ON THE SOLVABILITY OF SOME SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS WITH THE DOUBLE SCALE ANOMALOUS DIFFUSION IN HIGHER DIMENSIONS

Vitali Vougalter

Department of Mathematics, University of Toronto
Toronto, Ontario, M5S 2E4, Canada
e-mail: vitali@math.toronto.edu

Abstract: In the article we establish the existence of solutions of a system of integro-differential equations in the case of the double scale anomalous diffusion. Each equation of the system contains the sum of the two negative Laplace operators raised to two distinct fractional powers in $\mathbb{R}^d,\ d = 4, 5$. The proof of the existence of solutions relies on a fixed point technique. We use the solvability conditions for the non-Fredholm elliptic operators in unbounded domains.

AMS Subject Classification: 35R11, 35A01, 35P30, 47F05
Key words: integro-differential equations, non-Fredholm operators

1. Introduction

The present article is devoted to the studies of the existence of stationary solutions of the following system of the integro-differential equations in $\mathbb{R}^d,\ d = 4, 5$

$$\frac{\partial u_m}{\partial t} = -D_m[(-\Delta)^{s_{1,m}} + (-\Delta)^{s_{2,m}}]u_m +$$

$$+ \int_{\mathbb{R}^d} K_m(x - y)g_m(u(y, t))dy + f_m(x), \tag{1.1}$$

where $1 \le m \le N,\ 0 < s_{1,m} < s_{2,m} < 1$ and $\frac{3}{2} - \frac{d}{4} < s_{2,m} < 1$ appearing in the cell population dynamics. The results of the work are obtained in these particular ranges of the values of the powers of the negative Laplacians, which is based on the solvability of the linear Poisson type equations (1.13) and the applicability of the Sobolev inequality (1.7) for the fractional Laplace operator. The solvability of the system analogous to (1.1) containing a single fractional Laplacian in the diffusion term of each equation was covered in [29]. Note that the space variable x in our problem corresponds to the cell genotype, the functions $u_m(x,t)$ describe the cell density distributions for various groups of cells as functions of their genotype and time,

$$u(x,t) = (u_1(x,t), u_2(x,t), ..., u_N(x,t))^T.$$

The right side of system (1.1) describes the evolution of cell densities by virtue of the cell proliferation, mutations and cell influx or efflux. The double scale anomalous diffusion terms with positive coefficients D_m correspond to the change of genotype due to small random mutations, and the integral production terms describe large mutations. Functions $g_m(u)$ stand for the rates of cell birth depending on u (density dependent proliferation), and the kernels $K_m(x-y)$ express the proportions of newly born cells changing their genotype from y to x. Let us assume that they depend on the distance between the genotypes. The functions $f_m(x)$ designate the influxes or effluxes of cells for different genotypes.

The fractional Laplace operator describes a particular case of the anomalous diffusion actively studied in the context of the various applications in plasma physics and turbulence [7], [24], surface diffusion [19], [22], semiconductors [23] and so on. The anomalous diffusion can be understood as a random process of the particle motion characterized by the probability density distribution of the jump length. The moments of this density distribution are finite in the case of the normal diffusion, but this is not the case for the anomalous diffusion. The asymptotic behavior at the infinity of the probability density function determines the value of the power of the negative Laplacian (see [20]). Weak error for continuous time Markov chains related to fractional in time P(I)DEs was estimated in [17]. In the present article we discuss the case of $0 < s_{1,m} < s_{2,m} < 1, \ \frac{3}{2} - \frac{d}{4} < s_{2,m} < 1, \ 1 \le m \le N$ and $d = 4, 5$. The necessary conditions of the preservation of the nonnegativity of the solutions of a system of parabolic equations in the situation of the double scale anomalous diffusion were obtained in [13]. In the work [15] the authors consider the simultaneous inversion for the fractional exponents in the space-time fractional diffusion equation.

We set here all $D_m = 1$ and demonstrate the existence of solutions of the system of equations

$$-[(-\Delta)^{s_{1,m}} + (-\Delta)^{s_{2,m}}]u_m + \int_{\mathbb{R}^d} K_m(x-y)g_m(u(y))dy + f_m(x) = 0, \quad (1.2)$$

where $0 < s_{1,m} < s_{2,m} < 1, \ \frac{3}{2} - \frac{d}{4} < s_{2,m} < 1, \ 1 \le m \le N$ and $d = 4, 5$. Let us treat the case when the linear parts of the operators involved in our system fail to satisfy the Fredholm property. Consequently, the conventional methods of the nonlinear analysis may not be applicable. We use the solvability conditions for the non-Fredholm operators along with the method of contraction mappings.

Consider the problem

$$-\Delta u + V(x)u - au = f, \tag{1.3}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d),\ d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ is either zero in the whole space or tends to 0 at the infinity. Such model equation is discussed here in order to illustrate certain features of the problems without the Fredholm property, the techniques used to solve them and the preceding results. If $a \geq 0$, the essential spectrum of the operator $A : E \to F$, which corresponds to the left side of equation (1.3) contains the origin. Consequently, such operator does not satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present article deals with the studies of the certain properties of the operators of this kind. The elliptic equations containing non-Fredholm operators were studied actively in recent years. Approaches in weighted Sobolev and Hölder spaces were developed in [2], [3], [4], [5], [6]. The Schrödinger type operators without Fredholm property were treated with the methods of the spectral and the scattering theory in [12], [25], [30], [33]. The nonlinear non-Fredholm elliptic equations were covered in [12], [13], [29], [31], [32], [34]. The significant applications to the theory of reaction-diffusion type equations were developed in [9], [10]. Fredholm structures, topological invariants and applications were considered in [11]. The works [14] and [21] are important for the understanding of the Fredholm and properness properties of the quasilinear elliptic systems of the second order and of the operators of this kind on $\mathbb{R}^N$. The non-Fredholm operators arise also when considering the wave systems with an infinite number of localized traveling waves (see [1]). In particular, when $a = 0$ the operator A is Fredholm in some properly chosen weighted spaces (see [2], [3], [4], [5], [6]). However, the case of $a \neq 0$ is significantly different and the method developed in these articles cannot be applied. The front propagation equations with the anomalous diffusion were treated actively in recent years (see e.g. [26], [27]).

Let us set $K_m(x) = \varepsilon_m H_m(x)$, where $\varepsilon_m \geq 0$, so that

$$\varepsilon := \max_{1\leq m\leq N}\varepsilon_m, \quad s_2 := \max_{1\leq m\leq N} s_{2,m}, \tag{1.4}$$

where $\frac{3}{2} - \frac{d}{4} < s_2 < 1$ and assume the following.

Assumption 1.1. *Let* $1 \leq m \leq N,\ 0 < s_{1,m} < s_{2,m} < 1,\ \frac{3}{2} - \frac{d}{4} < s_m < 1$, *where* $d = 4, 5$, *the functions* $f_m : \mathbb{R}^d \to \mathbb{R}$ *do not vanish identically for some* m, *such that*

$$f_m \in L^1(\mathbb{R}^d), \quad (-\Delta)^{\frac{3}{2}-s_{2,m}} f_m \in L^2(\mathbb{R}^d).$$

Let us also assume that $H_m : \mathbb{R}^d \to \mathbb{R}$, *so that*

$$H_m \in L^1(\mathbb{R}^d), \quad (-\Delta)^{\frac{3}{2}-s_{2,m}} H_m \in L^2(\mathbb{R}^d).$$

Moreover,

$$H^2 := \sum_{m=1}^{N} \|H_m\|^2_{L^1(\mathbb{R}^d)} > 0 \tag{1.5}$$

and

$$Q^2 := \sum_{m=1}^{N} \|(-\Delta)^{\frac{3}{2}-s_{2,m}} H_m\|^2_{L^2(\mathbb{R}^d)} > 0. \tag{1.6}$$

We choose here the space dimensions $d = 4, 5$. This is related to the solvability conditions for the linear Poisson type equation (4.1) stated in Lemma 4.1 below. For the practical applications, the space dimensions are not limited to $d = 4, 5$, because the space variable here corresponds to the cell genotype but not to the usual physical space. Let us apply the Sobolev inequality for the fractional negative Laplacian (see Lemma 2.2 of [16], also [18]), namely

$$\|f_m\|_{L^{\frac{2d}{d-6+4s_{2,m}}}(\mathbb{R}^d)} \leq c_{s_{2,m},d} \|(-\Delta)^{\frac{3}{2}-s_{2,m}} f_m\|_{L^2(\mathbb{R}^d)}, \tag{1.7}$$

with $\frac{3}{2} - \frac{d}{4} < s_{2,m} < 1,\ d = 4, 5$ and $1 \leq m \leq N$. By virtue of the Assumption 1.1 above along with the standard interpolation argument, we arrive at

$$f_m \in L^2(\mathbb{R}^d), \quad d = 4, 5, \quad 1 \leq m \leq N. \tag{1.8}$$

Let us use the Sobolev spaces for the technical purposes, namely

$$H^{2s_{2,m}}(\mathbb{R}^d) := \{\phi : \mathbb{R}^d \to \mathbb{R} \mid \phi \in L^2(\mathbb{R}^d),\ (-\Delta)^{s_{2,m}} \phi \in L^2(\mathbb{R}^d)\}, \tag{1.9}$$

where $\frac{3}{2} - \frac{d}{4} < s_{2,m} < 1,\ 1 \leq m \leq N,\ d = 4, 5$.
Each space (1.9) is equipped with the norm

$$\|\phi\|^2_{H^{2s_{2,m}}(\mathbb{R}^d)} := \|\phi\|^2_{L^2(\mathbb{R}^d)} + \|(-\Delta)^{s_{2,m}} \phi\|^2_{L^2(\mathbb{R}^d)}. \tag{1.10}$$

For a vector function

$$u(x) = (u_1(x), u_2(x), ..., u_N(x))^T,$$

throughout the article we will use the norm

$$\|u\|^2_{H^3(\mathbb{R}^d,\mathbb{R}^N)} := \|u\|^2_{L^2(\mathbb{R}^d,\mathbb{R}^N)} + \sum_{m=1}^{N} \|(-\Delta)^{\frac{3}{2}} u_m\|^2_{L^2(\mathbb{R}^d)}, \tag{1.11}$$

with $d = 4, 5$ and

$$\|u\|^2_{L^2(\mathbb{R}^d,\mathbb{R}^N)} := \sum_{m=1}^{N} \|u_m\|^2_{L^2(\mathbb{R}^d)}.$$

We recall the Sobolev embedding in $\mathbb{R}^d, \ d=4,5$, namely

$$\|\phi\|_{L^\infty(\mathbb{R}^d)} \leq c_e \|\phi\|_{H^3(\mathbb{R}^d)}, \tag{1.12}$$

where $c_e > 0$ is the constant of the embedding. When all the nonnegative parameters ε_m are trivial, we arrive at the linear Poisson type equations

$$[(-\Delta)^{s_{1,m}} + (-\Delta)^{s_{2,m}}]u_m(x) = f_m(x), \quad 1 \leq m \leq N. \tag{1.13}$$

By virtue of Lemma 4.1 below under the stated assumptions each problem (1.13) admits a unique solution

$$u_{0,m} \in H^{2s_{2,m}}(\mathbb{R}^d), \quad \frac{3}{2} - \frac{d}{4} < s_{2,m} < 1, \quad 1 \leq m \leq N,$$

and no orthogonality conditions for the right side of (1.13) are required here. Obviously, for $1 \leq m \leq N$,

$$[(-\Delta)^{\frac{3}{2}-s_{2,m}+s_{1,m}} + (-\Delta)^{\frac{3}{2}}]u_{0,m} = (-\Delta)^{\frac{3}{2}-s_{2,m}} f_m \in L^2(\mathbb{R}^d) \tag{1.14}$$

via Assumption 1.1. It can be easily derived from (1.14) using the standard Fourier transform (2.1) that

$$(-\Delta)^{\frac{3}{2}} u_{0,m} \in L^2(\mathbb{R}^d), \ 1 \leq m \leq N.$$

Hence, each linear equation (1.13) possesses a unique solution $u_{0,m} \in H^3(\mathbb{R}^d)$. By means of the definition of the norm (1.11), we have

$$u_0(x) := (u_{0,1}(x), u_{0,2}(x), ..., u_{0,N}(x))^T \in H^3(\mathbb{R}^d, \mathbb{R}^N).$$

Let us look for the resulting solution of the nonlinear system of equations (1.2) as

$$u(x) = u_0(x) + u_p(x), \tag{1.15}$$

where

$$u_p(x) := (u_{p,1}(x), u_{p,2}(x), ..., u_{p,N}(x))^T.$$

Evidently, we easily obtain the perturbative system of equations

$$[(-\Delta)^{s_{1,m}} + (-\Delta)^{s_{2,m}}]u_{p,m}(x) = \varepsilon_m \int_{\mathbb{R}^d} H_m(x-y) g_m(u_0(y) + u_p(y)) dy, \tag{1.16}$$

where $0 < s_{1,m} < s_{2,m} < 1, \ \dfrac{3}{2} - \dfrac{d}{4} < s_{2,m} < 1, \ 1 \leq m \leq N, \ d = 4,5.$
We introduce a closed ball in our Sobolev space

$$B_\rho := \{u \in H^3(\mathbb{R}^d, \mathbb{R}^N) \mid \|u\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)} \leq \rho\}, \quad 0 < \rho \leq 1. \tag{1.17}$$

Let us seek the solution of problem (1.16) as the fixed point of the auxiliary nonlinear system

$$[(-\Delta)^{s_{1,m}} + (-\Delta)^{s_{2,m}}]u_m(x) = \varepsilon_m \int_{\mathbb{R}^d} H_m(x-y)g_m(u_0(y)+v(y))dy, \quad (1.18)$$

with $0 < s_{1,m} < s_{2,m} < 1, \ \frac{3}{2} - \frac{d}{4} < s_{2,m} < 1, \ 1 \leq m \leq N, \ d = 4, 5$
in ball (1.17). For a given vector function $v(y)$ this is a system of equations with respect to $u(x)$. The left side of the mth equation in (1.18) involves the operator which fails to satisfy the Fredholm property

$$l_m := (-\Delta)^{s_{1,m}} + (-\Delta)^{s_{2,m}} : H^{2s_{2,m}}(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \quad 1 \leq m \leq N. \quad (1.19)$$

We have (1.19) defined via the spectral calculus. It is the pseudo-differential operator with the symbol $|p|^{2s_{1,m}} + |p|^{2s_{2,m}}$, such that for $1 \leq m \leq N$

$$l_m\phi(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} (|p|^{2s_{1,m}} + |p|^{2s_{2,m}})\widehat{\phi}(p)e^{ipx}dp, \quad \phi \in H^{2s_{2,m}}(\mathbb{R}^d),$$

with the standard Fourier transform defined in (2.1). The essential spectrum of (1.19) fills the nonnegative semi-axis $[0, +\infty)$. Thus, this operator does not have a bounded inverse. The similar situation appeared in articles [31] and [32] but as distinct from the present case, the equations studied there required the orthogonality relations. The fixed point technique was applied in [28] to evaluate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger operator involved in the nonlinear problem there had the Fredholm property (see Assumption 1 of [28], also [8]). Let us introduce the closed ball in the space of N dimensions as

$$I := \{z \in \mathbb{R}^N \mid |z|_{\mathbb{R}^N} \leq c_e\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + c_e\}, \quad d = 4, 5. \quad (1.20)$$

Here and below $|.|_{\mathbb{R}^N}$ will denote the length of a vector in $\mathbb{R}^N$. The closed ball D_M in the space of $C^2(I, \mathbb{R}^N)$ vector functions is given by

$$\{g(z) := (g_1(z), g_2(z), ..., g_N(z)) \in C^2(I, \mathbb{R}^N) \mid \|g\|_{C^2(I,\mathbb{R}^N)} \leq M\}, \quad (1.21)$$

where $M > 0$. Here the norms

$$\|g\|_{C^2(I,\mathbb{R}^N)} := \sum_{m=1}^{N} \|g_m\|_{C^2(I)}, \quad (1.22)$$

$$\|g_m\|_{C^2(I)} := \|g_m\|_{C(I)} + \sum_{n=1}^{N} \left\|\frac{\partial g_m}{\partial z_n}\right\|_{C(I)} + \sum_{n,l=1}^{N} \left\|\frac{\partial^2 g_m}{\partial z_n \partial z_l}\right\|_{C(I)}, \quad (1.23)$$

where $\|g_m\|_{C(I)} := \max_{z\in I}|g_m(z)|$. We make the following technical assumption on the nonlinear part of the system of equations (1.2). From the perspective of the applications in biology, $g_m(z)$ can be, for example the quadratic functions, which describe the cell-cell interactions.

Assumption 1.2. *Let $1 \le m \le N$. Suppose that $g_m : \mathbb{R}^N \to \mathbb{R}$ is such that $g_m(0) = 0$ and $\nabla g_m(0) = 0$. We also assume that $g \in D_M$ and it does not vanish identically in the ball I.*

We use the technical Assumptions 1.1 and 1.2 above in the proofs of our main theorems. It is not clear at the moment if there is a more efficient way to analyze our system of equations which would enable us to weaken these conditions.
Let us introduce the operator T_g, such that $u = T_g v$, where u is a solution of the system of equations (1.18). Our first main statement is as follows.

Theorem 1.3. *Let Assumptions 1.1 and 1.2 hold. Then for every $\rho \in (0,1]$ system (1.18) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all*

$$0 < \varepsilon \le \frac{\rho}{M(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^2}\times$$

$$\times\left[\frac{H^2(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^{\frac{8s_2}{d}-2}d}{(d-4s_2)(2\pi)^{4S_2}}\left(\frac{|S^d|}{4S_2}\right)^{\frac{4S_2}{d}}+Q^2\right]^{-\frac{1}{2}}. \tag{1.24}$$

The unique fixed point u_p of this map T_g is the only solution of problem (1.16) in B_ρ.

Note that $\varepsilon,\ s_2,\ H,\ Q$ and S_2 are defined in formulas (1.4), (1.5), (1.6) and (2.6). Here and further down S^d stands for the unit sphere in the space of $d = 4, 5$ dimensions centered at the origin and $|S^d|$ denotes its Lebesgue measure.
Clearly, the resulting solution $u(x)$ of the system of equations (1.2) given by (1.15) will not vanish identically because the influx/efflux terms $f_m(x)$ are nontrivial for some $1 \le m \le N$ and all $g_m(0) = 0$ as we assume. Let us make use of the following elementary lemma.

Lemma 1.4. *Let $R \in (0, +\infty)$ and $d = 4, 5$. We consider the function*

$$\varphi(R) := \alpha R^{d-4s} + \frac{1}{R^{4s}}, \quad \frac{3}{2} - \frac{d}{4} < s < 1, \quad \alpha > 0.$$

It attains its minimal value at $R^* := \left(\frac{4s}{\alpha(d-4s)}\right)^{\frac{1}{d}}$*, which is given by*

$$\varphi(R^*) = \left(\frac{\alpha}{4s}\right)^{\frac{4s}{d}}\frac{d}{(d-4s)^{\frac{d-4s}{d}}}.$$

Our second main proposition deals with the continuity of the resulting solution of the system of equations (1.2) given by formula (1.15) with respect to the nonlinear vector function g. Let us use the following positive auxiliary expression

$$\sigma := M(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)\times$$
$$\times\left\{\frac{H^2(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^{\frac{8s_2}{d}-2}d}{(d-4s_2)(2\pi)^{4S_2}}\left(\frac{|S^d|}{4S_2}\right)^{\frac{4S_2}{d}}+Q^2\right\}^{\frac{1}{2}}. \tag{1.25}$$

Theorem 1.5. *Let $j=1,2$, the assumptions of Theorem 1.3 are valid, such that $u_{p,j}$ is the unique fixed point of the map $T_{g_j}: B_\rho \to B_\rho$, which is a strict contraction for all the values of ε satisfying (1.24) and the resulting solution of the system of equations (1.2) with $g(z)=g_j(z)$ is*

$$u_j(x) := u_0(x)+u_{p,j}(x). \tag{1.26}$$

Then for all the values of ε, which satisfy inequality (1.24), the bound

$$\|u_1-u_2\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} \le \frac{\varepsilon\sigma}{M(1-\varepsilon\sigma)}(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)\|g_1-g_2\|_{C^2(I,\mathbb{R}^N)} \tag{1.27}$$

holds.

Let us turn our attention to the proof of our first main result.

2. The existence of the perturbed solution

Proof of Theorem 1.3. We choose arbitrarily a vector function $v \in B_\rho$ and designate the terms involved in the integral expressions in the right side of the system of equations (1.18) as

$$G_m(x) := g_m(u_0(x)+v(x)), \quad 1 \le m \le N.$$

Let us use the standard Fourier transform throughout the article, namely

$$\widehat{\phi}(p) := \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\phi(x)e^{-ipx}dx, \quad d=4,5. \tag{2.1}$$

Obviously, the estimate from above

$$\|\widehat{\phi}\|_{L^\infty(\mathbb{R}^d)} \le \frac{1}{(2\pi)^{\frac{d}{2}}}\|\phi\|_{L^1(\mathbb{R}^d)} \tag{2.2}$$

is valid. We apply (2.1) to both sides of system (1.18) and arrive at

$$\widehat{u_m}(p) = \varepsilon_m(2\pi)^{\frac{d}{2}}\frac{\widehat{H_m}(p)\widehat{G_m}(p)}{|p|^{2s_{1,m}}+|p|^{2s_{2,m}}},$$

where $0 < s_{1,m} < s_{2,m} < 1,\ \dfrac{3}{2} - \dfrac{d}{4} < s_{2,m} < 1,\ 1 \le m \le N,\ d = 4, 5$. We obtain the expression for the norm given by

$$\|u_m\|^2_{L^2(\mathbb{R}^d)} = (2\pi)^d \varepsilon_m^2 \int_{\mathbb{R}^d} \frac{|\widehat{H_m}(p)|^2 |\widehat{G_m}(p)|^2}{[|p|^{2s_{1,m}} + |p|^{2s_{2,m}}]^2} dp. \tag{2.3}$$

As distinct from works [31] and [32] with the standard Laplacian in the diffusion term, here we do not try to control the norms

$$\left\| \frac{\widehat{H_m}(p)}{|p|^{2s_{1,m}} + |p|^{2s_{2,m}}} \right\|_{L^\infty(\mathbb{R}^d)}, \quad 1 \le m \le N.$$

Instead, we estimate the right side of (2.3) using the analog of bound (2.2) applied to functions H_m and G_m with $R \in (0, +\infty)$ as

$$(2\pi)^d \varepsilon_m^2 \int_{\mathbb{R}^d} \frac{|\widehat{H_m}(p)|^2 |\widehat{G_m}(p)|^2}{[|p|^{2s_{1,m}} + |p|^{2s_{2,m}}]^2} dp \le$$

$$\le (2\pi)^d \varepsilon_m^2 \Big[\int_{|p| \le R} \frac{|\widehat{H_m}(p)|^2 |\widehat{G_m}(p)|^2}{|p|^{4s_{2,m}}} dp + \int_{|p|>R} \frac{|\widehat{H_m}(p)|^2 |\widehat{G_m}(p)|^2}{|p|^{4s_{2,m}}} dp \Big] \le$$

$$\le \varepsilon_m^2 \|H_m\|^2_{L^1(\mathbb{R}^d)} \left\{ \frac{|S^d|}{(2\pi)^d} \|G_m\|^2_{L^1(\mathbb{R}^d)} \frac{R^{d-4s_{2,m}}}{d - 4s_{2,m}} + \frac{\|G_m\|^2_{L^2(\mathbb{R}^d)}}{R^{4s_{2,m}}} \right\}. \tag{2.4}$$

By means of norm definition (1.11) along with the triangle inequality and using the fact that $v \in B_\rho$, we easily derive

$$\|u_0 + v\|_{L^2(\mathbb{R}^d, \mathbb{R}^N)} \le \|u_0\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)} + 1, \quad d = 4, 5.$$

Sobolev embedding (1.12) yields

$$|u_0 + v|_{\mathbb{R}^N} \le c_e (\|u_0\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)} + 1).$$

Let the dot stand for the scalar product of two vectors in $\mathbb{R}^N$. Clearly,

$$G_m(x) = \int_0^1 \nabla g_m(t(u_0(x) + v(x))).(u_0(x) + v(x)) dt, \quad 1 \le m \le N.$$

We use the ball I introduced in (1.20). Hence,

$$|G_m(x)| \le \sup_{z \in I} |\nabla g_m(z)|_{\mathbb{R}^N} |u_0(x) + v(x)|_{\mathbb{R}^N} \le M |u_0(x) + v(x)|_{\mathbb{R}^N},$$

so that

$$\|G_m\|_{L^2(\mathbb{R}^d)} \le M \|u_0 + v\|_{L^2(\mathbb{R}^d, \mathbb{R}^N)} \le M(\|u_0\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)} + 1).$$

Evidently, for $t \in [0,1]$ and $1 \le m, j \le N$, we can write

$$\frac{\partial g_m}{\partial z_j}(t(u_0(x)+v(x))) = \int_0^t \nabla \frac{\partial g_m}{\partial z_j}(\tau(u_0(x)+v(x))).(u_0(x)+v(x))d\tau.$$

This implies that

$$\Big|\frac{\partial g_m}{\partial z_j}(t(u_0(x)+v(x)))\Big| \le \sup_{z\in I}\Big|\nabla \frac{\partial g_m}{\partial z_j}\Big|_{\mathbb{R}^N} |u_0(x)+v(x)|_{\mathbb{R}^N} \le$$

$$\le \sum_{n=1}^{N} \Big\|\frac{\partial^2 g_m}{\partial z_n \partial z_j}\Big\|_{C(I)} |u_0(x)+v(x)|_{\mathbb{R}^N}.$$

Therefore,

$$|G_m(x)| \le |u_0(x)+v(x)|_{\mathbb{R}^N} \sum_{n,j=1}^{N} \Big\|\frac{\partial^2 g_m}{\partial z_n \partial z_j}\Big\|_{C(I)} |u_{0,j}(x)+v_j(x)| \le$$

$$\le M|u_0(x)+v(x)|^2_{\mathbb{R}^N}.$$

Thus,

$$\|G_m\|_{L^1(\mathbb{R}^d)} \le M\|u_0+v\|^2_{L^2(\mathbb{R}^d,\mathbb{R}^N)} \le M(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^2. \tag{2.5}$$

This allows us to derive the upper bound for the right side of (2.4) given by

$$\varepsilon_m^2 M^2 \|H_m\|^2_{L^1(\mathbb{R}^d)} (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^2 \times$$

$$\times \left\{ \frac{|S^d|(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^2 R^{d-4s_{2,m}}}{(2\pi)^d(d-4s_{2,m})} + \frac{1}{R^{4s_{2,m}}} \right\},$$

with $R \in (0,+\infty)$. Lemma 1.4 yields the minimal value of the expression above, such that

$$\|u_m\|^2_{L^2(\mathbb{R}^d)} \le {\varepsilon_m}^2 M^2 \|H_m\|^2_{L^1(\mathbb{R}^d)} \times$$

$$\times (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^{2+\frac{8s_{2,m}}{d}} \left(\frac{|S^d|}{4s_{2,m}}\right)^{\frac{4s_{2,m}}{d}} \frac{d}{(d-4s_{2,m})(2\pi)^{4s_{2,m}}}.$$

We define

$$\left(\frac{|S^d|}{4S_2}\right)^{\frac{4S_2}{d}} \frac{1}{(2\pi)^{4S_2}} := \max_{1\le m\le N} \left(\frac{|S^d|}{4s_{2,m}}\right)^{\frac{4s_{2,m}}{d}} \frac{1}{(2\pi)^{4s_{2,m}}}, \tag{2.6}$$

where $\frac{3}{2} - \frac{d}{4} < S_2 < 1$. Hence, we obtain

$$\|u\|^2_{L^2(\mathbb{R}^d,\mathbb{R}^N)} \le$$

$$\leq \varepsilon^2 M^2 H^2(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^{2+\frac{8s_2}{d}} \frac{d}{d-4s_2}\left(\frac{|S^d|}{4S_2}\right)^{\frac{4S_2}{d}} \frac{1}{(2\pi)^{4S_2}}. \quad (2.7)$$

By means of (1.18),

$$[(-\Delta)^{\frac{3}{2}-s_{2,m}+s_{1,m}}+(-\Delta)^{\frac{3}{2}}]u_m(x)=\varepsilon_m(-\Delta)^{\frac{3}{2}-s_{2,m}}\int_{\mathbb{R}^d} H_m(x-y)G_m(y)dy$$

with $0<s_{1,m}<s_{2,m}<1,\ \dfrac{3}{2}-\dfrac{d}{4}<s_{2,m}<1,\ 1\leq m\leq N,\ d=4,5.$
We use the standard Fourier transform (2.1), the analog of upper bound (2.2) applied to function G_m along with (2.5) to derive

$$\|(-\Delta)^{\frac{3}{2}}u_m\|^2_{L^2(\mathbb{R}^d)}\leq \varepsilon_m^2\|G_m\|^2_{L^1(\mathbb{R}^d)}\|(-\Delta)^{\frac{3}{2}-s_{2,m}}H_m\|^2_{L^2(\mathbb{R}^d)}\leq$$

$$\leq \varepsilon^2 M^2(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^4\|(-\Delta)^{\frac{3}{2}-s_{2,m}}H_m\|^2_{L^2(\mathbb{R}^d)}.$$

Thus,

$$\sum_{m=1}^N \|(-\Delta)^{\frac{3}{2}}u_m\|^2_{L^2(\mathbb{R}^d)}\leq \varepsilon^2 M^2(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^4 Q^2. \quad (2.8)$$

Let us recall the definition of the norm (1.11). Bounds (2.7) and (2.8) give us that

$$\|u\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}\leq \varepsilon M(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^2\times$$

$$\times\left[\frac{H^2(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^{\frac{8s_2}{d}-2}d}{(d-4s_2)(2\pi)^{4S_2}}\left(\frac{|S^d|}{4S_2}\right)^{\frac{4S_2}{d}}+Q^2\right]^{\frac{1}{2}}\leq \rho \quad (2.9)$$

for all the values of ε, which satisfy (1.24). Hence, $u\in B_\rho$ as well.
Suppose that for a certain $v\in B_\rho$ there exist two solutions $u_{1,2}\in B_\rho$ of system (1.18). Clearly, their difference $w(x):=u_1(x)-u_2(x)\in H^3(\mathbb{R}^d,\mathbb{R}^N)$ satisfies the homogeneous system of equations

$$[(-\Delta)^{s_{1,m}}+(-\Delta)^{s_{2,m}}]w_m(x)=0,$$

where $0<s_{1,m}<s_{2,m}<1,\ \dfrac{3}{2}-\dfrac{d}{4}<s_{2,m}<1,\ 1\leq m\leq N,\ d=4,5.$
Each operator $l_m: H^{2s_{2,m}}(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ introduced in (1.19) does not have any nontrivial zero modes. Thus, $w(x)$ vanishes in $\mathbb{R}^d$. Therefore, problem (1.18) defines a map $T_g: B_\rho\to B_\rho$ for all ε satisfying bound (1.24).
Our goal is to show that this map is a strict contraction. Let us choose arbitrarily $v_1,v_2\in B_\rho$. By virtue of the argument above, $u_{1,2}:=T_g v_{1,2}\in B_\rho$ as well if ε satisfies (1.24). Obviously, by means of (1.18) we obtain for $1\leq m\leq N$

$$[(-\Delta)^{s_{1,m}}+(-\Delta)^{s_{2,m}}]u_{1,m}(x)=\varepsilon_m\int_{\mathbb{R}^d} H_m(x-y)g_m(u_0(y)+v_1(y))dy, \quad (2.10)$$

$$[(-\Delta)^{s_{1,m}} + (-\Delta)^{s_{2,m}}]u_{2,m}(x) = \varepsilon_m \int_{\mathbb{R}^d} H_m(x-y) g_m(u_0(y) + v_2(y))dy, \tag{2.11}$$

with $0 < s_{1,m} < s_{2,m} < 1,\ \frac{3}{2} - \frac{d}{4} < s_{2,m} < 1,\ d = 4, 5$. We define

$$G_{1,m}(x) := g_m(u_0(x) + v_1(x)), \quad G_{2,m}(x) := g_m(u_0(x) + v_2(x)), \quad 1 \le m \le N$$

and apply the standard Fourier transform (2.1) to both sides of systems (2.10) and (2.11). This gives us

$$\widehat{u_{1,m}}(p) = \varepsilon_m (2\pi)^{\frac{d}{2}} \frac{\widehat{H_m}(p)\widehat{G_{1,m}}(p)}{|p|^{2s_{1,m}} + |p|^{2s_{2,m}}}, \quad \widehat{u_{2,m}}(p) = \varepsilon_m (2\pi)^{\frac{d}{2}} \frac{\widehat{H_m}(p)\widehat{G_{2,m}}(p)}{|p|^{2s_{1,m}} + |p|^{2s_{2,m}}}.$$

Evidently,

$$\|u_{1,m} - u_{2,m}\|^2_{L^2(\mathbb{R}^d)} = \varepsilon_m^2 (2\pi)^d \int_{\mathbb{R}^d} \frac{|\widehat{H_m}(p)|^2 |\widehat{G_{1,m}}(p) - \widehat{G_{2,m}}(p)|^2}{[|p|^{2s_{1,m}} + |p|^{2s_{2,m}}]^2} dp. \tag{2.12}$$

Clearly, the right side of (2.12) can be estimated from above by means of inequality (2.2) as

$$\varepsilon_m^2 (2\pi)^d \left[\int_{|p| \le R} \frac{|\widehat{H_m}(p)|^2 |\widehat{G_{1,m}}(p) - \widehat{G_{2,m}}(p)|^2}{|p|^{4s_{2,m}}} dp + \right.$$

$$\left. + \int_{|p| > R} \frac{|\widehat{H_m}(p)|^2 |\widehat{G_{1,m}}(p) - \widehat{G_{2,m}}(p)|^2}{|p|^{4s_{2,m}}} dp \right] \le \varepsilon^2 \|H_m\|^2_{L^1(\mathbb{R}^d)} \times$$

$$\times \left\{ \frac{\|G_{1,m} - G_{2,m}\|^2_{L^1(\mathbb{R}^d)}}{(2\pi)^d} \frac{|S^d| R^{d-4s_{2,m}}}{d - 4s_{2,m}} + \frac{\|G_{1,m} - G_{2,m}\|^2_{L^2(\mathbb{R}^d)}}{R^{4s_{2,m}}} \right\}$$

with $R \in (0, +\infty)$. Obviously, we can express for $1 \le m \le N$

$$G_{1,m}(x) - G_{2,m}(x) = \int_0^1 \nabla g_m(u_0(x) + tv_1(x) + (1-t)v_2(x)).(v_1(x) - v_2(x))dt.$$

For $t \in [0, 1]$, we have

$$\|v_2 + t(v_1 - v_2)\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)} \le t\|v_1\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)} +$$

$$+ (1-t)\|v_2\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)} \le \rho.$$

Hence, $v_2 + t(v_1 - v_2) \in B_\rho$. We easily obtain the upper bound

$$|G_{1,m}(x) - G_{2,m}(x)| \le \sup_{z \in I} |\nabla g_m(z)|_{\mathbb{R}^N} |v_1(x) - v_2(x)|_{\mathbb{R}^N} \le M|v_1(x) - v_2(x)|_{\mathbb{R}^N},$$

so that

$$\|G_{1,m} - G_{2,m}\|_{L^2(\mathbb{R}^d)} \le M\|v_1 - v_2\|_{L^2(\mathbb{R}^d, \mathbb{R}^N)} \le M\|v_1 - v_2\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)}.$$

Let us write $\frac{\partial g_m}{\partial z_j}(u_0(x)+tv_1(x)+(1-t)v_2(x))$ for $1\le m,j\le N$ as

$$\int_0^1 \nabla \frac{\partial g_m}{\partial z_j}(\tau[u_0(x)+tv_1(x)+(1-t)v_2(x)]).[u_0(x)+tv_1(x)+(1-t)v_2(x)]d\tau.$$

Thus, for $t\in[0,1]$

$$\Big|\frac{\partial g_m}{\partial z_j}(u_0(x)+tv_1(x)+(1-t)v_2(x))\Big| \le$$

$$\le \sum_{n=1}^{N}\left\|\frac{\partial^2 g_m}{\partial z_n \partial z_j}\right\|_{C(I)}(|u_0(x)|_{\mathbb{R}^N}+t|v_1(x)|_{\mathbb{R}^N}+(1-t)|v_2(x)|_{\mathbb{R}^N}),$$

so that

$$|G_{1,m}(x)-G_{2,m}(x)|\le M|v_1(x)-v_2(x)|_{\mathbb{R}^N}\Big(|u_0(x)|_{\mathbb{R}^N}+\frac{1}{2}|v_1(x)|_{\mathbb{R}^N}+\frac{1}{2}|v_2(x)|_{\mathbb{R}^N}\Big).$$

By virtue of the Schwarz inequality, we derive the estimate from above for the norm $\|G_{1,m}-G_{2,m}\|_{L^1(\mathbb{R}^d)}$ as

$$M\|v_1-v_2\|_{L^2(\mathbb{R}^d,\mathbb{R}^N)}\Big(\|u_0\|_{L^2(\mathbb{R}^d,\mathbb{R}^N)}+\frac{1}{2}\|v_1\|_{L^2(\mathbb{R}^d,\mathbb{R}^N)}+\frac{1}{2}\|v_2\|_{L^2(\mathbb{R}^d,\mathbb{R}^N)}\Big)\le$$

$$\le M\|v_1-v_2\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1). \tag{2.13}$$

Therefore, the upper bound for the norm $\|u_{1,m}-u_{2,m}\|^2_{L^2(\mathbb{R}^d)}$ is given by

$$\varepsilon^2\|H_m\|^2_{L^1(\mathbb{R}^d)}M^2\|v_1-v_2\|^2_{H^3(\mathbb{R}^d,\mathbb{R}^N)}\Big\{\frac{(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^2|S^d|R^{d-4s_{2,m}}}{(2\pi)^d(d-4s_{2,m})}+\frac{1}{R^{4s_{2,m}}}\Big\}.$$

Let us minimize the expression above over $R\in(0,+\infty)$ using Lemma 1.4, such that

$$\|u_{1,m}-u_{2,m}\|^2_{L^2(\mathbb{R}^d)}\le\varepsilon^2\|H_m\|^2_{L^1(\mathbb{R}^d)}M^2\|v_1-v_2\|^2_{H^3(\mathbb{R}^d,\mathbb{R}^N)}\times$$

$$\times(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^{\frac{8s_{2,m}}{d}}\left(\frac{|S^d|}{4s_{2,m}}\right)^{\frac{4s_{2,m}}{d}}\frac{d}{(2\pi)^{4s_{2,m}}(d-4s_{2,m})}.$$

Then

$$\|u_1-u_2\|^2_{L^2(\mathbb{R}^d,\mathbb{R}^N)}\le\varepsilon^2H^2M^2\|v_1-v_2\|^2_{H^3(\mathbb{R}^d,\mathbb{R}^N)}\times$$

$$\times(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^{\frac{8s_2}{d}}\frac{d}{(2\pi)^{4S_2}(d-4s_2)}\left(\frac{|S^d|}{4S_2}\right)^{\frac{4S_2}{d}}. \tag{2.14}$$

By means of (2.10) and (2.11) with $1\le m\le N$, we have

$$[(-\Delta)^{\frac{3}{2}-s_{2,m}+s_{1,m}}+(-\Delta)^{\frac{3}{2}}](u_{1,m}(x)-u_{2,m}(x))=$$

$$= \varepsilon_m (-\Delta)^{\frac{3}{2}-s_{2,m}} \int_{\mathbb{R}^d} H_m(x-y)[G_{1,m}(y) - G_{2,m}(y)]dy.$$

Let us use the standard Fourier transform (2.1) along with upper bounds (2.2) and (2.13). Hence,

$$\|(-\Delta)^{\frac{3}{2}}(u_{1,m} - u_{2,m})\|^2_{L^2(\mathbb{R}^d)} \le$$

$$\le \varepsilon^2 \|G_{1,m} - G_{2,m}\|^2_{L^1(\mathbb{R}^d)} \|(-\Delta)^{\frac{3}{2}-s_{2,m}} H_m\|^2_{L^2(\mathbb{R}^d)} \le$$

$$\le \varepsilon^2 M^2 \|v_1 - v_2\|^2_{H^3(\mathbb{R}^d,\mathbb{R}^N)} (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^2 \|(-\Delta)^{\frac{3}{2}-s_{2,m}} H_m\|^2_{L^2(\mathbb{R}^d)}.$$

Therefore,

$$\sum_{m=1}^{N} \|(-\Delta)^{\frac{3}{2}}(u_{1,m} - u_{2,m})\|^2_{L^2(\mathbb{R}^d)} \le$$

$$\le \varepsilon^2 M^2 \|v_1 - v_2\|^2_{H^3(\mathbb{R}^d,\mathbb{R}^N)} (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^2 Q^2. \tag{2.15}$$

Inequalities (2.14) and (2.15) imply that the norm $\|u_1 - u_2\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}$ can be estimated from above by the expression $\varepsilon M(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)\times$

$$\times \left\{ \frac{H^2(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^{\frac{8s_2}{d}-2} d}{(d-4s_2)(2\pi)^{4S_2}} \left(\frac{|S^d|}{4S_2} \right)^{\frac{4S_2}{d}} + Q^2 \right\}^{\frac{1}{2}} \|v_1 - v_2\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}. \tag{2.16}$$

It can be trivially checked that for all the values of ε satisfying (1.24) the constant in the right side of (2.16) is less than one. Hence, the map $T_g : B_\rho \to B_\rho$ defined by the system of equations (1.18) is a strict contraction. Its unique fixed point u_p is the only solution of system (1.16) in the ball B_ρ. The resulting $u \in H^3(\mathbb{R}^d, \mathbb{R}^N)$ given by (1.15) solves problem (1.2). Obviously, by virtue of (2.9), u_p converges to zero in the $H^3(\mathbb{R}^d, \mathbb{R}^N)$ norm as $\varepsilon \to 0$. ∎

Let us proceed to the proof of the second main proposition of the work.

3. The continuity of the resulting solution

Proof of Theorem 1.5. Clearly, for all the values of ε satisfying (1.24)

$$u_{p,1} = T_{g_1} u_{p,1}, \quad u_{p,2} = T_{g_2} u_{p,2},$$

such that

$$u_{p,1} - u_{p,2} = T_{g_1} u_{p,1} - T_{g_1} u_{p,2} + T_{g_1} u_{p,2} - T_{g_2} u_{p,2}.$$

Thus,

$$\|u_{p,1} - u_{p,2}\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} \le \|T_{g_1} u_{p,1} - T_{g_1} u_{p,2}\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + \|T_{g_1} u_{p,2} - T_{g_2} u_{p,2}\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}.$$

Upper bound (2.16) gives us

$$\|T_{g_1} u_{p,1} - T_{g_1} u_{p,2}\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} \le \varepsilon\sigma \|u_{p,1} - u_{p,2}\|_{H^3(\mathbb{R}^2,\mathbb{R}^N)},$$

where σ is introduced in (1.25). We have $\varepsilon\sigma < 1$ because our map $T_{g_1} : B_\rho \to B_\rho$ is a strict contraction under the stated assumptions. Hence,

$$(1-\varepsilon\sigma)\|u_{p,1} - u_{p,2}\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} \le \|T_{g_1}u_{p,2} - T_{g_2}u_{p,2}\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}. \tag{3.1}$$

Evidently, for the fixed point we have $T_{g_2}u_{p,2} = u_{p,2}$. We denote $\eta(x) := T_{g_1}u_{p,2}(x)$. For $1 \le m \le N$, we obtain

$$[(-\Delta)^{s_{1,m}} + (-\Delta)^{s_{2,m}}]\eta_m(x) = \varepsilon_m \int_{\mathbb{R}^d} H_m(x-y)g_{1,m}(u_0(y) + u_{p,2}(y))dy, \tag{3.2}$$

$$[(-\Delta)^{s_{1,m}} + (-\Delta)^{s_{2,m}}]u_{p,2,m}(x) =$$

$$= \varepsilon_m \int_{\mathbb{R}^d} H_m(x-y)g_{2,m}(u_0(y) + u_{p,2}(y))dy, \tag{3.3}$$

with $0 < s_{1,m} < s_{2,m} < 1,\ \dfrac{3}{2} - \dfrac{d}{4} < s_{2,m} < 1,\ d = 4, 5$. Let us designate

$$G_{1,2,m}(x) := g_{1,m}(u_0(x) + u_{p,2}(x)), \quad G_{2,2,m}(x) := g_{2,m}(u_0(x) + u_{p,2}(x)).$$

We apply the standard Fourier transform (2.1) to both sides of systems (3.2) and (3.3) and arrive at

$$\widehat{\eta_m}(p) = \varepsilon_m (2\pi)^{\frac{d}{2}} \frac{\widehat{H_m}(p)\widehat{G_{1,2,m}}(p)}{|p|^{2s_{1,m}} + |p|^{2s_{2,m}}}, \quad \widehat{u_{p,2,m}}(p) = \varepsilon_m (2\pi)^{\frac{d}{2}} \frac{\widehat{H_m}(p)\widehat{G_{2,2,m}}(p)}{|p|^{2s_{1,m}} + |p|^{2s_{2,m}}}.$$

Then,

$$\|\eta_m - u_{p,2,m}\|^2_{L^2(\mathbb{R}^d)} =$$

$$= \varepsilon_m^2 (2\pi)^d \int_{\mathbb{R}^d} \frac{|\widehat{H_m}(p)|^2 |\widehat{G_{1,2,m}}(p) - \widehat{G_{2,2,m}}(p)|^2}{[|p|^{2s_{1,m}} + |p|^{2s_{2,m}}]^2} dp. \tag{3.4}$$

Let us derive the upper bound on the right side of (3.4) via (2.2) as

$$\varepsilon_m^2 (2\pi)^d \left[\int_{|p|\le R} \frac{|\widehat{H_m}(p)|^2 |\widehat{G_{1,2,m}}(p) - \widehat{G_{2,2,m}}(p)|^2}{|p|^{4s_{2,m}}} dp + \right.$$

$$\left. + \int_{|p|>R} \frac{|\widehat{H_m}(p)|^2 |\widehat{G_{1,2,m}}(p) - \widehat{G_{2,2,m}}(p)|^2}{|p|^{4s_{2,m}}} dp \right] \le \varepsilon^2 \|H_m\|^2_{L^1(\mathbb{R}^d)} \times$$

$$\times \left\{ \frac{|S^d|}{(2\pi)^d} \frac{\|G_{1,2,m} - G_{2,2,m}\|^2_{L^1(\mathbb{R}^d)} R^{d-4s_{2,m}}}{d - 4s_{2,m}} + \frac{\|G_{1,2,m} - G_{2,2,m}\|^2_{L^2(\mathbb{R}^d)}}{R^{4s_{2,m}}} \right\},$$

where $R \in (0, +\infty)$. Obviously, we can write

$$G_{1,2,m}(x) - G_{2,2,m}(x) = \int_0^1 \nabla[g_{1,m} - g_{2,m}](t(u_0(x) + u_{p,2}(x))).(u_0(x) + u_{p,2}(x))dt.$$

Hence,

$$|G_{1,2,m}(x) - G_{2,2,m}(x)| \le \|g_{1,m} - g_{2,m}\|_{C^2(I)} |u_0(x) + u_{p,2}(x)|_{\mathbb{R}^N}.$$

This yields

$$\|G_{1,2,m} - G_{2,2,m}\|_{L^2(\mathbb{R}^d)} \le \|g_{1,m} - g_{2,m}\|_{C^2(I)} \|u_0 + u_{p,2}\|_{L^2(\mathbb{R}^d,\mathbb{R}^N)} \le$$

$$\le \|g_{1,m} - g_{2,m}\|_{C^2(I)} (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1).$$

Let us use another representation formula with $1 \le m, j \le N$ and $t \in [0,1]$, namely

$$\frac{\partial}{\partial z_j}(g_{1,m} - g_{2,m})(t(u_0(x) + u_{p,2}(x))) =$$

$$= \int_0^t \nabla \Big[\frac{\partial}{\partial z_j}(g_{1,m} - g_{2,m})\Big](\tau(u_0(x) + u_{p,2}(x))).(u_0(x) + u_{p,2}(x)) d\tau.$$

Thus,

$$\Big|\frac{\partial}{\partial z_j}(g_{1,m} - g_{2,m})(t(u_0(x) + u_{p,2}(x)))\Big| \le$$

$$\le \sum_{n=1}^{N} \left\| \frac{\partial^2 (g_{1,m} - g_{2,m})}{\partial z_n \partial z_j} \right\|_{C(I)} |u_0(x) + u_{p,2}(x)|_{\mathbb{R}^N}.$$

Clearly,

$$|G_{1,2,m}(x) - G_{2,2,m}(x)| \le \|g_{1,m} - g_{2,m}\|_{C^2(I)} |u_0(x) + u_{p,2}(x)|^2_{\mathbb{R}^N},$$

so that

$$\|G_{1,2,m} - G_{2,2,m}\|_{L^1(\mathbb{R}^d)} \le \|g_{1,m} - g_{2,m}\|_{C^2(I)} \|u_0 + u_{p,2}\|^2_{L^2(\mathbb{R}^d,\mathbb{R}^N)} \le$$

$$\le \|g_{1,m} - g_{2,m}\|_{C^2(I)} (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^2. \tag{3.5}$$

This allows us to obtain the estimate from above for the norm $\|\eta_m - u_{p,2,m}\|^2_{L^2(\mathbb{R}^d)}$ as $\varepsilon^2 \|H_m\|^2_{L^1(\mathbb{R}^d)} (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^2 \times$

$$\times \|g_{1,m} - g_{2,m}\|^2_{C^2(I)} \left[(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^2 \frac{|S^d| R^{d-4s_{2,m}}}{(2\pi)^d (d - 4s_{2,m})} + \frac{1}{R^{4s_{2,m}}} \right].$$

We minimize this expression over $R \in (0, +\infty)$ via Lemma 1.4 and arrive at the inequality

$$\|\eta_m - u_{p,2,m}\|^2_{L^2(\mathbb{R}^d)} \le$$

$$\le \varepsilon^2 \|H_m\|^2_{L^1(\mathbb{R}^d)} (\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} + 1)^{2 + \frac{8s_{2,m}}{d}} \left(\frac{|S^d|}{4s_{2,m}} \right)^{\frac{4s_{2,m}}{d}} \frac{d \|g_{1,m} - g_{2,m}\|^2_{C^2(I)}}{(2\pi)^{4s_{2,m}} (d - 4s_{2,m})},$$

so that

$$\|\eta - u_{p,2}\|^2_{L^2(\mathbb{R}^d,\mathbb{R}^N)} \le$$

$$\le \varepsilon^2 H^2(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^{2+\frac{8s_2}{d}} \frac{d\|g_1-g_2\|^2_{C^2(I,\mathbb{R}^N)}}{(d-4s_2)(2\pi)^{4S_2}} \left(\frac{|S^d|}{4S_2}\right)^{\frac{4S_2}{d}}.$$

By virtue of formulas (3.2) and (3.3) with $1 \le m \le N$, we derive

$$[(-\Delta)^{\frac{3}{2}-s_{2,m}+s_{1,m}} + (-\Delta)^{\frac{3}{2}}]\eta_m(x) = \varepsilon_m(-\Delta)^{\frac{3}{2}-s_{2,m}} \int_{\mathbb{R}^d} H_m(x-y)G_{1,2,m}(y)dy,$$

$$[(-\Delta)^{\frac{3}{2}-s_{2,m}+s_{1,m}} + (-\Delta)^{\frac{3}{2}}]u_{p,2,m}(x) = \varepsilon_m(-\Delta)^{\frac{3}{2}-s_{2,m}} \int_{\mathbb{R}^d} H_m(x-y)G_{2,2,m}(y)dy,$$

where $0 < s_{1,m} < s_{2,m} < 1,\ \dfrac{3}{2} - \dfrac{d}{4} < s_{2,m} < 1,\ d = 4,5$.
By means of the standard Fourier transform (2.1) along with (2.2) and (3.5), the norm $\|(-\Delta)^{\frac{3}{2}}(\eta_m - u_{p,2,m})\|^2_{L^2(\mathbb{R}^d)}$ can be bounded from above by

$$\varepsilon^2\|G_{1,2,m} - G_{2,2,m}\|^2_{L^1(\mathbb{R}^d)}\|(-\Delta)^{\frac{3}{2}-s_{2,m}}H_m\|^2_{L^2(\mathbb{R}^d)} \le$$

$$\le \varepsilon^2\|g_{1,m} - g_{2,m}\|^2_{C^2(I)}(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^4\|(-\Delta)^{\frac{3}{2}-s_{2,m}}H_m\|^2_{L^2(\mathbb{R}^d)}.$$

Then

$$\sum_{m=1}^N \|(-\Delta)^{\frac{3}{2}}(\eta_m - u_{p,2,m})\|^2_{L^2(\mathbb{R}^d)} \le$$

$$\le \varepsilon^2\|g_1 - g_2\|^2_{C^2(I,\mathbb{R}^N)}(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^4 Q^2.$$

Therefore,

$$\|\eta - u_{p,2}\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)} \le \varepsilon\|g_1 - g_2\|_{C^2(I,\mathbb{R}^N)} \times$$

$$\times(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^2 \left[\frac{H^2(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^{\frac{8s_2}{d}-2}d}{(d-4s_2)(2\pi)^{4S_2}}\left(\frac{|S^d|}{4S_2}\right)^{\frac{4S_2}{d}} + Q^2\right]^{\frac{1}{2}}.$$

By virtue of (3.1), the norm $\|u_{p,1} - u_{p,2}\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}$ can be estimated from above by

$$\frac{\varepsilon}{1-\varepsilon\sigma}(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^2\times$$

$$\times\left[\frac{H^2(\|u_0\|_{H^3(\mathbb{R}^d,\mathbb{R}^N)}+1)^{\frac{8s_2}{d}-2}d}{(d-4s_2)(2\pi)^{4S_2}}\left(\frac{|S^d|}{4S_2}\right)^{\frac{4S_2}{d}} + Q^2\right]^{\frac{1}{2}} \|g_1 - g_2\|_{C^2(I,\mathbb{R}^N)}.$$

Let us use formulas (1.25) and (1.26) to complete the proof of the theorem. ■

4. Auxiliary results

We establish the solvability conditions for the linear Poisson type equation with a square integrable right side in the situation of the double scale anomalous diffusion

$$[(-\Delta)^{s_1} + (-\Delta)^{s_2}]\phi(x) = f(x), \quad x \in \mathbb{R}^d, \quad d = 4, 5, \quad 0 < s_1 < s_2 < 1. \tag{4.1}$$

This auxiliary statement was proved in the previous work [34] using the standard Fourier transform (2.1). Let us provide the argument below for the convenience of the readers.

Lemma 4.1. *Let* $0 < s_1 < s_2 < 1, \ f : \mathbb{R}^d \to \mathbb{R}, \ d = 4, 5$ *and* $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. *Then equation (4.1) admits a unique solution* $\phi \in H^{2s_2}(\mathbb{R}^d)$.

Proof. It can be trivially checked that if $\phi \in L^2(\mathbb{R}^d)$ is a solution of problem (4.1) with a square integrable right side, it will be contained in $H^{2s_2}(\mathbb{R}^d)$ as well. Indeed, if we apply the standard Fourier transform (2.1) to both sides of (4.1), we obtain

$$(|p|^{2s_1} + |p|^{2s_2})\widehat{\phi}(p) = \widehat{f}(p) \in L^2(\mathbb{R}^d).$$

Hence,

$$\int_{\mathbb{R}^d} [|p|^{2s_1} + |p|^{2s_2}]^2 |\widehat{\phi}(p)|^2 dp < \infty.$$

Clearly, the equality

$$\|(-\Delta)^{s_2}\phi\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |p|^{4s_2} |\widehat{\phi}(p)|^2 dp < \infty$$

holds, so that $(-\Delta)^{s_2}\phi \in L^2(\mathbb{R}^d)$. Let us recall the definition of the norm (1.10). Thus, $\phi \in H^{2s_2}(\mathbb{R}^d)$ as well.
To establish the uniqueness of solutions for problem (4.1), we suppose that our equation has two solutions $\phi_{1,2} \in H^{2s_2}(\mathbb{R}^d)$. Then their difference $w := \phi_1 - \phi_2 \in H^{2s_2}(\mathbb{R}^d)$ solves the homogeneous problem

$$[(-\Delta)^{s_1} + (-\Delta)^{s_2}]w = 0.$$

The operator

$$(-\Delta)^{s_1} + (-\Delta)^{s_2} : H^{2s_2}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

does not have any nontrivial zero modes. Therefore, $w(x)$ vanishes in $\mathbb{R}^d$.
Let us apply the standard Fourier transform (2.1) to both sides of equation (4.1). This yields

$$\widehat{\phi}(p) = \frac{\widehat{f}(p)}{|p|^{2s_1} + |p|^{2s_2}} \chi_{\{|p| \le 1\}} + \frac{\widehat{f}(p)}{|p|^{2s_1} + |p|^{2s_2}} \chi_{\{|p| > 1\}}. \tag{4.2}$$

In formula (4.2) and below χ_A will denote the characteristic function of a set $A \subseteq \mathbb{R}^d$.

Evidently, the second term in the right side of (4.2) can be estimated from above in the absolute value by $\frac{|\widehat{f}(p)|}{2} \in L^2(\mathbb{R}^d)$ due to the one of our assumptions.
The first term in the right side of (4.2) can be bounded from above in the absolute value by virtue of (2.2) by

$$\frac{\|f\|_{L^1(\mathbb{R}^d)}}{(2\pi)^{\frac{d}{2}}|p|^{2s_2}}\chi_{\{|p|\le 1\}}. \tag{4.3}$$

It can be easily verified that expression (4.3) with $d = 4, 5$ and $0 < s_2 < 1$ is contained in $L^2(\mathbb{R}^d)$. ■.

Note that in the auxiliary lemma above we establish the solvability of equation (4.1) in $H^{2s_2}(\mathbb{R}^d),\ d = 4, 5$ for all the values of the powers of the fractional Laplace operators $0 < s_1 < s_2 < 1$, such that no orthogonality conditions are needed for the right side $f(x)$. This is similar to the case when the Poisson type equation is studied with a single fractional Laplacian in the spaces of the same dimensions (see Theorem 1.1 of [33], also [29]). The solvability of the problem analogous to (4.1) containing a scalar potential was considered in [12].

Acknowledgements

The author is grateful to Michael Ruzhansky for the stimulating discussions and to Israel Michael Sigal for the partial support by the NSERC grant NA 7901.

References

[1] G.L. Alfimov, E.V. Medvedeva, D.E. Pelinovsky, *Wave systems with an infinite number of localized traveling waves*, Phys. Rev. Lett., **112** (2014), 054103, 5 pp.

[2] C. Amrouche, V. Girault, J. Giroire, *Dirichlet and Neumann exterior problems for the n-dimensional Laplace operator: an approach in weighted Sobolev spaces*, J. Math. Pures Appl. (9), **76** (1997), no. 1, 55–81.

[3] C. Amrouche, F. Bonzom, *Mixed exterior Laplace's problem*, J. Math. Anal. Appl., **338** (2008), no. 1, 124–140.

[4] P. Bolley, T.L. Pham, *Propriétés d'indice en théorie hölderienne pour des opérateurs différentiels elliptiques dans R^n*, J. Math. Pures Appl. (9), **72** (1993), no. 1, 105–119.

[5] P. Bolley, T.L. Pham, *Propriété d'indice en théorie Hölderienne pour le problème extérieur de Dirichlet*, Comm. Partial Differential Equations, **26** (2001), no. 1-2, 315–334.

[6] N. Benkirane, *Propriétés d'indice en théorie hölderienne pour des opérateurs elliptiques dans R^n*, C. R. Acad. Sci. Paris Ser. I Math., **307** (1988), no. 11, 577–580.

[7] B. Carreras, V. Lynch, G. Zaslavsky, *Anomalous diffusion and exit time distribution of particle tracers in plasma turbulence model*, Phys. Plasmas, **8** (2001), 5096–5103.

[8] S. Cuccagna, D. Pelinovsky, V. Vougalter, *Spectra of positive and negative energies in the linearized NLS problem*, Comm. Pure Appl. Math., **58** (2005), no. 1, 1–29.

[9] A. Ducrot, M. Marion, V. Volpert, *Systemes de réaction-diffusion sans propriété de Fredholm*, C. R. Math. Acad. Sci. Paris, **340** (2005), no. 9, 659–664.

[10] A. Ducrot, M. Marion, V. Volpert, *Reaction-diffusion problems with non-Fredholm operators*, Adv. Differential Equations, **13** (2008), no. 11-12, 1151–1192.

[11] M. Efendiev, *Fredholm structures, topological invariants and applications*, AIMS Ser. Differ. Equ. Dyn. Syst., **3**. American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2009. 205 pp.

[12] M. Efendiev, V. Vougalter, *Linear and nonlinear non-Fredholm operators and their applications*, Electron. Res. Arch., **30** (2022), no. 2, 515–534.

[13] M. Efendiev, V. Vougalter, *On the necessary conditions for preserving the nonnegative cone: double scale anomalous diffusion*, Adv. Math. Sci. Appl., **31** (2022), no. 1, 197–206.

[14] H.G. Gebran, C.A. Stuart, *Fredholm and properness properties of quasilinear elliptic systems of second order*, Proc. Edinb. Math. Soc. (2), **48** (2005), no. 1, 91–124.

[15] N. Guerngar, E. Nane, R. Tinaztepe, S. Ulusoy, H.W. Van Wyk, *Simultaneous inversion for the fractional exponents in the space-time fractional diffusion equation $\partial_t^\beta u = -(-\Delta)^{\frac{\alpha}{2}} u - (-\Delta)^{\frac{\gamma}{2}} u$*, Fract. Calc. Appl. Anal., **24** (2021), no. 3, 818–847.

[16] H. Hajaiej, X. Yu, Z. Zhai, *Fractional Gagliardo-Nirenberg and Hardy inequalities under Lorentz norms*, J. Math. Anal. Appl., **396** (2012), no. 2, 569–577.

[17] M. Kelbert, V. Konakov, S. Menozzi, *Weak error for continuous time Markov chains related to fractional in time P(I)DEs*, Stochastic Process. Appl., **126** (2016), no. 4, 1145–1183.

[18] E.H. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math. (2), **118** (1983), no. 2, 349–374.

[19] P. Manandhar, J. Jang, G.C. Schatz, M.A. Ratner, S. Hong, *Anomalous surface diffusion in nanoscale direct deposition processes*, Phys. Rev. Lett., **90** (2003), 4043–4052.

[20] R. Metzler, J. Klafter, *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep., **339** (2000), 1–77.

[21] P.J. Rabier, C.A. Stuart, *Fredholm and properness properties of quasilinear elliptic operators on* $\mathbb{R}^N$, Math. Nachr., **231** (2001), 129–168.

[22] J. Sancho, A. Lacasta, K. Lindenberg, I. Sokolov, A. Romero, *Diffusion on a solid surface: anomalous is normal*, Phys. Rev. Lett., **92** (2004), 250601.

[23] H. Scher, E. Montroll, *Anomalous transit-time dispersion in amorphous solids*, Phys. Rev. B, **12** (1975), 2455–2477.

[24] T. Solomon, E. Weeks, H. Swinney, *Observation of anomalous diffusion and Lévy flights in a two-dimensional rotating flow*, Phys. Rev. Lett., **71** (1993), 3975–3978.

[25] V. Volpert, Elliptic partial differential equations. Volume 1: Fredholm theory of elliptic problems in unbounded domains. Monogr. Math., **101**. Birkhäuser/Springer Basel AG, Basel, (2011), 639 pp.

[26] V.A. Volpert, Y. Nec, A.A. Nepomnyashchy, *Exact solutions in front propagation problems with superdiffusion*, Phys. D, **239** (2010), no. 3–4, 134–144.

[27] V.A. Volpert, Y. Nec, A.A. Nepomnyashchy, *Fronts in anomalous diffusion-reaction systems*, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., **371** (2013), no. 1982, 20120179, 18 pp.

[28] V. Vougalter, *On threshold eigenvalues and resonances for the linearized NLS equation*, Math. Model. Nat. Phenom., **5** (2010), no. 4, 448–469.

[29] V. Vougalter, *On the solvability of some systems of integro-differential equations with anomalous diffusion in higher dimensions*, Pure Appl. Funct. Anal., **8** (2023), no. 1, 357–372.

[30] V. Vougalter, V. Volpert, *Solvability conditions for some non-Fredholm operators*, Proc. Edinb. Math. Soc. (2), **54** (2011), no. 1, 249–271.

[31] V. Vougalter, V. Volpert, *On the existence of stationary solutions for some non-Fredholm integro-differential equations*, Doc. Math., **16** (2011), 561–580.

[32] V. Vougalter, V. Volpert, *Solvability conditions for some linear and nonlinear non-Fredholm elliptic problems*, Anal. Math. Phys., **2** (2012), no. 4, 473–496.

[33] V. Vougalter, V. Volpert, *On the solvability in the sense of sequences for some non-Fredholm operators related to the anomalous diffusion*. Trends Math., Birkhäuser / Springer, Cham, (2019), 229–257.

[34] V. Vougalter, V. Volpert, *Solvability of some integro-differential equations with the double scale anomalous diffusion in higher dimensions*, preprint (2023).