SOLVABILITY OF SOME QUADRATIC INTEGRAL EQUATIONS IN HIGHER DIMENSIONS

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ABSTRACT. The work deals with the existence of solutions of a certain quadratic integral equation in $H^3(\mathbb{R}^d),\ d=4,5$. The theory of quadratic integral equations has many significant applications in the mathematical physics, economics, biology. It is important for describing the real world problems. The proof of the existence of solutions is based on a fixed point technique in the Sobolev space in dimensions four and five.

1. Introduction

The present article is devoted to the existence of solutions of the following integral equation

$$u(x)=u_0(x)+[Tu(x)]\int_{\mathbb{R}^d}K(x-y)g(u(y))dy, \quad x\in\mathbb{R}^d, \quad d=4,5. \tag{1.1}$$

We generalize the results of the preceding works [16] and [17], in which the solvability of the problem analogous to (1.1) was established in $H^1(\mathbb{R})$ and $H^2(\mathbb{R}^d),\ d=2,3$ respectively. The exact conditions on the functions $u_0(x),\ g(u)$, the linear operator T and the kernel $K(x)$ will be specified further down. The second term in the right side of (1.1) is a product of $Tu(x)$ and the integral operator applied to the function $g(u)$. The sublinear growth for it will be shown in the proof of Theorem 1.3. below. Hence, the integral equation of this kind is called *quadratic*. The theory of the integral equations has many significant applications in describing the various events and problems of the real world. It is caused by the fact that this theory is often applicable in different branches of mathematics and in mathematical physics, economics, biology as well as for solving the real world problems. The quadratic integral equations are used in the theories of the radiative transfer, neutron transport, in the kinetic theory of gases, in the design of the bandlimited signals for the binary communication with the simple memoryless correlation detection, when the signals are disturbed by the additive white Gaussian noise (see e.g. [1], [5], [11] and the references therein). The work [1] deals with the solvability of a nonlinear quadratic integral equation in the Banach space of the real functions being defined and continuous

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on a bounded and closed interval via the fixed point technique. The articles [2] and [4] are devoted to the studies of the existence of solutions for the quadratic integral equations on unbounded intervals. The solvability of the quadratic integral inclusions was covered in [3]. In the work [10] the authors consider the nondecreasing solutions of a quadratic integral equation of Urysohn-Stieltjes type. The solvability of the quadratic integral equations in Orlicz spaces was discussed in [7], [8], [9]. The integro-differential equations containing either Fredholm or non-Fredholm operators appear in the mathematical biology when studying the systems with the nonlocal consumption of resources and the intra-specific competition (see [12], [13], [18], [19] and the references therein). The contraction argument was applied in [15] to evaluate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term were perturbed. The similar ideas were exploited to show the persistence of pulses for some reaction-diffusion type problems (see [6]). We suppose that the conditions below are fulfilled. They are needed for deriving the estimates on the solutions of our integral equations in the fixed point argument.

Assumption 1.1. *Let the kernel* $K(x) : \mathbb{R}^d \to \mathbb{R},\ d = 4, 5$ *be nontrivial, such that* $K(x), (-\Delta)^{\frac{3}{2}} K(x) \ \in L^1(\mathbb{R}^d)$. *The function* $u_0(x) : \mathbb{R}^d \to \mathbb{R}$ *does not vanish identically in* $\mathbb{R}^d$ *and* $u_0(x) \in H^3(\mathbb{R}^d)$. *Let us also suppose that the linear operator* $T : H^3(\mathbb{R}^d) \to H^3(\mathbb{R}^d)$ *is bounded, such that its norm* $0 < \|T\| < \infty$.

It can be easily verified that for the identity operator $T = I$ the conditions above are satisfied. But we do not use this special choice of the operator T in the argument. T can be any operator relevant to the applications, which satisfies the assumptions given above. Let us define the technical quantity

$$Q := \sqrt{\|K(x)\|^2_{L^1(\mathbb{R}^d)} + \|(-\Delta)^{\frac{3}{2}} K(x)\|^2_{L^1(\mathbb{R}^d)}}. \tag{1.2}$$

Evidently, under the conditions above we have $0 < Q < \infty$. We will use the Sobolev space

$$H^3(\mathbb{R}^d) := \big\{u(x) : \mathbb{R}^d \to \mathbb{R} \mid u(x) \in L^2(\mathbb{R}^d),\ (-\Delta)^{\frac{3}{2}} u(x) \in L^2(\mathbb{R}^d)\big\}, \tag{1.3}$$

where $d = 4, 5$. It is equipped with the norm

$$\|u\|^2_{H^3(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \|(-\Delta)^{\frac{3}{2}} u\|^2_{L^2(\mathbb{R}^d)}. \tag{1.4}$$

$(-\Delta)^{\frac{3}{2}}$ is defined via the spectral calculus. It is the pseudo-differential operator with the symbol $|p|^3$, such that

$$(-\Delta)^{\frac{3}{2}} u(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |p|^3 \widehat{u}(p) e^{ipx} dp,$$

where the standard Fourier transform is given by

$$\widehat{u}(p) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} u(x) e^{-ipx} dx. \tag{1.5}$$

Clearly, the upper bound

$$\|\widehat{u}(p)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^{\frac{d}{2}}} \|u(x)\|_{L^1(\mathbb{R}^d)} \tag{1.6}$$

holds. By means of the standard Sobolev embedding, we have

$$\|u(x)\|_{L^\infty(\mathbb{R}^d)} \leq c_e \|u(x)\|_{H^3(\mathbb{R}^d)}, \quad d = 4, 5. \tag{1.7}$$

Here $c_e > 0$ is the constant of the embedding. Let us recall the algebra property for the Sobolev space. Hence, for any $u(x), v(x) \in H^3(\mathbb{R}^d),\ d = 4, 5$

$$\|u(x)v(x)\|_{H^3(\mathbb{R}^d)} \leq c_a \|u(x)\|_{H^3(\mathbb{R}^d)} \|v(x)\|_{H^3(\mathbb{R}^d)}, \tag{1.8}$$

where $c_a > 0$ is a constant, so that $u(x)v(x) \in H^3(\mathbb{R}^d)$ as well.
The Young's inequality (see e.g. Section 4.2 of [14]) enables us to derive the estimate from above on the norm of the convolution as

$$\|u * v\|_{L^2(\mathbb{R}^d)} \leq \|u\|_{L^1(\mathbb{R}^d)} \|v\|_{L^2(\mathbb{R}^d)}. \tag{1.9}$$

Obviously, the upper bound

$$\left\| (-\Delta_x)^{\frac{3}{2}} \int_{\mathbb{R}^d} u(x-y) v(y) dy \right\|_{L^2(\mathbb{R}^d)} \leq \|(-\Delta)^{\frac{3}{2}} u\|_{L^1(\mathbb{R}^d)} \|v\|_{L^2(\mathbb{R}^d)} \tag{1.10}$$

can be easily obtained similarly to (1.9) using the standard Fourier transform (1.5) along with (1.6). Here and further down Δ_x will stand for the Laplace operator with respect to the x-variable.
We seek the resulting solution of nonlinear equation (1.1) as

$$u(x) = u_0(x) + u_p(x). \tag{1.11}$$

Evidently, we arrive at the perturbative problem

$$u_p(x) = [T(u_0(x) + u_p(x))] \int_{\mathbb{R}^d} K(x-y) g(u_0(y) + u_p(y)) dy \tag{1.12}$$

with $d = 4, 5$. Let us introduce a closed ball in the Sobolev space

$$B_\rho := \{u(x) \in H^3(\mathbb{R}^d) \mid \|u\|_{H^3(\mathbb{R}^d)} \leq \rho\}, \quad 0 < \rho \leq 1. \tag{1.13}$$

We look for the solution of equation (1.12) as the fixed point of the auxiliary nonlinear problem

$$u(x) = [T(u_0(x) + v(x))] \int_{\mathbb{R}^d} K(x-y) g(u_0(y) + v(y)) dy \tag{1.14}$$

in ball (1.13). Let us define the interval on the real line

$$I := [-c_e - c_e \|u_0\|_{H^3(\mathbb{R}^d)},\ c_e + c_e \|u_0\|_{H^3(\mathbb{R}^d)}] \tag{1.15}$$

along with the closed ball in the space of $C_1(I)$ functions, so that

$$D_M := \{g(z) \in C_1(I) \mid \|g\|_{C_1(I)} \leq M\}, \quad M > 0. \tag{1.16}$$

In this context the norm

$$\|g\|_{C_1(I)} := \|g\|_{C(I)} + \|g'\|_{C(I)} \tag{1.17}$$

with $\|g\|_{C(I)} := \max_{z\in I}|g(z)|$.

Assumption 1.2. *Let $g(z) : \mathbb{R} \to \mathbb{R}$, such that $g(0) = 0$. We also assume that $g(z) \in D_M$ and it does not vanish identically on the interval I.*

We introduce the operator t_g, such that $u = t_g v$, where u is a solution of equation (1.14). Our first main result is as follows.

Theorem 1.3. *Let Assumptions 1.1 and 1.2 hold and*

$$c_a\|T\|(\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^2 QM \le \frac{\rho}{2}. \tag{1.18}$$

Then for every $\rho \in (0, 1]$ equation (1.14) defines the map $t_g : B_\rho \to B_\rho$, which is a strict contraction. The unique fixed point $u_p(x)$ of this map t_g is the only solution of problem (1.12) in B_ρ.

Clearly, the resulting solution of equation (1.1) given by (1.11) will not vanish identically in $\mathbb{R}^d$, $d = 4, 5$, because $g(0) = 0$, the operator T is linear and the function $u_0(x)$ is nontrivial in the whole space as assumed.

We will use the technical quantity

$$\sigma := 2c_a(\|u_0\|_{H^3(\mathbb{R}^d)} + 1)\|T\|MQ > 0. \tag{1.19}$$

Our second major statement is about the continuity of the resulting solution of problem (1.1) given by (1.11) with respect to the function g.

Theorem 1.4. *Let $j = 1, 2$, the assumptions of Theorem 1.3 are valid, such that $u_{p,j}(x)$ is the unique fixed point of the map $t_{g_j} : B_\rho \to B_\rho$, which is a strict contraction since the upper bound (1.18) holds and the cumulative solution of equation (1.1) with $g(z) = g_j(z)$ is given by*

$$u_j(x) = u_0(x) + u_{p,j}(x). \tag{1.20}$$

Then the inequality

$$\|u_1(x) - u_2(x)\|_{H^3(\mathbb{R}^d)} \le \tag{1.21}$$

$$\le \frac{\sigma}{2M(1-\sigma)}(\|u_0\|_{H^3(\mathbb{R}^d)} + 1)\|g_1(z) - g_2(z)\|_{C_1(I)}$$

is valid.

Let us proceed to the proof of our first main proposition.

2. The existence of the perturbed solution

Proof of Theorem 1.3. We choose arbitrarily $v(x) \in B_\rho$. By virtue of (1.14) along with (1.8), we obtain the upper bound

$$\|u\|_{H^3(\mathbb{R}^d)} \leq$$

$$\leq c_a \|T(u_0(x)+v(x))\|_{H^3(\mathbb{R}^d)} \Big\| \int_{\mathbb{R}^d} K(x-y) g(u_0(y)+v(y))dy \Big\|_{H^3(\mathbb{R}^d)}. \tag{2.1}$$

Let us derive the estimate from above on the right side of (2.1). Obviously,

$$\|T(u_0(x)+v(x))\|_{H^3(\mathbb{R}^d)} \leq \|T\|(\|u_0(x)\|_{H^3(\mathbb{R}^d)}+1). \tag{2.2}$$

By means of inequality (1.9), we have

$$\Big\| \int_{\mathbb{R}^d} K(x-y) g(u_0(y)+v(y))dy \Big\|_{L^2(\mathbb{R}^d)} \leq \tag{2.3}$$
$$\leq \|K\|_{L^1(\mathbb{R}^d)} \|g(u_0(x)+v(x))\|_{L^2(\mathbb{R}^d)}.$$

Similarly, (1.10) yields

$$\Big\| (-\Delta_x)^{\frac{3}{2}} \int_{\mathbb{R}^d} K(x-y) g(u_0(y)+v(y))dy \Big\|_{L^2(\mathbb{R}^d)} \leq$$

$$\leq \|(-\Delta)^{\frac{3}{2}} K\|_{L^1(\mathbb{R}^d)} \|g(u_0(x)+v(x))\|_{L^2(\mathbb{R}^d)}. \tag{2.4}$$

By virtue of bounds (2.3) and (2.4),

$$\Big\| \int_{\mathbb{R}^d} K(x-y) g(u_0(y)+v(y))dy \Big\|_{H^3(\mathbb{R}^d)} \leq$$

$$\leq Q \|g(u_0(x)+v(x))\|_{L^2(\mathbb{R}^d)}. \tag{2.5}$$

Let us express

$$g(u_0(x)+v(x)) = \int_0^{u_0(x)+v(x)} g'(z)dz. \tag{2.6}$$

For $v(x) \in B_\rho$ by means of (1.7) we easily arrive at

$$|u_0+v| \leq c_e(\|u_0\|_{H^3(\mathbb{R}^d)}+1). \tag{2.7}$$

Hence,

$$|g(u_0(x)+v(x))| \leq \max_{z\in I}|g'(z)||u_0(x)+v(x)| \leq M|u_0(x)+v(x)|, \tag{2.8}$$

where the interval I defined in (1.15). Therefore,

$$\|g(u_0(x)+v(x))\|_{L^2(\mathbb{R}^d)} \leq M(\|u_0\|_{H^3(\mathbb{R}^d)}+1). \tag{2.9}$$

Thus, we obtain

$$\|u(x)\|_{H^3(\mathbb{R}^d)} \leq c_a\|T\|(\|u_0\|_{H^3(\mathbb{R}^d)}+1)^2 QM. \tag{2.10}$$

By virtue of (1.18), we have $\|u(x)\|_{H^3(\mathbb{R}^d)} \leq \rho$. This means that the function $u(x)$, which is uniquely determined by (1.14) belongs to B_ρ as well, such that under the given conditions problem (1.14) defines a map $t_g : B_\rho \to B_\rho$.

It remains to establish that under the stated assumptions this map is a strict contraction. Let us choose arbitrarily $v_{1,2}(x) \in B_\rho$. By means of the argument above, $u_{1,2} := t_g v_{1,2} \in B_\rho$. By virtue of (1.14),

$$u_1(x) = [T(u_0(x)+v_1(x))]\int_{\mathbb{R}^d} K(x-y)g(u_0(y)+v_1(y))dy, \tag{2.11}$$

$$u_2(x) = [T(u_0(x)+v_2(x))]\int_{\mathbb{R}^d} K(x-y)g(u_0(y)+v_2(y))dy. \tag{2.12}$$

From system (2.11), (2.12) we easily deduce that

$$u_1(x)-u_2(x) = [Tv_1(x)-Tv_2(x)]\int_{\mathbb{R}^d} K(x-y)g(u_0(y)+v_1(y))dy+ \tag{2.13}$$

$$+[T(u_0(x)+v_2(x))]\int_{\mathbb{R}^d} K(x-y)[g(u_0(y)+v_1(y))-g(u_0(y)+v_2(y))]dy.$$

By means of (2.13) along with (1.8),

$$\|u_1(x)-u_2(x)\|_{H^3(\mathbb{R}^d)} \le c_a\|Tv_1(x)-Tv_2(x)\|_{H^3(\mathbb{R}^d)}\times$$

$$\times\Big\|\int_{\mathbb{R}^d} K(x-y)g(u_0(y)+v_1(y))dy\Big\|_{H^3(\mathbb{R}^d)} + c_a\|T(u_0(x)+v_2(x))\|_{H^3(\mathbb{R}^d)}\times$$

$$\times\Big\|\int_{\mathbb{R}^d} K(x-y)[g(u_0(y)+v_1(y))-g(u_0(y)+v_2(y))]dy\Big\|_{H^3(\mathbb{R}^d)}. \tag{2.14}$$

Let us obtain the upper bound on the right side of (2.14). Evidently,

$$\|Tv_1(x)-Tv_2(x)\|_{H^3(\mathbb{R}^d)} \le \|T\|\|v_1(x)-v_2(x)\|_{H^3(\mathbb{R}^d)}. \tag{2.15}$$

Inequality (1.9) gives us

$$\Big\|\int_{\mathbb{R}^d} K(x-y)g(u_0(y)+v_1(y))dy\Big\|_{L^2(\mathbb{R}^d)} \le$$

$$\le \|K\|_{L^1(\mathbb{R}^d)}\|g(u_0(x)+v_1(x))\|_{L^2(\mathbb{R}^d)}. \tag{2.16}$$

We use (1.10) to derive

$$\Big\|(-\Delta_x)^{\frac{3}{2}}\int_{\mathbb{R}^d} K(x-y)g(u_0(y)+v_1(y))dy\Big\|_{L^2(\mathbb{R}^d)} \le$$

$$\le \|(-\Delta)^{\frac{3}{2}}K\|_{L^1(\mathbb{R}^d)}\|g(u_0(x)+v_1(x))\|_{L^2(\mathbb{R}^d)}. \tag{2.17}$$

By virtue of the estimates from above (2.16) and (2.17),

$$\Big\|\int_{\mathbb{R}^d} K(x-y)g(u_0(y)+v_1(y))dy\Big\|_{H^3(\mathbb{R}^d)} \le$$

$$\le Q\|g(u_0(x)+v_1(x))\|_{L^2(\mathbb{R}^d)}. \tag{2.18}$$

Clearly,

$$g(u_0(x)+v_1(x)) = \int_0^{u_0(x)+v_1(x)} g'(z)dz. \tag{2.19}$$

From (2.19) we easily obtain that

$$|g(u_0(x)+v_1(x))| \le \max_{z\in I}|g'(z)||u_0(x)+v_1(x)| \le M|u_0(x)+v_1(x)|. \tag{2.20}$$

Then

$$\|g(u_0(x)+v_1(x))\|_{L^2(\mathbb{R}^d)} \le M(\|u_0\|_{H^3(\mathbb{R}^d)}+1). \tag{2.21}$$

Thus, the first term in the right side of (2.14) can be bounded from above by

$$c_a\|T\|\|v_1(x)-v_2(x)\|_{H^3(\mathbb{R}^d)}QM(\|u_0\|_{H^3(\mathbb{R}^d)}+1). \tag{2.22}$$

Therefore, it remains to derive the estimate from above on the second term in the right side of inequality (2.14). Obviously,

$$\|T(u_0(x)+v_2(x))\|_{H^3(\mathbb{R}^d)} \le \|T\|(\|u_0\|_{H^3(\mathbb{R}^d)}+1). \tag{2.23}$$

By means of (1.9),

$$\Big\|\int_{\mathbb{R}^d} K(x-y)[g(u_0(y)+v_1(y))-g(u_0(y)+v_2(y))]dy\Big\|_{L^2(\mathbb{R}^d)} \le$$

$$\le \|K\|_{L^1(\mathbb{R}^d)}\|g(u_0(x)+v_1(x))-g(u_0(x)+v_2(x))\|_{L^2(\mathbb{R}^d)}. \tag{2.24}$$

Upper bound (1.10) gives us

$$\Big\|(-\Delta_x)^{\frac{3}{2}}\int_{\mathbb{R}^d} K(x-y)[g(u_0(y)+v_1(y))-g(u_0(y)+v_2(y))]dy\Big\|_{L^2(\mathbb{R}^d)} \le$$

$$\le \|(-\Delta)^{\frac{3}{2}}K\|_{L^1(\mathbb{R}^d)}\|g(u_0(x)+v_1(x))-g(u_0(x)+v_2(x))\|_{L^2(\mathbb{R}^d)}. \tag{2.25}$$

By virtue of (2.24) and (2.25),

$$\Big\|\int_{\mathbb{R}^d} K(x-y)[g(u_0(y)+v_1(y))-g(u_0(y)+v_2(y))]dy\Big\|_{H^3(\mathbb{R}^d)} \le$$

$$\le Q\|g(u_0(x)+v_1(x))-g(u_0(x)+v_2(x))\|_{L^2(\mathbb{R}^d)}. \tag{2.26}$$

Let us write

$$g(u_0(x)+v_1(x))-g(u_0(x)+v_2(x)) = \int_{u_0(x)+v_2(x)}^{u_0(x)+v_1(x)} g'(z)dz. \tag{2.27}$$

By means of (2.27), we have

$$|g(u_0(x)+v_1(x))-g(u_0(x)+v_2(x))| \le \max_{z\in I}|g'(z)||v_1(x)-v_2(x)| \le$$

$$\le M|v_1(x)-v_2(x)|. \tag{2.28}$$

Formula (2.28) yields that

$$\|g(u_0(x)+v_1(x))-g(u_0(x)+v_2(x))\|_{L^2(\mathbb{R}^d)} \le M\|v_1(x)-v_2(x)\|_{H^3(\mathbb{R}^d)}. \tag{2.29}$$

Thus, the second term in the right side of bound (2.14) can be estimated from above by expression (2.22) as well. Therefore,

$$\|u_1(x)-u_2(x)\|_{H^3(\mathbb{R}^d)} \le$$

$$\leq 2c_a(\|u_0\|_{H^3(\mathbb{R}^d)}+1)\|T\|MQ\|v_1(x)-v_2(x)\|_{H^3(\mathbb{R}^d)}. \tag{2.30}$$

By virtue of (2.30) along with definition (1.19), we derive that

$$\|t_g v_1(x)-t_g v_2(x)\|_{H^3(\mathbb{R}^d)} \leq \sigma\|v_1(x)-v_2(x)\|_{H^3(\mathbb{R}^d)}. \tag{2.31}$$

It can be easily verified using (1.18) that for the constant in the right side of (2.31), we have

$$\sigma < 1. \tag{2.32}$$

This implies that the map $t_g : B_\rho \to B_\rho$ defined by (1.14) is a strict contraction under the given conditions. Its unique fixed point $u_p(x)$ is the only solution of equation (1.12) in the ball B_ρ. The resulting $u(x)$ given by (1.11) solves problem (1.1). □

Let us conclude the article by establishing the validity of the second main statement.

3. The continuity of the resulting solution with respect to the function g

Proof of Theorem 1.4. Obviously, under the stated assumptions, we have

$$u_{p,1}=t_{g_1}u_{p,1}, \quad u_{p,2}=t_{g_2}u_{p,2}. \tag{3.1}$$

Thus,

$$u_{p,1}-u_{p,2}=t_{g_1}u_{p,1}-t_{g_1}u_{p,2}+t_{g_1}u_{p,2}-t_{g_2}u_{p,2}, \tag{3.2}$$

such that

$$\|u_{p,1}-u_{p,2}\|_{H^3(\mathbb{R}^d)} \leq$$

$$\leq \|t_{g_1}u_{p,1}-t_{g_1}u_{p,2}\|_{H^3(\mathbb{R}^d)}+\|t_{g_1}u_{p,2}-t_{g_2}u_{p,2}\|_{H^3(\mathbb{R}^d)}. \tag{3.3}$$

By means of (2.31), we have the upper bound

$$\|t_{g_1}u_{p,1}-t_{g_1}u_{p,2}\|_{H^3(\mathbb{R}^d)} \leq \sigma\|u_{p,1}-u_{p,2}\|_{H^3(\mathbb{R}^d)}, \tag{3.4}$$

where σ is defined in (1.19) and (2.32) holds. Hence, we obtain

$$(1-\sigma)\|u_{p,1}-u_{p,2}\|_{H^3(\mathbb{R}^d)} \leq \|t_{g_1}u_{p,2}-t_{g_2}u_{p,2}\|_{H^3(\mathbb{R}^d)}. \tag{3.5}$$

Evidently, for our fixed point $t_{g_2}u_{p,2}=u_{p,2}$. Let us introduce $\eta(x) := t_{g_1}u_{p,2}$. Therefore,

$$\eta(x)=[T(u_0(x)+u_{p,2}(x))]\int_{\mathbb{R}^d}K(x-y)g_1(u_0(y)+u_{p,2}(y))dy, \tag{3.6}$$

$$u_{p,2}(x)=[T(u_0(x)+u_{p,2}(x))]\int_{\mathbb{R}^d}K(x-y)g_2(u_0(y)+u_{p,2}(y))dy. \tag{3.7}$$

By virtue of system (3.6), (3.7),

$$\eta(x)-u_{p,2}(x)=[T(u_0(x)+u_{p,2}(x))]\times$$

$$\times \int_{\mathbb{R}^d} K(x-y)[g_1(u_0(y)+u_{p,2}(y)) - g_2(u_0(y)+u_{p,2}(y))]dy. \tag{3.8}$$

Let us recall (1.8). Thus,

$$\|\eta(x) - u_{p,2}(x)\|_{H^3(\mathbb{R}^d)} \le c_a \|T(u_0(x)+u_{p,2}(x))\|_{H^3(\mathbb{R}^d)} \times$$

$$\times \Big\| \int_{\mathbb{R}^d} K(x-y)[g_1(u_0(y)+u_{p,2}(y)) - g_2(u_0(y)+u_{p,2}(y))]dy \Big\|_{H^3(\mathbb{R}^d)}. \tag{3.9}$$

Clearly, the estimate from above

$$\|T(u_0(x)+u_{p,2}(x))\|_{H^3(\mathbb{R}^d)} \le \|T\|(\|u_0\|_{H^3(\mathbb{R}^d)}+1) \tag{3.10}$$

is valid. By means of (1.9), we have

$$\Big\| \int_{\mathbb{R}^d} K(x-y)[g_1(u_0(y)+u_{p,2}(y)) - g_2(u_0(y)+u_{p,2}(y))]dy \Big\|_{L^2(\mathbb{R}^d)} \le$$

$$\le \|K\|_{L^1(\mathbb{R}^d)} \|g_1(u_0(x)+u_{p,2}(x)) - g_2(u_0(x)+u_{p,2}(x))\|_{L^2(\mathbb{R}^d)}. \tag{3.11}$$

Similarly, inequality (1.10) implies that

$$\Big\| (-\Delta_x)^{\frac{3}{2}} \int_{\mathbb{R}^d} K(x-y)[g_1(u_0(y)+u_{p,2}(y)) - g_2(u_0(y)+u_{p,2}(y))]dy \Big\|_{L^2(\mathbb{R}^d)} \le$$

$$\le \|(-\Delta)^{\frac{3}{2}} K\|_{L^1(\mathbb{R}^d)} \|g_1(u_0(x)+u_{p,2}(x)) - g_2(u_0(x)+u_{p,2}(x))\|_{L^2(\mathbb{R}^d)}. \tag{3.12}$$

Let us use upper bounds (3.11) and (3.12) to derive

$$\Big\| \int_{\mathbb{R}^d} K(x-y)[g_1(u_0(y)+u_{p,2}(y)) - g_2(u_0(y)+u_{p,2}(y))]dy \Big\|_{H^3(\mathbb{R}^d)} \le$$

$$\le Q \|g_1(u_0(x)+u_{p,2}(x)) - g_2(u_0(x)+u_{p,2}(x))\|_{L^2(\mathbb{R}^d)}. \tag{3.13}$$

Obviously,

$$g_1(u_0(x)+u_{p,2}(x)) - g_2(u_0(x)+u_{p,2}(x)) = \tag{3.14}$$

$$= \int_0^{u_0(x)+u_{p,2}(x)} [g_1'(z) - g_2'(z)]dz.$$

It follows easily from (3.14) that

$$|g_1(u_0(x)+u_{p,2}(x)) - g_2(u_0(x)+u_{p,2}(x))| \le$$

$$\le \max_{z \in I} |g_1'(z) - g_2'(z)| |u_0(x)+u_{p,2}(x)| \le$$

$$\le \|g_1(z) - g_2(z)\|_{C_1(I)} |u_0(x)+u_{p,2}(x)|. \tag{3.15}$$

Then

$$\|g_1(u_0(x)+u_{p,2}(x)) - g_2(u_0(x)+u_{p,2}(x))\|_{L^2(\mathbb{R}^d)} \le$$

$$\le \|g_1(z) - g_2(z)\|_{C_1(I)} (\|u_0\|_{H^3(\mathbb{R}^d)}+1). \tag{3.16}$$

We combine estimates from above (3.9), (3.10), (3.13), (3.16) and arrive at

$$\|\eta(x) - u_{p,2}(x)\|_{H^3(\mathbb{R}^d)} \le$$

$$\leq c_a \|T\| (\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^2 Q \|g_1(z) - g_2(z)\|_{C_1(I)}. \tag{3.17}$$

Inequalities (3.5) and (3.17) imply that

$$\|u_{p,1}(x) - u_{p,2}(x)\|_{H^3(\mathbb{R}^d)} \leq$$

$$\leq \frac{c_a}{1-\sigma} \|T\| (\|u_0\|_{H^3(\mathbb{R}^d)} + 1)^2 Q \|g_1(z) - g_2(z)\|_{C_1(I)}. \tag{3.18}$$

By virtue of formula (1.20) along with upper bound (3.18) and definition (1.19), estimate (1.21) holds. □

Remark 3.1. *The results of the present work will be generalized to the case of the systems of the coupled equations of this kind in the consecutive articles.*

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