SOLVABILITY IN THE SENSE OF SEQUENCES FOR SOME NON- FREDHOLM OPERATORS WITH A DRIFT AND THE CUBED LAPLACIAN

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Abstract. We study the solvability of certain linear nonhomogeneous elliptic problems and demonstrate that under the given technical assumptions the convergence in L^2 of their right-hand sides implies the existence and the convergence in H^6 of the solutions. The equations contain the sixth order differential operators with or without the Fredholm property, particularly the sixth derivative operator, on the whole real line or on a finite interval with periodic boundary conditions. We establish that the drift term involved in these problems provides the regularization of the solutions.

1. Introduction

Consider the problem

$$(-\Delta + V(x))u - au = f, \tag{1.1}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant, and the function $V(x)$ tends to 0 at the infinity. If $a \geq 0$, then the essential spectrum of the operator $A : E \to F$, which corresponds to the left side of equation (1.1) contains the origin. As a consequence, such operator does not satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present article deals with the studies of the certain properties of the operators of this kind. Note that the elliptic equations involving the non-Fredholm operators were treated extensively in the recent years (see [10], [11], [12], [17], [18], [20], [21], [22], [23], [24], [25], [26], [27], also [3]) along with their potential applications to the theory of reaction-diffusion problems (see [7], [8]). Fredholm structures, topological invariants and their applications were discussed in [9]. The articles [13] and [16] are devoted to the understanding of the Fredholm and properness properties of the quasilinear elliptic systems of the second order and of the operators of this kind on $\mathbb{R}^N$. The exponential decay and Fredholm properties in the second-order quasilinear elliptic systems of equations were considered in [14]. In particular, if the constant $a = 0$, our operator A satisfies the Fredholm property

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in the certain properly chosen weighted spaces (see [1], [2], [3], [5], [6]). However, the case when a does not vanish is significantly different and the method developed in these works cannot be used.

One of the significant questions about the problems involving the non-Fredholm operators is their solvability. We address it in the following setting. Let f_n be a sequence of functions in the image of the operator A, so that $f_n \to f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$. Denote by u_n a sequence of functions from $H^2(\mathbb{R}^d)$ such that

$$Au_n = f_n, \ n \in \mathbb{N}.$$

Because the operator A does not satisfy the Fredholm property, the sequence u_n may not be convergent. We call a sequence u_n, such that $Au_n \to f$ in $L^2(\mathbb{R}^d)$ a solution in the sense of sequences of the equation $Au = f$ (see [17]). If such sequence converges to a function u_0 in the norm of the space E, then u_0 is a solution of this problem. The solution in the sense of sequences is equivalent in this sense to the usual solution. However, in the case of non-Fredholm operators this convergence may not hold or it can occur in some weaker sense. In this case, the solution in the sense of sequences may not imply the existence of the usual solution. In the present work we will find the sufficient conditions of equivalence of solutions in the sense of sequences and the usual solutions. In the other words, we will determine the conditions on sequences f_n under which the corresponding sequences u_n are strongly convergent.

In the first part of the work we treat the problem with the transport term

$$-\frac{d^6u}{dx^6} - b\frac{du}{dx} - au = f(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where $a \geq 0$ and $b \in \mathbb{R}, \ b \neq 0$ are the constants and the right side is square integrable. The equation with the drift in the context of the Darcy's law describing the fluid motion in the porous medium was studied in [23]. The transport term is important when studying the emergence and propagation of patterns arising in the theory of speciation (see [19]). Nonlinear propagation phenomena for the reaction-diffusion type problems containing the drift term were covered in [4]. Solvability conditions for a linearized Cahn-Hilliard equation of sixth order were determined in [24]. Solvability in the sense of sequences for some non Fredholm operators with drift and superdiffusion was covered in [27]. Evidently, the operator involved in the left side of (1.2)

$$L_{a,\ b} := -\frac{d^6}{dx^6} - b\frac{d}{dx} - a : \quad H^6(\mathbb{R}) \to L^2(\mathbb{R}) \tag{1.3}$$

is non-selfadjoint. By virtue of the standard Fourier transform

$$\widehat{f}(p) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ipx}dx, \quad p \in \mathbb{R} \tag{1.4}$$

it can be trivially obtained that the essential spectrum of the operator $L_{a,\ b}$ is given by

$$\lambda_{a,\ b}(p) := p^6 - a - ibp, \quad p \in \mathbb{R}.$$

Obviously, for $a > 0$ our operator $L_{a,\ b}$ is Fredholm, since the origin does not belong to its essential spectrum. But when $a = 0$, the operator $L_{0,\ b}$ fails to satisfy the Fredholm property because its essential spectrum contains the origin.

Note that in the absence of the drift term we are dealing with the self-adjoint operator

$$-\frac{d^6}{dx^6} - a : \quad H^6(\mathbb{R}) \to L^2(\mathbb{R}), \quad a \geq 0,$$

which is non-Fredholm.

We write down the corresponding sequence of the approximate equations with $m \in \mathbb{N}$, namely

$$-\frac{d^6 u_m}{dx^6} - b\frac{du_m}{dx} - au_m = f_m(x), \quad x \in \mathbb{R}, \tag{1.5}$$

where $a \geq 0$ and $b \in \mathbb{R},\ b \neq 0$ are the constants. The right sides of (1.5) converge to the right side of (1.2) in $L^2(\mathbb{R})$ as $m \to \infty$.

Let us define the inner product of two functions

$$(f(x), g(x))_{L^2(\mathbb{R})} := \int_{-\infty}^{\infty} f(x)\bar{g}(x)dx, \tag{1.6}$$

with a slight abuse of notations when these functions are not square integrable. Evidently, if $f(x) \in L^1(\mathbb{R})$ and $g(x)$ is bounded, then the integral considered above is well defined, like for instance in the case of the functions involved in the orthogonality conditions (1.8) and (1.9) of Theorems 1.1 and 1.2 below.

For our problem (1.2) on the finite interval $I := [0, 2\pi]$ with periodic boundary conditions (see (1.14)), we will use the inner product analogous to (1.6), replacing the real line with I. In the first part of the present work we will consider the space $H^6(\mathbb{R})$ equipped with the norm

$$\|u\|^2_{H^6(\mathbb{R})} := \|u\|^2_{L^2(\mathbb{R})} + \left\|\frac{d^6 u}{dx^6}\right\|^2_{L^2(\mathbb{R})}. \tag{1.7}$$

When dealing with the norm $H^6(I)$ later on, we will replace $\mathbb{R}$ with I in formula (1.7). Our first main proposition is as follows.

Theorem 1.1. *Let the constants $a \geq 0,\ b \in \mathbb{R},\ b \neq 0$ and $f(x) \in L^2(\mathbb{R})$.*

a) If $a > 0$, then problem (1.2) possesses a unique solution $u(x) \in H^6(\mathbb{R})$.

b) If $a = 0$ and $xf(x) \in L^1(\mathbb{R})$, then equation (1.2) has a unique solution $u(x) \in H^6(\mathbb{R})$ if and only if the orthogonality relation

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \tag{1.8}$$

holds.

Obviously, the expression in the left side of (1.8) is well defined by means of the trivial reasoning analogous to the proof of Fact 1 of [21].
Note that the argument in the case a) of the theorem above is not based on the orthogonality relations. But when the transport term is absent and we are dealing with the non-Fredholm operator, we need the orthogonality conditions to establish the solvability of the problem in the lower dimensions (see e.g. Lemmas 5 and 6 of [25] for the case of the standard Laplacian). Therefore, the introduction of the drift term into our equation provides the regularization of the solutions.
Our next result is about the solvability in the sense of sequences for our problem on whole the real line.

Theorem 1.2. *Let the constants* $a \geq 0$, $b \in \mathbb{R}$, $b \neq 0$ *and* $m \in \mathbb{N}$, $f_m(x) \in L^2(\mathbb{R})$, *such that* $f_m(x) \to f(x)$ *in* $L^2(\mathbb{R})$ *as* $m \to \infty$.

a) If $a > 0$, *then problems (1.2) and (1.5) admit unique solutions* $u(x) \in H^6(\mathbb{R})$ *and* $u_m(x) \in H^6(\mathbb{R})$ *respectively, so that* $u_m(x) \to u(x)$ *in* $H^6(\mathbb{R})$ *as* $m \to \infty$.

b) If $a = 0$, *let* $xf_m(x) \in L^1(\mathbb{R})$, *such that* $xf_m(x) \to xf(x)$ *in* $L^1(\mathbb{R})$ *as* $m \to \infty$. *Moreover,*

$$(f_m(x), 1)_{L^2(\mathbb{R})} = 0, \quad m \in \mathbb{N} \tag{1.9}$$

holds. Then equations (1.2) and (1.5) possess unique solutions $u(x) \in H^6(\mathbb{R})$ *and* $u_m(x) \in H^6(\mathbb{R})$ *respectively, so that* $u_m(x) \to u(x)$ *in* $H^6(\mathbb{R})$ *as* $m \to \infty$.

The second part of the article deals with the studies of our problem on the finite interval with the periodic boundary conditions (see (1.14)), i.e. $I := [0,\ 2\pi]$, namely

$$-\frac{d^6u}{dx^6} - b\frac{du}{dx} - au = f(x), \quad x \in I, \tag{1.10}$$

where $a \geq 0$ and $b \in \mathbb{R}$, $b \neq 0$ are the constants and the right side of (1.10) is continuous and periodic. Evidently,

$$\|f\|_{L^1(I)} \leq 2\pi \|f\|_{C(I)} < \infty, \quad \|f\|_{L^2(I)} \leq \sqrt{2\pi}\|f\|_{C(I)} < \infty. \tag{1.11}$$

Hence, $f(x) \in L^2(I)$ as well. Let us use the Fourier transform

$$f_n := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x)e^{-inx}dx, \quad n \in \mathbb{Z}, \tag{1.12}$$

such that

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \frac{e^{inx}}{\sqrt{2\pi}}.$$

Clearly, the non-selfadjoint operator contained in the left side of (1.10)

$$l_{a,\ b} := -\frac{d^6}{dx^6} - b\frac{d}{dx} - a : \quad H^6(I) \to L^2(I) \tag{1.13}$$

is Fredholm. By virtue of (1.12), it can be easily verified that the spectrum of $l_{a,\ b}$ is given by

$$\lambda_{a,\ b}(n) := n^6 - a - ibn, \quad n \in \mathbb{Z}$$

and the corresponding eigenfunctions are the Fourier harmonics

$$\frac{e^{inx}}{\sqrt{2\pi}}, \ n \in \mathbb{Z}.$$

The eigenvalues of the operator $l_{a,\ b}$ are simple, as distinct from the case without the drift term, when the eigenvalues corresponding to $n \neq 0$ are twofold degenerate. The appropriate function space here $H^6(I)$ is given by

$$\{u(x) : I \to \mathbb{C} \mid u(x), u^{(VI)}(x) \in L^2(I), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$
$$u''(0) = u''(2\pi), \quad u'''(0) = u'''(2\pi), \quad u^{(IV)}(0) = u^{(IV)}(2\pi),$$
$$u^{(V)}(0) = u^{(V)}(2\pi)\}. \tag{1.14}$$

For the technical purposes, we introduce the following auxiliary constrained subspace

$$H_0^6(I) = \{u(x) \in H^6(I) \mid (u(x), 1)_{L^2(I)} = 0\}. \tag{1.15}$$

Note that (1.15) is a Hilbert spaces as well (see e.g. Chapter 2.1 of [15]). Evidently, if $a > 0$, the kernel of the operator $l_{a,\ b}$ is trivial. If $a = 0$, we consider

$$l_{0,\ b} : \quad H_0^6(I) \to L^2(I).$$

Obviously, this operator has a trivial kernel as well.
Let us write down the corresponding sequence of the approximate equations with $m \in \mathbb{N}$, namely

$$-\frac{d^6 u_m}{dx^6} - b\frac{du_m}{dx} - au_m = f_m(x), \quad x \in I, \tag{1.16}$$

where $a \geq 0, \ b \in \mathbb{R}, \ b \neq 0$ are the constants. The right sides of (1.16) are continuous, periodic and tend to the right side of (1.10) in $C(I)$ as $m \to \infty$. The goal of Theorems 1.3 and 1.4 below is to demonstrate the formal similarity of the results on the finite interval with periodic boundary conditions to the ones derived for the whole real line situation in Theorems 1.1 and 1.2 above.

Theorem 1.3. *Let the constants $a \geq 0, \ b \in \mathbb{R}, \ b \neq 0$ and $f(0) = f(2\pi), \ f(x) \in C(I)$.*

a) If $a > 0$, then problem (1.10) has a unique solution $u(x) \in H^6(I)$.

b) If $a = 0$, then equation (1.10) admits a unique solution $u(x) \in H_0^6(I)$ if and only if the orthogonality relation

$$(f(x), 1)_{L^2(I)} = 0 \tag{1.17}$$

is valid.

The final main statement of the article deals with the solvability in the sense of sequences for our equation on the finite interval I.

Theorem 1.4. *Let the constants $a \geq 0$, $b \in \mathbb{R}$, $b \neq 0$ and $m \in \mathbb{N}$, such that $f_m(0) = f_m(2\pi)$. Moreover, $f_m(x) \in C(I)$ and $f_m(x) \to f(x)$ in $C(I)$ as $m \to \infty$.*

a) If $a > 0$, then problems (1.10) and (1.16) possess unique solutions $u(x) \in H^6(I)$ and $u_m(x) \in H^6(I)$ respectively, such that $u_m(x) \to u(x)$ in $H^6(I)$ as $m \to \infty$.

b) If $a = 0$, let

$$(f_m(x), 1)_{L^2(I)} = 0, \quad m \in \mathbb{N}. \tag{1.18}$$

Then equations (1.10) and (1.16) admit unique solutions $u(x) \in H_0^6(I)$ and $u_m(x) \in H_0^6(I)$ respectively, such that $u_m(x) \to u(x)$ in $H_0^6(I)$ as $m \to \infty$.

Note that in the cases a) of Theorems 1.3 and 1.4 above the reasoning does not rely on the orthogonality conditions. When there are no drift terms in our equations, the situation is more singular (see formulas (3.1) and (3.7) below with $a = n_0^6$, $n_0 \in \mathbb{N}$).

Remark 1.5. *It is strongly believed that the approach developed in the present work to study the solvability of the sixth order equation can be used to treat the problem containing a shallow, short-range potential if we use the generalized Fourier transform with the functions of the continuous spectrum of the corresponding Schrödinger type operator. This issue will be considered in the consecutive work.*

2. The whole real line case

Proof of Theorem 1.1. First we establish that it would be sufficient to solve our problem in $L^2(\mathbb{R})$. Indeed, if $u(x)$ is a square integrable solution of (1.2) on the whole real line, directly from this equation under the stated assumptions we obtain that

$$-\frac{d^6u}{dx^6} - b\frac{du}{dx} \in L^2(\mathbb{R})$$

as well. Using the standard Fourier transform (1.4), we derive

$$(p^6 - ibp)\widehat{u}(p) \in L^2(\mathbb{R}).$$

Hence $\int_{-\infty}^{\infty} p^{12}|\widehat{u}(p)|^2 dp < \infty$, such that $\frac{d^6u}{dx^6} \in L^2(\mathbb{R})$. By means of the definition of the norm (1.7), we derive that $u(x) \in H^6(\mathbb{R})$ as well.
Let us demonstrate that the uniqueness of solutions for equation (1.2) holds. Suppose that

$$u_1(x),\ u_2(x) \in H^6(\mathbb{R})$$

solve (1.2). Then their difference $w(x) := u_1(x) - u_2(x) \in H^6(\mathbb{R})$ satisfies

$$-\frac{d^6 w}{dx^6} - b\frac{dw}{dx} - aw = 0.$$

Because the operator $L_{a,\ b}$ defined in (1.3) does not have any nontrivial zero modes in $H^6(\mathbb{R})$, the function $w(x)$ vanished identically on the real line.
We apply the standard Fourier transform (1.4) to both sides of problem (1.2). This gives us

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{p^6 - a - ibp}, \tag{2.1}$$

so that

$$\|u\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} \frac{|\widehat{f}(p)|^2}{(p^6 - a)^2 + b^2p^2} dp. \tag{2.2}$$

Let us first consider the case a) of the theorem. Formula (2.2) yields

$$\|u\|^2_{L^2(\mathbb{R})} \le \frac{1}{C}\|f\|^2_{L^2(\mathbb{R})} < \infty$$

as assumed. Here and below C will stand for a finite, positive constant.
Then we turn our attention to the situation when the parameter a is trivial. By means of (2.1), we easily write

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{p^6 - ibp}\chi_{\{|p|\le 1\}} + \frac{\widehat{f}(p)}{p^6 - ibp}\chi_{\{|p|>1\}}. \tag{2.3}$$

Here and further down χ_A will designate the characteristic function of a set $A \subseteq \mathbb{R}$.
Clearly, the second term in the right side of (2.3) can be bounded from above in the absolute value by $\frac{|\widehat{f}(p)|}{|b|} \in L^2(\mathbb{R})$ since $f(x)$ is square integrable on the whole real line due to our assumption.
We express

$$\widehat{f}(p) = \widehat{f}(0) + \int_0^p \frac{d\widehat{f}(s)}{ds} ds.$$

Thus, the first term in the right side of (2.3) is given by

$$\frac{\widehat{f}(0)}{p^6 - ibp}\chi_{\{|p|\le 1\}} + \frac{\int_0^p \frac{d\widehat{f}(s)}{ds} ds}{p^6 - ibp}\chi_{\{|p|\le 1\}}. \tag{2.4}$$

By virtue of definition (1.4) of the standard Fourier transform, we have

$$\left|\frac{d\widehat{f}(p)}{dp}\right| \le \frac{1}{\sqrt{2\pi}}\|xf(x)\|_{L^1(\mathbb{R})}.$$

Hence, the second term in (2.4) can be estimated from above in the absolute value by

$$\frac{1}{\sqrt{2\pi}}\frac{\|xf(x)\|_{L^1(\mathbb{R})}}{|b|}\chi_{\{|p|\le 1\}} \in L^2(\mathbb{R}).$$

Obviously, the first term in (2.4) belongs to $L^2(\mathbb{R})$ if and only if $\widehat{f}(0) = 0$. This is equivalent to orthogonality condition (1.8). $\square$

We proceed to establishing the solvability in the sense of sequences for our equation on whole the real line.

Proof of Theorem 1.2. Let us first suppose that problems (1.2) and (1.5) have unique solutions $u(x) \in H^6(\mathbb{R})$ and $u_m(x) \in H^6(\mathbb{R}),\ m \in \mathbb{N}$ respectively, such that $u_m(x) \to u(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$. This will yield that $u_m(x)$ also tends to $u(x)$ in $H^6(\mathbb{R})$ as $m \to \infty$. Evidently, from (1.2) and (1.5) we easily derive

$$\left\| -\frac{d^6}{dx^6}(u_m - u) - b\frac{d(u_m - u)}{dx} \right\|_{L^2(\mathbb{R})} \leq \|f_m - f\|_{L^2(\mathbb{R})} + a\|u_m - u\|_{L^2(\mathbb{R})}. \tag{2.5}$$

The right side of (2.5) converges to zero as $m \to \infty$ via our assumptions. By virtue of the standard Fourier transform (1.4), we obtain that

$$\int_{-\infty}^{\infty} p^{12} |\widehat{u}_m(p) - \widehat{u}(p)|^2 dp \to 0, \quad m \to \infty.$$

Hence, $\dfrac{d^6 u_m}{dx^6} \to \dfrac{d^6 u}{dx^6}$ in $L^2(\mathbb{R})$ as $m \to \infty$. Let us use the definion of the norm (1.7) to deduce that $u_m(x) \to u(x)$ in $H^6(\mathbb{R})$ as $m \to \infty$ as well.
We apply the standard Fourier transform (1.4) to both sides of (1.5). This yields

$$\widehat{u}_m(p) = \frac{\widehat{f}_m(p)}{p^6 - a - ibp}, \quad m \in \mathbb{N}. \tag{2.6}$$

Let us first treat the case a) of the theorem. By means of the result of the part a) of Theorem 1.1, problems (1.2) and (1.5) have unique solutions $u(x) \in H^6(\mathbb{R})$ and $u_m(x) \in H^6(\mathbb{R}),\ m \in \mathbb{N}$ respectively. By virtue of (2.6) along with (2.1), we arrive at

$$\|u_m - u\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} \frac{|\widehat{f}_m(p) - \widehat{f}(p)|^2}{(p^6 - a)^2 + b^2 p^2} dp.$$

Thus,

$$\|u_m - u\|_{L^2(\mathbb{R})} \leq \frac{1}{C} \|f_m - f\|_{L^2(\mathbb{R})} \to 0, \quad m \to \infty$$

as we assume. This means that in the situation when $a > 0$ we have $u_m(x) \to u(x)$ in $H^6(\mathbb{R})$ as $m \to \infty$ due to the reasoning above.
We conclude the proof of our theorem by dealing with the case when the parameter a vanishes. By virtue of the result of the part a) of Lemma 3.3 of [20], under the given conditions

$$(f(x), 1)_{L^2(\mathbb{R})} = 0 \tag{2.7}$$

holds. Then by means of the part b) of Theorem 1.1, equations (1.2) and (1.5) admit unique solutions $u(x) \in H^6(\mathbb{R})$ and $u_m(x) \in H^6(\mathbb{R})$, $m \in \mathbb{N}$ respectively when $a = 0$. Let us use (2.6) and (2.1) to express

$$\widehat{u}_m(p) - \widehat{u}(p) = \frac{\widehat{f}_m(p) - \widehat{f}(p)}{p^6 - ibp}\chi_{\{|p|\le 1\}} + \frac{\widehat{f}_m(p) - \widehat{f}(p)}{p^6 - ibp}\chi_{\{|p|>1\}}. \tag{2.8}$$

Evidently, the second term in the right side of (2.8) can be estimated from above in the absolute value by $\dfrac{|\widehat{f}_m(p) - \widehat{f}(p)|}{|b|}$, such that

$$\left\| \frac{\widehat{f}_m(p) - \widehat{f}(p)}{p^6 - ibp}\chi_{\{|p|>1\}} \right\|_{L^2(\mathbb{R})} \le \frac{1}{|b|}\|f_m - f\|_{L^2(\mathbb{R})} \to 0, \quad m \to \infty$$

as we assume. Orthogonality relations (2.7) and (1.9) imply that

$$\widehat{f}(0) = 0, \quad \widehat{f}_m(0) = 0, \quad m \in \mathbb{N}.$$

Then we can write

$$\widehat{f}(p) = \int_0^p \frac{d\widehat{f}(s)}{ds}ds, \quad \widehat{f}_m(p) = \int_0^p \frac{d\widehat{f}_m(s)}{ds}ds, \quad m \in \mathbb{N}, \tag{2.9}$$

such that it remains to obtain the bound on the norm of the expression

$$\frac{\int_0^p [\frac{d\widehat{f}_m(s)}{ds} - \frac{d\widehat{f}(s)}{ds}]ds}{p^6 - ibp}\chi_{\{|p|\le 1\}}.$$

By virtue of the definition of the standard Fourier transform (1.4), we easily derive that

$$\left|\frac{d\widehat{f}_m(p)}{dp} - \frac{d\widehat{f}(p)}{dp}\right| \le \frac{1}{\sqrt{2\pi}}\|xf_m(x) - xf(x)\|_{L^1(\mathbb{R})}.$$

Thus,

$$\left|\frac{\int_0^p [\frac{d\widehat{f}_m(s)}{ds} - \frac{d\widehat{f}(s)}{ds}]ds}{p^6 - ibp}\chi_{\{|p|\le 1\}}\right| \le \frac{\|xf_m(x) - xf(x)\|_{L^1(\mathbb{R})}}{\sqrt{2\pi}|b|}\chi_{\{|p|\le 1\}},$$

so that

$$\left\|\frac{\int_0^p [\frac{d\widehat{f}_m(s)}{ds} - \frac{d\widehat{f}(s)}{ds}]ds}{p^6 - ibp}\chi_{\{|p|\le 1\}}\right\|_{L^2(\mathbb{R})} \le \frac{\|xf_m(x) - xf(x)\|_{L^1(\mathbb{R})}}{\sqrt{\pi}|b|} \to 0$$

as $m \to \infty$ via the one of our assumptions.
This implies that $u_m(x) \to u(x)$ in $L^2(\mathbb{R})$ as $m \to \infty$. By means of the argument above, $u_m(x) \to u(x)$ in $H^6(\mathbb{R})$ as $m \to \infty$ in the case b) of the theorem as well. □

3. The problem on the finite interval

Proof of Theorem 1.3. First we demonstrate that it would be sufficient to solve our equation in $L^2(I)$. Indeed, if $u(x)$ is a square integrable solution of (1.10), periodic on I along with its derivatives up to the fifth order inclusively, directly from our problem under the stated assumptions we obtain that

$$-\frac{d^6u}{dx^6} - b\frac{du}{dx} \in L^2(I).$$

By virtue of (1.12), we have $(n^6 - ibn)u_n \in l^2$. This means that

$$\sum_{n=-\infty}^{\infty} n^{12}|u_n|^2 < \infty,$$

so that $\frac{d^6u}{dx^6} \in L^2(I)$. Hence, $u(x) \in H^6(I)$ as well.
To establish the uniqueness of solutions of (1.10), we consider the situation when $a > 0$. If $a = 0$, we are able to apply the similar reasoning in the constrained subspace $H_0^6(I)$. Let us suppose that $u_1(x),\ u_2(x) \in H^6(I)$ satisfy (1.10). Then their difference $w(x) := u_1(x) - u_2(x) \in H^6(I)$ is a solution of the homogeneous problem

$$-\frac{d^6w}{dx^6} - b\frac{dw}{dx} - aw = 0.$$

Because the operator $l_{a,\,b}$ given by (1.13) does not possess any nontrivial $H^6(I)$ zero modes, the function $w(x) \equiv 0$ in I.
Let us apply the Fourier transform (1.12) to both sides of equation (1.10). This gives us

$$u_n = \frac{f_n}{n^6 - a - ibn}, \quad n \in \mathbb{Z}, \tag{3.1}$$

such that

$$\|u\|^2_{L^2(I)} = \sum_{n=-\infty}^{\infty} \frac{|f_n|^2}{(n^6 - a)^2 + b^2n^2}. \tag{3.2}$$

First we consider the case a) of the theorem. By means of (3.2), we have

$$\|u\|^2_{L^2(I)} \leq \frac{1}{C}\|f\|^2_{L^2(I)} < \infty$$

as we assume (see (1.11)). By virtue of the reasoning above, $u(x) \in H^6(I)$ as well.
We conclude the proof of our theorem by discussing the situation when a vanishes. From (3.1) we easily derive that

$$u_n = \frac{f_n}{n^6 - ibn}, \quad n \in \mathbb{Z}. \tag{3.3}$$

Evidently, the right side of (3.3) belongs to l^2 if and only if

$$f_0 = 0, \tag{3.4}$$

such that

$$\|u\|^2_{L^2(I)} = \sum_{n\in\mathbb{Z},\ n\neq 0} \frac{|f_n|^2}{n^{12}+b^2n^2} \le \frac{1}{b^2}\|f\|^2_{L^2(I)} < \infty,$$

via the one of our assumptions along with (1.11). The reasoning above yields that $u(x) \in H_0^6(I)$ as well. Clearly, (3.4) is equivalent to orthogonality relation (1.17). ☐

Let us discuss the solvability in the sense of sequences for our equation on the interval I with periodic boundary conditions.

Proof of Theorem 1.4. Using the given conditions, we have

$$|f(0)-f(2\pi)| \le |f(0)-f_m(0)| + |f_m(2\pi)-f(2\pi)| \le 2\|f_m - f\|_{C(I)} \to 0$$

as $m\to\infty$. Thus, $f(0)=f(2\pi)$.
Since $f_m(x),\ f(x)\in C(I),\ m\in\mathbb{N}$, they belong to $L^1(I)\cap L^2(I)$ by virtue of (1.11). Formula (1.11) also implies that

$$\|f_m(x)-f(x)\|_{L^1(I)} \le 2\pi\|f_m(x)-f(x)\|_{C(I)} \to 0, \quad m\to\infty. \tag{3.5}$$

Hence, $f_m(x)\to f(x)$ in $L^1(I)$ as $m\to\infty$. Similarly, (1.11) yields

$$\|f_m(x)-f(x)\|_{L^2(I)} \le \sqrt{2\pi}\|f_m(x)-f(x)\|_{C(I)} \to 0, \quad m\to\infty. \tag{3.6}$$

Thus, $f_m(x)\to f(x)$ in $L^2(I)$ as $m\to\infty$ as well.
Let us apply the Fourier transform (1.12) to both sides of (1.16). We arrive at

$$u_{m,n} = \frac{f_{m,n}}{n^6 - a - ibn}, \quad m\in\mathbb{N}, \quad n\in\mathbb{Z}. \tag{3.7}$$

First we treat the case a) of the theorem. By means of the part a) of Theorem 1.3, equations (1.10) and (1.16) admit unique solutions $u(x)\in H^6(I)$ and $u_m(x)\in H^6(I),\ m\in\mathbb{N}$ respectively. By virtue of formulas (3.1), (3.6) and (3.7),

$$\|u_m - u\|^2_{L^2(I)} = \sum_{n=-\infty}^{\infty} \frac{|f_{m,n}-f_n|^2}{(n^6-a)^2+b^2n^2} \le \frac{1}{C}\|f_m - f\|^2_{L^2(I)} \to 0, \quad m\to\infty.$$

Hence, $u_m(x)\to u(x)$ in $L^2(I)$ as $m\to\infty$.
Let us demonstrate that $u_m(x)$ tends to $u(x)$ in $H^6(I)$ as $m\to\infty$. Indeed, by means of (1.10) and (1.16), we obtain

$$\left\| -\frac{d^6}{dx^6}(u_m-u) - b\frac{d(u_m-u)}{dx}\right\|_{L^2(I)} \le \|f_m - f\|_{L^2(I)} + a\|u_m-u\|_{L^2(I)}.$$

The right side of this bound converges to zero as $m\to\infty$ via (3.6). Using the Fourier transform (1.12), we derive that

$$\sum_{n=-\infty}^{\infty} n^{12}|u_{m,n}-u_n|^2 \to 0, \quad m\to\infty.$$

Thus, $\dfrac{d^6u_m}{dx^6} \to \dfrac{d^6u}{dx^6}$ in $L^2(I)$ as $m \to \infty$. This means that $u_m(x) \to u(x)$ in $H^6(I)$ as $m \to \infty$ in the situation when $a > 0$.

We conclude the article with discussing the case when the parameter a is trivial. By virtue of (1.18) along with (3.5), we have

$$|(f(x), 1)_{L^2(I)}| = |(f(x) - f_m(x), 1)_{L^2(I)}| \le \|f_m - f\|_{L^1(I)} \to 0, \quad m \to \infty,$$

such that the limiting orthogonality relation

$$(f(x), 1)_{L^2(I)} = 0 \tag{3.8}$$

holds. By means of the part b) of Theorem 1.3 above problems (1.10) and (1.16) possess unique solutions $u(x) \in H_0^6(I)$ and $u_m(x) \in H_0^6(I)$, $m \in \mathbb{N}$ respectively if $a = 0$. By virtue of formulas (3.1) and (3.7), we obtain that

$$u_{m,n} - u_n = \frac{f_{m,n} - f_n}{n^6 - ibn}, \quad m \in \mathbb{N}, \quad n \in \mathbb{Z}. \tag{3.9}$$

Orthogonality conditions (3.8) and (1.18) imply

$$f_0 = 0, \quad f_{m,0} = 0, \quad m \in \mathbb{N}.$$

Let us derive the estimate from above for the norm as

$$\|u_m - u\|_{L^2(I)} = \sqrt{\sum_{n=-\infty,\ n\neq 0}^{\infty} \frac{|f_{m,n} - f_n|^2}{n^{12} + b^2n^2}} \le \frac{\|f_m - f\|_{L^2(I)}}{|b|} \to 0, \quad m \to \infty$$

due to (3.6). Thus, $u_m(x) \to u(x)$ in $L^2(I)$ as $m \to \infty$. Therefore, $u_m(x) \to u(x)$ in $H_0^6(I)$ as $m \to \infty$ as well by virtue of the reasoning analogous to the one above in the proof of the case a) of our theorem. □

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