THE ROLE OF ALLEE EFFECTS FOR GAUSSIAN AND LÉVY DISPERSALS IN AN ENVIRONMENTAL NICHE

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Abstract. In the study of biological populations, the Allee effect detects a critical density below which the population is severely endangered and at risk of extinction. This effect supersedes the classical logistic model, in which low densities are favorable due to lack of competition, and includes situations related to deficit of genetic pools, inbreeding depression, mate limitations, unavailability of collaborative strategies due to lack of conspecifics, etc.

The goal of this paper is to provide a detailed mathematical analysis of the Allee effect. After recalling the ordinary differential equation related to the Allee effect, we will consider the situation of a diffusive population. The dispersal of this population is quite general and can include the classical Brownian motion, as well as a Lévy flight pattern, and also a "mixed" situation in which some individuals perform classical random walks and others adopt Lévy flights (which is also a case observed in nature).

We study the existence and nonexistence of stationary solutions, which are an indication of the survival chance of a population at the equilibrium.

We also analyze the associated evolution problem, in view of monotonicity in time of the total population, energy consideration, and long-time asymptotics.

Furthermore, we also consider the case of an "inverse" Allee effect, in which low density populations may access additional benefits.

1. Introduction

1.1. **The biological scenario.** A classical topic in quantitative biology focuses on the detection of thresholds leading to the survival or the extinction of a given population. The topic becomes of major significance for small populations, or populations with small density, since this is the case closer to the disappearance of the population and in which small environmental modifications could play a major role.

The analysis of small and declining populations is particularly important in conservation biology related to rare or threatened species, see e.g. [ARB+07], but has also great financial implications, e.g. in fisheries management, see e.g. [Hut15].

The Allee effect (named after Warder Clyde Allee [All31, AB32, All38]) aims precisely at detecting how survival rates are affected by the size of the population and what the correlation is between individual fitness and population density.

The classical mathematical theory of population dynamics based on logistic equations would predict some advantage of low density populations due to the reduced competition for resources. Indeed, typically, the growth of the density u of the population in these systems is modeled according to a term of the form $(m - \mu u)u$, with $m > 0$ describing the birth rate or the availability of resources and $\mu > 0$ accounting for the death rate or the competition for resources among overcrowded populations and therefore the "per capita" growth rate $m - \mu u$ appears to be decreasing in u (hence favoring low densities).

The effect discovered by Allee, after experimenting survival rates of goldfish in a tank, is instead that aggregation could improve the chance of persistence in adverse conditions (e.g. by social thermoregulation, or by stimulating reproduction, or by enhancing protection from toxic

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agents) while "undercrowding" could present deleterious consequences which are not accurately described by the logistic model alone.

For instance, a common cause for a detrimental effect in low density populations is provided by inbreeding depression, i.e. the reduced ability to survive or procreate induced by low genetic variations in which recessive deleterious alleles manifest themselves in the offspring of related mates. The appearance of these deleterious mutations may generate a genetic load in small populations and increase the extinction risk.

Interestingly, Charles Darwin was a pioneer[1] in the study of the adverse effects of inbreeding depression, probably also in view of his personal concerns: he was married to his first cousin, Emma Wedgwood, and the consanguineous marriage may have contributed to the high childhood mortality experienced by the Darwin/Wedgwood progeny (three of their ten children died at age ten or younger), see [BAC10] for a specific study on this case.

Deleterious mutations in heterozygote state arising from inbreeding in small and isolated populations have also been studied in animals, also thanks to modern technologies allowing for relatively inexpensive sequencing of genomes, see for instance [RRV$^+$19, SE23] for the case of wolves.

Other instances in which individuals may benefit from the presence of conspecifics, as well as suffer from their absence, occur in view of collaborative patterns and collaborative behaviors in which working together is beneficial for the community. Examples of these situations are increased vigilance and group defense against predators, also thanks to synchronic movements to confuse the enemies, or hunting strategies in packs (which, conversely, could not take place in smaller groups).

The scarcity of population may also produce mate limitations (roughly speaking, in sparse populations organisms may become unable to find suitable mates). This factor can be amplified by concurrent phenomena, such as the lack of demographic stochasticity in small populations: namely, if the number of individuals is low, random fluctuations can more easily produce a scarcity in one sex and the consequential lack of access of mates for the opposite sex, decreasing the birth rate and the survival chances.

While some Allee effects arise spontaneously in nature, others are anthropogenic, i.e. human-induced: these include exploitation of biological populations by economic markets in which the price of rare, threatened, or endangered species justifies the expenses undertaken to hunt them, as well as human-caused environmental disturbances (e.g., alterations of climate and ecosystem).

These are however just exemplificative situations and we certainly do not aim at exhausting the complexity of density-dependent survival patterns. See e.g. [SSF99, CBG08, KBD18, FJS20] and the references therein for further information about the Allee effect. We stress however that no unique definition of Allee effect is unanimously accepted (e.g., about whether the main effect comes from the smallness of the population size or by its low density, or about whether the Allee effect is the cumulative outcome of heterogeneous causes) and in fact the magnitude and presence of Allee effects in biological systems is often debatable, due to the practical difficulties in sampling low-density populations, see e.g. [GBBC10]. In this sense, rigorous mathematics can be extremely beneficial to state clearly the model under consideration, together with its limitations and extents.

1.2. **The mathematical setting.** The quantitative setting of the Allee effect leverages a critical population density, which we denote here by a, providing a threshold value below which the population suffers a concrete risk of extinction.

[1]As a matter of fact, half a century before Allee's experiments detecting decreased fitness in small populations, Darwin had already pointed out that "a large stock of individuals of the same species [...] is absolutely necessary for its preservation", see [Dar72].

The growth of the population is also controlled by an intrinsic carrying capacity, that we denote by ρ here: roughly speaking, this carrying capacity should indicate the steady state of the population above the critical threshold a.

Specifically, the classical Allee effect is typically modeled by an evolution of the population density u dictated by a source term of the form $(u-a)(\rho-u)u$. Though we will allow more general assumptions, the range of parameters more commonly[2] chosen is $0<a<\rho$. In this framework, the growth rate becomes negative when the population density falls below the threshold a, and also when it overcomes the carrying capacity ρ.

The case $a=0<\rho$ is often called a "weak" Allee effect: in this situation, the growth rate is positive, as it happens for the purely logistic case, but it presents an important quantitative difference. Indeed, for the purely logistic case the growth rate for small densities becomes linear, while in the weak Allee case it is of quadratic type. In this sense, the weak Allee effect still allows population growth subject to small densities, but this growth is significantly smaller than that predicted by the logistic equation.

The case of $\rho<0<a$ is, in our opinion, of some interest as well. In this situation, an "inverse Allee effect" takes place, since the growth rate becomes positive for small densities below the threshold a and negative above this threshold, with a linear growth for small densities which is similar to the logistic one, but quantitatively different since the "per capita" growth rate $m-\mu u$ of the logistic model is decreasing in u while the one of the inverse Allee effect is of the type $(u-a)(\rho-u)=(a-u)(|\rho|+u)=a|\rho|+(a-|\rho|)u-u^2$ is either decreasing, when $|\rho|>a$, or increasing, when $a>|\rho|$.

This inverse Allee effect also corresponds to a situation in which the threatened population develops new strategies at low density to avoid extinction and the critical threshold a plays in this setting the role of a carrying capacity.

In this paper, we give a detailed analysis of these Allee effects in different situations. First of all, in Section 1.3 we recall the ordinary differential equation setting in which the Allee effect can be easily modeled and analyzed: we will present there some elementary results which can serve as a guidance for more complicated settings.

Then, in Sections 1.4 and 1.5 we will analyze the steady states associated with random dispersals (specifically, the constant coefficient case will be treated in Section 1.4 and the variable coefficient case in Section 1.5, where we also allow the possible presence of an additional "pollination" term of nonlocal type). The setting that we will consider is that of a biological niche in which individuals are confined in view of a zero-flux condition of Neumann type.

The dispersal considered can be Gaussian (corresponding to a diffusion modeled by the classical Laplacian), or induced by Lévy flights (hence, modeled by a fractional Laplacian). We also allow the dispersal to be of "mixed" type, i.e. the diffusion can correspond to a superposition of classical and fractional Laplace operators.

The generality of the diffusion studied in these sections has a deep biological motivation. Indeed, on the one hand, classical diffusion induced by standard random walks has been traditionally the main paradigm to study animal dispersal (at least, in the absence of additional phenomena such as drift of chemotaxis). On the other hand, the recent literature has often opened the possibility of accounting for more general types of biological diffusion based on Lévy flights, in which very long excursions occur rarely but with a fat-tail distribution. These anomalous types of diffusion may present strategic advantages with respect to the one induced by the standard Brownian motion, especially for foragers looking for sparse, randomly distributed, replaceable targets, since it avoids oversampling in confined and already explored patches.

The detection of these Lévy flights occurred in concrete biological scenarios, but it is also highly controversial (see e.g. [KDLD09, Pyk15]), given the inherent difficulties of modeling animal behaviors and intentions, as well as the dependence of the results on the rapidly evolving

[2]Note that the case $a>\rho$ can be reduced to the case $a<\rho$ by swapping the roles of the parameters: indeed, setting $\tilde{a}:=\rho$ and $\tilde{\rho}:=a$ the nonlinearity $(u-a)(\rho-u)u$ takes the form $(u-\tilde{a})(\tilde{\rho}-u)u$.

technologies utilized for the experiments, and the mathematical complications to distinguish Lévy flights from superpositions of Brownian patterns with different exponential tails. In any case, one of the proposals put forth is that, even within conspecifics, some individuals may explore the space through Brownian random walks while others may adopt Lévy flights.

To recall, among the many, a concrete example, in [VAB+96] five wandering albatrosses were monitored by attaching a salt-water immersion logger to one of their legs. This device measured the time intervals between landing on the ocean and the data gave the impression that the albatrosses were performing Lévy flights. This conclusion was however revisited in [EPW+07] by a new set of high-resolution data, which indicated the absence of Lévy flights and revealed spurious data in the previous analysis, due to birds sitting on their nest rather flying for searching food. The experiment was also reconsidered in [HWQ+12], through an accurate GPS monitoring, which indicated both Lévy and Brownian movement patterns for individual albatrosses.

This scenario naturally leads to the study of "mixed order" diffusive operators in which the classical and the fractional Laplacians coexist, and our analysis is general enough to include this case too.

We also stress that the analysis of the Lévy flights and that of the mixed operators require the specifications of new, appropriate zero-flux conditions for the ecological niche, as put forth in [DROV17, DPLV23].

After completing the analysis of the steady states, we will devote Section 1.6 to the detailed analysis of the full evolutionary problem. In doing so, we will also exploit some Maximum Principles for mixed local/nonlocal operators, that have been obtained in [DPLV].

1.3. **ODE analysis.** We now recall explicitly the main mathematical setting to describe the Allee effect through an ordinary differential equation. Namely, given a, $b \in \mathbb{R}$, one considers the evolutionary system for the density $u : [0, +\infty) \to \mathbb{R}$ described by

$$\begin{cases} u_t(t) = (u(t) - a)(\rho - u(t))u(t) & \text{for } t \in (0, +\infty), \\ u(0) = u_0. \end{cases} \tag{1.1}$$

The Allee effect is thus characterized by the following result:

Theorem 1.1. *Let $0 \leqslant a < \rho$. Let u be a solution of problem* (1.1).
Then,

(*i*) *if $0 \leqslant u_0 < a$,*
$$\lim_{t\to+\infty} u(t) = 0;$$
(*ii*) *if $a < u_0 \leqslant \rho$,*
$$\lim_{t\to+\infty} u(t) = \rho;$$
(*iii*) *if $u_0 > \rho$,*
$$\lim_{t\to+\infty} u(t) = \rho.$$

See Figure 1 for a graphical sketch of Theorem 1.1.

We point out that Theorem 1.1(i) is consistent with the notion of critical population density mentioned in Section 1.2, since the value a provides a threshold below which the population gets extinct.

Also, Theorem 1.1(ii)-(iii) aligns with the notion of carrying capacity for ρ, as described in Section 1.2.

We also stress that the weak Allee effect $a = 0$ is included in Theorem 1.1(ii)-(iii).

The inverse Allee effect mentioned in Section 1.2 and corresponding to $\rho < 0$ is discussed in the following result:

Theorem 1.2. *Let $\rho < 0 \leqslant a$. Let u be a solution of problem* (1.1).
Then,

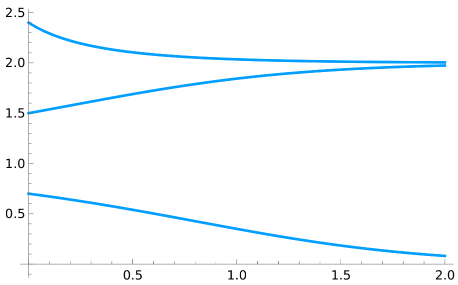

FIGURE 1. Description of the trajectories detected in Theorem 1.1 (in this diagram, $a = 1$ and $\rho = 2$).

(i) *if* $0 \leqslant u_0 < a$,

$$\lim_{t \to +\infty} u(t) = a;$$

(ii) *if* $u_0 > a$,

$$\lim_{t \to +\infty} u(t) = a.$$

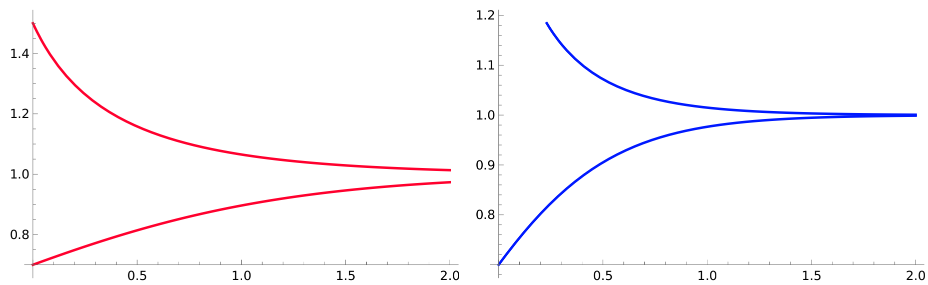

FIGURE 2. Description of the trajectories detected in Theorem 1.2 (in these diagrams, $a = 1$ and $\rho = -1/2$, and $a = 1$ and $\rho = -2$).

See Figure 2 for a visual sketch of the trajectories in Theorem 1.2. We observe that for this inverse Allee effect the threshold a becomes a carrying capacity in Theorem 1.2, consistently with the explanation given in Section 1.2.

1.4. **Elliptic problem with constant coefficients.** We now consider the steady states of a biological population subject to an Allee effect and dispersing with a superposition of classical diffusion and Lévy flights. In this setting, the elliptic operator under consideration takes the form

$$-\alpha\Delta u + \beta(-\Delta)^s, \tag{1.2}$$

with α, $\beta \in [0, +\infty)$, $\alpha + \beta \neq 0$, and $s \in (0, 1)$. The fractional Laplacian is defined as

$$(-\Delta)^s u(x) := \frac{1}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}}\, dy, \tag{1.3}$$

up to a normalization constant that we neglect here since it does not play a role in our analysis (see e.g. [DNPV12]).

Let also Ω be a bounded domain of $\mathbb{R}^n$ with C^1 boundary. The natural reflection conditions for mixed operators of the type (1.2) were introduced in [DPLV23] and correspond to these alternatives:

$$
\begin{cases}
\mathscr{N}_s u(x) = 0 \text{ for all } x \in \mathbb{R}^n \setminus \Omega, & \text{when } \alpha = 0, \\[2ex]
\partial_\nu u(x) = 0 \text{ for all } x \in \partial\Omega, & \text{when } \beta = 0, \\[2ex]
\begin{array}{l} \mathscr{N}_s u(x) = 0 \text{ for all } x \in \mathbb{R}^n \setminus \overline{\Omega} \\ \text{and } \partial_\nu u(x) = 0 \text{ for all } x \in \partial\Omega, \end{array} & \text{when } \alpha \neq 0 \text{ and } \beta \neq 0.
\end{cases} \tag{1.4}
$$

Here above and in what follows, we use the notation

$$
\mathscr{N}_s u(x) := \int_\Omega \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy. \tag{1.5}
$$

Some remarks on this setting are in order. First of all, the expression in (1.5) has been introduced in [DROV17] as a natural extension of the Neumann boundary condition in the nonlocal setting. The rationale of the condition $\mathscr{N}_s u(x) = 0$ is that it corresponds to a particle reflection of a Lévy process: roughly speaking, a particle following a Lévy process leaves a given bounded domain almost surely in a finite time and the condition $\mathscr{N}_s u(x) = 0$ corresponds to an instantaneous reentering into the domain by following the same process distribution (see also [Von21]).

Notice that this is indeed a nonlocal counterpart of the classical feature of the Neumann condition in Gaussian diffusion, whose role corresponds to a "billiard" reflection at the boundary.

In this sense, the first two conditions in (1.4) correspond to the nonlocal Neumann condition of [DROV17] when only the Lévy-type anomalous diffusion takes place and to the classical Neumann condition when only the Gaussian diffusion is present.

The third condition in (1.4) instead addresses the case in which both classical and anomalous diffusions are present. This is the case in which part of the biological population follows a classical random walk and part follows a Lévy flight. In this situation, the population is subject to both the classical and the nonlocal dispersion processes (with coefficients α and β depending on the proportion of the total population following either process) and the Neumann condition that arises is, maybe quite surprisingly, the prescription of both the classical and the nonlocal constraints. We stress that this prescription is not an "overdetermined" condition (indeed, overdetermined problems typically do not have solutions, apart from very special configurations): instead, the "double condition" given in (1.4) gives rise to well-posed problems and describes an ecological niche endowed with a zero-flux condition, see [DV21].

The main question that we address is whether or not the environmental niche is suited for the survival of the population, and what is the role of the Allee effect in this scenario. To this end, given a, $\rho \in \mathbb{R}$, we consider the problem

$$
\begin{cases}
-\alpha\Delta u + \beta(-\Delta)^s u = (u-a)(\rho-u)u & \text{in } \Omega, \\
\text{with } (\alpha,\beta) - \text{Neumann conditions},
\end{cases} \tag{1.6}
$$

where the latter expression is a short notation for (1.4).

Here above and in the rest of the paper, we assume that α and β are nonnegative constants, not both of them equal to zero, and s is a fractional diffusive parameter in $(0,1)$ (see e.g., [AV19] for a basic introduction to the fractional Laplacian). Note that the case of the classical diffusion is included in (1.6) by the choice $\beta := 0$, while the case of pure Lévy flights corresponds to the choice $\alpha := 0$.

We observe that problem (1.6) has the constant solutions $u \equiv 0$, $u \equiv a$ and $u \equiv \rho$. The next results discuss the existence (and nonexistence) of constant positive solutions for the above problem and their energetic properties, in dependence of the structural parameters of the model. We consider the Allee effect here in Theorem 1.3 and the inverse Allee effect in Theorem 1.4.

Theorem 1.3. *Let $0 \leqslant a \leqslant \rho$.*
Then,

(*i*) *if $a = 0 < \rho$, problem* (1.6) *admits the solution $u \equiv \rho$ as an absolute minimum of the energy functional;*
(*ii*) *if $a \in (0, \rho/2]$, problem* (1.6) *admits the solution $u_1 \equiv a$ as a saddle point and the solution $u_2 \equiv \rho$ as an absolute minimum of the energy functional;*
(*iii*) *if $a \in (\rho/2, \rho)$, problem* (1.6) *admits the solution $u_1 \equiv a$ as a saddle point and the solution $u_2 \equiv \rho$ as a relative minimum of the energy functional;*
(*iv*) *if $a = \rho$, problem* (1.6) *admits the solution $u \equiv \rho$ which is neither a local maximum nor a local minimum of the energy functional.*

Theorem 1.4. *Let $\rho < 0 \leqslant a$.*
Then,

(*i*) *if $a = 0$, problem* (1.6) *only admits the trivial solution $u \equiv 0$.*
(*ii*) *if $a > 0$, problem* (1.6) *admits the solution $u \equiv a$ as an absolute minimum of the energy functional.*

1.5. **Elliptic problem with variable coefficients.** Here we consider a generalization of the model in (1.6). Specifically, we allow the density threshold a and the carrying capacity ρ to depend on space. We also permit the presence of an additional nonlocal term of pollination type.

Namely, we look at the equation

$$\begin{cases} -\alpha\Delta u + \beta(-\Delta)^s u = (u-a)(\rho-u)u + J\star u & \text{in } \Omega, \\ \text{with } (\alpha,\beta) - \text{Neumann conditions.} \end{cases} \tag{1.7}$$

Here, a and ρ are functions of the variable x, and we assume that $a(x) \geqslant 0$ for all $x \in \Omega$.

Also, we suppose that $J \in L^1(\mathbb{R}^n, [0,+\infty))$ with $J(x) = J(-x)$ and

$$\int_{\mathbb{R}^n} J(x)\,dx =: \tau. \tag{1.8}$$

Here above, we have used the convolution notation

$$J\star u(x) := \int_\Omega J(x-y)\,u(y)\,dy.$$

This convolution describes a pollination term, providing an additional population growth due to the presence of nearby individuals.

It is also useful to introduce the auxiliary exponents

$$\overline{s} := \begin{cases} 1 & \text{if } \alpha \neq 0, \\ s & \text{if } \alpha = 0, \end{cases}$$

and

$$\begin{aligned} \underline{q} &:= \begin{cases} \dfrac{2^*_{\overline{s}}}{2^*_{\overline{s}}-2} & \text{if } n > 2\overline{s}, \\ 1 & \text{if } n \leqslant 2\overline{s}, \end{cases} \\ &= \begin{cases} \dfrac{n}{2\overline{s}} & \text{if } n > 2\overline{s}, \\ 1 & \text{if } n \leqslant 2\overline{s}, \end{cases} \end{aligned} \tag{1.9}$$

where $2^*_{\overline{s}} := \frac{2n}{n-2\overline{s}}$ is the Sobolev critical exponent related to the exponent $\overline{s}$.

We will also use the customary notation for the positive and negative parts of a function, i.e.

$$f_+(x) := \max\{f(x),0\} \qquad \text{and} \qquad f_-(x) := \max\{-f(x),0\}.$$

In this setting, the classical Allee effect is described by the condition $\int_\Omega \rho(x)\,dx \geqslant 0$, which is a natural generalization of the positivity of ρ assumed when ρ was just a constant. To detect the existence of nontrivial solutions, we present the following result:

Theorem 1.5. *Suppose that*

$$\int_\Omega \rho(x)\,dx \geqslant 0. \tag{1.10}$$

Assume also that $\rho a \in L^q(\Omega)$ for some $q \in (\underline{q}, +\infty]$, and that

$$(\rho + a)_+^4 \in L^1(\Omega) \quad \text{and} \quad (\tau - \rho a)_+^2 \in L^1(\Omega).$$

If $(\rho + a)_+ \not\equiv 0$, suppose in addition that $n < 6\overline{s}$ and that $(\rho + a)_+^r \in L^1(\Omega)$ for some $r > \frac{2n}{6\overline{s}-n}$. Then,

- *(i) if $J \equiv 0$ and $a(x) = \rho(x)$ is not a constant, then the only solution of (1.7) is the one identically zero;*
- *(ii) if*

$$\int_\Omega \big(\rho(x)a(x) - J \star 1(x)\big)\,dx < \frac{2}{9|\Omega|}\left(\int_\Omega \big(\rho(x) + a(x)\big)\,dx\right)^2 \tag{1.11}$$

then (1.7) admits a nonnegative solution $u \not\equiv 0$.

For variable coefficients, the inverse Allee case is encoded by the condition $\int_\Omega \rho(x)\,dx < 0$, which is a natural generalization of the negative sign assumption of ρ in the constant coefficient case. In this setting, we have:

Theorem 1.6. *Suppose that*

$$\int_\Omega \rho(x)\,dx < 0. \tag{1.12}$$

Assume also that $\rho a \in L^q(\Omega)$ for some $q \in (\underline{q}, +\infty]$, and that

$$(\rho + a)_+^4 \in L^1(\Omega) \quad \text{and} \quad (\tau - \rho a)_+^2 \in L^1(\Omega).$$

If $(\rho + a)_+ \not\equiv 0$, suppose in addition that $n < 6\overline{s}$ and that $(\rho + a)_+^r \in L^1(\Omega)$ for some $r > \frac{2n}{6\overline{s}-n}$. Then,

- *(i) if $J \equiv 0$ and $a(x) = \rho(x) \equiv 0$, then the only solution of (1.7) is the one identically zero;*
- *(ii) if either*

$$\begin{aligned} &\int_\Omega \big(\rho(x)a(x) - J \star 1(x)\big)\,dx < \frac{2}{9|\Omega|}\left(\int_\Omega \big(\rho(x) + a(x)\big)\,dx\right)^2 \\ &\text{when } \int_\Omega \big(\rho(x) + a(x)\big)\,dx \geqslant 0 \end{aligned} \tag{1.13}$$

or

$$\begin{aligned} &\int_\Omega \big(\rho(x)a(x) - J \star 1(x)\big)\,dx < 0 \\ &\text{when } \int_\Omega \big(\rho(x) + a(x)\big)\,dx < 0, \end{aligned} \tag{1.14}$$

then (1.7) admits a nonnegative solution $u \not\equiv 0$.

Remark 1.7. Notice that large pollination terms are advantageous for the survival of the species, as specified by Theorem 1.5(*ii*) and 1.6(*ii*).

Interestingly, when ρ and a are constants and $J \equiv 0$, the condition

$$\int_\Omega \big(\rho(x)a(x) - J \star 1(x)\big)\,dx \leqslant \frac{2}{9|\Omega|}\left(\int_\Omega \big(\rho(x) + a(x)\big)\,dx\right)^2$$

reduces to

$$2(\rho - 2a)\left(\rho - \frac{a}{2}\right) \geqslant 0.$$

Recalling that $a > 0$, the above inequality holds true if either $\rho > 0$ and $a \in [0, \rho/2]$, or $\rho < 0$. It is worth observing that these conditions are in agreement respectively with the assumptions in Theorems 1.3 and 1.4, in which one obtains that the absolute minimum is nontrivial. In this sense, Theorems 1.5 and 1.6 are the natural counterpart of Theorems 1.3 and 1.4 that includes the case of variable coefficients and pollination/conflict terms.

1.6. **Parabolic problem.** Now we consider the evolutionary problem associated with the constant coefficients case discussed in Section 1.4. Here, a, $\rho \in \mathbb{R}$ and we look at the parabolic equation

(1.15)
$$\begin{cases} \partial_t u(x,t) - \alpha\Delta u(x,t) + \beta(-\Delta)^s u(x,t) = (u(x,t) - a)(\rho - u(x,t))u(x,t) & \text{in } \Omega \times (0, +\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ \text{with } (\alpha, \beta) - \text{Neumann conditions} & \text{in } (0, +\infty). \end{cases}$$

Without further notice, we assume here that we are dealing with classical solutions, e.g. for all $t \in [0,T]$, $u(\cdot,t) \in L^\infty(\mathbb{R}^n) \cap C(\overline{\Omega}) \cap C^2(\Omega)$ and, for all $x \in \Omega$, $u(x,\cdot) \in C^1((0,T]) \cap C([0,T])$, with the (α,β)-Neumann conditions satisfied pointwise in $\mathbb{R}^n \setminus \Omega$ and $\partial\Omega$ respectively.

We remark that, to the best of our knowledge, a complete regularity theory for mixed operators with mixed Neumann conditions is not available yet and we plan to deal with it in a forthcoming project (see [AFNRO23] for a regularity theory in the elliptic setting with nonlocal Neumann conditions, where the authors prove that solutions are C^α up to the boundary of the domain Ω).

Now, we recall that an open set Ω is said to satisfy the interior ball condition if for any $x \in \partial\Omega$ there exists a ball $B \subset \Omega$ such that $x \in \partial B$.

The first result thet we present in this setting is related to the evolution of the total population (corresponding to the integral of the density function over the space). We have that the critical threshold a highlights the different monotonic behavior in time of the population, namely, in the presence of the Allee effect, when the initial density lies below a the total population decreases in time, while when the initial density is above a but below ρ the total population increases, and when the initial density is above ρ the total population decreases:

Theorem 1.8. *Let $0 \leqslant a \leqslant \rho$. Let $u(x,t)$ be a classical solution of* (1.15) *satisfying*

$$\partial_t u \in L^\infty_{\text{loc}}\big((0,T),\, L^1(\Omega)\big) \tag{1.16}$$

and, for every $t \in (0,T]$,

$$\alpha|\Delta u(\cdot,t)| + \beta|(-\Delta)^s u(\cdot,t)| \in L^1(\Omega). \tag{1.17}$$

Assume that

- *if $\beta = 0$, then Ω satisfies the interior ball condition,*
- *if $\beta \neq 0$, then, for every $t \in (0,T]$,*

$$\text{the function } \Omega \times (\mathbb{R}^n \setminus \Omega) \ni (x,y) \mapsto \frac{u(x,t) - u(y,t)}{|x-y|^{n+2s}} \text{ belongs to } L^1\big(\Omega \times (\mathbb{R}^n \setminus \Omega)\big), \tag{1.18}$$

- *if $\alpha \neq 0$, then, for every $t \in (0,T]$,*

$$u(\cdot,t) \in C^1(\overline{\Omega}), \tag{1.19}$$

- *if $\alpha\beta \neq 0$, then there exist $\mu > 0$ and $\theta > 2s$ such[3] that*

$$u \in L^\infty\big([0,T],\, C^\theta(\Omega_\mu)\big) \qquad \text{with} \qquad \Omega_\mu := \bigcup_{x \in \Omega} B_\mu(x). \tag{1.20}$$

[3]As customary, if $\theta > 1$, the notation $u \in C^\theta(\Omega_\mu)$ means $u \in C^{k,\vartheta}(\Omega_\mu)$ with $k \in \mathbb{N}$, $\vartheta \in (0,1]$ and $\theta = k + \vartheta$.

Then,

(*i*) *if* $0 < u_0(x) \leqslant a$ *for all* $x \in \Omega$, *with* $0 < \inf_\Omega u_0$ *and* $u_0 \not\equiv a$, *then for* $t_1 < t_2$ *we have that*
$$\int_\Omega u(x,t_1)\,dx > \int_\Omega u(x,t_2)\,dx,$$
namely the total population is strictly decreasing in time $t > 0$*;*

(*ii*) *if* $a \leqslant u_0(x) < \rho$ *for all* $x \in \Omega$, *with* $\sup_\Omega u_0 < \rho$ *and* $u_0 \not\equiv a$, *then for* $t_1 < t_2$ *we have that*
$$\int_\Omega u(x,t_1)\,dx < \int_\Omega u(x,t_2)\,dx,$$
namely the total population is increasing in time $t > 0$*;*

(*iii*) *if* $\inf_\Omega u_0 > \rho$, *then for* $t_1 < t_2$ *we have that*
$$\int_\Omega u(x,t_1)\,dx > \int_\Omega u(x,t_2)\,dx,$$
namely the total population is strictly decreasing in time $t > 0$.

The energy associated with the parabolic problem in (1.15) is also decreasing in time, as pointed out in the next result:

Theorem 1.9. *Let* $0 \leqslant a \leqslant \rho$. *Let* $u(x,t)$ *be a classical and nonnegative solution of* (1.15) *satisfying*

$$\alpha\,\nabla u(\cdot,t)\,\nabla\partial_t u(\cdot,t) \in L^1(\Omega), \tag{1.21}$$

$$\text{the map } \mathcal{Q} \ni (x,y) \mapsto \frac{\beta(u(x,t)-u(y,t))(\partial u_t(x,t)-\partial_t u(y,t))}{|x-y|^{n+2s}} \text{ belongs to } L^1(\mathcal{Q}), \tag{1.22}$$

with $\mathcal{Q} = \mathbb{R}^{2n} \setminus (\mathbb{R}^n \setminus \Omega)^2$, *and*

$$\partial_t u(\cdot,t)\Big(\alpha|\Delta u(\cdot,t)| + \beta|(-\Delta u)^s(\cdot,t)|\Big) \in L^1(\Omega), \tag{1.23}$$

for every $t \in (0,T]$.

Assume that

- *if* $\beta \neq 0$, *then, for every* $t \in (0,T]$,
$$(-\Delta)^s u(\cdot,t)\,\partial_t u(\cdot,t) \in L^1(\Omega) \tag{1.24}$$
and
$$(\mathbb{R}^n \setminus \Omega) \times \Omega \ni (x,y) \mapsto \frac{\big(u(x,t)-u(y,t)\big)\partial_t u(x,t)}{|x-y|^{n+2s}} \text{ belongs to } L^1((\mathbb{R}^n \setminus \Omega) \times \Omega), \tag{1.25}$$
- *if* $\alpha \neq 0$, *then, for every* $t \in (0,T]$, *the assumption in* (1.19) *is satisfied.*

Then, the energy
$$\begin{aligned} E(t) := {}& \frac{\alpha}{2}\int_\Omega |\nabla u(x,t)|^2\,dx + \frac{\beta}{4}\iint_{\mathcal{Q}} \frac{|u(x,t)-u(y,t)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &+ \int_\Omega \left(\frac{1}{4}u(x,t)^4 - \frac{\rho+a}{3}u(x,t)^3 + \frac{\rho a}{2}u(x,t)^2\right)dx \end{aligned}$$
is decreasing in time $t > 0$, *unless* u *is constant in time.*

Here above and in what follows, we have used the notation $\mathcal{Q} := \mathbb{R}^{2n} \setminus (\mathbb{R}^n \setminus \Omega)^2$.

The asymptotic of the diffusive Allee model is also dictated by the roles of the critical density threshold a and the carrying capacity ρ. More precisely, as shown in the next result, a population with initial density below the critical density will perish, while a large population will asymptotically approach the carrying capacity:

Theorem 1.10. *Let $0 \leqslant a \leqslant \rho$. Let $u(x,t)$ be a classical solution of* (1.15) *satisfying* (1.16) *and* (1.17), *for every $t \in (0,T]$.*

Assume that

- *if $\beta = 0$, then Ω satisfies the interior ball condition,*
- *if $\beta \neq 0$, then, for every $t \in (0,T]$, the assumption in* (1.18) *is satisfied,*
- *if $\alpha \neq 0$, then, for every $t \in (0,T]$, the assumption in* (1.19) *is satisfied,*
- *if $\alpha\beta \neq 0$, then there exist $\mu > 0$ and $\theta > 2s$ such that the assumption in* (1.20) *is satisfied.*

Assume also that, for all $x \in \Omega$, the following limit exists

$$\lim_{t\to+\infty} u(x,t)$$

and it is uniform.

Then,

(*i*) *if $0 < u_0(x) \leqslant a$ for all $x \in \Omega$, with $0 < \inf_\Omega u_0$ and $u_0 \not\equiv a$, then, for a.e. $x \in \Omega$,*

$$\lim_{t\to+\infty} u(x,t) = 0;$$

(*ii*) *if $a \leqslant u_0(x) < \rho$ for all $x \in \Omega$, with $\sup_\Omega u_0 < \rho$ and $u_0 \not\equiv a$, then, for a.e. $x \in \Omega$,*

$$\lim_{t\to+\infty} u(x,t) = \rho;$$

(*iii*) *if $\inf_\Omega u_0 > \rho$, then, for a.e. $x \in \Omega$,*

$$\lim_{t\to+\infty} u(x,t) = \rho.$$

We stress that Theorem 1.10 can be seen as the diffusive counterpart of the ODE analysis recalled in Theorem 1.1.

Now we deal with the case of the inverse Allee effect. In this case, small populations can better fight for survival and indeed the total population corresponding to low initial densities is increasing in time (differently from the case presented in Theorem 1.8):

Theorem 1.11. *Let $\rho < 0 \leqslant a$. Let $u(x,t)$ be a classical solution of* (1.15) *satisfying* (1.16) *and* (1.17), *for every $t \in (0,T]$.*

Assume that

- *if $\beta = 0$, then Ω satisfies the interior ball condition,*
- *if $\beta \neq 0$, then, for every $t \in (0,T]$, the assumption in* (1.18) *is satisfied,*
- *if $\alpha \neq 0$, then, for every $t \in (0,T]$, the assumption in* (1.19) *is satisfied,*
- *if $\alpha\beta \neq 0$, then there exist $\mu > 0$ and $\theta > 2s$ such that the assumption in* (1.20) *is satisfied.*

Then,

(*i*) *if $0 \leqslant u_0(x) < a$ for a.e. $x \in \Omega$, with $\sup_\Omega u_0 < a$ and $u_0 \not\equiv 0$, then for $t_1 < t_2$ we have that*

$$\int_\Omega u(x,t_1)\,dx < \int_\Omega u(x,t_2)\,dx,$$

namely the total population is increasing in time $t > 0$;

(*ii*) *if $\sup u_0 > a$, then for $t_1 < t_2$ we have that*

$$\int_\Omega u(x,t_1)\,dx > \int_\Omega u(x,t_2)\,dx,$$

namely the total population is decreasing in time $t > 0$.

Differently from the previous results, the energetic analysis of the inverse Allee effect appears to be similar to that of the classical Allee effect, at least from the point of view of energy monotonicity, as shown in the next result (to be compared with Theorem 1.9):

Theorem 1.12. *Let $\rho < 0 \leqslant a$. Let $u(x,t)$ be a classical and nonnegative solution of* (1.15) *satisfying* (1.21), (1.22) *and* (1.23), *for every $t \in (0,T]$.*

Assume that

- *if $\beta \neq 0$, then, for every $t \in (0,T]$, the assumptions in* (1.24) *and* (1.25) *are satisfied,*
- *if $\alpha \neq 0$, then, for every $t \in (0,T]$, the assumption in* (1.19) *is satisfied.*

Then, the energy

$$\begin{aligned} E(t) := \frac{\alpha}{2}\int_\Omega |\nabla u(x,t)|^2\,dx - \frac{\beta}{4}\iint_\Omega \frac{|u(x,t)-u(y,t)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ + \int_\Omega \left(\frac{1}{4}u(x,t)^4 - \frac{\rho+a}{3}u(x,t)^3 + \frac{\rho a}{2}u(x,t)^2\right)dx \end{aligned}$$

is decreasing in time $t > 0$.

For the inverse Allee effect, the diffusive problem exhibits the threshold a as its natural carrying capacity, as stated below:

Theorem 1.13. *Let $\rho < 0 \leqslant a$. Let $u(x,t)$ be a classical solution of* (1.15) *satisfying* (1.16) *and* (1.17), *for every $t \in (0,T]$.*

Assume that

- *if $\beta = 0$, then Ω satisfies the interior ball condition,*
- *if $\beta \neq 0$, then, for every $t \in (0,T]$, the assumption in* (1.18) *is satisfied,*
- *if $\alpha \neq 0$, then, for every $t \in (0,T]$, the assumption in* (1.19) *is satisfied,*
- *if $\alpha\beta \neq 0$, then there exist $\mu > 0$ and $\theta > 2s$ such that the assumption in* (1.20) *is satisfied.*

Assume also that, for all $x \in \Omega$, the following limit exists

$$\lim_{t\to+\infty} u(x,t)$$

and it is uniform.

Then,

(*i*) *if $0 \leqslant u_0(x) < a$ for a.e. $x \in \Omega$, with $\sup_\Omega u_0 < a$ and $u_0 \not\equiv 0$, then*

$$\lim_{t\to+\infty} u(x,t) = a;$$

(*ii*) *if $\sup u_0 > a$, then*

$$\lim_{t\to+\infty} u(x,t) = a.$$

We stress that, on the one hand, the phenomenon in Theorem 1.13 for the inverse Allee effect differs from that showcased in Theorem 1.10 for the classical Allee effect. On the other hand, Theorem 1.13 can be considered as the natural counterpart of the ODE analysis for the inverse Allee effect which was carried out in Theorem 1.2.

1.7. **Organization of the paper.** The rest of this paper is devoted to the proofs of the main results presented so far. Specifically, Section 2 contains the elementary treatment of the ordinary differential equation and the proofs of Theorems 1.1 and 1.2, Section 3 deals with the steady states corresponding to constant coefficients and showcases the proofs of Theorems 1.3 and 1.4, while Section 4 focuses on the case of variable coefficients and possible pollination, with the proofs of Theorems 1.5 and 1.6.

In Section 5 we deal with the parabolic case and prove Theorems 1.8, 1.9, 1.10, 1.11, 1.12, and 1.13.

2. ODE theory: proofs of Theorems 1.1 and 1.2

This section deals with the elementary results related to ordinary differential equations. For the facility of the reader, we provide the proofs of the results stated in Section 1.3.

Proof of Theorem 1.1. We give the proof for case (i), the other two cases being analogous.

Let $0 \leqslant u_0 < a$. If $u_0 = 0$, then $u(t) = 0$ for every $t > 0$, so we focus on the case $0 < u_0 < a$. Since $v_1 \equiv 0$ and $v_2 \equiv a$ are constant solutions of the equation in (1.1), we know that $0 < u(t) < a$ for every $t > 0$, and thus, by the differential equation in (1.1), $u_t(t) < 0$ for every $t > 0$.

Then, there exists $\ell \in [0, u_0)$ such that

$$\lim_{t\to+\infty} u(t) = \ell \qquad \text{and} \qquad \lim_{t\to+\infty} u_t(t) = 0.$$

Moreover, again by (1.1),

$$0 = \lim_{t\to+\infty} u_t(t) = (\ell - a)(\rho - \ell)\ell,$$

which yields that $\ell = 0$, as desired. □

Proof of Theorem 1.2. The arguments are the same as in the proof of Theorem 1.1. □

3. Elliptic problem with constant coefficients: proof of Theorems 1.3 and 1.4

Here we deal with the steady states corresponding to the constant coefficient case presented in Section 1.4. Before dealing with the proofs of Theorems 1.3 and 1.4, we recall some notions on the functional space in which we work.

First, we recall the space H^s_Ω introduced in [DROV17], defined as

$$H^s_\Omega := \left\{ u : \mathbb{R}^n \to \mathbb{R} \text{ s.t. } u \in L^2(\Omega) \text{ and } \iint_{\mathcal{Q}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}}\, dx\, dy < +\infty \right\}, \tag{3.1}$$

where $\mathcal{Q} := \mathbb{R}^{2n} \setminus (\mathbb{R}^n \setminus \Omega)^2$, and the space $X_{\alpha,\beta}$ as introduced in [DPLV23], that is

$$X_{\alpha,\beta} := \begin{cases} H^1(\Omega) & \text{if } \beta = 0, \\ H^s_\Omega & \text{if } \alpha = 0, \\ H^1(\Omega) \cap H^s_\Omega & \text{if } \alpha\beta \neq 0. \end{cases}$$

Furthermore, we recall that $X_{\alpha,\beta}$ is a Hilbert space with respect to the scalar product

$$(u, v) := \int_\Omega u(x)v(x)\, dx + \alpha \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}}\, dx\, dy$$

for every u, $v \in X_{\alpha,\beta}$.

It is also useful to define the seminorm

$$[u]^2 := \alpha \int_\Omega |\nabla u(x)|^2\, dx + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}}\, dx\, dy. \tag{3.2}$$

We say that $u \in X_{\alpha,\beta}$ is a (weak) solution of (1.6) if

$$\begin{aligned} \alpha \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx &+ \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}}\, dx\, dy \\ &= \int_\Omega (u(x) - a)(\rho - u(x))u(x)v(x)\, dx \end{aligned}$$

for all functions $v \in X_{\alpha,\beta}$.

In light of this definition, we seek nonnegative solutions of (1.6) as critical points of the functional

$$E(u) := \frac{1}{2}[u]^2 + \frac{1}{4}\int_\Omega u^4\, dx - \frac{\rho + a}{3}\int_\Omega |u|^3\, dx + \frac{\rho a}{2}\int_\Omega u^2\, dx. \tag{3.3}$$

We are now able to give the proof of Theorem 1.3, which covers the case $0 \leqslant a \leqslant \rho$.

Proof of Theorem 1.3. Taking $v \equiv 1$ and $t > 0$, we see that the functional E, as defined in (3.3), evaluated on constant functions, is

$$E(tv) = \int_\Omega \left(\frac{t^4}{4} - \frac{\rho+a}{3} t^3 + \frac{\rho a}{2} t^2 \right) dx = \left(\frac{t^4}{4} - \frac{\rho+a}{3} t^3 + \frac{\rho a}{2} t^2 \right) |\Omega|. \tag{3.4}$$

Additionally,

$$(E'(tv), tv) = \int_\Omega t^2 \big(t^2 - (\rho+a)t + \rho a \big)\, dx = t^2 \big(t^2 - (\rho+a)t + \rho a \big)\, |\Omega|, \tag{3.5}$$

which vanishes if and only if $t \in \{0, a, \rho\}$.

Now, for all $u \in X_{\alpha,\beta}$, we have that

$$\begin{aligned} E(tv + \varepsilon u) = E(tv) &+ \varepsilon \int_\Omega t(t^2 - (\rho+a)t + \rho a) u\, dx \\ &+ \varepsilon^2 \left[\frac{1}{2}[u]^2 + \int_\Omega \left(\frac{3}{2} t^2 - (\rho+a)t + \frac{\rho a}{2} \right) u^2\, dx \right] \\ &+ \varepsilon^3 \int_\Omega \left(t - \frac{\rho+a}{3} \right) u^3\, dx + \varepsilon^4 \int_\Omega \frac{t^4}{4}\, dx. \end{aligned}$$

Accordingly, the cases $t := a$ and $t := \rho$ give, respectively, that

$$E(av + \varepsilon u) = E(av) + \varepsilon^2 \left[\frac{1}{2}[u]^2 - \frac{a}{2}(\rho - a) \int_\Omega u^2\, dx \right] + o(\varepsilon^2) \tag{3.6}$$

$$\text{and} \quad E(\rho v + \varepsilon u) = E(\rho v) + \varepsilon^2 \left[\frac{1}{2}[u]^2 + \frac{\rho}{2}(\rho - a) \int_\Omega u^2\, dx \right] + o(\varepsilon^2) \tag{3.7}$$

as $\varepsilon \to 0$.

We first consider the case in which $0 < a < \rho$. In this case, we recall that the mixed operator $-\alpha\Delta + \beta(-\Delta)^s$ has an unbounded sequence of eigenvalues $(\lambda_k)_k$ (see [DPLV22]), and therefore there exists $\overline{k} \in \mathbb{N}$ such that

$$\lambda_k \leqslant a(\rho - a) \text{ for } k = \{0, ..., \overline{k}\} \qquad \text{and} \qquad \lambda_k > a(\rho - a) \text{ for every } k > \overline{k},$$

with $\lambda_0 = 0$. This and (3.6) entail that av is a saddle point for the functional E.

Furthermore, (3.7) says that ρv is a minimum for E. We compute

$$E(\rho v) = \frac{\rho^3}{6} \left(a - \frac{\rho}{2} \right) |\Omega|. \tag{3.8}$$

Hence,

- if $a \in (0, \rho/2)$, we have that $E(\rho v) < 0 = E(0)$, and therefore ρv is an absolute minimum for E;
- if $a = \rho/2$, we have that $E(\rho v) = E(0) = 0$, and therefore both 0 and ρv are absolute minima for E;
- if $a \in (\rho/2, \rho)$, we have that $E(\rho v) > 0 = E(0)$, and therefore ρv is a relative minimum for E.

When $a = 0$, we deduce from (3.7) and (3.8) that

$$E(\rho v + \varepsilon u) = -\frac{\rho^4}{12}|\Omega| + \varepsilon^2 \left[\frac{1}{2}[u]^2 + \frac{\rho^2}{2} \int_\Omega u^2\, dx \right] + o(\varepsilon^2)$$

as $\varepsilon \to 0$, which entails that ρv is an absolute minimum for E.

Finally, when $a = \rho$, we see that

$$E(\rho v + \varepsilon u) = \frac{\rho^4}{12} + \frac{\varepsilon^2}{2}[u]^2 + \frac{\varepsilon^3}{3} \int_\Omega |u|^3\, dx + o(\varepsilon^3)$$

as $\varepsilon \to 0$. In particular, if u is a positive constant c, then

$$E(\rho v + \varepsilon u) = \frac{\rho^4}{12} + \frac{\varepsilon^3 c^3 |\Omega|}{3} + o(\varepsilon^3).$$

Therefore, there exists $\varepsilon_0 > 0$ small enough such that, if $\varepsilon \in (0, \varepsilon_0)$, then

$$E(\rho v - \varepsilon u) < E(\rho v) < E(\rho v + \varepsilon u),$$

hence ρv is neither a local maximum nor a local minimum. □

Now, we deal with the case $\rho < 0 \leqslant a$.

Proof of Theorem 1.4. When $a = 0$, the energy functional in (3.3) boils down to

$$E(u) = \frac{1}{2}[u]^2 + \frac{1}{4}\int_\Omega u^4\,dx - \frac{\rho}{3}\int_\Omega |u|^3\,dx \geqslant 0,$$

for every $u \in X_{\alpha,\beta}$, so 0 is an absolute minimum for E.

From the computation in (3.5) we see that, in this case, ρ is a critical point for E, but it would give a negative solution.

Instead, when $a > 0$, from (3.5) we obtain that $t = 0$ and $t = a$ are the only nonnegative constant critical points of E. Taking $v \equiv 1$, we use (3.4) to compute

$$E(av) = -\frac{a^3}{12}(a - 2\rho) < 0$$

Hence, from (3.6) we find that

$$E(av + \varepsilon u) = -\frac{a^3}{12}(a - 2\rho) + \varepsilon^2 \left(\frac{1}{2}[u]^2 + \frac{a}{2}(a - \rho)\int_\Omega u^2\,dx\right) + o(\varepsilon^2)$$

as $\varepsilon \to 0$, which gives that av is an absolute minimum for E. □

4. Elliptic problem with variable coefficients: proofs of Theorems 1.5 and 1.6

Now we delve into the steady states of the general setting described in Section 1.5. The energy functional associated with problem (1.7) is

$$\mathscr{E}(u) := \frac{1}{2}[u]^2 + \frac{1}{4}\int_\Omega u^4\,dx - \frac{1}{3}\int_\Omega (\rho + a)|u|^3\,dx + \frac{1}{2}\int_\Omega \rho a\, u^2\,dx - \frac{1}{2}\int_\Omega u(J \star u)\,dx, \tag{4.1}$$

where the seminorm $[u]$ has been defined in (3.2).

We say that $u \in X_{\alpha,\beta}$ is a (weak) solution of (1.7) if

$$\begin{aligned} &\alpha \int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx + \frac{\beta}{2}\iint_{\mathbb{Q}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}}\,dx\,dy \\ &\quad = \int_\Omega \Big((u(x) - a(x))(\rho(x) - u(x))u(x) + J \star u(x)\Big)v(x)\,dx \end{aligned} \tag{4.2}$$

for all functions $v \in X_{\alpha,\beta}$ (in what follows, we will not insist in the subtle difference between weak and strong solutions, assuming, whenever needed, that the solutions possess all the necessary regularity to carry out the calculations).

We have the following useful inequalities:

Lemma 4.1. *Let v, $w \in L^2(\Omega)$.*

Then,

$$\int_\Omega |v(x)|\,|(J \star w)(x)|\;dx \leqslant \tau \|v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)}. \tag{4.3}$$

Proof. By the Cauchy-Schwarz Inequality we have

$$\int_\Omega |v(x)|\,|(J\star w)(x)|\,dx \leqslant \|v\|_{L^2(\Omega)}\|J\star w\|_{L^2(\Omega)}. \tag{4.4}$$

Then, using the Young Inequality for convolutions with exponents 1 and 2 (see e.g. Theorem 9.1 in [WZ15]) together with (1.8), we get

$$\|J\star w\|_{L^2(\Omega)} = \|J*(w\chi_\Omega)\|_{L^2(\Omega)} \leqslant \|J\|_{L^1(\Omega)}\|w\chi_\Omega\|_{L^2(\Omega)} \leqslant \tau\|w\|_{L^2(\Omega)},$$

where the symbol "$*$" denotes the standard convolution over $\mathbb{R}^n$. From this and (4.4) we obtain (4.3). □

Now, we provide a minimization argument for the functional $\mathscr{E}$ in (4.1).

Proposition 4.2. *Assume that $\rho a \in L^q(\Omega)$ for some $q \in (\underline{q}, +\infty]$, where $\underline{q}$ was introduced in* (1.9), *and that*

$$(\rho+a)_+^4 \in L^1(\Omega) \qquad \textit{and} \qquad (\tau-\rho a)_+^2 \in L^1(\Omega). \tag{4.5}$$

If $(\rho+a)_+ \not\equiv 0$, suppose in addition that $n < 6\overline{s}$ and

$$(\rho+a)_+^r \in L^1(\Omega) \textit{ for some } r > \tfrac{2n}{6\overline{s}-n}. \tag{4.6}$$

Then, the functional $\mathscr{E}$ defined in (4.1) *attains a nonnegative minimum in $X_{\alpha,\beta}$.*

Proof. Let $q \in (\underline{q}, +\infty]$ and define $p := \dfrac{2q}{q-1}$, then $p \in \left[2, \dfrac{2\underline{q}}{\underline{q}-1}\right)$ and

$$\frac{2}{p} + \frac{1}{q} = 1. \tag{4.7}$$

From the inequality in (4.3) of Lemma 4.1, we have that

$$\frac{1}{2}\int_\Omega u(J\star u)\,dx \leqslant \frac{\tau}{2}\int_\Omega u^2\,dx. \tag{4.8}$$

Furthermore, using the Young Inequality with exponents 4/3 and 4, we get

$$\frac{(\rho+a)_+}{3}\,|u|^3 = \frac{|u|^3}{3^{\frac{3}{2}}}\cdot\frac{(\rho+a)_+}{3^{-\frac{1}{2}}} \leqslant \frac{u^4}{12} + \frac{9}{4}(\rho+a)_+^4, \tag{4.9}$$

while, using it with exponents 2 and 2,

$$\frac{\tau-\rho a}{2}\,u^2 \leqslant \frac{(\tau-\rho a)_+}{2}\,u^2 = \frac{u^2}{2^{\frac{1}{2}}3^{\frac{1}{2}}}\cdot\frac{(\tau-\rho a)_+}{2^{\frac{1}{2}}3^{-\frac{1}{2}}} \leqslant \frac{u^4}{12} + \frac{3}{4}(\tau-\rho a)_+^2.$$

From this, (4.8) and (4.9), we conclude that

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{6}u^4 - \frac{\rho+a}{3}|u|^3 + \frac{\rho a}{2}u^2 - \frac{1}{2}u(J\star u)\right)dx\\
&\qquad\geqslant \int_\Omega \left(\frac{1}{6}u^4 - \frac{\rho+a}{3}|u|^3 + \frac{\rho a}{2}u^2 - \frac{\tau}{2}u^2\right)dx\\
&\qquad\geqslant -\frac{3}{4}\int_\Omega \Big(3(\rho+a)_+^4 + (\tau-\rho a)_+^2\Big)\,dx =: -\kappa.
\end{aligned} \tag{4.10}$$

We note that, from (4.5), the quantity κ is finite and does not depend on u.

Hence, recalling the definition of $\mathscr{E}$ in (4.1), we deduce from (4.10) that

$$\mathscr{E}(u) \geqslant \frac{\alpha}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\beta}{4}\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega \frac{u^4}{12}\,dx - \kappa. \tag{4.11}$$

Now, taking a minimizing sequence u_j, we can assume, in light of Theorem 2.1 in [DPLV23], that

$$\mathscr{N}_s u_j = 0 \text{ in } \mathbb{R}^n \setminus \overline{\Omega}, \text{ for every } j \in \mathbb{N}. \tag{4.12}$$

In addition, we can suppose that

$$\begin{aligned}0 &= \mathscr{E}(0) \geqslant \mathscr{E}(u_j)\\ &\geqslant \frac{\alpha}{2}\int_\Omega |\nabla u_j|^2\,dx + \frac{\beta}{4}\iint_\Omega \frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega \frac{u_j^4}{12}\,dx - \kappa,\end{aligned}$$

where we used (4.11) in the last inequality.

This gives that

$$\frac{\alpha}{2}\int_\Omega |\nabla u_j|^2\,dx + \frac{\beta}{4}\iint_\Omega \frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega \frac{u_j^4}{12}\,dx \leqslant \kappa.$$

In particular, using the Cauchy-Schwarz Inequality, we have that

$$\|u_j\|^2_{L^2(\Omega)} \leqslant \left(\int_\Omega u_j^4\,dx\right)^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} \leqslant 12^{\frac{1}{2}}|\Omega|^{\frac{1}{2}}\kappa.$$

In light of these observations, one can employ compactness arguments and obtain that, up to a subsequence, u_j converges to some $u \in L^m(\Omega)$ for every $m \in [1, 2^*_s)$ (see Corollary 7.2 in [DNPV12]) and a.e. in Ω, and also that $|u_j| \leqslant h$ for some $h \in L^m(\Omega)$ for every $j \in \mathbb{N}$ (see Theorem IV.9 in [Bre83]).

Then, for $x \in \mathbb{R}^n \setminus \overline{\Omega}$, we can utilize the Dominated Convergence Theorem, obtaining that

$$\lim_{j\to+\infty}\int_\Omega \frac{u_j(y)}{|x-y|^{n+2s}}\,dy = \int_\Omega \frac{u(y)}{|x-y|^{n+2s}}\,dy.$$

Moreover, recalling (4.12), for $x \in \mathbb{R}^n \setminus \overline{\Omega}$ we have that

$$\lim_{j\to+\infty} u_j(x) = \lim_{j\to+\infty} \frac{\displaystyle\int_\Omega \frac{u_j(y)}{|x-y|^{n+2s}}\,dy}{\displaystyle\int_\Omega \frac{dy}{|x-y|^{n+2s}}} = \frac{\displaystyle\int_\Omega \frac{u(y)}{|x-y|^{n+2s}}\,dy}{\displaystyle\int_\Omega \frac{dy}{|x-y|^{n+2s}}} =: u(x),$$

yielding that u_j converges a.e. in $\mathbb{R}^n$ to a function u that satisfies $\mathscr{N}_s u = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$.

In order to pass to the limit the cubic term, we observe that if $(\rho+a)_+ \equiv 0$, then, by Fatou's Lemma,

$$\liminf_{j\to+\infty}\left(-\int_\Omega (\rho+a)|u_j|^3\,dx\right) \geqslant -\int_\Omega (\rho+a)|u|^3\,dx.$$

If instead $(\rho+a)_+ \not\equiv 0$, we see that

$$\begin{aligned}\liminf_{j\to+\infty}\left(-\int_\Omega (\rho+a)|u_j|^3\,dx\right) &= \liminf_{j\to+\infty}\left(-\int_\Omega (\rho+a)_+|u_j|^3\,dx + \int_\Omega (\rho+a)_-|u_j|^3\,dx\right)\\ &\geqslant \liminf_{j\to+\infty}\left(-\int_\Omega (\rho+a)_+|u_j|^3\,dx\right) + \liminf_{j\to+\infty}\left(\int_\Omega (\rho+a)_-|u_j|^3\,dx\right)\\ &\geqslant \liminf_{j\to+\infty}\left(-\int_\Omega (\rho+a)_+|u_j|^3\,dx\right) + \int_\Omega (\rho+a)_-|u|^3\,dx,\end{aligned} \tag{4.13}$$

where we have used again Fatou's Lemma in the last line.

Now, we observe that $(\rho+a)_+|u_j|^3 \leqslant (\rho+a)_+h^3$ and we claim that

$$(\rho+a)_+h^3 \in L^1(\Omega). \tag{4.14}$$

Indeed, we recall (4.6) and we utilize the Hölder Inequality with exponents r and $\frac{r}{r-1}$ to obtain that

$$\int_\Omega (\rho+a)_+h^3\,dx \leqslant \left(\int_\Omega (\rho+a)_+^r\,dx\right)^{\frac{1}{r}} \left(\int_\Omega h^{\frac{3r}{r-1}}\,dx\right)^{\frac{r-1}{r}}. \tag{4.15}$$

We point out that

$$\frac{3r}{r-1} = 3 + \frac{3}{r-1} < 3 + \frac{3}{\frac{2n}{n-6\overline{s}} - 1} = \frac{6n}{n+6\overline{s}} < 2^*_{\overline{s}}.$$

This implies that the integrals in (4.15) are finite, thus giving (4.14).

Thanks to (4.14) and the pointwise convergence, we can use the Dominated Convergence Theorem and obtain from (4.13) that

$$\liminf_{j\to+\infty}\left(-\int_\Omega(\rho+a)|u_j|^3\,dx\right) \geqslant -\int_\Omega(\rho+a)_+|u|^3\,dx + \int_\Omega(\rho+a)_-|u|^3\,dx = -\int_\Omega(\rho+a)|u|^3\,dx.$$

Furthermore, to deal with the quadratic term, we recall (4.7), observe that $p < 2^*_{\overline{s}}$ and use the convergenge in $L^p(\Omega)$ to see that

$$\begin{aligned}\limsup_{j\to+\infty}\left|\int_\Omega \rho a(u_j^2-u^2)\,dx\right| &\leqslant \limsup_{j\to+\infty}\int_\Omega \big|\rho a(u_j^2-u^2)\big|\,dx\\ &= \limsup_{j\to+\infty}\int_\Omega \big|\rho a(u_j-u)(u_j+u)\big|\,dx\\ &\leqslant \limsup_{j\to+\infty}\|\rho a\|_{L^q(\Omega)}\|u_j-u\|_{L^p(\Omega)}\|u_j+u\|_{L^p(\Omega)} = 0.\end{aligned}$$

As for the "pollination" term, we notice that

$$\left|\int_\Omega \big(u_j(J\star u_j) - u(J\star u)\big)\,dx\right| \leqslant \left|\int_\Omega (u_j-u)(J\star u_j)\,dx\right| + \left|\int_\Omega (J\star u_j - J\star u)u\,dx\right|.$$

In light of (4.3), used here with $v := u_j - u$ and $w := u_j$,

$$\limsup_{j\to+\infty}\int_\Omega |u_j-u|\big|J\star u_j\big|\,dx \leqslant (\tau_+ + \tau_-)\limsup_{j\to+\infty}\|u_j-u\|_{L^2(\Omega)}\|u_j\|_{L^2(\Omega)} = 0.$$

Also, by (4.3) used now with $v := u$ and $w := u_j - u$,

$$\begin{aligned}\limsup_{j\to+\infty}\int_\Omega \big|J\star u_j - J\star u\big||u|\,dx &= \limsup_{j\to+\infty}\int_\Omega \big|J\star(u_j-u)\big||u|\,dx\\ &\leqslant (\tau_+ + \tau_-)\limsup_{j\to+\infty}\|u_j-u\|_{L^2(\Omega)}\|u_j\|_{L^2(\Omega)} = 0.\end{aligned}$$

Gathering these pieces of information, we conclude that

$$\lim_{j\to+\infty}\int_\Omega\Big(u_j(J\star u_j) - u(J\star u)\Big)\,dx = 0.$$

Additionally, from Fatou's Lemma and the lower semicontinuity of the L^2-norm, one obtains that

$$\liminf_{j\to+\infty}\iint_Q \frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \geqslant \iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$

$$\liminf_{j\to+\infty}\int_\Omega |\nabla u_j|^2\,dx \geqslant \int_\Omega |\nabla u|^2\,dx$$

and

$$\liminf_{j\to+\infty}\int_\Omega \frac{u_j^4}{12}\,dx \geqslant \int_\Omega \frac{u^4}{12}\,dx.$$

Putting together all these observations, we conclude that

$$\liminf_{j\to+\infty}\mathscr{E}(u_j) \geqslant \mathscr{E}(u),$$

which entails that u is a minimum for $\mathscr{E}$.

Finally, since $\mathscr{E}(|u|) \leqslant \mathscr{E}(u)$, we can suppose that the minimum u is nonnegative. □

We now give the proof of Theorem 1.5, which covers the case $\int_\Omega \rho(x)\,dx > 0$.

Proof of Theorem 1.5. Thanks to Proposition 4.2 there exists a nonnegative minimizer of $\mathscr{E}$, hence there exists a nonnegative solution of (1.7).

We now prove the claim in (i). In order to do this, we assume that $J \equiv 0$ and $a(x) = \rho(x)$ is not a constant function. We suppose by contradiction that there exists a nontrivial solution u of (1.7). Taking $v := u$ in (4.2), we obtain that

$$\begin{aligned} 0 \leqslant \alpha \int_\Omega |\nabla u(x)|^2\,dx + \frac{\beta}{2}\iint_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ = -\int_\Omega u^2(x)\big(u(x)-a(x)\big)^2\,dx \leqslant 0. \end{aligned}$$

This implies that $u(x) = a(x)$ for a.e. $x \in \Omega$ and that

$$\alpha \int_\Omega |\nabla a(x)|^2\,dx + \frac{\beta}{2}\iint_\Omega \frac{|a(x)-a(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = 0,$$

which is a contradiction since the function $a(x)$ in not constant, hence the claim in (i) is proved.

Now we deal with the claim in (ii). Thanks to Proposition 4.2, we just need to show that 0 is not a minimizer for $\mathscr{E}$. In order to do this, we look for a function $u \in X_{\alpha,\beta}$ such that $\mathscr{E}(u) < 0$. Taking $t > 0$ and $v \equiv 1$, we have

$$\mathscr{E}(tv) = t^2\left(\frac{|\Omega|}{4}t^2 - \frac{t}{3}\int_\Omega(\rho+a)\,dx + \frac{1}{2}\int_\Omega \rho a\,dx - \frac{1}{2}\int_\Omega (J\star 1)\,dx\right) =: g(t).$$

We also compute the derivative of the above function with respect to t, obtaining that

$$g'(t) = t\Big(t^2|\Omega| - t\int_\Omega(\rho+a)\,dx + \int_\Omega \rho a\,dx - \int_\Omega (J\star 1)\,dx\Big).$$

Therefore, g achieves its minimum at

$$t_{min} := \frac{1}{2|\Omega|}\left[\int_\Omega(\rho+a)\,dx + \sqrt{\left(\int_\Omega(\rho+a)\,dx\right)^2 - 4|\Omega|\left(\int_\Omega \rho a\,dx - \int_\Omega (J\star 1)\,dx\right)}\,\right] \tag{4.16}$$

provided that

$$\int_\Omega \rho a\,dx - \int_\Omega (J\star 1)\,dx \leqslant \frac{1}{4|\Omega|}\left(\int_\Omega(\rho+a)\,dx\right)^2. \tag{4.17}$$

We point out that, in light of the assumption in (1.10), we have that

$$\int_\Omega(\rho+a)\,dx \geqslant 0, \tag{4.18}$$

and therefore $t_{min} > 0$.

Furthermore, we remark that the assumption in (1.11) implies, in particular, that (4.17) holds true. Thus, in this case, we plug the explicit expression of t_{min}, as given by (4.16), into the functional and we see that

$$
\begin{aligned}
&\mathscr{E}(t_{min}v) \\
&= t_{min}^2\Bigg[\frac{1}{16|\Omega|}\left(\int_\Omega(\rho+a)\,dx+\sqrt{\left(\int_\Omega(\rho+a)\,dx\right)^2-4|\Omega|\left(\int_\Omega\rho a\,dx-\int_\Omega(J\star 1)\,dx\right)}\right)^2 \\
&\quad-\frac{1}{6|\Omega|}\int_\Omega(\rho+a)\,dx\left(\int_\Omega(\rho+a)\,dx+\sqrt{\left(\int_\Omega(\rho+a)\,dx\right)^2-4|\Omega|\left(\int_\Omega\rho a\,dx-\int_\Omega(J\star 1)\,dx\right)}\right) \\
&\quad+\frac12\int_\Omega\rho a\,dx-\frac12\int_\Omega(J\star 1)\,dx\Bigg] \\
&= -\frac{t_{min}^2}{4}\Bigg[\frac{1}{6|\Omega|}\left(\int_\Omega(\rho+a)\,dx\right)^2 \\
&\quad+\frac{1}{6|\Omega|}\int_\Omega(\rho+a)\,dx\sqrt{\left(\int_\Omega(\rho+a)\,dx\right)^2-4|\Omega|\left(\int_\Omega\rho a\,dx-\int_\Omega(J\star 1)\,dx\right)} \\
&\quad-\int_\Omega\rho a\,dx+\int_\Omega(J\star 1)\,dx\Bigg]
\end{aligned}
\tag{4.19}
$$

Thanks to (4.18) and the condition in (1.11), we conclude that

$$
\begin{aligned}
&\frac{1}{6|\Omega|}\left(\int_\Omega(\rho+a)\,dx\right)^2 \\
&\qquad+\frac{1}{6|\Omega|}\int_\Omega(\rho+a)\,dx\sqrt{\left(\int_\Omega(\rho+a)\,dx\right)^2-4|\Omega|\left(\int_\Omega\rho a\,dx-\int_\Omega(J\star 1)\,dx\right)} \\
&\qquad-\int_\Omega\rho a\,dx+\int_\Omega(J\star 1)\,dx \\
&>\frac{1}{6|\Omega|}\left(\int_\Omega(\rho+a)\,dx\right)^2+\frac{1}{6|\Omega|}\int_\Omega(\rho+a)\,dx\sqrt{\left(\int_\Omega(\rho+a)\,dx\right)^2-\frac89\left(\int_\Omega(\rho+a)\,dx\right)^2} \\
&\qquad-\frac{2}{9|\Omega|}\left(\int_\Omega(\rho+a)\,dx\right)^2 \\
&=\frac{1}{|\Omega|}\left(\frac16+\frac1{18}-\frac29\right)\left(\int_\Omega(\rho+a)\,dx\right)^2=0.
\end{aligned}
\tag{4.20}
$$

Plugging this information into (4.19), we infer that $\mathscr{E}(t_{min}v)<0$.

Thus, if u is a minimizer of $\mathscr{E}$, we have that $\mathscr{E}(u)\leqslant\mathscr{E}(t_{min}v)<0=\mathscr{E}(0)$. Accordingly, u is a nontrivial minimizer for $\mathscr{E}$ and hence a nontrivial solution of (1.7). □

Now, we give the proof of Theorem 1.6, dealing with the case $\int_\Omega\rho(x)\,dx<0$.

Proof of Theorem 1.6. The arguments are similar to those in the proof of Theorem 1.5, we provide the details here for the facility of the reader.

First of all, Proposition 4.2 gives the existence of a nonnegative minimizer of $\mathscr{E}$, and thus of a nonnegative solution of (1.7).

We first prove the claim in (i), and we assume that $J\equiv 0$ and $a(x)=\rho(x)\equiv 0$. We suppose by contradiction that there exists a nontrivial solution u of (1.7). Taking $v:=u$ in (4.2) we obtain

$$0 \leqslant \alpha \int_\Omega |\nabla u(x)|^2\,dx + \frac{\beta}{2}\iint_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = -\int_\Omega u^4\,dx < 0,$$

which is a contradiction, hence the claim in (i) is proved.

Now we deal with the claim in (ii). Reasoning as in the proof of Theorem 1.5 we get that the functional $\mathscr{E}$ attains a minimum in $t_{min}v$ with $v \equiv 1$ and t_{min} defined as in (4.16).

Notice that, in this case, in light of the assumption in (1.12), it is not guaranteed that $t_{min} > 0$. Thus, we distinguish two cases, when $\int_\Omega(\rho+a)\,dx \geqslant 0$ and when $\int_\Omega(\rho+a)\,dx < 0$.

In the first case, we have that $t_{min} > 0$, and therefore, recalling the computation in (4.19) and (4.20), we conclude that $\mathscr{E}(t_{min}v) < 0$ as long as (1.13) holds true.

Hence, if u is a minimizer, then $\mathscr{E}(u) \leqslant \mathscr{E}(t_{min}v) < 0 = \mathscr{E}(0)$, and therefore u is a notrivial solution of (1.7). This establishes the claim in (ii) under the assumptions in (1.13).

If instead $\int_\Omega(\rho+a)\,dx < 0$, then we see from (4.16) that $t_{min} > 0$ if and only if

$$\int_\Omega \rho a\,dx - \int_\Omega (J\star 1)\,dx < 0.$$

In this case, we observe that

$$\begin{aligned}\mathscr{E}(tv) &= t^2\left(\frac{|\Omega|}{4}t^2 - \frac{t}{3}\int_\Omega(\rho+a)\,dx + \frac{1}{2}\int_\Omega \rho a\,dx - \frac{1}{2}\int_\Omega (J\star 1)\,dx\right)\\ &= c_1t^2 + c_2t^3 + c_3t^4,\end{aligned}$$

with

$$\begin{aligned} c_1 &:= \frac{1}{2}\left(\int_\Omega \rho a\,dx - \int_\Omega (J\star 1)\,dx\right),\\ c_2 &:= -\frac{1}{3}\int_\Omega(\rho+a)\,dx\\ \text{and}\qquad c_3 &:= \frac{|\Omega|}{4}.\end{aligned}$$

In particular, if the assumption in (1.14) is in force, we see that $c_1 < 0$. As a result, we have that $\mathscr{E}(tv) < 0$ for sufficiently small values of $t > 0$.

Hence, if u is a minimizer, then $\mathscr{E}(u) \leqslant \mathscr{E}(t_{min}v) \leqslant \mathscr{E}(tv) < 0 = \mathscr{E}(0)$, for all $t > 0$ sufficiently small, and therefore u is a nontrivial solution of (1.7). This establishes the claim in (ii) under the assumptions in (1.14), and the proof of Theorem 1.6 is thereby complete. □

5. Parabolic problem: proofs of Theorems 1.8, 1.9, 1.10, 1.11, 1.12, and 1.13

Now we address the evolutionary problem presented in Section 1.6.

In what follows we will use the nonlocal counterparts of the Divergence Theorem and the integration by parts formula, as established, respectively, in Lemmata 3.2 and 3.3 in [DROV17]. Since we will use these results in a slightly different setting, we provide the proofs here for completeness.

Lemma 5.1. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be such that $u \in C^2(\Omega)$ and*

$$(-\Delta)^s u \in L^1(\Omega). \tag{5.1}$$

Assume that the function

$$\Omega \times (\mathbb{R}^n \setminus \Omega) \ni (x,y) \mapsto \frac{u(x)-u(y)}{|x-y|^{n+2s}} \text{ belongs to } L^1(\Omega \times (\mathbb{R}^n\setminus\Omega)). \tag{5.2}$$

Then,

$$\int_\Omega (-\Delta)^s u(x)\,dx = -\int_{\mathbb{R}^n\setminus\Omega} \mathscr{N}_s u(x)\,dx. \tag{5.3}$$

Proof. Notice that the assumptions (5.1) and (5.2) guarantee that the integrals defined in formula (5.3) are finite.

Moreover, since $u \in C^2(\Omega)$, for every $\varepsilon > 0$, we have that

$$\iint_{\substack{\Omega\times\Omega\\ \{|x-y|\geqslant\varepsilon\}}} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dx\,dy = \iint_{\substack{\Omega\times\Omega\\ \{|x-y|\geqslant\varepsilon\}}} \frac{u(y)-u(x)}{|x-y|^{n+2s}}\,dx\,dy,$$

and therefore

$$\iint_{\substack{\Omega\times\Omega\\ \{|x-y|\geqslant\varepsilon\}}} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dx\,dy = 0.$$

Thus, taking the limit as $\varepsilon \searrow 0$,

$$\iint_{\Omega\times\Omega} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dx\,dy = 0.$$

From this, and using also (5.2) to exchange the order of integration, we get that

$$\begin{aligned}\int_\Omega (-\Delta)^s u(x)\,dx &= \int_\Omega\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy\,dx = \int_\Omega\int_{\mathbb{R}^n\setminus\Omega} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy\,dx\\ &= \int_{\mathbb{R}^n\setminus\Omega}\int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dx\,dy = -\int_{\mathbb{R}^n\setminus\Omega} \mathscr{N}_s u(y)\,dy,\end{aligned}$$

as desired. □

Lemma 5.2. *Let u, $v : \mathbb{R}^n \to \mathbb{R}$ be such that $u \in C^2(\Omega)$ and*

$$\mathcal{Q} \ni (x,y) \mapsto \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} \quad \textit{belongs to } L^1(\mathcal{Q}), \tag{5.4}$$

with $\mathcal{Q} = \mathbb{R}^{2n} \setminus (\mathbb{R}^n\setminus\Omega)^2$.

Assume that

$$(-\Delta)^s u\, v \in L^1(\Omega) \tag{5.5}$$

and that the function

$$(\mathbb{R}^n\setminus\Omega)\times\Omega \ni (x,y) \mapsto \frac{\big(u(x)-u(y)\big)v(x)}{|x-y|^{n+2s}} \quad \textit{belongs to } L^1((\mathbb{R}^n\setminus\Omega)\times\Omega). \tag{5.6}$$

Then,

$$\frac12\iint_{\mathcal{Q}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy = \int_\Omega (-\Delta)^s u(x)\,v(x)\,dx + \int_{\mathbb{R}^n\setminus\Omega} \mathscr{N}_s u(x)\,v(x)\,dx. \tag{5.7}$$

We point out that, by taking $v \equiv 1$ in Lemma 5.2, we recover the assumptions on u in Lemma 5.1 and the "nonlocal" Divergence Theorem as given by formula (5.3).

Proof of Lemma 5.2. We point out that the assumptions in (5.4), (5.5) and (5.6) guarantee that the integrals in (5.7) are finite.

Now we notice that $\mathcal{Q} = \big(\Omega\times\mathbb{R}^n\big) \cup \big((\mathbb{R}^n\setminus\Omega)\times\Omega\big)$ and therefore we can write

$$\begin{aligned}\frac12\iint_{\mathcal{Q}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy &= \iint_{\mathcal{Q}} \frac{(u(x)-u(y))v(x)}{|x-y|^{n+2s}}\,dx\,dy\\ &= \int_\Omega\int_{\mathbb{R}^n} \frac{(u(x)-u(y))v(x)}{|x-y|^{n+2s}}\,dx\,dy + \int_{\mathbb{R}^n\setminus\Omega}\int_\Omega \frac{(u(x)-u(y))v(x)}{|x-y|^{n+2s}}\,dx\,dy.\end{aligned}$$

Hence,

$$\begin{aligned}&\frac12\iint_{\mathcal{Q}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy\\ &= \int_\Omega v(x)\left(\frac12\int_{\mathbb{R}^n} \frac{2(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}\,dy\right)dx + \int_{\mathbb{R}^n\setminus\Omega} v(x)\left(\int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy\right)dx\end{aligned}$$

$$= \int_{\Omega} v(x)(-\Delta)^s u(x)\,dx + \int_{\mathbb{R}^n\setminus\Omega} v(x)\mathscr{N}_s u(x)\,dx,$$

where we used (1.3) and (1.5). This is the desired result in (5.7). □

Since it will be used several times throughout this section, it is also useful to recall the following (weak) Maximum Principle (see Theorem 1.1 in [DPLV]).

Theorem 5.3. *Let Ω be a bounded domain of $\mathbb{R}^n$ with C^1 boundary, $T>0$, $c\in\mathbb{R}$ and $f\in L^\infty(\Omega\times(0,T))$. Assume that u is a classical solution of*

$$\begin{cases} \partial_t u(x,t) - \alpha\Delta u(x,t) + \beta(-\Delta)^s u(x,t) = (u(x,t)-c)f(x,t) & \text{in } \Omega\times(0,T),\\ u(x,0)=u_0(x) & \text{in } \Omega,\\ \text{with } (\alpha,\beta)-\text{Neumann conditions} & \text{in } (0,T). \end{cases}$$

Assume also that

- *if $\beta=0$, then Ω satisfies the interior ball condition,*
- *if $\alpha\beta\neq 0$, then there exist $\mu>0$ and $\theta>2s$ such that*

$$u\in L^\infty\big([0,T],\,C^\theta(\Omega_\mu)\big) \qquad \text{with} \quad \Omega_\mu := \bigcup_{x\in\Omega} B_\mu(x).$$

Then,[4]

(i) if $u_0(x)\geqslant c$ for any $x\in\mathbb{R}^n$, then $u(x,t)\geqslant c$ for any $x\in\mathbb{R}^n$ and $t\in[0,T)$;
(ii) if $u_0(x)\leqslant c$ for any $x\in\mathbb{R}^n$, then $u(x,t)\leqslant c$ for any $x\in\mathbb{R}^n$ and $t\in[0,T)$.

We start with the case $0\leqslant a\leqslant\rho$. First, we prove the statement on the monotonicity of the total population, as given in Theorem 1.8.

Proof of Theorem 1.8. We start with the proof of (i). We observe that we are in the position of exploiting Theorem 5.3 (with $f(x,t):=(\rho-u(x,t))u(x,t)$ and $f(x,t):=(u(x,t)-a)(\rho-u(x,t))$) and we have that $0\leqslant u(x,t)\leqslant a$ for any $x\in\mathbb{R}^n$ and $t>0$.

Then, by (1.16) and the Dominated Convergence Theorem,

$$\frac{d}{dt}\int_\Omega u\,dx = \int_\Omega \partial_t u\,dx = \alpha\int_\Omega \Delta u\,dx - \beta\int_\Omega (-\Delta)^s u\,dx + \int_\Omega (u-a)(\rho-u)u\,dx.$$

Now, in light of (1.17), (1.18), (1.19), (5.3), the classical Divergence Theorem and the (α,β)-Neumann conditions, we find that

$$\alpha\int_\Omega \Delta u\,dx - \beta\int_\Omega(-\Delta)^s u\,dx = \alpha\int_{\partial\Omega}\frac{\partial u}{\partial\nu} + \beta\int_{\mathbb{R}^n\setminus\Omega}\mathscr{N}_s u\,dx = 0,$$

and therefore

$$\frac{d}{dt}\int_\Omega u\,dx = \int_\Omega (u-a)(\rho-u)u\,dx \leqslant 0. \tag{5.8}$$

In particular, this, together with the assumptions on u_0, says that

$$\int_\Omega u(x,t)\,dx \leqslant \int_\Omega u_0(x)\,dx < \int_\Omega a\,dx \leqslant \int_\Omega \rho\,dx.$$

This implies that equality in (5.8) holds if and only if $u\equiv 0$ at some point in time.

Hence, we conclude that either the quantity $\int_\Omega u\,dx$ is decreasing in time $t>0$ or

$$\text{there exists } \bar t>0 \text{ such that } u(x,\bar t)=0 \text{ for all } x\in\mathbb{R}^n. \tag{5.9}$$

We now show that the latter is not possible (which would complete the proof of case (i)).

[4]We point out that, here and throughout the paper, we use the notation "for all $x\in\mathbb{R}^n$" even when $\beta=0$. As a matter fact, in this case it suffices to write "for all $x\in\Omega$" in the statement of Theorem 5.3, but we use this more compact notation for the sake of readability.

To this end, we notice that, since $\inf_\Omega u_0 > 0$, we can take a real number $v_0 \in (0, \inf_\Omega u_0)$ and consider a function $v = v(t)$ that solves

$$\begin{cases} v_t(t) = (v(t) - a)(\rho - v(t))v(t) & \text{for } t \in (0, +\infty), \\ v(0) = v_0. \end{cases} \tag{5.10}$$

Thus, setting $w := u - v$, we have that

$$\begin{aligned} \partial_t w - \alpha \Delta w + \beta(-\Delta)^s w &= (u-a)(\rho - u)u - (v-a)(\rho - v)v \\ &= (u-v)(-u^2 - uv - v^2 + (\rho + a)(u+v) - \rho a) \\ &= w\widetilde{f}, \end{aligned}$$

where we have set

$$\widetilde{f}(x,t) := -u^2(x,t) - u(x,t)v(x,t) - v^2(x,t) + (\rho + a)\big(u(x,t) + v(x,t)\big) - \rho a.$$

We point out that $\widetilde{f}$ is bounded, since so are u and v.

Notice that, if $\beta \neq 0$, for all $x \in \mathbb{R}^n \setminus \Omega$,

$$u_0(x) = \frac{\displaystyle\int_\Omega \frac{u_0(y)}{|x-y|^{n+2s}}\, dy}{\displaystyle\int_\Omega \frac{dy}{|x-y|^{n+2s}}} \geqslant \inf_\Omega u_0.$$

Accordingly, for all $x \in \mathbb{R}^n$, we have that

$$w(x,0) = u(x,0) - v(0) = u_0(x) - v_0 > u_0(x) - \inf_\Omega u_0 \geqslant 0.$$

Gathering these pieces of information, we deduce that we are in the position of applying Theorem 5.3 to w, obtaining that $w(x,t) \geqslant 0$ for any $x \in \mathbb{R}^n$ and $t > 0$. This gives that $u(x,t) \geqslant v(t) > 0$ for any $x \in \mathbb{R}^n$ and $t > 0$, and therefore (5.9) cannot hold true. This completes the proof of case (i).

Similarly, in case (ii) we use Theorem 5.3 to deduce that $a \leqslant u(x,t) \leqslant \rho$ for any $x \in \mathbb{R}^n$ and $t > 0$. Therefore, we see that

$$\frac{d}{dt} \int_\Omega u \, dx = \int_\Omega (u-a)(\rho - u)u \, dx \geqslant 0$$

and equality holds if and only if there exists $\bar{t} > 0$ such that $u(x, \bar{t}) = \rho$ for all $x \in \mathbb{R}^n$. One can show that this is not possible taking v that solves (5.10) with $v_0 \in (\sup_\Omega u_0, \rho)$ and applying Theorem 5.3 to $w := v - u$. In this way, one gets that $\rho > v(t) \geqslant u(x,t)$ for any $x \in \mathbb{R}^n$ and $t > 0$.

Finally, in case (iii) we use Theorem 5.3 to deduce that $u(x,t) \geqslant \rho$ for any $x \in \mathbb{R}^n$ and $t > 0$. Thus, we have that

$$\frac{d}{dt} \int_\Omega u \, dx = \int_\Omega (u-a)(\rho - u)u \, dx \leqslant 0$$

and equality holds if and only if there exists $\bar{t} > 0$ such that $u(x, \bar{t}) = \rho$ for all $x \in \mathbb{R}^n$. Now, taking v that solves (5.10) with $v_0 \in (\rho, \inf_\Omega u_0)$, one can apply Theorem 5.3 to $w := u - v$ and get that $u(x,t) \geqslant v(t) > \rho$ for any $x \in \mathbb{R}^n$ and $t > 0$. This concludes the proof of Theorem 1.8. $\square$

Next, we prove that the energy introduced in Theorem 1.9 is decreasing in time.

Proof of Theorem 1.9. The strategy is to compute $E'(t)$ and show that it is nonpositive. More precisely, recalling (1.21) and (1.22),

$$\begin{aligned} &E'(t) \\ &= \alpha \int_\Omega \nabla u(x,t) \cdot \nabla \partial_t u(x,t)\,dx + \frac{\beta}{2} \iint_\Omega \frac{(u(x,t)-u(y,t))(\partial_t u(x,t) - \partial_t u(y,t))}{|x-y|^{n+2s}}\,dx\,dy \\ &\quad + \int_\Omega \Big[u(x,t)^3 \partial_t u(x,t) - (\rho + a) u(x,t)^2 \partial_t u(x,t) + \rho a u(x,t) \partial_t u(x,t) \Big]\,dx. \end{aligned} \tag{5.11}$$

In light of (1.19), (1.21), (1.22), (1.23), (1.24) and (1.25), from (5.7), the classical integration by parts formula and the (α, β)-Neumann boundary conditions, we get

$$\begin{aligned} &\alpha \int_\Omega \nabla u(x,t) \cdot \nabla \partial_t u(x,t)\,dx + \frac{\beta}{2} \iint_\Omega \frac{(u(x,t)-u(y,t))(\partial_t u(x,t) - \partial_t u(y,t))}{|x-y|^{n+2s}}\,dx\,dy \\ &\quad = -\alpha \int_\Omega \partial_t u(x,t) \Delta u(x,t)\,dx + \beta \int_\Omega \partial_t u(x,t) (-\Delta)^s u(x,t)\,dx \\ &\quad = -\int_\Omega \partial_t u(x,t) \Big(\alpha \Delta u(x,t) - \beta (-\Delta)^s u(x,t) \Big)\,dx. \end{aligned}$$

Thus (5.11) becomes

$$\begin{aligned} E'(t) &= -\int_\Omega \partial_t u(x,t) \Big(\alpha \Delta u(x,t) - \beta (-\Delta)^s u(x,t) + (u(x,t) - a)(\rho - u(x,t)) u(x,t) \Big)\,dx \\ &= -\int_\Omega |\partial_t u(x,t)|^2\,dx \leqslant 0, \end{aligned}$$

with equality holding if and only if u is constant in time. This concludes the proof of Theorem 1.9. ☐

Now we show the behavior of the solutions of (1.15) when $t \to +\infty$, as presented in Theorem 1.10.

Proof of Theorem 1.10. We only provide the proof of (i), being the ones of the claims in (ii) and (iii) analogous with obvious modifications.

To establish (i), we observe that, if $\beta \neq 0$, for every $x \in \mathbb{R}^n \setminus \Omega$,

$$u_0(x) = \frac{\displaystyle\int_\Omega \frac{u_0(y)}{|x-y|^{n+2s}}\,dy}{\displaystyle\int_\Omega \frac{dy}{|x-y|^{n+2s}}},$$

which, in particular, implies that $0 \leqslant u_0(x) \leqslant a$ for all $x \in \mathbb{R}^n$. Accordingly, one can deduce from Theorem 5.3 that

$$0 \leqslant u(x,t) \leqslant a \text{ for all } x \in \mathbb{R}^n \text{ and } t > 0. \tag{5.12}$$

Now, we define, for all $t > 0$,

$$A(t) := \int_\Omega u(x,t)\,dx.$$

Notice that $A(t) \geqslant 0$ for all $t > 0$, thanks to (5.12). Additionally, in light of Theorem 1.8(i) we know that A is decreasing in time, and therefore there exists $\ell \in [0, a|\Omega|)$ such that

$$\lim_{t \to +\infty} A(t) = \ell.$$

Moreover, in light of (1.16), (1.17), (1.18), (1.19), we can repeat the computation in (5.8) and deduce that

$$A'(t) = \frac{d}{dt} \int_\Omega u\,dx = \int_\Omega \big(u(x,t) - a\big)\big(\rho - u(x,t)\big) u(x,t)\,dx \leqslant 0.$$

Exploiting again (5.12), we have that $\big(u(x,t)-a\big)\big(\rho-u(x,t)\big)u(x,t)\leqslant 0$ for all $x\in\Omega$ and $t>0$. Moreover, by hypotesis we know that $u(x,t)$ admits limit as $t\to+\infty$ for a.e $x\in\Omega$, and this limit is finite in view of (5.12). Combining these pieces of information, we conclude that, for a.e $x\in\Omega$,

$$\lim_{t\to+\infty}\big(u(x,t)-a\big)\big(\rho-u(x,t)\big)u(x,t)=0. \tag{5.13}$$

Now, we set

$$L(x):=\lim_{t\to+\infty}u(x,t).$$

With this, (5.13) reads as

$$\big(L(x)-a\big)\big(\rho-L(x)\big)L(x)=0.$$

Therefore, we have that $L(\Omega):=\{L(x)\,:\,x\in\Omega\}\subset\{0,a,\rho\}$. Accordingly, for every connected component $\widetilde\Omega$ of Ω, we have that $L(\widetilde\Omega)\subset\{0,a,\rho\}$.

Moreover, we observe that $0\leqslant L(x)\leqslant a$, thanks to (5.12). Thus, since L is continuous in Ω by assumption, either $L(\widetilde\Omega)=\{0\}$ or $L(\widetilde\Omega)\subset\{a,\rho\}$.

We claim that

$$L(x)=0 \qquad \text{for all } x\in\widetilde\Omega. \tag{5.14}$$

To this end, we argue by contradiction and suppose that

$$L(x)=a \qquad \text{for all } x\in\widetilde\Omega \tag{5.15}$$

(notice indeed that $L(x)=\rho$ only if $\rho=a$, in light of (5.12)).

For all $t>0$, we set

$$\widetilde A(t):=\int_{\widetilde\Omega}u(x,t)\,dx.$$

Notice that $\widetilde A(t)\geqslant 0$ for all $t>0$, thanks to (5.12). Moreover, since $u(x,t)$ converges uniformly as $t\to+\infty$, we conclude that there exists $\widetilde\ell\in[0,a|\widetilde\Omega|)$ such that

$$\lim_{t\to+\infty}\widetilde A(t)=\widetilde\ell.$$

Furthermore, thanks to (1.16), (1.17), (1.18) and (1.19), we can use the classical Divergence Theorem and (5.3) to get

$$\begin{aligned}
\widetilde A'(t)&=\frac{d}{dt}\int_{\widetilde\Omega}u(x,t)\,dx=\int_{\widetilde\Omega}\partial_t u(x,t)\,dx\\
&=\alpha\int_{\widetilde\Omega}\Delta u(x,t)\,dx-\beta\int_{\widetilde\Omega}(-\Delta)^s u(x,t)\,dx+\int_{\widetilde\Omega}\big(u(x,t)-a\big)\big(\rho-u(x,t)\big)u(x,t)\,dx\\
&=\alpha\int_{\partial\widetilde\Omega}\frac{\partial u}{\partial\nu}+\beta\int_{\mathbb{R}^n\setminus\widetilde\Omega}\mathscr{N}_s u(x,t)\,dx+\int_{\widetilde\Omega}\big(u(x,t)-a\big)\big(\rho-u(x,t)\big)u(x,t)\,dx\\
&=\beta\int_{\Omega\setminus\widetilde\Omega}\mathscr{N}_s u(x,t)\,dx+\int_{\widetilde\Omega}\big(u(x,t)-a\big)\big(\rho-u(x,t)\big)u(x,t)\,dx\\
&=\beta\iint_{(\Omega\setminus\widetilde\Omega)\times\widetilde\Omega}\frac{u(x,t)-u(y,t)}{|x-y|^{n+2s}}\,dx\,dy+\int_{\widetilde\Omega}\big(u(x,t)-a\big)\big(\rho-u(x,t)\big)u(x,t)\,dx.
\end{aligned}$$

Hence,

$$\begin{aligned}
0&=\lim_{t\to+\infty}\widetilde A'(t)\\
&=\lim_{t\to+\infty}\left[\beta\iint_{(\Omega\setminus\widetilde\Omega)\times\widetilde\Omega}\frac{u(x,t)-u(y,t)}{|x-y|^{n+2s}}\,dx\,dy+\int_{\widetilde\Omega}\big(u(x,t)-a\big)\big(\rho-u(x,t)\big)u(x,t)\,dx\right]\\
&=\beta\iint_{(\Omega\setminus\widetilde\Omega)\times\widetilde\Omega}\frac{L(x)-L(y)}{|x-y|^{n+2s}}\,dx\,dy+\int_{\widetilde\Omega}\big(L(x)-a\big)\big(\rho-L(x)\big)L(x)\,dx.
\end{aligned}$$

Thus, recalling (5.15),

$$0=\beta\iint_{(\Omega\setminus\widetilde{\Omega})\times\widetilde{\Omega}}\frac{L(x)-a}{|x-y|^{n+2s}}\,dx\,dy.$$

Since $L(x)\leqslant a$ for all $x\in\Omega$, this implies that $L(x)=a$ for all $x\in\Omega$.

On the other hand, recalling the assumptions on u_0, we can say that there exists $\varepsilon_0>0$ such that

$$\int_\Omega u_0(x)\,dx<|\Omega|(a-\varepsilon_0).$$

Therefore, exploting again Theorem 1.8(i),

$$\int_\Omega u(x,t)\,dx\leqslant\int_\Omega u_0(x)\,dx<|\Omega|(a-\varepsilon_0),$$

which gives that

$$|\Omega|a=\int_\Omega L(x)\,dx\leqslant|\Omega|(a-\varepsilon_0).$$

This provides the desired contradiction and thereby establishes (5.14).

By repeating the same argument on every connected component of Ω, we obtain the desired result in (i). □

The arguments to establish Theorems 1.11, 1.12 and 1.13 in the case $\rho<0\leqslant a$ are similar to those provided to obtain Theorems 1.8, 1.9 and 1.10, and therefore we omit them.

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