On Proving A Special Decomposition of Numbers Using Complex Variable Theory

Abhinandan Dubey

Stony Brook University

Abstract

We revisit the special decomposition presented by L. Euler in his correspondences with C. Goldbach. We present a proof to a general result of this decomposition using complex variable theory.

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1. Introduction

Almost all of the correspondences between L. Euler and C. Goldbach [1] were published by P.H Fuss [2] in 1845. Apart from much of the correspondence between Leonhard Euler and Christian Goldbach, it also contains correspondence between Nicolas Fuss, and several members of the Bernoulli family (Johann (I), Nicolas, and Daniel). [3]

Here, we are concerned with a special decomposition of numbers discussed in one such correspondence, where Euler has discussed a less general case of the theorem we prove here. Consider the expression;

$$\sum_{i=1}^{n} \left(x_i^r \Big/ \prod_{\substack{1 \leq k \leq n \\ k \neq i}} (x_i - x_k) \right)$$

The pattern in the above expression can be visualized by substituting $r = 3$ and obtaining

$$x_1^3 \big/ (x_1 - x_2)(x_1 - x_3) + x_2^3 \big/ (x_2 - x_1)(x_2 - x_3) + x_3^3 \big/ (x_3 - x_1)(x_3 - x_2)$$

which evaluates to $x_1 + x_2 + x_3$. We state a general formula for evaluation of the above expression and also produce a proof using complex variable theory.

Theorem 1. *Let $x_1, x_2, \ldots, x_n$ be n distinct numbers,*

$$\sum_{i=1}^{n} \left(x_i^r \Big/ \prod_{\substack{1 \leq k \leq n \\ k \neq i}} (x_i - x_k) \right) = \begin{cases} 0 & 0 \leq r < n-1 \\ 1 & r = n-1 \\ \sum_{i=1}^{n} x_i & r = n \end{cases}$$

The above generalized result has been mentioned in [4]. The result can be proved using a strong induction on the variable n. This is immediately satisfied when we observe the resulting expression as a polynomial in x_i. However, as suggested in [5], we provide a more elegant proof of the above theorem using complex variable theory. We start by providing some results we will be using in the course of our proof.

Theorem 2. *(Cauchy's Residue Theorem) [6]*
Let $\mathfrak{W}$ be a simply connected open subset of the complex plane containing a finite list of points $a_1, \ldots, a_n$, and f a function holomorphic on $\mathfrak{W}/\{a_1, ..., a_n\}$.

Let γ be a closed rectifiable curve in $\mathfrak{W}$ which does not meet any of the a_k, and denote the winding number of γ around a_k by $\mathrm{I}(\gamma, a_k)$. Then line integral of f around γ is equal to $2\pi i$ times the sum of residues of f at the points, each counted as many times as γ winds around the point:

$$\oint_\gamma f(z)\, dz = 2\pi i \sum_{k=1}^{n} \mathrm{I}(\gamma, a_k)\, \mathrm{Res}(f, a_k).$$

If γ is a positively oriented simple closed curve, $\mathrm{I}(\gamma, a_k) = 1$ if a_k is in the interior of γ, and 0 if not, so

$$\oint_\gamma f(z)\, dz = 2\pi i \sum \mathrm{Res}(f, a_k)$$

with the sum over those a_k inside γ.

Definition 1. *[7] The Laurent series for a complex function $f(z)$ about a point c is given by:*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-c)^n$$

where the a_n and c are constants, defined by a line integral which is a generalization of Cauchy's integral formula:

$$a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)\, \mathrm{d}z}{(z-c)^{n+1}}$$

The path of integration γ is counterclockwise around a closed, rectifiable path containing no self-intersections, enclosing c and lying in an annulus A in which $f(z)$ is holomorphic (analytic).

In the above definition, consider the case when $n = -1$, it immediately satisfied that,

$$a_{-1} = \frac{1}{2\pi i} \oint_\gamma f(z) dz$$

$$\Rightarrow \oint_\gamma f(z) dz = 2\pi i a_{-1} = 2\pi i \sum_{k=1}^{n} \mathtt{Res}(f(z), z_k)$$

Hence, the argument of theorem 2 is immediately satisfied.

Definition 2. ***Residue of a Simple Pole***

Consider a function f, defined on $\mathfrak{W}/\{z_o\}$ as

$$f(z) = \frac{f_o(z)}{z - z_o} = \frac{a_{-1}}{z - z_o} + a_o + \dots$$

We are interested in finding the value of a_{-1}, the residue of f

$$\Rightarrow (z - z_o)f(z) = f_o(z) = a_{-1} + a_o(z - z_o) + a_1(z - z_o)^2 + \dots$$

now since $z \neq z_o$, we need to take limits;

$$\lim_{z \to z_o} (z - z_o)f(z) = a_{-1} = \texttt{Res}(f(z), z_o)$$

Definition 3. ***Residue of a Higher Order Pole***

Consider a function f, defined on $\mathfrak{W}/\{z_o\}$ as

$$f(z) = \frac{f_o(z)}{(z - z_o)^n} = \frac{a_{-n}}{(z - z_o)^n} + \dots + \frac{a_{-1}}{(z - z_o)} + a_o + a_1(z - z_o) + \dots$$

again, we are interested in finding the value of a_{-1}, the residue of f

$$\Rightarrow (z - z_o)^n f(z) = a_{-n} + \dots + a_{-1}(z - z_o)^{n-1} + a_o(z - z_o)^n + \dots$$

now since $z \neq z_o$, we need to take limits;

$$\frac{1}{(n-1)!} \lim_{z \to z_o} (z - z_o)^n f(z) = a_{-1} = \texttt{Res}(f(z), z_o)$$

2. Proving The Decomposition

As stated in Section 1, consider the expression written on the left side of Theorem 1,

$$\sum_{i=1}^{n} \left(x_i^r \Big/ \prod_{\substack{1 \leq k \leq n \\ k \neq i}} (x_i - x_k) \right)$$

Using the Residue theorem (Theorem 2), we arrive at,

$$\sum_{i=1}^{n} \left(x_i^r \Big/ \prod_{\substack{1 \leq k \leq n \\ k \neq i}} (x_i - x_k) \right) = \frac{1}{2\pi i} \int_{|z| = \mathbb{R}} \frac{z^r dz}{(z - x_1) \dots (z - x_n)}$$

where $R > |x_1|, \dots, |x_n|$. Consider the integrand in the above equation. We define

$$f(z) = \frac{z^r}{(z - x_1) \dots (z - x_n)} = z^{r-n} \left(\frac{1}{1 - {x_1}/{z}} \right) \left(\frac{1}{1 - {x_2}/{z}} \right) \cdots \left(\frac{1}{1 - {x_n}/{z}} \right)$$

we are interested in the laurent series expansion of $f(z)$,

$$z^{r-n}\left(\frac{1}{1-{}^{x_1}/_z}\right)\left(\frac{1}{1-{}^{x_2}/_z}\right)\cdots\left(\frac{1}{1-{}^{x_n}/_z}\right)=z^{r-n}\prod_{i=1}^{n}\left(\frac{1}{1-{}^{x_i}/_z}\right)$$

$$\Rightarrow z^{r-n}\prod_{i=1}^{n}\left(\frac{1}{1-{}^{x_i}/_z}\right)=z^{r-n}\prod_{i=1}^{n}\sum_{m\geq 0}\frac{{x_i}^m}{z}=z^{r-n}\sum_{m_1\geq 0}\left(\frac{x_1}{z}\right)^{m_1}\sum_{m_2\geq 0}\left(\frac{x_2}{z}\right)^{m_2}\cdots\sum_{m_n\geq 0}\left(\frac{x_n}{z}\right)^{m_n}$$

$$\Rightarrow z^{r-n}\sum_{n\geq 0}\frac{1}{z^n}\sum_{m_1+m_2+\cdots+m_n=n}x_1^{m_1}.x_2^{m_2}.\dots.x_n^{m_n}$$

Thus,

$$f(z)=z^{r-n}+(x_1+\cdots+x_n)z^{r-n-1}+({x_1}^2+x_1x_2+\dots)z^{r-n-2}+\dots$$

Integrating the above terms, we observe that only the following coefficient survives:

$$\sum_{\substack{j_1+\dots j_n=r-n+1\\ j_1,\dots j_n\geq 0}}x_1^{j_1}.x_2^{j_2}.\dots.x_n^{j_n}=\sum_{1\leq j_1\leq\cdots\leq j_{r-n+1}\leq n}x_{j_1}.x_{j_2}.\dots.x_{j_n}$$

Consequently, when $r=n$, the above expression simplifies to $\sum_{j=1}^{n}x_j$. Clearly, when $0\leq r<n-1$, the expression evaluates to zero and when $r=n-1$, it evaluates to 1. ■

Appendix A. References

[1] L. Euler, Institutionum Calculi Integralis, volume 2, 1769.

[2] P.-H. Fuss, Correspondance Mathmatique et Physique de quelques clbres gomtres du XVIIIme Sicle, volume 2, St. Petersbourg, 1845.

[3] The Mathematical Association of America, The Euler Archive, `http://eulerarchive.maa.org/correspondence/correspondents/Goldbach.html`, 1993. Accessed: 2016-08-30.

[4] D. E. Knuth, The Art of Computer Programming, volume 1, Addison-Wesley Longman, p. 136.

[5] D. E. Knuth, The Art of Computer Programming, volume 1, Addison-Wesley Longman, pp. 472–473.

[6] K. Knopp, Theory of Functions Parts I and II, Two Volumes Bound as One, Part I, Dover, New York, pp. 129–134.

[7] S.-G. Krantz, "Laurent Series", in: Handbook of Complex Variables, Birkhuser, Boston, MA, 1999, p. 43.