

# **Boundary knot method: A meshless, exponential convergence, integration-free, and boundary-only RBF technique**

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## **Abstract**

Based on the radial basis function (RBF), non-singular general solution and dual reciprocity method (DRM), this paper presents an inherently meshless, exponential convergence, integration-free, boundary-only collocation techniques for numerical solution of general partial differential equation systems. The basic ideas behind this methodology are very mathematically simple and generally effective. The RBFs are used in this study to approximate the inhomogeneous terms of system equations in terms of the DRM, while non-singular general solution leads to a boundary-only RBF formulation. The present method is named as the boundary knot method (BKM) to differentiate it from the other numerical techniques. In particular, due to the use of non-singular general solutions rather than singular fundamental solutions, the BKM is different from the method of fundamental solution in that the former does not need to introduce the artificial boundary and results in the symmetric system equations under certain conditions. It is also found that the BKM can result in linear analogization formulations of nonlinear partial differential equations with linear boundary conditions if only boundary knots are used. The efficiency and utility of this new technique are validated through a number of typical numerical examples. Some promising developments of the BKM are also discussed.

**Key words:** boundary knot method, dual reciprocity method, BEM, method of fundamental solution, radial basis function

## 1. Introduction

It has long been claimed that the boundary element method (BEM) is a viable alternative to the domain-type finite element method (FEM) and finite difference method (FDM) due to its advantages in dimensional reducibility and suitability to infinite domain problems. However, the practical situations are that the FEM and FDM are still dominant techniques now used in science and engineering computations. The major bottlenecks in performing the BEM analysis have long been its weakness in handling inhomogeneous terms such as time-dependent and nonlinear problems. The recent introduction of the dual reciprocity BEM (DRBEM) by Nardini and Brebbia [1] eliminates or at least eases to great extent these inefficiencies in the BEM. Notwithstanding, as was pointed out in [2], the method is still more mathematically complex and requires strenuous manual effort compared with the FEM and FDM. In particular, the handling of singular integration is not easy to non-expert users and is often computationally expensive. The use of the low order polynomial approximation in the BEM also causes the slow convergence. More importantly, just like the FEM, surface mesh (or re-mesh) requires costly computation, especially for moving boundary and nonlinear problems. The method of fundamental solution (MFS) is shown an emerging technique to alleviate these drawbacks and is getting increasing attraction especially due to some recent works by Golberg, Chen and Muleskov [2-5]. The MFS affords advantages of integration-free (of course, no singular integration), spectral convergence, and meshless (meshfree). However, the use of artificial boundary outside physical domain has been a major limitation of the MFS, which may cause severe ill-conditioning of the resulting equations, especially for complex boundary geometry. For some relative details see Kitigawa [6,7]. Additionally, the present author noted that the MFS loses the symmetric matrix structure even for self-adjoint operators subject to the single type of boundary condition. These

inefficiencies of the MFS motivate us to search an alternative technique, which keeps all merits of the method but removes the shortcomings undermining its attractiveness.

Recently, Golberg et al. [4] established the DRBEM on the firm mathematical theory of the radial basis function (RBF). The DRBEM can be regarded as a two-step methodology. In terms of dual reciprocity method (DRM), the RBF is applied at first to approximate the particular solution of inhomogeneous terms, and then the standard BEM is used to discretize the remaining homogeneous equation. Chen et al [3] and Golberg et al [2,4] extended this RBF approximation of particular solution to the MFS, which greatly enhances its applicability. In fact, the MFS itself can also be considered a special RBF collocation approach, where the fundamental solution of the governing equation is taken as the radial basis function. From the above discussions, one can conclude that the RBF is a centerpiece of the DRBEM and MFS. Therefore, there is necessity to introduce more on the RBF approach before involving the innovative work being presented in this paper.

Besides the boundary-type RBF technique of the MFS, the domain-type RBF collocation is also now under intense study since Kansa [8] first introduced the RBF to directly solve various types of partial differential system equations in 1990. The most impressive charming in the RBF-type technique is its meshless inherence. The construction of mesh in high dimension is not a trivial work. The meshless feature is very desirable for numerical computation. Unlike the recently developed meshless FEM with moving least square, the RBF approach is truly simple meshless technique without any difficulty applying boundary conditions. On the other hand, Kansa and Hon [9] discovered that the convergence speed of the RBF approximation is dimensionally increased. Namely, the order of convergence rate grows linearly as the dimension increases. Therefore, the RBF avoids the dimension curse to other numerical techniques and requires relatively much fewer nodes to achieve accurate solutions in higher

dimension cases. The meshless merit and dimension increasing of convergent order make the RBF approach very suitable for high-dimensional free boundary problems such as American option pricing [10]. It is obvious that the RBF is an essential factor to construct a viable numerical technique.

The purpose of this study is to introduce a novel boundary knot method (BKM) to remedy the above-mentioned disadvantages in the MFS through the use of the non-singular general solution instead of the singular fundamental solution. In this way, the inherent inefficiency of the MFS due to the use of the fictitious boundary is alleviated handily. This leads to tremendous improvement in computational stability and produces the symmetric matrix structure under certain conditions. The preliminary study of this paper shows that the BKM is a very competitive alternative to both the MFS and BEM including DRBEM in terms of efficiency, accuracy, simplicity and stability. The new method can be defined as a combination of the DRM, RBF and no-singular general solution. The name of the BKM is due to its essential meshless property, namely, the BKM does not need any discretization (grid) for any dimension problem and only uses knot point. An amazing finding of this study is that the truly boundary-only technique such as the BKM can produce linear modeling formulations of problems with nonlinear governing equations and linear boundary conditions, which circumvents all perplexing concerns in the solution of nonlinear equations. Also time-dependent non-singular general solutions are derived to avoid the calculation of the temporal derivative in the BKM solution of transient problems. The mesh, singular integration, lower convergence rate, and mathematical complexity have been the major disadvantages of the BEM. The BKM readily eliminates all these inefficiencies yet retains the boundary-only merit for general problems. This paper also discusses the promise to generate sparse banded symmetric matrix of the BKM by using local boundary collocation and compactly-supported RBF in the DRM, which circumvents another drawback of the BEM, full and asymmetric matrix structure.

This paper is organized as follows. Section 2 involves the procedure of the DRM and RBF approximation to particular solution. In particular, different from normal approach in Muleskov et al [5], we use a very simple reverse method to get the analytical approximate particular solution for complicated operators and RBFs. In section 3, we introduce non-singular general solution and derive the resulting algebraic equations of the BKM. Numerical results are provided and assessed in section 4 to establish the validity and accuracy of the method. Some promising potentials of the BKM are explored in section 5, which include iteration-free nonlinear computation, non-singular general solution of some often-encountered 2D and 3D operators, time-dependent problem, localization, and conservation of symmetric matrix structure of the BKM. Finally, some concluding remarks are given based on the results reported here.

## **2. RBF approximation to particular solution**

Like the DRBEM and MFS, the BKM can be viewed as a two-step numerical scheme, namely, DRM and RBF approximation to particular solution and the evaluation of homogeneous solution. The latter is the emphasis of this paper, which will be detailed in section 3. The former has been well developed by recent works of Chen and Golberg [2-4]. However, their approach [5] to determine analytical approximate solutions is rather complicated and not suitable for general case. By analogy with ideas in [11,12], we propose a viable alternative scheme to eliminate this problem very easily.

In the following, we outline the basic methodology to employ the RBF to approximate particular solution. Let us consider the differential equation

$$L\{u(x)\} = f(x), \quad x \in \Omega \tag{1}$$

with boundary conditions

$$u(x) = b_1(x), \quad x \in \Gamma_u, \quad (2)$$

$$\frac{\partial u(x)}{\partial n} = b_2(x), \quad x \in \Gamma_T, \quad (3)$$

where  $L$  is a differential operator,  $f(x)$  is a known forcing function.  $n$  is the unit outward normal. The above simple forms of boundary conditions are to simplify description. More complicated boundary condition matters little in the present methodology.  $x \in R^d$ ,  $d$  is the dimension of geometry domain, which is bounded by a piece-wise smooth boundary  $\Gamma = \Gamma_u + \Gamma_T$ . It is assumed here that operator  $L$  includes Laplace operator, namely,

$$L\{u\} = \nabla^2 u + L_1\{u\}. \quad (4)$$

In section 5.2, we will point out that this is not necessary. Eq. (1) can be restated as

$$\nabla^2 u + u = f(x) + u - L_1\{u\}. \quad (5)$$

The solution of the above equation (5) can be expressed as

$$u = v + u_p, \quad (6)$$

where  $v$  and  $u_p$  are the general (homogeneous) and particular (inhomogeneous) solutions, respectively.

The latter satisfies equation

$$\nabla^2 u_p + u_p = f(x) + u - L_1\{u\}, \quad (7)$$

but does not necessarily satisfy boundary conditions (2) and (3).  $v$  is the homogeneous solution of the Helmholtz equation

$$\nabla^2 v + v = 0, \quad x \in \Omega, \quad (8)$$

$$v(x) = b_1(x) - u_p, \quad x \in \Gamma_u, \quad (9)$$

$$\frac{\partial v(x)}{\partial n} = b_2(x) - \frac{\partial u_p(x)}{\partial n}, \quad x \in \Gamma_T. \quad (10)$$

The first step in the BKM is to evaluate the particular solution  $u_p$  by the DRM and RBF. After this, Eq. (8)-(10) can be solved by the boundary collocation RBF methodology using non-singular general solution proposed in later section 3.

Unless the left side of Eq. (7) is rather simple, it is practically impossible to get analytical particular solution particularly in case of nonlinear and time-dependent problems. In addition, even if the analytical solutions for some problems are available, their forms are usually rather complicated and are not easy to use. Therefore, we prefer to approximate these inhomogeneous terms by a general numerical approach. Among them, the quasi-Monte Carlo method [13] and RBF in terms of dual reciprocity method [2-5] are two approaches applied successfully. In this paper, we use the latter.

The DRM analogizes the particular solution by the use of a series of approximate particular solution at all specified nodes of boundary and domain. The left side of Eq. (7) is approximated by the RBF approach, namely,

$$f(x) + u - L_1\{u\} \cong \sum_{j=1}^{N+L} \alpha_j \phi(\|x - x_j\|) + \sum_{k=1}^d \alpha_{N+L+k} x^{(k)} + \alpha_{N+L+d+1}, \quad (11)$$

where  $\alpha_j$  are the unknown coefficients.  $N$  and  $L$  are respectively the numbers of knots on boundary and domain.  $d$  denotes the dimension of the geometry. It is noted that the use of the nodes inside domain is not necessary as will discussed in later section 5.1.  $\|\bullet\|$  represents the Euclidean norm,  $\phi(\|x - x_j\|)$  is the RBF function of one-dimensional distance variable. The first-order polynomial interpolation in the second term of Eq. (11) is set up to assure nonsingularity of interpolation matrix, since popular globally support multiquadratics (MQ) and thin plate spline (TPS) are only conditionally positive definite [14]. Note that  $x^{(k)}$  indicates different independent variables.

By forcing Eq. (11) to exactly satisfy Eq. (7) at all nodes together with the  $d+1$  side conditions

$$\sum_{j=1}^{N+L} \alpha_{N+L+j} x_j + \alpha_{N+L+d+1} = 0, \quad (12)$$

we can uniquely determine

$$\alpha = A_\phi^{-1} \begin{Bmatrix} f(x) + u - L_1\{u\} \\ 0 \end{Bmatrix}, \quad (13)$$

where  $A_\phi$  means nonsingular interpolation matrix and holds symmetric structure. Similar to interpolation formula (11), applying the RBF to approximation of function  $u$  yields

$$u \cong \sum_{j=1}^{N+L} \lambda_j \phi(\|x - x_j\|) + \sum_{k=1}^d \lambda_{N+L+k} x^{(k)} + \lambda_{N+L+d+1}, \quad (14)$$

namely,

$$\begin{Bmatrix} u \\ 0 \end{Bmatrix} = A_\phi \lambda. \quad (15)$$

For operator  $L_1$ , we have

$$L_1 \begin{Bmatrix} u \\ 0 \end{Bmatrix} = L_1 \{A_\phi\} \lambda. \quad (16)$$

So

$$L_1 \begin{Bmatrix} u \\ 0 \end{Bmatrix} = L_1 \{A_\phi\} A_\phi^{-1} \begin{Bmatrix} u \\ 0 \end{Bmatrix}. \quad (17)$$

Substituting Eq. (17) into Eq. (13), we get

$$\alpha = A_\phi^{-1} \left[ \begin{Bmatrix} f(x) \\ 0 \end{Bmatrix} + (I - L_1 \{A_\phi\} A_\phi^{-1}) \begin{Bmatrix} u \\ 0 \end{Bmatrix} \right], \quad (18)$$

where  $I$  is the unite matrix.

In the normal procedure, the RBF  $\phi$  in Eqs. (11) and (14) is first selected, and then corresponding approximate particular solutions  $\varphi$  are determined by analytically integrating equation

$$\nabla^2 \varphi + \varphi = \begin{Bmatrix} \phi \\ x^{(k)} \\ 1 \end{Bmatrix}, \quad j = 1, \dots, N+L, \quad k = 1, \dots, d. \quad (19)$$

This methodology performs easily only for simple Laplace operator and linear RBF

$$\phi(x, x_j) = \phi(r_j) = r_j, \quad (20)$$

where  $r_j = \|x - x_j\|$ , rather than the other more popular RBFs including the MQ

$$\phi(x, x_j) = \phi(r_j) = \sqrt{r_j^2 + c_j^2}, \quad (21)$$

where  $c_j$  is the shape parameter, and the TPS

$$\phi(x, x_j) = \phi(r_j) = r_j^{2m} \log(r_j), \quad (22)$$

where  $m$  is order of the TPS. Muleskov et al. [5] recently worked out the TPS analytical approximate particular solutions of Helmholtz operator by using a sophisticated mathematical approach, and the forms of their solutions are rather complicated. In addition, the analytical approximate particular solutions for general cases such as the MQ and other differential operators are not yet available now due to great difficulty involved.

In this study, we use an alternative reverse approach to eliminate this difficulty in a much easier fashion. Inverse to the normal scheme, the proposed approach is to determine the approximate particular solution  $\varphi$  at first, and then we can very easily evaluate the corresponding RBF  $\phi$  through substituting the specified  $\varphi$  into Eq. (19). The principle used here to choose approximate solution for the Helmholtz operator is to increase two-order power of the original RBFs and to avoid the singularity at  $r=0$ . For the MQ, the second-order TPS ( $m=2$ ) and linear RBFs, the chosen approximate particular solution are respectively

$$\varphi(r_j) = (r_j^2 + c_j^2)^{3/2}, \quad (23a)$$

$$\varphi(r_j) = r_j^6 \log(r_j), \quad (23b)$$

$$\varphi(r_j) = r_j^3. \quad (23c)$$

The corresponding RBFs are

$$\phi(r_j) = 6(r_j^2 + c_j^2) + \frac{3r_j^2}{\sqrt{r_j^2 + c_j^2}} + (r_j^2 + c_j^2)^{3/2}, \quad (24a)$$

$$\phi(r_j) = 36r_j^4 \log(r_j) + 12r_j^4 + r_j^6 \log(r_j), \quad (24b)$$

$$\phi(r_j) = 9r_j + r_j^3. \quad (24c)$$

The present scheme can also be easily extended to the 2D, 3D biharmonic or convection diffusion

operators other than the present 2D Helmholtz operator. The numerical experiments given in section 4 demonstrate that this approach works well and robustly. For example, we observe of the super convergence of the MQ clearly.

Finally, we can get particular solutions at any point by summation of approximate particular solutions at all nodes, namely,

$$u_p = \sum_{j=1}^M \alpha_j \varphi(\|x - x_j\|), \quad (25)$$

where  $M = N + L + d + 1$ . Substituting Eq. (18) into Eq. (25) yields

$$u_p = \Phi A_\phi^{-1} \left[ \begin{array}{c} f(x) \\ 0 \end{array} \right] + (I - L_1 \{A_\phi\} A_\phi^{-1}) \left[ \begin{array}{c} u \\ 0 \end{array} \right], \quad (26)$$

where  $\Phi$  is known matrix comprised of  $\varphi(\|x - x_j\|)$ . Eq. (26) will be used in the boundary collocation equations introduced in next section 3.

### 3. Non-singular general solution and boundary knot method

It is well known that the Laplace operator of the simplest among all the known operators has not nonsingular general solution. Golberg and Chen [2] pointed out that the placement of source points outside domain in the MFS is to avoid the singularities. However, we found through numerical experiments that even if the source and response points on physical boundary are chosen differently to circumvent the singularities, the solutions were still degraded evidently compared with those by the standard MFS, where the source points are placed only on the fictitious auxiliary boundary. Moreover, the more distant the source points are located from physical domain, the more accurate MFS solutions are obtained [2]. However, unfortunately, the resulting equations become extremely ill-conditioned which in some cases deteriorates the solution [2,6,7].

To illustrate the basic idea of the boundary collocation using non-singular general solution, we take the 2D Helmholtz operator as an illustrative example. There do exist non-singular general solutions of the other differential operators. The reason for this choice is that the Helmholtz operator is the simplest among various often-encountered operators having non-singular general solution.

The 2D homogeneous Helmholtz equation (8) has two general solutions, namely,

$$v(r) = c_1 J_0(r) + c_2 Y_0(r), \quad (27)$$

where  $J_0(r)$  and  $Y_0(r)$  are the zero-order Bessel functions of the first and second kinds, respectively. In the standard BEM and MFS, the Hankel function

$$H(r) = J_0(r) + iY_0(r) \quad (28)$$

is applied as the fundamental solution. It is noted that  $Y_0(r)$  encounters logarithm singularity, which causes the major technique difficulty in applying the BEM and MFS. Many special techniques have developed to solve or circumvent this troublesome singular problem.

The crucial process in the present BKM scheme is to discard the singular general solution  $Y_0(r)$  and to only use  $J_0(r)$  as the radial basis function to collocate the boundary condition equations (9) and (10). It is noted that  $J_0(r)$  exactly satisfies the Helmholtz equation and we can therefore get a boundary-only RBF collocation scheme. Unlike the MFS, all collocation nodes are placed only on physical boundary and can be used as either source or response points. The major drawbacks in the MFS disappear naturally in the present BKM.

Let  $\{x_k\}_{k=1}^N$  denote a set of nodes on the physical boundary, the solution  $v(x)$  of Eq. (8) is approximated in a standard RBF collocation fashion

$$v(x) = \sum_{k=1}^N \beta_k J_0(r_k), \quad (29)$$

where  $r_k = \|x - x_k\|$ .  $k$  is index of source points.  $N$  is the total number of boundary knots.  $\beta_k$  are the desired coefficients. In terms of the collocation of Eqs. (9) and (10), we have

$$\sum_{k=1}^N \beta_k J_0(r_{ik}) = b_1(x_i) - u_p(x_i), \quad (30)$$

$$\sum_{k=1}^N \beta_k \frac{\partial J_0(r_{jk})}{\partial n} = b_2(x_j) - \frac{\partial u_p(x_j)}{\partial n}, \quad (31)$$

where  $i$  and  $j$  indicate boundary response knots respectively corresponding to two types of boundary conditions. If internal nodes are used, we need to constitute another set of supplement equations. It is evident that the following equations at interior knots

$$\sum_{k=1}^N \beta_k J_0(r_{lk}) = u_l - u_p(x_l), \quad l = 1, \dots, L \quad (32)$$

can be established, where  $l$  indicates the internal response knots and  $L$  is the total number of interior points used. Substituting Eq. (26) into Eqs. (30), (31) and (32), we get  $N+L$  resulting simultaneous algebraic equations. The unknown variables of these equations are boundary collocation coefficients  $\beta_k$  and values of domain-inside dependent variable  $u_l$ . After solving these equations, we need compute coefficient  $\alpha$  by Eq. (18) so that we can get the approximate solution at any points by

$$u(x) = \sum_{k=1}^N \beta_k J_0(\|x - x_k\|) + u_p, \quad (33)$$

where  $u_p$  is evaluated by Eq. (26).

It is stressed that the use of interior points is not necessary in the BKM. The term boundary-only is used here in the sense as in the DRBEM and MFS that only boundary knots are required, although internal knots can improve solution accuracy in some cases. If only boundary knots are employed, Eq. (32) should be omitted and  $u_p$  and  $\alpha$  are determined directly by Eq. (18) and (26) before the solution of resulting algebraic Eqs. (30) and (31). From the above analysis, one can find that the present BKM

is rather mathematically simple and easy-to-program. The extension of this methodology to the other partial differential operators is very straightforward.

Before proceeding the numerical experiments, we consider to choose the radial basis function. In general, RBFs are globally defined basis functions and lead to a dense matrix, which becomes highly ill-conditioned if very smooth radial basis functions are used with large number of interpolation nodes [14]. This causes severe stability problems and computationally inefficiency for large size problem. A number of approaches have been proposed to remedy this problem such as domain decomposition and compactly supported RBFs (CS-RBFs). The latter is very recently developed by Wendland [15], Wu [16] and Schaback [17]. The CS-RBFs with simple and explicit algebraic form can result in sparse banded matrices and is therefore preferred in many cases. Golberg et al. [2] and Wong et al. [18] respectively applied the CS-RBFs to the MFS and RBF collocation method successfully. An excellent comprehensive research on this topic can be found in Kansa and Hon [9]. Chen et al [19] recently also used the CS-RBFs to the DRBEM. However, in this study, we will do not use the CS-RBFs to focus on the illustration of the basic idea of the BKM with simpler globally-supported RBFs. Potential benefits of the CS-RBFs to the BKM will be discussed in later section 5.4.

The MQ, TPS and linear RBF are simple and most widely used now, which were first introduced respectively by Hardy [20], Duchon [21], and Nardini and Brebbia [1]. Among them, it is well known that the MQ ranks the best in accuracy [22] and is only the RBF with desirable merit of spectral convergence for general case [4]. However, its accuracy is greatly influenced by the choice of the shape parameter [23,24]. So far the optimal choice of shape parameter is still an open research topic. Despite this problem, the MQ is still most widely used in the RBF solution of various differential systems. For numerical example of Laplace equation shown in next section 4.2, the linear and TPS

RBFs show evidently slower convergence rate than the MQ. For example, the average relative error is 0.91% for the Linear RBF with 11 knots, -0.39% for the TPS with 11 knots and -0.023% for the MQ (shape parameter 2) with 9 knots. We even got the very accurate solutions, 0.50% relative average error, by the MQ with only 3 knots. Also the use of the other types of RBFs such as the TPS is satisfactory in many cases. To simplify the presentation, this paper only uses the MQ to approximate the particular solution.

## 4. Numerical results and discussions

In this paper, all numerical examples unless otherwise specified are taken from [25]. The geometry of test problem is all an ellipse featured with semi-major axis of length 2 and semi-minor axis of length 1 as shown in Figs. 1 and 2. These examples are chosen since their analytical and numerical solutions are obtainable so that the prowess of the new BKM technique can explicitly exposed through comparison. More complex problems can be handled in the same fashion of the BKM without any extra difficulty. The Bessel function of the first kind of zero order is evaluated via a short subroutine given in [26]. The 2D Cartesian co-ordinates (x,y) system is used as in [25].

### 4.1. Helmholtz equation

The governing equation of the 2D Helmholtz type is given by

$$\nabla^2 u + \lambda^2 u = 0. \tag{34}$$

In this case, we let  $\lambda = 1$  and pose inhomogeneous boundary condition

$$u = \sin x. \tag{35}$$

It is obvious that Eq. (35) is also a particular solution of Eq. (34). The BKM collocation of this test example is shown schematically in Fig. 1. The DRBEM discretization with 16 boundary and 17 interior points is illustrated in Fig. 2. Numerical results by the present BKM is displayed in Table 1

together with those by the DRBEM for comparison.

The numbers in brackets of Table 1 mean the total nodes used. It is found that the present BKM converges very quickly. This demonstrates that the BKM enjoys the super-convergent property as in the other types of collocation methods [27]. The BKM solutions using 7 nodes are adequately accurate. In contrast, the standard BEM has only low order of convergence speed due to its lower order polynomial approximation [2]. Please note that in this case, there is no particular solution to be approximated by using the RBF and DRM in the BKM since the inhomogeneous term does not appear.

#### 4.2. Laplace equation

Reader may argue that it is somehow unfair to choose the homogeneous Helmholtz equation to compare the BKM and DRBEM which uses the Laplace fundamental solution in the above example. To further justify the exponential convergence of the BKM through comparison with the BEM, in the following we exam Laplace equation

$$\nabla^2 u = 0 \tag{36}$$

with boundary condition

$$u = x + y. \tag{37}$$

Eq. (37) is easily found to be a particular solution of Eq. (36). This homogeneous problem is typically well suited to be solved by the standard BEM technique. In contrast, there appears inhomogeneous term in the BKM formulation to apply the non-singular solution of the Helmholtz operator. Therefore, this is a persuasive example to show up the accuracy and efficiency of the BKM vis-a-vis the BEM.

Eq. (36) is rewritten as

$$\nabla^2 u + u = u. \tag{38}$$

The left inhomogeneous term  $u$  is approximated by the MQ in terms of the DRM as shown in the previous section 2. The numerical results are displayed in Table 2, where the BEM solutions come from [25].

The MQ shape parameter  $c$  is set 25 for both 3 and 5 boundary knots in the BKM. It is observed that the solutions are not sensitive to the parameter  $c$ . It is surprising to see from Table 2 that the BKM solutions using 3 boundary knots achieve the accuracy of four significant digits and are far more accurate than the standard BEM solution using 16 boundary nodes. This striking accuracy of the BKM with very few knots again validates its spectral convergence. In this case only boundary points are employed to approximate the particular solution by the DRM and RBF. It is interesting to note that the coefficient matrix of the BEM and BKM are both fully populated. Unlike the BEM, however, the BKM has symmetric coefficient matrix for all self-adjoint operators with only one type of boundary conditions as this case. In next section 5.4, we will propose some possible strategy to conserve this symmetric matrix structure in the BKM to the problems with several types of boundary conditions.

### 4.3. Convection diffusion equation

The FEM and FDM encounter some difficulty to produce accurate solution to the systems involving the first order derivative of convection term. Special care need be taken to handle this problem. It is claimed that the BEM does not suffer such accuracy problem. In particular, the DRBEM was said to be very suitable for this type problem [25]. As a starting point applying the BKM to general system equation, we consider the convection diffusion equation

$$\nabla^2 u = -\partial u / \partial x, \tag{39}$$

which is given in [25] to test the DRBEM. The boundary condition is stated as

$$u = e^{-x}, \tag{40}$$

which also constitutes a particular solution of this problem. By adding  $u$  on dual sides of Eq. (39), we have

$$\nabla^2 u + u = u - \partial u / \partial x. \quad (41)$$

The results by both the DRBEM and BKM are listed in Table 3.

The MQ shape parameter is chosen 4 in this calculation. The BKM employs 7 boundary knots and 8 or 11 internal knots. In contrast, the DRBEM [25] applied 16 boundary and 17 inner nodes. It is noted that in this case the use of the interior points can improve the solution accuracy evidently. This is due to the fact that the governing equation has convection-diffusion domain-dominant solution. Only by using boundary nodes, the Helmholtz operator in the present BKM and Laplace operator in the DRBEM [25] can not capture well convection effects of the system equation. In next section 5.1, we will give some further discussions to this type of problems. It is found from Table 3 that both the BKM and DRBEM achieve the salient accurate solutions. The BKM with fewer nodes outperforms the DRBEM in computational efficiency due to the super-convergent features of the MQ interpolation and global BKM collocation scheme.

Furtherly consider equation

$$\nabla^2 u = -\partial u / \partial x - \partial u / \partial y \quad (42)$$

with boundary conditions

$$u = e^{-x} + e^{-y}, \quad (43)$$

which is also a particular solution of Eq. (42). The numerical results are summarized in Table 4.

In the BKM, the MQ shape parameter is taken 5.5. We employ 8 or 11 inner points and 7 boundary knots in the BKM compared with 16 boundary nodes and 17 inner points in the DRBEM [25]. It is

seen that the BKM works equally in this case. It is seen that the BKM with fewer points produced almost the same accurate solutions as in the DRBEM. Considering the extremely simple and easy-to-use merits of the BKM, the method is obviously superior to the DRBEM.

## **5. Further developments**

In the foregoing section 4, we only dealt with the two-dimensional linear stationary problems. By its very basis, the BKM can be handily extended to the nonlinear, three-dimensional, time-dependent, and any types of partial differential systems. This section will brief some relative theoretical and numerical results we have so far finished. The purpose is to expose promising potentials of the BKM in some important aspects.

### **5.1. Nonlinear problems**

One of the main challenges in computational engineering and science is to solve nonlinear problems. The standard solution procedures of nonlinear modeling equations are mostly originated from the Newton method and its variants [28]. For these methods, the proper determination of initial guess solution and evaluation of Jacobian matrix and its inverse are either difficult or very computationally expensive tasks. It is highly desirable to circumvent or ease to some extent these difficulties. It is noted that most of nonlinear systems are comprised of nonlinear governing equations and linear boundary conditions. This reminds us that the boundary-only technique may provide such a possible strategy to develop one-step nonlinear analysis technique. In particular, the BKM, MFS and DRBEM are feasible choice of the method due to their truly boundary-only merit for general nonlinear problems compared with normal BEM techniques, which are often cumbersome in solving nonlinear system due to the fact that it is nearly impossible to find fundamental solution of most practically significant nonlinear problems.

The basic methodology presented here is that we should as possible not use the internal points to formulate linear BKM modeling equations of nonlinear problems at boundary knots. To clearly state this idea, consider the following nonlinear equation in two dimensions

$$\nabla^2 u + u_{xx} u = 2 + 2x^2 \quad (44)$$

with boundary condition

$$u = x^2, \quad (45)$$

which is also the solution of this problem. This case is not from [25]. Adding  $u$  to both sides of Eq. (44), we have

$$\nabla^2 u + u = u - u_{xx} u + 2 + 2x^2. \quad (46)$$

Before modeling, we here introduce concept of the Hadmard matrix product to simplify the nonlinear formulation expression [29].

**Definition 5.1** Let matrices  $A=[a_{ij}]$  and  $B=[b_{ij}] \in C^{N \times M}$ , the Hadamard product of matrices is defined as  $A \circ B = [a_{ij} b_{ij}] \in C^{N \times M}$ . where  $C^{N \times M}$  denotes the set of  $N \times M$  real matrices.

Following the standard BKM procedure, we have

$$\alpha = A_\phi^{-1} \left[ u - \left( \frac{\partial^2 A_\phi}{\partial x^2} A_\phi^{-1} u \right) \circ u + 2x^2 + 2 \right] \quad (47)$$

and

$$u_p = \Phi \alpha,$$

where  $A_\phi$  and  $\phi$  are respectively the RBF interpolation matrix and approximate particular solutions as given in Eq. (26). If only boundary knots are employed, we have

$$Jc = x^2 - \Phi \left\{ A_\phi^{-1} \left[ x^2 - \frac{2}{x} \circ \left( \frac{\partial^2 A_\phi}{\partial x^2} A_\phi^{-1} x^2 \right) + 2x^2 + 2 \right] \right\} = b(x), \quad (48)$$

where  $J$  and  $c$  are boundary interpolation matrix of the Bessel function and the corresponding unknown coefficient vector as in Eq. (30). Eq. (48) is linear algebraic equations of boundary modeling, which can be solved readily by a linear solver. Table 5 displays some computed results. The MQ shape parameter is taken 2. The average relative error is 0.91%. It is seen from Table 5 that solutions at all knots are sufficiently accurate.

However, the present scheme does not work equally well for any cases. For example, let us consider the Berger equation of steady-state situation

$$\nabla^2 u + u_x u = 0. \quad (49)$$

Here we still use data given in [25], where the boundary condition is given by

$$u = 2/x. \quad (50)$$

Eq. (50) is also a particular solution of Berger equation (49). Note that the origin of the Cartesian coordinates system in Fig. 1 is displaced to the node (3,0) to circumvent singularity at  $x=0$ . The solutions with the BKM linear modeling only at boundary knots are listed in Table 6.

The shape parameter is taken 1 in the MQ. The average relative absolute error is 3.97%. It is worth stressing that the Berger equation has the structure of domain-dominant convection diffusion equation and  $u$  in Eq. (49) is in fact a velocity. We also found in the previous linear convection diffusion cases 4.3 that the average relative error is 0.56% if with 8 interior points and is degraded to 10.7% if without interior points. The similar problem also happened to the DRBEM solution of the convection diffusion problem using the Laplace fundamental solution [25]. Therefore, it is the convection terms rather than nonlinear constitutions which cause the inaccuracy of the solution if not using inner points. As was pointed out in [25], the internal nodes used in DRM analysis is not in general a condition to obtain accurate solution. In most cases, the use of some interior nodes is useful to improve the solution

accuracy. The inaccuracy in the present case is mainly due to the inefficiency in the RBF approximation of particular solution with convection characteristics if without using interior nodes. If the proper RBFs are applied to capture the system convection effects accurately in approximation of particular solution, the internal points will not be necessary to improve accuracy. By analogy with multiple reciprocity BEM [30], it may be feasible to apply the zero-order Bessel function of the first kind repeatedly as the RBF to approximation of particular solution (high-order general solution), which may be superior to the MQ in this case. In general, it may be a significant research topic to combine the multiple reciprocity method and RBF to approximate the particular solution in the BKM. We will investigate this scheme in next stage of work.

It is expected that if inner points are used in the BKM solution of Berger case, the accuracy will be improved greatly as in the DRBEM [25]. However, the use of internal node will lead to nonlinear systems of algebraic equations although its size may be minimal. In the DRBEM, it is reported [25] that the use of the fundamental solution of convection diffusion equation can significantly improve the solution accuracy of linear transient convection diffusion problem without using any inner point. This suggests us that a non-singular general solution of convection-diffusion equation may be much more suitable for this type of problem. For two-dimensional steady-state convection-diffusion equation

$$D\nabla^2\phi + v_x \frac{\partial\phi}{\partial x} + v_y \frac{\partial\phi}{\partial y} + k\phi = 0, \quad (51)$$

where  $v_x$  and  $v_y$  are the components of the velocity vector  $v$ ,  $D$  is the diffusivity coefficient and  $k$  represents the reaction coefficient. We have non-singular general solution

$$\phi^* = \frac{1}{2\pi D} e^{-\frac{v \cdot r}{2D}} J_0(\mu r), \quad (52)$$

where

$$\mu = \left[ \left( \frac{|v|}{2D} \right)^2 + \frac{k}{D} \right]^{\frac{1}{2}} \quad (53)$$

and marked  $\mathbf{r}$  denotes the distance vector between the source and response knots.  $J_0$  is the zero-order Bessel function of the first kind. The above general solution (52) is in fact the non-singular part of the fundamental solution of convection diffusion equation. According to the experience in the DRBEM [25], alternative non-singular general solution (52), which is consistent with the convection behavior, may make the use of the interior points unnecessary in the BKM for the previous linear convection diffusion case 4.3 and the present nonlinear Burger equation. It is worth pointing out that since in nonlinear cases velocity values in non-singular general solution (52) may vary at different boundary knots, the RBFs used on distinct boundary knots may differ in velocity coefficients. Namely, knot-dependent RBFs should be used. In this way, the resulting formulation of Burger equation will be linear algebraic equations if the only boundary nodes are applied.

The convection diffusion is not an easy problem to handle. The oscillation and damping manifest in the FEM and FDM solution of this type of problems. Even if the interior knots are necessarily used in the BKM to assure enough accurate solution in such case, the number of the knots will be much fewer than the domain-type methods. Consequently, the BKM will results in much smaller nonlinear algebraic equations. Moreover, the solution by the linear BKM modeling without interior nodes can provide a reliable initial guess solution for nonlinear iteration computation. The research on these interesting issues is now under way and is left to address in detail in the subsequent paper. On the other hand, it is conceivable that we can use the same strategy to handle nonlinear boundary condition problem. Namely, inverse to the present BKM scheme, we choose the RBFs, which satisfy the nonlinear boundary equation. Then we have a domain-type linear RBF collocation modeling.

Recently Chen et al. [29] demonstrated that the special matrix product approach is very promising and powerful techniques to nonlinear stability analysis, uncoupling computation, evaluation of Jacobian matrix, and construction of simple, efficient and robust iteration formulas. The Hadamard product and SJT product are two types of special matrix product very useful to all point-approximation techniques, which include the finite difference, finite volume, collocation, and RBF-related techniques (DRBEM, MFS and the present BKM). In this section, we only used the Hadamard product to simplify the representation of the RBF formulation. A significant research topic will be to explore the potential of a combined use of the Hadamard and SJT products technique and the BKM to solve the nonlinear problems.

## 5.2. Non-singular general solutions for some differential operators and others

We here present non-singular general solutions of some often-used 2D and 3D operators. In the following equations  $r$  means distance variable and  $c$  is coefficients. For 2D equation

$$\nabla^2 u - \lambda^2 u = 0, \quad (54)$$

the non-singular particular solution is

$$u^* = c I_0(\lambda r), \quad (55)$$

where  $I_0$  is the zero-order modified Bessel function of the first kind. For the 3D Helmholtz-like operator,

$$\nabla^2 u \pm \lambda^2 u = 0, \quad (56)$$

we have the non-singular general solution

$$u^* = c \frac{\sin(\lambda r)}{r}, \quad (57)$$

and

$$u^* = c \frac{\sinh(\lambda r)}{r}, \quad (58)$$

where  $\sinh$  denotes the hyperbolic function. For the 2D biharmonic operator

$$\nabla^4 w - \lambda^2 w = 0, \quad (59)$$

we have the non-singular general solution

$$w^* = c_1 J_0(\lambda r) + c_2 I_0(\lambda r), \quad (60)$$

where  $J_0$  and  $I_0$  are respectively the Bessel and modified Bessel functions of the first kind of the zero-order. The non-singular general solution of the 3D biharmonic operator is given by

$$w^* = c_1 \frac{\sin(\lambda r)}{r} + c_2 \frac{\sinh(\lambda r)}{r}, \quad (61)$$

Katsikadelis and Nerantzaki [31] proposed a simple strategy to apply the DRBEM to any partial differential equations which do not include standard Laplace or biharmonic operators. The strategy can be readily extended to the BKM calculation of any linear or nonlinear systems yet keeping its boundary-only merit.

### 5.3. Time-dependent problems

For time-dependent problem, a numerical time integrator is often employed to approximate time derivative after spatial modeling is done. The use of marching scheme of time integrator involves some difficult issues relating to stability and accuracy [32]. Another methodology is the mode analysis. However, the method is not very applicable for many cases such as shock. We here hope to introduce an alternative technique free from computing temporal derivative within the BKM frame. To explain clearly, let us consider 2D heat and diffusion governing equation

$$\Delta u = \frac{1}{k} \frac{\partial u}{\partial t}. \quad (62)$$

By separation, we have

$$u = \phi(r)\varphi(t). \quad (63)$$

Substituting the above equation into governing equation (62) yields

$$\varphi\Delta\phi = \frac{1}{k}\phi\frac{\partial\varphi}{\partial t}. \quad (64)$$

Let

$$\phi = J_0(r), \quad (65)$$

where  $J_0$  is the Bessel function of the first kind of zero order, we have

$$\frac{d\varphi}{dt} + k\varphi = 0. \quad (66)$$

Its general solution is

$$\varphi(t) = Ae^{-kt}. \quad (67)$$

Therefore, we have time-dependent general solution of transient heat and diffusion equation

$$u^*(r, t) = Ae^{-kt}J_0(r), \quad (68)$$

where  $A$  is unknown constant determined by boundary and initial conditions. Furthermore, consider wave equation

$$\Delta u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \quad (69)$$

Similarly, we get the general solution

$$u^*(r, t) = [C\cos(ct) + D\sin(ct)]J_0(r), \quad (70)$$

where  $C$  and  $D$  are two constants. By using the above non-singular time-dependent general solutions (68) and (70) as the RBFs, we can obtain the BKM boundary-only formulation for time-dependent diffusion problems without time derivative. The strategy is straightforward and in fact equivalent to the BEM using time-dependent fundamental solution. The difficulty in applying this scheme may lie in how to satisfy the initial conditions inside domain as in the time-dependent BEM.

Recently Zhu et al. [33] used the Laplace transforms to eliminate temporal derivative in the DRBEM solution of time-dependent problems. The accuracy and efficiency of their method are satisfactory.

More recently Golberg and Chen [2] combined this approach with the MFS to solve the transient diffusion problems successfully. The key to efficient use of the Laplace transforms is to implement the Stehfest  $s$  algorithm [34] for the accurate numerical inversion of Laplace transform. It is very obvious that such Laplace transforms methodology can be extended to the BKM solution of transient problems.

#### **5.4. Symmetricity, localization and system conditioning**

The global support BKM can produce the symmetric modeling equations of self-adjoint operators subject to one kind of boundary condition. However, its coefficient matrix is dense and loses symmetricity due to mixed boundary conditions, while the normal BEM and global support MFS also always encounter full and asymmetric matrix. Solving equations with large full asymmetric coefficient matrix is rather computationally costly. It is practically significant for the BKM to preserve the symmetric structure in modeling self-adjoint operator and to result in a sparse banded coefficient matrix.

Like the BKM, the RBF domain collocation method also loses the symmetric matrix structure for self-adjoint operators having more than one type of boundary conditions. A Hermite interpolation RBF has been proposed to preserve the symmetric structure of the resultant coefficient matrix. The strategy is also applicable to the BKM. Namely, the boundary collocation in the BKM will be performed locally within the same type of boundary condition separately. These collocation equations in different local boundary zones will then be matched through keeping  $C_0$  continuity at interface knots in such a way that we get a sparse banded symmetric matrix of the boundary modeling. As was pointed out in [35], the symmetric structure can also assure matrix well-conditioned. Another issue relating to the full coefficient matrix is globally-supported RBF to approximate particular solutions. Recently, Chen et al.

[19] applied compactly-supported RBF to obtain the sparse banded system matrix in the approximation of particular solution in the MFS and DRBEM. Obviously, this methodology is applicable to the particular solution approximate by RBF and DQM within the frame of the BKM.

A combines use of the CS-RBF approximate particular solution and local boundary collocation builds the BKM a local numerical modeling technique as a whole. The localization eliminates the ill-conditioning of the global BKM when applied to large system problem with a huge number of nodes. Numerical investigation will be subject of further study along this line.

## **6. Concluding remarks**

The present BKM can be regarded one kind of the Trefftz method [36], where the trial function is required to satisfy governing equation. The BKM distinguishes the other Trefftz techniques in that the RBF is used through non-singular general solution. The BKM also looks like the MFS in many senses except that the non-singular general solution is used instead of the singular fundamental solution. The shortcomings of the MFS using fictitious boundary are eliminated in the present BKM and the robustness of solution is greatly improved. The term BKM can be interpreted a mixed boundary-only modeling technique combining several powerful methodologies of the DRM, RBF, and non-singular general solution. The method may be especially attractive for three-dimensional problem due to the RBF's dimensionally-increased order of convergent rate recently found by Kansa and Hon [9].

The proper choice of the RBF to approximate the particular solution is one of pivotal issues affecting the BKM efficiency. Although the MQ holds exponential convergence merit, the problem-dependent shape parameter degrades its attractiveness. Among all currently available RBFs, the CS-RBF and TPS are more practically useful in general. We plan to further combine these two types of RBFs with

the BKM to investigate some benchmark problems. In particular, the CS-RBF can lead to a banded systematic interpolation matrix and is therefore preferred. On the other hand, it is very significant to develop some problem-dependent RBFs which at least partly comply the intrinsic characteristics of the targeted problems as done successfully in [12,13] for the DRBEM solution of axisymmetric and infinite diffusion problems. For example, even in the domain-type RBF method, it is expected that the zero-order Bessel of the first kind or non-singular general solution of convection diffusion equation may be the RBF of the choice in the collocation of various wave, convection, heat, and diffusion problems due to their conformation with governing equations at least in part. The same care should also be taken in the DRM and RBF approximation of particular solution within the BKM.

In conclusion, the presented BKM is inherently possesses desirable numerical merits which include meshless, boundary-only, integration-free, exponential convergence, and mathematical simplicity. The use of the method is remarkably easy, especially for complicated shapes and higher dimensions. Beside these, by using the local boundary collocation and CS-RBFs, the resulting BKM approximate matrix as a whole is expected to have a symmetric sparse banded structure for self-adjoint operator problems with any boundary conditions. The prospect developing linear BKM boundary modeling of some nonlinear problems is especially attractive, which will be emphasis of the near future work. In addition, more numerical experiments to test the BKM will be beneficial. This paper can be regarded a starting point of a series of work afterward.

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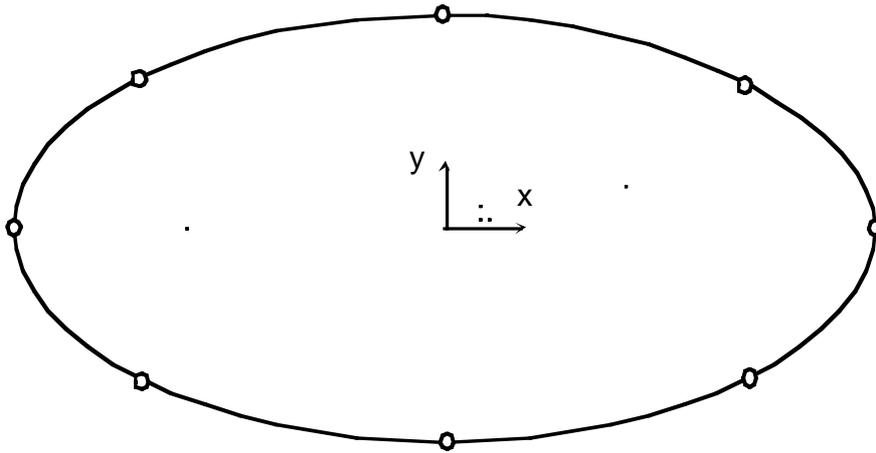


Fig. 1. TheBKM boundary knots

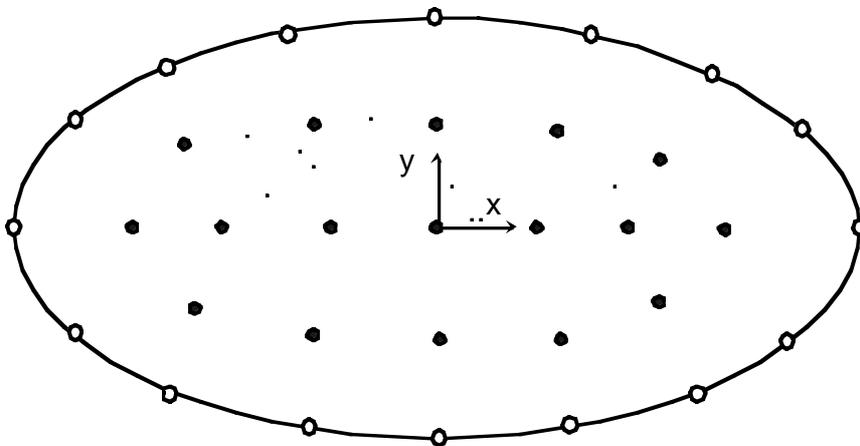


Fig. 2. TheDRBEM boundary and internal knots

**Table 1. Results for Helmholtz equation**

x	y	Exact	DRBEM (33)	BKM (7)	BKM (11)
1.5	0.0	0.997	0.994	0.999	0.997
1.2	-0.35	0.932	0.928	0.931	0.932
0.6	-0.45	0.565	0.562	0.557	0.565
0.0	0.0	0.0	0.0	0.0	0.0
0.9	0.0	0.783	0.780	0.779	0.783
0.3	0.0	0.296	0.294	0.289	0.296
0.0	0.0	0.0	0.0	0.0	0.0

**Table 2. Results for Laplace equation.**

x	y	Exact	BEM (16)	BKM (3)	BKM (5)
1.5	0.0	1.500	1.507	1.500	1.500
1.2	-0.35	0.850	0.857	0.850	0.850
0.6	-0.45	0.150	0.154	0.150	0.150
0.0	0.0	-0.450	-0.451	-0.450	-0.450
0.9	0.0	0.900	0.913	0.900	0.900
0.3	0.0	0.300	0.304	0.300	0.300
0.0	0.0	0.0	0.0	0.0	0.0

**Table 3. Results for  $\nabla^2 u = -\partial u/\partial x$** 

x	y	Exact	DRBEM (33)	BKM(15)	BKM (18)
1.5	0.0	0.223	0.229	0.229	0.224
1.2	-0.35	0.301	0.307	0301	0.305
0.0	-0.45	1.000	1.003	1.010	1.000
-0.6	-0.45	1.822	1.819	1.822	1.818
-1.5	0.0	4.482	4.489	4.484	4.477
0.3	0.0	0.741	0.745	0.744	0.743
-0.3	0.0	1.350	1.348	1.353	1.354
0.0	0.0	1.000	1.002	1.003	1.004

**Table 4. Results for  $\nabla^2 u = -\partial u/\partial x - \partial u/\partial y$**

<b>x</b>	<b>y</b>	<b>Exact</b>	<b>DRBEM (33)</b>	<b>BKM(15)</b>	<b>BKM (18)</b>
1.5	0.0	1.223	1.231	1.225	1.224
1.2	-0.35	1.720	1.714	1.725	1.723
0.0	-0.45	2.568	2.557	2.546	2.551
-0.6	-0.45	3.390	3.378	3.403	3.405
-1.5	0.0	5.482	5.485	5.490	5.491
0.3	0.0	1.741	1.731	1.729	1.731
-0.3	0.0	2.350	2.335	2.349	2.350
0.0	0.0	2.000	1.989	1.992	1.993

**Table 5. Results for nonlinear equation (44)**

<b>x</b>	<b>y</b>	<b>Exact</b>	<b>BKM (5)</b>	<b>Relative error %</b>
4.5	0.0	20.25	20.34	-0.44
4.2	-0.35	17.64	17.74	-0.58
3.6	-0.45	12.96	13.07	-0.86
3.0	-0.45	9.00	9.09	-0.98
2.4	-0.45	5.76	5.82	-0.94
1.8	-0.35	3.24	3.26	-0.77
1.5	0.0	2.25	2.25	-0.18
3.9	0.0	15.21	15.36	-0.99
3.3	0.0	10.89	11.03	-1.28
3.0	0.0	9.00	9.12	-1.32
2.7	0.0	7.29	7.38	-1.31
2.1	0.0	4.41	4.46	-1.14

**Table 6. Results for Burger equation**

<b>x</b>	<b>y</b>	<b>Exact</b>	<b>BKM (5)</b>	<b>Relative error %</b>
4.5	0.0	0.444	0.479	-7.9
4.2	-0.35	0.476	0.515	-8.2
3.6	-0.45	0.555	0.585	-5.4
3.0	-0.45	0.666	0.666	0.15
2.4	-0.45	0.833	0.808	3.1
1.8	-0.35	1.111	1.089	2.0
1.5	0.0	1.333	1.300	2.5
3.9	0.0	0.512	0.563	-9.7
3.3	0.0	0.606	0.632	-4.2
3.0	0.0	0.666	0.672	-7.3
2.7	0.0	0.740	0.725	2.0
2.1	0.0	0.952	0.918	3.6

Fig. 1. The BKM boundary only knots

Fig. 2. The DRBEM boundary and internal knots.