the Birkhoff property to conclude that we also have a finite slab contained on the infimal minimizer, which again can be shown to be a contradiction with the fact that it is the infimal minimizer.

The argument given in Remark 8.10 also works in this context and is perhaps clearer. One just needs to substitute cubes for fundamental domains. Cubes of side 2 are substituted by the union of the fundamental domain and the action of the generators of Π_1 and cubes of side 3 is the union of the fundamental domain, the action of the generators and the action of the product of two generators.

If we still use the names *cubes* for the object above, the argument goes word by word. Note that the density estimates show that if there is an intersection of a cube of size 2 with the boundary of the minimizer, then, the cube of size 3 with the same center gives a contribution to the functional bounded from below.

On the other hand, taking into account that the set $\tilde{H}$ is a normal subgroup, then the set $\Gamma^{\varphi,[0,M]}$ contains $M\#\tilde{H}$ cubes and $MC\#\tilde{H}$ cubes of side 3, where C is a constant that depends only on the relations of length less than three.

If we take the level set of the cocycle, we obtain as before an upper bound that is also of the form $B\#\tilde{H}$ where B depends only on the metric.

As before, by comparing the upper and the lower bound, we conclude that there is a cube of side 2 that does not intersect the boundary of the minimizer and, therefore, using the Birkhoff property that the minimizer is unconstrained.

We note that the theory developed in [dlL00] for elliptic differential operators considers the minimization of functionals defined on functions $u : M \to \mathbb{R}$. The graph of these functions should be considered similar to our surfaces. If we want to compare the geometric assumptions of both results we note that $M \times \mathbb{T}^1 = L$. Hence, $\Pi_1(L) = \Pi_1(M) \times \mathbb{Z}$. So that the geometric assumptions of [dlL00] are stronger than the assumptions here. On the other hand, if we consider the functionals in [dlL00] as functionals of the graph, they fail to satisfy H4, H2'. Formulating the results of [dlL00] on pseudo-differential operators or on subelliptic operators in terms of functionals on Caccioppoli sets is not so clear.

Of course, the generalization of Theorem 4.1 to other manifolds that $\mathbb{R}^d$ can be done at the same time as the generalization of adding a volume term.

12. Appendix: Summary of the theory of finite perimeter sets.

In this appendix, we collect some of the results that we will use from the theory of sets of finite perimeter (also called Caccioppoli sets.

All of the notions and results stated in this section are in the literature. Our main references are [Giu84], and [DGCP72]. We have also found useful [EG92], [MM84], [HS86]. We refer to these places for the proofs and for references to the original literature.

Definition 12.1. *We say that a Borel set $E \subset \mathbb{R}^d$ is a Caccioppoli set (or a set with finite perimeter) when*

a) *For every $\Omega \subset \mathbb{R}^d$ open*

$$\mathrm{Per}(E \cap \Omega) \equiv \sup\{\int_{E\cap\Omega} \operatorname{div} g \, dx^d \in C_0^1(\mathbb{R}^d), ||g||_{C^0} \leq 1\} < \infty. \tag{43}$$

Equivalently:

b) *There exist a vector valued Radon measure ω_E with locally finite variation such that*

$$\int_E \operatorname{div} g \, dx^d = \int g \cdot d\omega_E \tag{44}$$

By the Radon-Nikodim theorem we can write

$$\omega_E = \nu_E |\omega_E| \tag{45}$$

where ν_E is a function taking values on the unit ball in $\mathbb{R}^d$ and $|\omega_E|$ is the total variation of ω_E.

Note that the equation (43) — which is obviously local – combined with (44) show that ω_E is depends only on the local properties of the set. That is if $E_1 \cap B_r(x) = E_2 \cap B_r(x)$ then, $\omega_{E_1} = \omega_{E_2}$ restricted to any ball $B_r'(x)$ $r' < r$. (See [DGCP72] p. 16.)

Remark 12.2. For smooth enough sets – that is sets whose boundary is a smooth manifold –, $\omega_E = D\chi_E$, where χ_E is the characteristic function.

Also, for smooth enough sets, $\mathrm{Per}(E)$ is the perimeter of E. That is, the $(d-1)$ measure of the boundary and ω_E is the *normal boundary element* of infinitesimal calculus and ν_E is the unit inner normal.

When E is irregular, the perimeter may not agree with the $d-1$ measure of the boundary. See [Giu84] p. 7 for an example. □

Remark 12.3. It is easy to show that $\mathrm{supp}(\omega_E) \subset \partial E$. Where we denote by ∂E the boundary of E. □

Remark 12.4. The formula (44) for smooth enough sets reduces to the Gauss-Green formula.

One could, therefore consider Definition 12.1 as a characterization of the sets for which a generalized Gauss-Green formula holds. □

The Definition 12.1 can be generalized without difficulty to any Riemannian manifold with a volume form. Having a metric and a volume form allows us to define the divergence of a vector field. Of course, since a metric induces a a volume form in a canonical way, given any metric we can obtain a notion of finite perimeter. Given any two equivalent C^1 metrics, even if the perimeter changes, the sets with finite perimeter in one metric remain of finite perimeter in the other.

12.1. **Some general properties of Finite perimeter sets.** Caccioppoli sets enjoy many properties that make them well suited for variational problems. On the one hand, their regularity properties are weak enough that it is easy to establish existence of limits, compactness properties, etc. On the other hand, they can be well approximated by smooth sets, define traces, etc. so that it is possible to control them reasonably well and, in particular to develop regularity theories for them when they satisfy extremal properties.

Some of the properties that we are going to use follow. Again, we refer to [Giu84], [EG92] and [DGCP72] for proofs and more details.

The first property is the isoperimetric inequality. (See [Giu84] 1.2.9.)

Lemma 12.5. *Let $E \subset \mathbb{R}^d$ be a bounded Caccioppoli set of finite perimeter. Then, for some c_1, c_2 which depend only on d, we have, denoting by $|E|$ the Lebesgue measure of a set and by $|\omega_E|$ the total variation of ω_E and by B any ball.*

$$|E|^{(d-1)/d} \le c_1(d)\,\mathrm{Per}(E) \tag{46}$$

$$\min\{|E \cap B|, |(\mathbb{R}^d - E) \cap B|\}^{(d-1)/d} \le c_2 \int_B |\omega_E| \tag{47}$$

Of fundamental importance in what follows is the De Giorgi reduced boundary (See [Giu84] ch. 3.)

Definition 12.6. *Given a Caccioppoli set E, we say that $x \in \partial^* E$ ($\partial^* E$ is termed the reduced boundary) if*

i) $\int_{B_\rho(x)} |\omega_E| > 0$ *for all* $r > 0$.
ii) *The limit* $\nu(x) = \lim_{\rho \to 0^+} \int_{B_\rho(x)} \omega_E / \int_{B_\rho(x)} |\omega_E|$ *exists.*
ii) $|\nu(x)| = 1$.

In case that ∂E is a C^1 hypersurface, $\partial E = \partial^* E$

In general: we have. (See [Giu84] p. 43 ff. and [DGCP72]. p. 11)

Lemma 12.7. *If E is a Caccioppoli set*

$$|\omega_E|(\partial E - \partial^* E) = 0 \tag{48}$$

Moreover if $x \in \partial^* E$ we have:

$$\begin{aligned} &\lim_{r\to 0} r^{-(d-1)} \int_{B_r(x)} d|\omega_E| = C(x,\nu,d-1) \\ &\lim_{r\to 0} r^{-d} |B_r(x) \cap E| = C(x,d)/2 \end{aligned} \tag{49}$$

In case that we are considering the standard metric in $\mathbb{R}^d$, we have $C(d)$ is the volume of the unit ball in $\mathbb{R}^d$. In particular the $C(x,\nu,d-1)$ in the first of (49) is independent of ν and is the volume of the $d-1$ dimensional ball.

The fact that we can write $\omega_E = \nu|\omega_E|$ is just the Radon-Nykodim theorem. The fact that the limit in ii) in Lemma 12.6 exists is a consequence of Besicovitch theorem on differentiation of Radon measures. Note that the second part of the Lemma makes very precise the idea that in small scales, we can compare the boundaries with hyperplanes.

Changing sets in sets of measure zero does not alter the perimeter nor the value of the functionals that we are considering in this paper.

Nevertheless, changing sets in sets of measure zero allows to have sets for which the topological properties are more closely related to the measure theoretic ones. (See [DGCP72] p. 14 and [Giu84] p. 42)

Proposition 12.8. *Given a Caccioppoli set E, it is possible to find another set $\tilde{E}$ differing from E in a set of zero Lebesgue measure and such that*

$$\partial \tilde{E} = \overline{\partial^* E} = \overline{\partial^* \tilde{E}} \tag{50}$$

For every Borel set L we can find another Borel set $\tilde{L}$ differing from in in a set of zero Lebesgue measure and such that for all $x \in \partial \tilde{L}$, we have

$$0 < |\tilde{E} \cap B_r(x)| \le |B_r(x)| \tag{51}$$

In view of the above, it is customary to use equivalence classes of sets differing in sets of measure zero and refer to them still as sets (In the same way that we speak of functions in L^1). Using this convention, we can assume that all the sets that we are considering satisfy (50) and (51).

Closely related to sets of finite perimeter are functions of bounded variation.

Definition 12.9. *Let $\Omega \subset \mathbb{R}^d$ be an open set.*

We say that a function $f \in L^1(\Omega)$ is of bounded variation when

$$\mathrm{Var}_\Omega(f) \equiv \sup\{\int_\Omega f \operatorname{div} g\, dx^d | g \in C^1_0(\Omega, \mathbb{R}^d), ||g||_{C^0} \le 1\} \tag{52}$$

If $f \in C^\infty$, we have $\mathrm{Var}(f) = \int \nabla(f)$.

The following results summarize properties of the space of bounded functions. Note the compactness in L^1 of functions whose total variation is uniformly estimated.

Lemma 12.10. *With the definitions above we have;*

a) *The set* $\mathrm{BV}(\Omega)$ *of functions of bounded variation is a separable Banach space when endowed with the norm*

$$||f||_{\mathrm{BV}(\Omega)} = ||f||_{L^1(\Omega)} + \mathrm{Var}_\Omega(f)$$

b) *If* f_j *are functions of bounded variation* $f_j \xrightarrow{L^1} f$,

$$\liminf \mathrm{Var}(f_j) \geq \mathrm{Var}(f)$$

c) *If* $f \in \mathrm{BV}(\Omega)$, *then there exists* $\{f_j\}_{j\in\mathbb{N}} \subset C_0^\infty$ *such that* $f_j \xrightarrow{L^1} f$, $\mathrm{Var}(f_j) \to \mathrm{Var}(f)$.

d) *If* Ω *is bounded and with* C^1 *boundary, then, given a set* $\mathcal{S}$ *of functions in* $\mathrm{BV}(\Omega)$ *such that* $\sup_{f\in\mathcal{S}i} \mathrm{Var}(f) = M < \infty$ *we can extract a sequence* f_j *such that* $f_j \xrightarrow{L^1} f$. *Moreover* $\mathrm{Var}(f) \leq M$.

If a set is a Caccioppoli set, then, it is clear that its characteristic function is a function of Bounded Variation. Conversely, if the characteristic function of a set is a function of bounded variation, then, the set – or another set which differs from it in a set of measure zero – is a Caccioppoli set.

In what follows, we will follow standard practice and identify Caccioppoli sets that differ by sets of zero Lebesgue measure. Hence, we will say that a sequence E_j of sets converges in L^1 (resp. in BV) when their characteristic functions converge in L^1 (resp. in BV).

Of fundamental importance in the theory are the approximation properties of functions of bounded variation and the approximation properties of Caccioppoli sets for polygonal ones. Indeed, the later appeared already in [Cac52a, Cac52b].

Lemma 12.11. *If* $E \subset \mathbb{R}^d$ *has finite perimeter, there exists a sequence of polygonal domains* $\{E_n\}$ *such that* $E_n \xrightarrow{L^1} E$ *and* $\mathrm{Per}(E_n) \to Per(E)$.

Given a function $f \in BV(\Omega)$. *Then, there exist a sequence* $\{f_n\} \subset C_0^\infty(\Omega)$ $||f - f_j||_{L^1(\Omega)} \to 0$, $\mathrm{Var}(f_j) \to \mathrm{Var}(f)$.

Lemma 12.12. *If* $E_j \to E$ *in* L^1, *then* $\mathrm{Per}(E) \leq \liminf \mathrm{Per}(E_j)$

The proof is rather easy if we consider the definition (43) of perimeter as the supremum of a functional on a set of functions. (See [Giu84] p. 7.) Note also that Lemma 5.1 is a generalization of this fact.

Lemma 12.13. *If* E *and* L *are Caccioppoli sets, then:*

$$\mathrm{Per}(E) + \mathrm{Per}(L) \geq \mathrm{Per}(E \cup L) + \mathrm{Per}(E \cap L) \tag{53}$$

Note, of course, that the inequality could be strict even for smooth sets if the sets have a common boundary.

Proof. This is Theorem 3.5 of [DGCP72] p.18.

The proof makes use of the fact that the inequality is obvious for polygonal approximations since polygonal sets satisfy that

$$\partial E \cup \partial L = \partial(E \cup L) \cup \partial(E \cap L) \tag{54}$$

the perimeter satisfies the desired inequality.

If we apply this to the polygonal approximations such as those constructed in Lemma 12.11 and use Lemma 12.12, we obtain the desired result. □

Lemma 6.1 is a version of this result for more general boundary functionals, in a class of sets with more regularity. In such a class of sets, one can still construct polygonal approximations to our functionals.

13. Appendix: Proof of some technical lemmas.

In this Appendix, we collect proofs of the technical lemmas that we have used in the argument. Most of them are related to other arguments that exist in the literature.

13.1. **Proof of Lemma 5.1.** We present two arguments. The first one is shorter and more elementary. The second one gives a more detailed local information.

Similar arguments happen in [Fed69] Chapter 5 in the context of currents. (See the proof of Theorem 5.1.5 p. 519)

Since $F(x, \nu)$ is convex and homogeneous of degree 1 in ν

We can write $F(x, \nu) = \sup_{l \in \mathcal{C}_x} l \cdot \nu$ where $\mathcal{C}_x$ is the convex support of the function $F(x, \cdot)$.

We first discuss the case when the convex support is finite at every point. (That is, the set $\{\nu \quad | \quad F(x, \nu) = 1\}$ is a polyhedron wit a finite number of faces.)

In such a case, $\mathcal{C}_x$ is finite and it depends continuously on the point.

Hence we can write $F(x, \nu) = \sup_{l(x) \in \mathcal{R}} l(x) \cdot \nu$ where $\mathcal{R}$ is a set of continuous functions.

Given a polygonal Caccioppoli set, we can write

$$\mathcal{J}(E) = \int \sup_{l \in \mathcal{R}} l(x) \cdot \nu_E d|\omega_E|$$

Since in this case, the ν_E are discrete, we can write

$$\mathcal{J}(E) = \sup \int l(x) \cdot \nu_E d|\omega_E| = \int l(x) \cdot d\omega_E \tag{55}$$

Once one has (55), the desired result follows rather rapidly.

In [Giu84] p. 222, it is shown that if a sequence of sets of finite perimeter E_j converge in L^1 to another set E, the corresponding measures $d\omega_{E_j}$ converge in the weak-* topology to $d\omega_E$.

Hence if $E_j \xrightarrow{L^1} E$ and $\mathcal{J}(E) \geq \int l(x) \cdot d\omega_E - \epsilon$, for any subsequence of E_j, we can find a further subsequence we can find j_k such that $\mathcal{J}(E_{j_k}) \geq \int l(x) \cdot d\omega_{E_{j_k}} - 2\epsilon$.

This shows that the lim inf has to be greater or equal than the value.

To prove the general case, the only thing that we need to do is to establish the formula (55) for a general set. This follows from the fact given a function F satisfying H1,H2,H3,H4 it can be approximated uniformly by a function whose convex support is finite. Also, Caccioppoli sets can be approximated by polygonal ones. (See Lemma 12.11.)

□

A second proof that also gives some local information can be obtained by observing that, by Besicovitch theorem on differentiation of Radon measures, (see [EG92] p. 30), Lemma 5.1 follows from a local version near differentiability points of $|\omega_E|$. Namely:

Proposition 13.1. *In the conditions of Lemma 5.1, Let x_0 be a differentiation point for $|\omega_E|$. Given $\epsilon > 0$, there exists $r > 0$ such that*

$$\mathcal{J}_{B_r(x_0)}(E) \le r^{d-1}\epsilon + \liminf \mathcal{J}_{B_r(x_0)}(E_j) \tag{56}$$

Proof. To simplify the notation, choose coordinates in such a way that $\nu(x_0) = e_d$.

We write:

$$\begin{aligned}
\int_{B_r(x_0)} F(x, \nu_{E_j}) d|\omega_{E_j}| - \int_{B_r(x_0)} F(x, \nu_E) d|\omega_E| \\
= \int_{B_r(x_0)} F(x_0, \nu_E) d|\omega_E| - \int_{B_r(x_0)} F(x, \nu_E) d|\omega_E| \\
+ \int_{B_r(x_0)} F(x_0, e_d) d|\omega_E| - \int_{B_r(x_0)} F(x_0, \nu_E) d|\omega_E| \\
+ \int_{B_r(x_0)} F(x_0, \nu_{E_j}) d|\omega_{E_j}| - \int_{B_r(x_0)} F(x_0, e_d) d|\omega_{E_j}| \\
+ \int_{B_r(x_0)} F(x_0, e_d) d|\omega_{E_j}| - \int_{B_r(x_0)} F(x_0, e_d) d|\omega_E|
\end{aligned} \tag{57}$$

The absolute value of first line in the R.H.S. of (57) can be estimated by above by $\sigma(r)r^{d-1}$, where $\sigma(r)$ is a function that decreases to zero when r decreases to zero, using the uniform continuity of F in the first variable and the fact that, because x_0 is a differentiation point, $|\omega_E|B_r(x_0) \le Cr^{d-1}$.

The absolute value of the second line can be estimated by above by $\sigma(r)r^{d-1}$ because x_0 is a differentiation point.

Hence, to establish the desired result, it suffices to show that the lim inf of the last two lines of (57) is greater or equal than zero plus a term $\sigma(r)r^{d-1}$.

Note that, if we consider a subsequence where the lim inf is reached, we can (see [Giu84] p. 222) pass to a further subsequence when $d\omega_{E_j}$ converges as a Radon measure to $d\omega_E$ and $d|\omega_{E_j}|$ converges, also as a Radon measure, to a positive measure $d\mu$. In general, $\mu(A) \ge |\omega_E|(A)$ and the inequality can be strict in some examples.

Since F is convex in the second argument, we can bound

$$F(x_0, \nu_{E_j}) d|\omega_{E_j}| - F(x_0, e_d) d|\omega_{E_j}| \ge \nabla_- F(x_0, e_d)(\nu_{E_j} - e_d) d|\omega_{E_j}| \tag{58}$$

where by $\nabla_- F(x_0, e_d)$ we denote the subdifferential of the convex function $F(x_0, \cdot)$.

Recalling that $\nu_{E_j} d|\omega_{E_j}| = d\omega_{E_j}$ and that we arranged that $d\omega_{E_j}$ converges to $d\omega_E$, and that $d|\omega_{E_j}|$ converges to $d\mu$, we have, using (58) that the limit of the last two lines of (57) is greater or equal than:

$$\begin{aligned} &\nabla_- F(x_0, e_d)[\omega_E(B_r(x_0)) - e_d \mu(B_r(x_0))] \\ &\qquad + F(x_0, e_d)[\mu(B_r(x_0)) - |\omega_E|(B_r(x_0))] \end{aligned} \tag{59}$$

Again we use that because x_0 is a differentiation point,

$$| \, \omega_E(B_r(x_0)) - e_d |\omega_E|(B_r(x_0)) \, | \le \sigma(r) r^{d-1} \tag{60}$$

Hence, the desired result reduces to showing that

$$\begin{aligned} &\nabla_- F(x_0, e_d) e_d [|\omega_E|(B_r(x_0)) - \mu(B_r(x_0))] \\ &\qquad + F(x_0, e_d)[\mu(B_r(x_0)) - |\omega_E|(B_r(x_0))] \end{aligned} \tag{61}$$

is non-negative.

Using the homogeneity of the function F in the second argument, we can rewrite (61) as:

$$\begin{aligned} &F(x_0, \mu(B_r(x_0)) \, e_d) - F(x_0, |\omega_E|(B_r(x_0)) \, e_d) \\ &\qquad - \nabla_- F(x_0, e_d)[\mu(B_r(x_0)) e_d - |\omega_E|(B_r(x_0)) e_d] \end{aligned} \tag{62}$$

The positivity of (61) is a property of the subdifferential of a convex function.

This finishes the proof of Lemma 5.1. □

13.2. **Proof of Proposition 6.1.** The proof will follow the same argument as in Lemma 12.12. We need an improvement of Lemma 12.11. The proof of this approximation lemma is more or less along the same lines that lead to approximations of the set that also approximate the perimeter. (See [Giu84] p. 22)

Proposition 13.2. *Given a finite perimeter set E and a functional satisfying H1,H3 we can find a sequence of polygonal sets (resp. sets with C^∞ boundary) E_j such that $E_j \xrightarrow{L^1} E$, $\mathcal{J}(E_j) \to E$.*

Proof. Clearly, it suffices to produce C^∞ approximations satisfying the conclusions since C^∞ sets can be approximated by polygonal ones.

The idea of the proof (similar to arguments used frequently in the theory of Caccioppoli sets) is that, by Besicovitch differentiation theorem, we can approximate the boundary in a neighborhood of a differentiation point by a plane. The way of making sure that the planes thus defined indeed bound a set which is close to the set we started with is to construct the planes as a the level sets of a function which is an smoothed out version of the characteristic function.

An important device to approximate functions – in particular the characteristic functions of sets – is symmetric molifiers (see [Giu84]. p. 10 ff.)

Definition 13.3. *A function $\eta : \mathbb{R}^d \to \mathbb{R}^+$ is a positive symmetric mollifier when*

1. $\eta \in C_0^\infty$
2. *Vanishes outside the unit ball*
3. $\int \eta(x)dx = 1$
4. $\eta(x) = \tilde{\eta}(|x|)$

Given such a function, and a positive ϵ, we denote by $\eta_\epsilon(x) = \epsilon^{-d}\eta(x/\epsilon)$ and for any function $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ we set $f_\epsilon = g_\epsilon * f$.

Let x_0 be a differentiation point for $d\omega_E$. By Lemma 12.7, (See also [Giu84] p. 50-51.) and the definition of differentiation point, given any $\delta > 0$ we can find a sufficiently small r in such a way that:

$$
\begin{aligned}
&|\omega_E|(B_r(x_0) \cap \{x \cdot \nu < 0\} \cap E) \le \delta r^n \\
&|\omega_E|((B_r(x_0) - E) \cap \{x \cdot \nu > 0\}) \le \delta r^n \\
&|B_r(x_0)|(1/2 - \delta) \ge |E \cap B_r(x_0)| \ge |B_r(x_0)|(1/2 + \delta) \\
&\left| (|\omega_E|(B_r(x_0))^{-1} \int_{B_r(x_0)} d\omega_E - \nu_{x_0} \right| \le \delta
\end{aligned} \tag{63}
$$

From this, it follows that for ϵ small enough (roughly a fixed multiple of r) and for any $0 < t < 1$, the set $E_t^\epsilon = \{f_\epsilon \ge t\}$ satisfies

1. $|(E_t^\epsilon - E) \cup (E - E_t^\epsilon)| \to 0$ as $\epsilon \to 0$.
2. $|\nabla f_\epsilon / |\nabla f_\epsilon - \nu| \le \delta |B_r(x_0)|$.

The level sets of the function f will gives us the desired approximation. □

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