SPECTRAL ASYMPTOTICS OF THE DIRICHLET LAPLACIAN ON A GENERALIZED PARABOLIC LAYER

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ABSTRACT. We perform quantitative spectral analysis of the self-adjoint Dirichlet Laplacian H on an unbounded, radially symmetric (generalized) parabolic layer $\mathcal{P} \subset \mathbb{R}^3$. It was known before that H has an infinite number of eigenvalues below the threshold of its essential spectrum. In the present paper, we find the discrete spectrum asymptotics for H by means of a consecutive reduction to the analogous asymptotic problem for an effective one-dimensional Schrödinger operator on the half-line with the potential the behaviour of which far away from the origin is determined by the geometry of the layer $\mathcal{P}$ at infinity.

1. Introduction

1.1. State of the art and motivation

The topic of this paper is the spectral asymptotics for the Dirichlet Laplacian in domains of the form of a generalized parabolic layer. By the layer, we understand the Euclidean domain consisting of the points, whose distance from a hypersurface, in general unbounded one, is less than some given constant. The self-adjoint Laplace operator on unbounded layers subject to the Dirichlet boundary condition has been considered in numerous papers, *e.g.* [CEK04, DOR15, DEK01, DLO17, ET10, KL14, LL07, LR12], see also [EK, Chap. 4] and the references therein.

Unbounded layers belong to the class of *quasi-cylindrical domains* in the classification [EE, Dfn. X.6.1 (ii)] introduced by Glazman, because they do not contain arbitrarily large cubes, but they do contain infinitely many disjoint cubes of a fixed size. For this reason, as a rule, the essential spectrum is non-empty and its threshold is strictly positive; *cf.* [DEK01, KL14] and [EK, Chap. 4]. A challenging question is to prove the existence of bound states below this threshold and to find their properties. Various general results on the bound states can be found in [CEK04, DEK01, LL07, LR12] and in [EK, Chap. 4]. However, there is no result general enough to claim that this problem is completely understood. In this perspective, more precise analysis of various particular cases deserves attention. Examples of layers treated so far include mildly curved layers [BEGK01, EKr01], conical layers [DOR15, ET10, OP17], and so-called octant (or Fichera) layers [DLO17].

In the present paper, we initiate an investigation of *generalised parabolic layers* by performing the analysis of radially symmetric layers of this kind. With the indicated layer geometry in mind, the previously known general results yield an explicit formula for the threshold of the essential spectrum and imply the existence of infinitely many bound

states below it. Our aspiration here is to understand better the nature of this discrete spectrum and to obtain its quantitative properties.

1.2. **Geometric setting**

Here and in what follows, we use the shorthand notations $\mathbb{R}_+ := (0, \infty)$, $\overline{\mathbb{R}}_+ := [0, \infty)$, and $\mathbb{I}_a := (-a, a)$ for $a > 0$. We also introduce conventional Cartesian coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$ in the Euclidean space $\mathbb{R}^3$.

Given the parameters $k > 0$, $\alpha > 1$, and $R \geq 0$, we consider the class of real-valued functions $f \in \mathcal{C}^\infty(\overline{\mathbb{R}}_+)$ such that $f(0) = \dot{f}(0) = 0$ and that $f(x) = kx^\alpha$ holds for all $x \geq R$. The *generalized symmetric paraboloid* determined by such a function f is defined by

$$(1.1) \qquad \Sigma = \Sigma(f) := \left\{ \big(\boldsymbol{x}, f(|\boldsymbol{x}|)\big) \in \mathbb{R}^3 \colon \boldsymbol{x} \in \mathbb{R}^2 \right\}.$$

The principal curvatures of Σ will be denoted by κ_1, κ_2. Note that the usual paraboloid corresponds to the choice $\alpha = 2$ and $R = 0$ in the above definition.

The configuration space of our problem is a fixed-width layer built over Σ, in other words, the a-tubular neighborhood of Σ defined as[1]

$$(1.2) \qquad \mathcal{P} = \mathcal{P}(f, a) := \left\{ \boldsymbol{x} \in \mathbb{R}^3 \colon \operatorname{dist}\big(\boldsymbol{x}, \Sigma(f)\big) < a \right\}.$$

The domain $\mathcal{P} = \mathcal{P}(f, a)$ will be called generalized radially symmetric parabolic layer, for brevity we will speak of a *generalized parabolic layer* as there is no danger of confusion.

In the following, $\nu(\boldsymbol{x})$ stands for the unit normal vector to the surface Σ at the point $(\boldsymbol{x}, f(|\boldsymbol{x}|))$. The mapping $\mathbb{R}^2 \ni \boldsymbol{x} \mapsto \nu(\boldsymbol{x})$ is smooth and the orientation of $\nu(\boldsymbol{x})$ is fixed so that its projection on the x_3-axis is positive for $|\boldsymbol{x}| \geq R$.

In Subsection 2.3, we show the existence of a constant $a_\star = a_\star(f) > 0$ satisfying $a_\star \|\kappa_j\|_\infty < 1$ for $j = 1, 2$ such that the restriction of the map

$$(1.3) \qquad \mathcal{L} \colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^3, \qquad \mathcal{L}(\boldsymbol{x}, u) = (\boldsymbol{x}, f(|\boldsymbol{x}|))^\top + \nu(\boldsymbol{x})u,$$

onto $\mathbb{R}^2 \times \mathbb{I}_a$ is injective for all $a < a_\star$. Consequently, the generalized parabolic layer of a half-width $a < a_\star$ can alternatively be represented as the image of $\mathbb{R}^2 \times \mathbb{I}_a$ under the map $\mathcal{L}$. Everywhere in this paper we consider only the generalized parabolic layers the half-width of which satisfies the condition $a < a_\star$.

1.3. **Definition of the operator**

We define a non-negative, symmetric quadratic form on the Hilbert space $L^2(\mathcal{P})$ by

$$(1.4) \qquad \mathfrak{Q}[\psi] := \|\nabla \psi\|^2_{L^2(\mathcal{P}; \mathbb{C}^3)}, \qquad \operatorname{dom} \mathfrak{Q} := H^1_0(\mathcal{P}).$$

According to [Dav95, Lem. 6.1.1 and Lem. 6.1.3], the quadratic form $\mathfrak{Q}$ is closed and densely defined. This allows us to introduce the main object of this paper, namely, the self-adjoint operator H in $L^2(\mathcal{P})$ associated with the form $\mathfrak{Q}$ via the first representation theorem [Ka, Thm. VI 2.1]. The operator H is nothing but the *Dirichlet Laplacian* on the

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[1]$\operatorname{dist}(\boldsymbol{x}, E) := \inf_{\boldsymbol{y} \in E} |\boldsymbol{x} - \boldsymbol{y}|_{\mathbb{R}^d}$ is the distance between a point $\boldsymbol{x} \in \mathbb{R}^d$ and a set $E \subset \mathbb{R}^d$.

generalized parabolic layer $\mathcal{P}$, the name referring to the fact that the operator H acts as follows,

$$(1.5) \qquad \mathsf{H}\psi = -\Delta\psi, \qquad \operatorname{dom}\mathsf{H} = \big\{\psi \in H_0^1(\mathcal{P})\colon \Delta\psi \in L^2(\mathcal{P})\big\}.$$

1.4. **Notation and the main result**

Before stating the main result of this paper, we introduce a few further notations. For measurable functions $\mathsf{f}_1, \mathsf{f}_2\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{E\to 0+}\mathsf{f}_j(E) = +\infty$, $j = 1, 2$, and $\lim_{E\to 0+}\frac{\mathsf{f}_1(E)}{\mathsf{f}_2(E)} = 1$, we agree to write

$$\mathsf{f}_1(E) \underset{E\searrow 0}{\sim} \mathsf{f}_2(E).$$

Furthermore, let T be the semi-bounded self-adjoint operator associated with a quadratic form $\mathfrak{T}$. We denote by $\sigma_{\mathrm{ess}}(\mathsf{T})$ and $\sigma_{\mathrm{d}}(\mathsf{T})$ the essential and the discrete spectrum of T, respectively; by $\sigma(\mathsf{T})$ we denote the (entire) spectrum of T, *i.e.* $\sigma(\mathsf{T}) = \sigma_{\mathrm{ess}}(\mathsf{T}) \cup \sigma_{\mathrm{d}}(\mathsf{T})$. We set $E_\infty(\mathsf{T}) := \inf \sigma_{\mathrm{ess}}(\mathsf{T})$ and, for a given $j \in \mathbb{N}$, $E_j(\mathsf{T})$ denotes the jth-eigenvalue of T in the interval $(-\infty, E_\infty(\mathsf{T}))$. These eigenvalues are arranged in the ascending order and repeated according to their multiplicities. We define the counting function of T as

$$\mathcal{N}_E(\mathsf{T}) := \#\big\{j \in \mathbb{N}\colon E_j(\mathsf{T}) < E\big\}, \qquad E \le E_\infty(\mathsf{T}).$$

When working with the quadratic form $\mathfrak{T}$, we use the notations $\sigma_{\mathrm{ess}}(\mathfrak{T})$, $\sigma_{\mathrm{d}}(\mathfrak{T})$, $\sigma(\mathfrak{T})$, $E_\infty(\mathfrak{T})$, $E_j(\mathfrak{T})$, and $\mathcal{N}_E(\mathfrak{T})$ instead. If it is clear from the context, we will not indicate the dependence of the eigenvalues and the threshold of the essential spectrum on the operator, *i.e.* we will write E_j and E_∞ instead of $E_j(\mathsf{T})$ and $E_\infty(\mathsf{T})$, respectively.

First, we recall spectral properties of the Hamiltonian H, which follow from the previously known general results after checking the appropriate geometric assumptions.

Proposition 1.1. *For any $a \in (0, a_\star)$, the following claims hold.*

(i) $\sigma_{\mathrm{ess}}(\mathsf{H}) = [E_\infty, \infty)$, *where* $E_\infty = \left(\frac{\pi}{2a}\right)^2$.

(ii) $\#\sigma_{\mathrm{d}}(\mathsf{H}) = \infty$.

To be specific, one has to verify that both the principal curvatures of Σ are bounded, and that the mean and the Gauss curvatures of Σ vanish at infinity, then the item (i) of the above proposition follows from [EK, Prop. 4.2.1]; see also [KL14]. In addition, checking that the total Gauss curvature of Σ does not vanish and making use of the rotational symmetry for $\mathcal{P}$ we derive item (ii) of the above proposition from [EK, Cor. 4.2.2].

In this paper we give an alternative proof of the fact that $\#\sigma_{\mathrm{d}}(\mathsf{H}) = \infty$ and, what is more important, we obtain as the main result the spectral asymptotics of H.

Theorem 1.2. *For any $a \in (0, a_\star)$, the counting function of H satisfies*

$$\mathcal{N}_{E_\infty - E}(\mathsf{H}) \underset{E\searrow 0}{\sim} \frac{1}{2\pi}\frac{\alpha k}{2^\alpha}\frac{\mathsf{B}\left(\frac{3}{2}, \frac{\alpha}{2} - \frac{1}{2}\right)}{E^{\frac{\alpha}{2} - \frac{1}{2}}},$$

where $\mathsf{B}(\cdot, \cdot)$ is the Euler beta function.

Remark 1.3. In the particular case when the layer is 'genuinely' parabolic, $\alpha = 2$, possibly outside a compact, the asymptotic expression on the right-hand side simplifies to $\frac{k}{8\sqrt{E}}$. We note that for larger values of α the accumulation is slower. Moreover, using Stirling formula [AS, 6.1.37] we find that

$$\lim_{\alpha \to \infty} \frac{1}{2\pi} \frac{\alpha k}{2^\alpha} \frac{\mathsf{B}\left(\frac{3}{2}, \frac{\alpha}{2} - \frac{1}{2}\right)}{E^{\frac{\alpha}{2} - \frac{1}{2}}} = \lim_{\alpha \to \infty} \frac{k}{\sqrt{2\pi\alpha} 2^\alpha E^{\frac{\alpha}{2} - \frac{1}{2}}} = \infty$$

holds for any fixed $E < 1/4$. This might formally correspond to the fact that as α grows, the layer is becoming closer to the one with annular cylindrical end for which the essential spectrum extends below E_∞; *cf.* [EK, Prob. 4.10]. On the other hand, for the (possibly locally deformed) conical layer, $\alpha = 1$, where the discrete spectrum is also infinite [ET10] one might ask whether the accumulation rate is obtained, upon passing to the limit $\alpha \to 1^+$ in the above asymptotic expression, but this not the case as it is obvious from the fact that the limit $\frac{k}{4\pi}$ is independent of E. The correct spectral asymptotics for the conical layer [DOR15, Thm. 1.4] is given by $\mathcal{N}_{E_\infty - E}(\mathsf{H}) \sim \frac{k}{4\pi} |\ln E|$ as $E \to 0^+$. Thus, an abrupt transition from the power-type accumulation rate of eigenvalues to an exponential one occurs upon switching from $\alpha > 1$ to $\alpha = 1$.

In the proof of Theorem 1.2 we first proceed as in [DOR15, Thm. 1.4]. Namely, we decompose the Hamiltonian H into an infinite orthogonal sum of the fiber operators, acting on the two-dimensional meridian domain each. In our geometric setting, the meridian domain is the generalized parabolic half-strip. Such a reduction is possible in view of the rotational invariance of the domain $\mathcal{P}$ with respect to the x_3-axis. The 's-wave' fiber corresponding to radially symmetric functions turns out to produce an infinite discrete spectrum below the threshold E_∞, while the remaining fibers in the decomposition of H altogether produce at most a finite number of eigenvalues below E_∞, and thus they do not influence the accumulation rate in the vicinity of the threshold. Recall in this respect the conical layer case, where one is able to make a stronger claim, namely that *none* of the fibers except the 's-wave' one contributes to the discrete spectrum.

The spectral analysis of the 's-wave' fiber is reduced to the well-known properties of the one-dimensional Schrödinger operators corresponding to the differential expression $-\frac{d^2}{ds^2} + q$ on the positive semi-axis, where the bounded potential $q: \mathbb{R}_+ \to \mathbb{R}$ asymptotically behaves as

$$q(s) \sim -\frac{1}{4} \left(\frac{k}{s}\right)^{\frac{2}{\alpha}} \quad \text{as} \quad s \to +\infty.$$

Note that for the conventional paraboloid ($\alpha = 2$), the potential q behaves at infinity as the Coulomb potential. This reduction is the core of the paper. As an intermediate step, it contains straightening of the underlying generalized parabolic half-strip in suitably chosen curvilinear coordinates, in the spirit of [EŠ89, DE95]; see also [EK, Chap. 1]. This step is new compared to the conical layer, where the meridian domain is straight right from the beginning.

2. **Preliminaries**

In this preparatory section we collect the auxiliary material, which is needed in the proof of the main result. First, in Subsection 2.1, we provide the standard decomposition of the self-adjoint operator H into fibers. Furthermore, in Subsection 2.2 we derive the arc-length parametrization for the graph of $f(\cdot)$, define its signed curvature and introduce curvilinear coordinates in the associated generalized parabolic strip. Then, in Subsection 2.3, we check that the generalized paraboloid Σ satisfies some of the assumptions in [EK, Chap. 4] obtaining in this way a proof of Proposition 1.1. Finally, we recall in Subsection 2.4 several standard spectral results on a class of one-dimensional Schrödinger operators we will need in the following.

2.1. **The fiber decomposition of** H

In view of the rotational symmetry, it is convenient to express the Hamiltonian H in the cylindrical coordinates. Let $\mathbb{R}_+^2$ be the positive half-plane $\mathbb{R}_+ \times \mathbb{R}$. We consider the cylindrical coordinates $(r, z, \theta) \in \mathbb{R}_+^2 \times \mathbb{S}^1$ linked with the Cartesian coordinates (x_1, x_2, x_3) via the following conventional relations

$$(2.1) \qquad x_1 = r\cos\theta, \qquad x_2 = r\sin\theta, \qquad x_3 = z.$$

We denote the graph of the function $x \mapsto f(x)$ by

$$(2.2) \qquad \Gamma = \Gamma(f) := \big\{(x, f(x))\colon x > 0\big\} \subset \mathbb{R}_+^2.$$

In order to describe in cylindrical coordinates the generalized parabolic layer $\mathcal{P} = \mathcal{P}(f, a)$ of the half-width $a \in (0, a_\star)$, we first define its meridian domain $\mathcal{G} = \mathcal{G}(f, a) \subset \mathbb{R}_+^2$ by

$$\mathcal{G} = \mathcal{G}(f, a) := \big\{\boldsymbol{x} \in \mathbb{R}_+^2\colon \operatorname{dist}\big(\boldsymbol{x}, \Gamma(f)\big) < a\big\}.$$

Note that the domain $\mathcal{G}$ can be viewed as a generalized parabolic half-strip. The generalized parabolic layer $\mathcal{P}$ in (1.2) is defined in the cylindrical coordinates (2.1) by

$$(2.3) \qquad \mathcal{P} = \mathcal{P}(f, a) := \mathcal{G}(f, a) \times \mathbb{S}^1.$$

The layer $\mathcal{P}$ can be seen as a sub-domain of $\mathbb{R}^3$ constructed via rotation of the meridian domain $\mathcal{G}$ around the x_3-axis. For later purposes, we split the boundary $\partial\mathcal{G}$ of $\mathcal{G}$ into two disjoint parts,

$$\partial_0\mathcal{G} := \{0\} \times \mathbb{I}_a \qquad \text{and} \qquad \partial_1\mathcal{G} := \partial\mathcal{G} \setminus \overline{\partial_0\mathcal{G}}.$$

Next, we introduce the usual cylindrical L^2-space and the first order cylindrical Sobolev space on $\mathcal{P}$ as

$$L_{\mathrm{cyl}}^2(\mathcal{P}) := L^2(\mathcal{G} \times \mathbb{S}^1; r\mathrm{d}r\mathrm{d}z\mathrm{d}\theta), \qquad H_{\mathrm{cyl}}^1(\mathcal{P}) := \big\{\psi\colon \psi, \partial_r\psi, \partial_z\psi, r^{-1}\partial_\theta\psi \in L_{\mathrm{cyl}}^2(\mathcal{P})\big\}.$$

The space $H_{\mathrm{cyl}}^1(\mathcal{P})$ is endowed with the norm $\|\cdot\|_{H_{\mathrm{cyl}}^1(\mathcal{P})}$, defined for all $\psi \in H_{\mathrm{cyl}}^1(\mathcal{P})$ by

$$\|\psi\|_{H_{\mathrm{cyl}}^1(\mathcal{P})}^2 := \|\psi\|_{L_{\mathrm{cyl}}^2(\mathcal{P})}^2 + \int_{\mathcal{P}} \left(|\partial_r\psi|^2 + |\partial_z\psi|^2 + \frac{|\partial_\theta\psi|^2}{r^2}\right) r\mathrm{d}r\mathrm{d}z\mathrm{d}\theta.$$

The expression for the quadratic form $\mathfrak{Q}$ associated with H can be rewritten in the cylindrical coordinates as follows

$$\mathfrak{Q}_{\mathrm{cyl}}[\psi] = \int_{\mathbb{S}^1} \int_{\mathcal{G}} \left(|\partial_r \psi|^2 + |\partial_z \psi|^2 + \frac{|\partial_\theta \psi|^2}{r^2} \right) r \mathrm{d}r \mathrm{d}z \mathrm{d}\theta.$$

Following the strategy of [DOR15, KLO17, ET10], we consider an orthonormal basis of the Hilbert space $L^2(\mathbb{S}^1)$ given by

$$(2.4) \qquad\qquad v_m(\theta) = (2\pi)^{-1/2} e^{\mathrm{i}m\theta}, \qquad m \in \mathbb{Z}.$$

For any $m \in \mathbb{Z}$, we define the mapping

$$(2.5) \qquad \Pi_m \colon L^2_{\mathrm{cyl}}(\mathcal{P}) \to L^2(\mathcal{G}; r\mathrm{d}r\mathrm{d}z), \qquad (\Pi_m \psi)(r, z, \theta) = \big(\psi(r, z, \cdot), v_m\big)_{L^2(\mathbb{S}^1)}.$$

Performing a 'partial wave' decomposition [RS-II, App. to X.1], see also [DOR15, KLO17, LO16], with respect to the basis in (2.4), we obtain

$$(2.6) \qquad\qquad \mathsf{H} \cong \bigoplus_{m \in \mathbb{Z}} \mathsf{F}_m,$$

where the symbol $\cong$ stands for the unitary equivalence relation and, for all $m \in \mathbb{Z}$, the operators F_m acting on $L^2(\mathcal{G}; r\mathrm{d}r\mathrm{d}z)$ are called the *fibers* of H. They are associated through the first representation theorem with closed, densely defined, symmetric, and non-negative quadratic forms in the Hilbert space $L^2(\mathcal{G}; r\mathrm{d}r\mathrm{d}z)$

$$(2.7) \qquad \begin{aligned} \mathfrak{F}_m[\psi] &:= \mathfrak{Q}_{\mathrm{cyl}}[\psi \otimes v_m] = \int_{\mathcal{G}} \left(|\partial_r \psi|^2 + |\partial_z \psi|^2 + \frac{m^2}{r^2}|\psi|^2 \right) r \mathrm{d}r \mathrm{d}z, \\ \mathrm{dom}\, \mathfrak{F}_m &:= \Pi_m\big(H_0^1(\mathcal{P})\big). \end{aligned}$$

The domain of the operator F_m can be deduced from the quadratic form $\mathfrak{F}_m$ in the standard way via the first representation theorem.

Next, we introduce the unitary operator

$$\mathsf{U} \colon L^2(\mathcal{G}; r\mathrm{d}r\mathrm{d}z) \to L^2(\mathcal{G}), \qquad \mathsf{U}\psi := \sqrt{r}\psi.$$

This unitary operator allows us to transform the quadratic forms $\mathfrak{F}_m$ into equivalent ones expressed in a flat metric. Indeed, the quadratic form $\mathfrak{F}_m$ is unitarily equivalent via U to the form in the Hilbert space $L^2(\mathcal{G})$ defined as

$$(2.8) \qquad \begin{aligned} \mathfrak{f}_m[\psi] &:= \mathfrak{F}_m[\mathsf{U}^{-1}\psi] = \int_{\mathcal{G}} \left(|\partial_r \psi|^2 + |\partial_z \psi|^2 + \frac{m^2 - \frac{1}{4}}{r^2}|\psi|^2 \right) \mathrm{d}r\mathrm{d}z, \\ \mathrm{dom}\, \mathfrak{f}_m &:= \mathsf{U}(\mathrm{dom}\, \mathfrak{F}_m). \end{aligned}$$

In fact, it is not difficult to check that $\mathcal{C}_0^\infty(\mathcal{G})$ is a core for the form $\mathfrak{f}_m$ and that for all $m \neq 0$ its form domain satisfies

$$(2.9) \qquad\qquad \mathrm{dom}\, \mathfrak{f}_m = \mathsf{U}(\mathrm{dom}\, \mathfrak{F}_m) = H_0^1(\mathcal{G}).$$

Finally, for the sake of convenience, we introduce for $m \in \mathbb{Z}$ the following shorthand notation for frequently used integrands:

$$(2.10)\ \ \mathcal{E}_m[\psi] := |\partial_r \psi|^2 + |\partial_z \psi|^2 + \frac{m^2}{r^2}|\psi|^2 \qquad \text{and} \qquad \mathcal{F}_m[\psi] := |\partial_r \psi|^2 + |\partial_z \psi|^2 + \frac{m^2 - \frac{1}{4}}{r^2}|\psi|^2.$$

2.2. The arc-length parametrization of Γ and associated coordinates on $\mathcal{G}$

The arc-length parametrization for the curve $\Gamma = \Gamma(f)$ in (2.2) is given by

$$\overline{\mathbb{R}}_+ \ni s \mapsto \big(\phi(s), f(\phi(s))\big) \in \mathbb{R}^2,$$

where $\phi \colon \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is a monotonously increasing function such that $\phi(0) = 0$ and that $\lim_{s \to \infty} \phi(s) = \infty$, and which satisfies the ordinary differential equation

$$(2.11) \qquad \dot{\phi}^2 \big(1 + \dot{f}^2(\phi)\big) = 1.$$

On the interval $[R, \infty)$, the above equation reduces to $\dot{\phi}^2 \big(1 + \alpha^2 k^2 \phi^{2\alpha-2}\big) = 1$. The equation (2.11) combined with $\phi(0) = 0$ and $\dot{f}(0) = 0$ immediately implies

$$(2.12) \qquad \dot{\phi}(0) = 1 \qquad \text{and} \qquad 0 < \dot{\phi} \le 1.$$

The signed curvature of Γ is given by

$$(2.13) \qquad \gamma = \big(\dot{f}(\phi)\ddot{\phi} + \ddot{f}(\phi)\dot{\phi}^2\big)\dot{\phi} - \dot{f}(\phi)\dot{\phi}\ddot{\phi} = \ddot{f}(\phi)\dot{\phi}^3.$$

On the interval $[R, \infty)$, the above expression simplifies to $\gamma = \alpha(\alpha - 1)k\dot{\phi}^3\phi^{\alpha-2}$. Asymptotic properties of ϕ and γ that will be needed in the following are obtained in Appendix A.

The unit tangential vector to Γ at the point $(\phi(s), f(\phi(s)))$ is given by $(\dot{\phi}(s), \dot{f}(\phi(s))\dot{\phi}(s))^\top$. Consequently, the unit normal vector to Γ at the same point can be expressed as

$$(2.14) \qquad \nu_\Gamma(s) = \big(-\dot{f}(\phi(s))\dot{\phi}(s), \dot{\phi}(s)\big)^\top.$$

Next, on the half-strip $\Omega := \mathbb{R}_+ \times \mathbb{I}_a$, we introduce the map $\tau \colon \Omega \to \mathcal{G}$ by

$$(2.15) \qquad \tau(s, u) = \begin{pmatrix} \tau_1(s, u) \\ \tau_2(s, u) \end{pmatrix} := \begin{pmatrix} \phi(s) \\ f(\phi(s)) \end{pmatrix} + u\nu_\Gamma(s) = \begin{pmatrix} \phi(s) - u\dot{f}(\phi(s))\dot{\phi}(s) \\ f(\phi(s)) + u\dot{\phi}(s) \end{pmatrix}.$$

This map defines convenient curvilinear coordinates (s, u) on the generalized parabolic strip $\mathcal{G}$. Note also that the Jacobian $\mathcal{J} = \mathcal{J}(s, u)$ of the map τ is given by

$$(2.16) \qquad \begin{aligned} \mathcal{J} &= \begin{vmatrix} \partial_s \tau_1 & \partial_u \tau_1 \\ \partial_s \tau_2 & \partial_u \tau_2 \end{vmatrix} = \begin{vmatrix} \dot{\phi} - u\ddot{f}(\phi)\dot{\phi}^2 - u\dot{f}(\phi)\ddot{\phi} & -\dot{f}(\phi)\dot{\phi} \\ \dot{f}(\phi)\dot{\phi} + u\ddot{\phi} & \dot{\phi} \end{vmatrix} \\ &= \dot{\phi}^2 - u\ddot{f}(\phi)\dot{\phi}^3 + \dot{f}^2(\phi)\dot{\phi}^2 = 1 - u\gamma, \end{aligned}$$

where (2.11) was used in the last step. We also set

$$(2.17) \qquad g(s, u) := (\mathcal{J}(s, u))^2 = (1 - u\gamma(s))^2.$$

2.3. Parametrization and curvatures of Σ

In this subsection we provide the natural parametrization of Σ as a surface of revolution in the sense [dC, §3.3, Example 4]. Using this parametrization we obtain formulæ for

the curvatures of Σ and a parametrization for the generalized parabolic layer $\mathcal{P}$. Finally, we check the assumptions on layers [EK, Chap. 4] and eventually prove Proposition 1.1.

The generalized paraboloid Σ can be alternatively parametrized as

$$(2.18) \qquad \mathbb{R}_+ \times \mathbb{S}^1 \ni (s,\theta) \mapsto \sigma(s,\theta) := \big(\phi(s)\cos\theta, \phi(s)\sin\theta, f(\phi(s))\big)^\top,$$

where the functions f and ϕ are as in Subsection 1.2 and Subsection 2.2, respectively. According to [dC, §3.3, Example 4], see also [EK, §4.2.2], the principal curvatures of Σ at the point $x = \sigma(s,\theta)$ are explicitly given by

$$(2.19) \qquad \begin{aligned} \kappa_1(s,\theta) &= -\frac{\dot{f}(\phi(s))\dot{\phi}(s)}{\phi(s)} = -\frac{\dot{f}(\phi(s))}{\phi(s)(1+\dot{f}^2(\phi(s)))^{1/2}}, \\ \kappa_2(s,\theta) &= -\gamma(s) = -\ddot{f}(\phi(s))\dot{\phi}^3(s) = -\frac{\ddot{f}(\phi(s))}{\big(1+\dot{f}^2(\phi(s))\big)^{3/2}}. \end{aligned}$$

Hence, the mean $M = \frac{1}{2}(\kappa_1 + \kappa_2)$ and the Gauss $K = \kappa_1\kappa_2$ curvatures of Σ can be evaluated as

$$(2.20) \qquad \begin{aligned} M(s,\theta) &= -\frac{\dot{f}(\phi(s)) + \dot{f}^3(\phi(s)) + \ddot{f}(\phi(s))\phi(s)}{2\phi(s)(1+\dot{f}^2(\phi(s)))^{3/2}}, \\ K(s,\theta) &= \frac{\dot{f}(\phi(s))\ddot{f}(\phi(s))}{\phi(s)(1+\dot{f}^2(\phi(s)))^2}. \end{aligned}$$

It follows from [EK, Sec. 4.2.2], using $\lim_{s\to\infty}\dot{\phi}(s) = 0$ (*cf.* Proposition A.2 (ii)) that the total Gauss curvature of Σ is given by $\mathcal{K} := \int_\Sigma K = 2\pi$. Furthermore, the normal vector $\nu_\Sigma(s,\theta)$ to Σ at $x = \sigma(s,\theta)$ with $(s,\theta) \in \mathbb{R}_+ \times \mathbb{S}^1$ can be expressed as

$$(2.21) \qquad \nu_\Sigma(s,\theta) = \big(-\dot{f}(\phi(s))\dot{\phi}(s)\cos\theta, -\dot{f}(\phi(s))\dot{\phi}(s)\sin\theta, \dot{\phi}(s)\big)^\top,$$

thus the layer $\mathcal{P}$ is the image of $\mathbb{R}_+ \times \mathbb{S}^1 \times \mathbb{I}_a$ under the map

$$(2.22) \quad \mathbb{R}_+ \times \mathbb{S}^1 \times \mathbb{R} \ni (s,\theta,u) \mapsto \pi(s,\theta,u) := \big(\phi(s)\cos\theta, \phi(s)\sin\theta, f(\phi(s))\big)^\top + u\nu_\Sigma(s,\theta).$$

Now, we have all the tools to prove Proposition 1.1.

Proof of Proposition 1.1. First, recall that Σ is a surface of revolution with a non-zero total Gauss curvature $\mathcal{K} = 2\pi$. Furthermore, we immediately notice that both the principal curvatures $\kappa_j(s,\theta)$, $j = 1,2$, of Σ computed in (2.19) are $\mathcal{C}^\infty$-smooth on $\overline{\mathbb{R}}_+ \times \mathbb{S}^1$, pointwise positive, and vanish in the limit $s \to \infty$; *cf.* Lemma A.1 and Proposition A.3 (i). Consequently, the minimal normal curvature radius of Σ defined in [EK, Chap. 4, Assumption (ii)] and given by

$$\rho_{\mathrm{m}} := \big(\max\{\|\kappa_1\|_\infty, \|\kappa_2\|_\infty\}\big)^{-1}$$

is positive and finite. In addition, the layer $\mathcal{P}$ is asymptotically planar in the sense of [EK, §4.1.2, Assumption (iii)], because both $M(s,\theta)$ and $K(s,\theta)$ vanish in the limit $s \to \infty$.

The remaining step is more technical: we have to check that there exists $a_\star \in (0,\rho_{\mathrm{m}})$ such that the generalized parabolic layer $\mathcal{P} = \mathcal{P}(f,a)$ in (1.2) is not self-intersecting for

all $a \in (0, a_\star)$ in the sense of [EK, Chap. 4, Assumption (i)]. The latter is equivalent to showing that the restriction of the map $\pi(\cdot)$ in (2.22) onto $\mathbb{R}_+ \times \mathbb{S}^1 \times \mathbb{I}_a$ is injective for all $a \in (0, a_\star)$. Upon checking this last assumption, [EK, Thm. 4.4] yields $\sigma_{\mathrm{ess}}(\mathsf{H}) = [E_\infty, \infty)$ and $\#\sigma_{\mathrm{d}}(\mathsf{H}) = \infty$.

We will proceed by *reductio ad absurdum*. If such an $a_\star > 0$ did not exist, then it would be possible to find $(s_j, \theta_j, u_j) \in \mathbb{R}_+ \times \mathbb{S}^1 \times \mathbb{I}_{\rho_m}$, $j = 1, 2$, such that $(s_1, \theta_1, u_1) \neq (s_2, \theta_2, u_2)$, $s_1 \leq s_2$, and at the same time $\pi(s_1, \theta_1, u_1) = \pi(s_2, \theta_2, u_2)$ with $u_1, u_2 \in (-\rho_{\mathrm{m}}, \rho_{\mathrm{m}})$ having arbitrarily small absolute values. Without loss of generality we can assume that both u_1 and u_2 are positive, because the surface Σ splits $\mathbb{R}^3$ into its hypograph and epigraph, respectively. Using the explicit expression of the normal vector in (2.21) together with the fact that $\pi(s_1, \theta_1, u_1) - \pi(s_2, \theta_2, u_2)$ has trivial projections onto all the axes, we get

$$(2.23) \quad \begin{cases} \dot{f}(\phi(s_1)) \cos\theta_1 c_1 - \dot{f}(\phi(s_2)) \cos\theta_2 c_2 = \phi(s_1) \cos\theta_1 - \phi(s_2) \cos\theta_2, \\ \dot{f}(\phi(s_1)) \sin\theta_1 c_1 - \dot{f}(\phi(s_2)) \sin\theta_2 c_2 = \phi(s_1) \sin\theta_1 - \phi(s_2) \sin\theta_2, \\ c_1 - c_2 = f(\phi(s_2)) - f(\phi(s_1)), \end{cases}$$

where $c_j := u_j \dot{\phi}(s_j)$ for $j = 1, 2$. The above system of linear equations with respect to c_1 and c_2 has a non-trivial solution if, and only if

$$\Delta := \begin{vmatrix} 1 & -1 & f(\phi(s_2)) - f(\phi(s_1)) \\ \dot{f}(\phi(s_1)) \cos\theta_1 & -\dot{f}(\phi(s_2)) \cos\theta_2 & \phi(s_1) \cos\theta_1 - \phi(s_2) \cos\theta_2 \\ \dot{f}(\phi(s_1)) \sin\theta_1 & -\dot{f}(\phi(s_2)) \sin\theta_2 & \phi(s_1) \sin\theta_1 - \phi(s_2) \sin\theta_2 \end{vmatrix} = 0.$$

The determinant Δ can be explicitly computed, giving

$$\Delta = \begin{vmatrix} -\dot{f}(\phi(s_2)) \cos\theta_2 & \phi(s_1) \cos\theta_1 - \phi(s_2) \cos\theta_2 \\ -\dot{f}(\phi(s_2)) \sin\theta_2 & \phi(s_1) \sin\theta_1 - \phi(s_2) \sin\theta_2 \end{vmatrix}$$

$$+ \begin{vmatrix} \dot{f}(\phi(s_1)) \cos\theta_1 & \phi(s_1) \cos\theta_1 - \phi(s_2) \cos\theta_2 \\ \dot{f}(\phi(s_1)) \sin\theta_1 & \phi(s_1) \sin\theta_1 - \phi(s_2) \sin\theta_2 \end{vmatrix}$$

$$+ \left(f(\phi(s_2)) - f(\phi(s_1)) \right) \begin{vmatrix} \dot{f}(\phi(s_1)) \cos\theta_1 & -\dot{f}(\phi(s_2)) \cos\theta_2 \\ \dot{f}(\phi(s_1)) \sin\theta_1 & -\dot{f}(\phi(s_2)) \sin\theta_2 \end{vmatrix}$$

$$= \sin(\theta_2 - \theta_1) \left(\dot{f}(\phi(s_2))\phi(s_1) - \dot{f}(\phi(s_1))\phi(s_2) + \dot{f}(\phi(s_1))\dot{f}(\phi(s_2)) \left[f(\phi(s_1)) - f(\phi(s_2)) \right] \right).$$

Hence, the requirement $\Delta = 0$ can only be satisfied in two disjoint cases:

(i) $\theta := \theta_1 = \theta_2$ and $s_1 < s_2$;

(ii) $\theta_1 \neq \theta_2$ and $\dot{f}(\phi(s_2))\phi(s_1) - \dot{f}(\phi(s_1))\phi(s_2) + \dot{f}(\phi(s_1))\dot{f}(\phi(s_2)) \left[f(\phi(s_1)) - f(\phi(s_2)) \right] = 0$.

These cases require separate analysis.

Case (i). Pick $R_1 > R$ such that $\alpha k \phi(R_1)^{\alpha-1} > 1$. There are two sub-cases to distinguish. If $s_1 < R_1$, then for simple geometric reasons there is a constant $R_2 = R_2(f, R_1) > R_1$ such that necessarily $s_2 < R_2$ holds. Hence, [BEHL16, Prop. B.2], applied for the image of $(0, R_2) \times \mathbb{S}^1$ under the mapping in (2.18), implies that there exists $a_1 \in (0, \rho_{\mathrm{m}})$ such that the condition $u_1, u_2 \in (0, a_1)$ is incompatible with $s_1 < s_2$.

In the second sub-case ($s_1 \geq R_1$), we reduce the linear system (2.23) to

$$\begin{cases} \dot{f}(\phi(s_1))c_1 - \dot{f}(\phi(s_2))c_2 = \phi(s_1) - \phi(s_2), \\ c_1 - c_2 = f(\phi(s_2)) - f(\phi(s_1)). \end{cases}$$

Solving it and returning to the initial variables u_j, $j = 1, 2$, we find

$$u_1 = \frac{\phi(s_1) - \phi(s_2) + \dot{f}(\phi(s_2))[f(\phi(s_2)) - f(\phi(s_1))]}{\dot{\phi}(s_1)[\dot{f}(\phi(s_2)) - \dot{f}(\phi(s_1))]},$$

$$u_2 = \frac{\phi(s_1) - \phi(s_2) + \dot{f}(\phi(s_1))[f(\phi(s_2)) - f(\phi(s_1))]}{\dot{\phi}(s_2)[\dot{f}(\phi(s_2)) - \dot{f}(\phi(s_1))]}.$$

For the sake of brevity, set $x_1 := \phi(R_1)$. We estimate u_1 using $\dot{\phi} \leq 1$ and applying Cauchy's mean-value theorem

$$u_1 \geq \inf_{x \in [x_1, \infty)} \frac{\dot{f}(x_1)\dot{f}(x) - 1}{\ddot{f}(x)} = \inf_{x \in [x_1, \infty)} \frac{\alpha^2 k^2 (xx_1)^{\alpha-1} - 1}{\alpha(\alpha-1)kx^{\alpha-2}}$$

$$\geq \inf_{x \in [x_1, \infty)} \frac{x(\alpha^2 k^2 x_1^{\alpha-1} - x^{1-\alpha})}{\alpha(\alpha-1)k} = \frac{x_1(\alpha^2 k^2 x_1^{\alpha-1} - x_1^{1-\alpha})}{\alpha(\alpha-1)k} > 0.$$

Hence, there exists $a_2 \in (0, \rho_{\mathrm{m}})$ such that $u_1, u_2 \in (0, a_2)$ is incompatible with $s_1 < s_2$.

Case (ii). The linear system (2.23) reduces to

(2.24) $$\begin{cases} \dot{f}(\phi(s_1))\cos\theta_1 c_1 - \dot{f}(\phi(s_2))\cos\theta_2 c_2 = \phi(s_1)\cos\theta_1 - \phi(s_2)\cos\theta_2, \\ \dot{f}(\phi(s_1))\sin\theta_1 c_1 - \dot{f}(\phi(s_2))\sin\theta_2 c_2 = \phi(s_1)\sin\theta_1 - \phi(s_2)\sin\theta_2. \end{cases}$$

Solving it and returning to the initial variables u_j, $j = 1, 2$, we find

$$u_j = \frac{\phi(s_j)}{\dot{\phi}(s_j)\dot{f}(\phi(s_j))}, \qquad j = 1, 2.$$

In view of Lemma A.1, we conclude that there exists $a_3 \in (0, \rho_{\mathrm{m}})$ such that the condition $u_1, u_2 \in (0, a_3)$ can not be satisfied together with $\theta_1 \neq \theta_2$.

Merging the outcome of the analysis for the Cases (i) and (ii), we get the claim for $a_\star = \min\{a_1, a_2, a_3\}$. $\qquad\square$

Note that the principal curvatures κ_1, κ_2 of Σ computed in (2.19) do not depend on θ. Using this observation we introduce the functions

(2.25)
$$\xi_p := a \sup_{s \in [p,\infty)} \max\{|\kappa_1(s)|, |\kappa_2(s)|\} = a \sup_{s \in [p,\infty)} \max\left\{|\gamma(s)|, \frac{|\dot{f}(\phi(s))|\dot{\phi}(s)}{\phi(s)}\right\},$$

$$\zeta_p := \left(\frac{1 - \xi_p}{1 + \xi_p}\right)^2.$$

In view of the condition $a\|\kappa_j\|_\infty < 1$ for $j = 1, 2$, Lemma A.1 and Proposition A.3 (i) yield that the functions $\overline{\mathbb{R}}_+ \ni p \mapsto \xi_p, \zeta_p$ satisfy $0 < \xi_p < 1$, $\lim_{p\to\infty} \xi_p = 0$, and $\lim_{p\to\infty} \zeta_p = 1$.

2.4. **Spectral properties of auxiliary one-dimensional Schrödinger operators**

Consider the one-dimensional Schrödinger operator

$$(2.26) \qquad \mathsf{h}_q\psi = -\psi'' + q\psi, \qquad \operatorname{dom} \mathsf{h}_q = H^2(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+),$$

acting in the Hilbert space $L^2(\mathbb{R}_+)$ with a real-valued potential $q \in L^\infty(\mathbb{R}_+)$. The operator h_q is obviously self-adjoint, and the corresponding quadratic form will be denoted by $\mathfrak{h}_q$. The next proposition describes relations between the spectral properties of h_q and the behavior of the potential $q(s)$ as $s \to \infty$. It covers only the classes of potentials required for the proof of our main result.

Proposition 2.1. *Let the self-adjoint operator* h_q *be as in* (2.26) *with the potential* $q\colon \mathbb{R}_+ \to \mathbb{R}$ *satisfying*

$$(2.27) \qquad |q(s)| \le \frac{C}{(1+s)^\beta} \qquad \text{and} \qquad \lim_{s\to\infty} s^\beta q(s) = -c,$$

with $\beta \in (0, 2]$, $C > 0$, *and* $c \ge 0$. *Then the following claims hold.*

  (i) $\sigma_{\mathrm{ess}}(\mathsf{h}_q) = [0, \infty)$.

 (ii) *For* $\beta = 2$ *and* $c = 0$, *it holds* $\mathcal{N}_0(\mathsf{h}_q) < \infty$.

(iii) *For* $\beta \in (0, 2)$ *and* $c > 0$, *it holds* $\mathcal{N}_0(\mathsf{h}_q) = \infty$ *and*

$$\mathcal{N}_{-E}(\mathsf{h}_q) \underset{E\searrow 0}{\sim} \frac{c^{\frac{1}{\beta}} \mathsf{B}\left(\frac{3}{2}, \frac{1}{\beta} - \frac{1}{2}\right)}{\pi \beta E^{\frac{1}{\beta} - \frac{1}{2}}},$$

   where $\mathsf{B}(\cdot, \cdot)$ *is the Euler beta-function.*

Proof. The claim (i) follows from [Te, Lem. 9.35], because the potential q satisfies the condition

$$\int_n^{n+1} |q(s)| \,\mathsf{d}s \le \int_n^{n+1} \frac{C \,\mathsf{d}s}{(1+s)^\beta} \le \frac{C}{(1+n)^\beta} \to 0, \qquad n \to \infty.$$

The claims (ii) and (iii) follow from [BKRS09, Lem 4.9], see also [RS-IV, Thm. XIII.9 (a)] and [RS-IV, Thm. XIII.82]. We remark that the original claim of [BKRS09, Lem 4.9] is formulated for operators on the full line, but the result for half-line operators follows easily with an additional factor $\frac{1}{2}$ in the asymptotic formula for the eigenvalue counting function. $\qquad\square$

3. **Bracketing lemma**

Pick an arbitrary $p > 0$ and define the line segment $\mathcal{S} \subset \mathcal{G}$ by

$$\mathcal{S} := \tau(\{p\} \times \mathbb{I}_a),$$

where the mapping τ is as in (2.15). For the sake of brevity, we mostly avoid indicating the dependence on p as long as there is no danger of misunderstanding. We define the

open sets $\Omega_0 := (0,p) \times \mathbb{I}_a$ and $\Omega_1 := (p, \infty) \times \mathbb{I}_a$. Clearly, the line segment $\mathcal{S}$ splits the parabolic strip $\mathcal{G}$ into two disjoint sub-domains $\mathcal{G}_j := \tau(\Omega_j)$, $j = 0, 1$. In particular, we have the orthogonal decompositions

$$L^2(\mathcal{G}) = L^2(\mathcal{G}_0) \oplus L^2(\mathcal{G}_1) \qquad \text{and} \qquad L^2(\mathcal{G}; r\mathrm{d}r\mathrm{d}z) = L^2(\mathcal{G}_0; r\mathrm{d}r\mathrm{d}z) \oplus L^2(\mathcal{G}_1; r\mathrm{d}r\mathrm{d}z).$$

Next, we introduce for $m \in \mathbb{Z}$ the following symmetric, densely defined quadratic forms in $L^2(\mathcal{G}_1)$,

(3.1a) $$\mathfrak{f}_{m1}^{\mathrm{N}}[\psi] := \int_{\mathcal{G}_1} \mathcal{F}_m[\psi]\,\mathrm{d}r\mathrm{d}z, \qquad \mathrm{dom}\,\mathfrak{f}_{m1}^{\mathrm{N}} := \{\Psi|_{\mathcal{G}_1} : \Psi \in H_0^1(\mathcal{G})\},$$

(3.1b) $$\mathfrak{f}_{m1}^{\mathrm{D}}[\psi] := \int_{\mathcal{G}_1} \mathcal{F}_m[\psi]\,\mathrm{d}r\mathrm{d}z, \qquad \mathrm{dom}\,\mathfrak{f}_{m1}^{\mathrm{D}} := H_0^1(\mathcal{G}_1),$$

where the integrand $\mathcal{F}_m[\psi]$ is as in (2.10). It is easy to see that both the forms $\mathfrak{f}_{m1}^{\mathrm{N}}$ and $\mathfrak{f}_{m1}^{\mathrm{D}}$ are closed and semi-bounded. Now we are in position to formulate the bracketing lemma which we employ in the following.

Lemma 3.1. *For any $p > 0$, $m \in \mathbb{Z}$, and all $E > 0$ the following claims hold.*

(i) $\mathcal{N}_{E_\infty - E}(\mathfrak{F}_m) \geq \mathcal{N}_{E_\infty - E}(\mathfrak{f}_{m1}^{\mathrm{D}})$.

(ii) $\mathcal{N}_{E_\infty - E}(\mathfrak{F}_m) \leq \mathcal{N}_{E_\infty - E}(\mathfrak{f}_{m1}^{\mathrm{N}}) + C_m^{\mathrm{N}}$ *for some constant* $C_m^{\mathrm{N}} = C_m^{\mathrm{N}}(p) > 0$.

Proof. (i) First, we define the restriction of the quadratic form $\mathfrak{F}_m$, $m \in \mathbb{Z}$, to functions vanishing on $\mathcal{S}$,

(3.2) $$\mathfrak{F}_m^{\mathcal{S}}[\psi] := \mathfrak{F}_m[\psi], \qquad \mathrm{dom}\,\mathfrak{F}_m^{\mathcal{S}} := \{\psi \in \mathrm{dom}\,\mathfrak{F}_m : \psi|_{\mathcal{S}} = 0\}.$$

The quadratic form $\mathfrak{F}_m^{\mathcal{S}}$ can be naturally decomposed into the orthogonal sum $\mathfrak{F}_m^{\mathcal{S}} = \mathfrak{F}_{m0}^{\mathcal{S}} \oplus \mathfrak{F}_{m1}^{\mathcal{S}}$, where the quadratic forms $\mathfrak{F}_{m0}^{\mathcal{S}}$ and $\mathfrak{F}_{m1}^{\mathcal{S}}$ are defined in the Hilbert spaces $L^2(\mathcal{G}_0; r\mathrm{d}r\mathrm{d}z)$ and $L^2(\mathcal{G}_1; r\mathrm{d}r\mathrm{d}z)$, respectively. Using the ordering $\mathfrak{F}_m \prec \mathfrak{F}_m^{\mathcal{S}}$ of the forms and the min-max principle we obtain

$$\mathcal{N}_{E_\infty - E}(\mathfrak{F}_m) \geq \mathcal{N}_{E_\infty - E}(\mathfrak{F}_m^{\mathcal{S}}) = \mathcal{N}_{E_\infty - E}(\mathfrak{F}_{m0}^{\mathcal{S}}) + \mathcal{N}_{E_\infty - E}(\mathfrak{F}_{m1}^{\mathcal{S}}), \qquad \forall\, E > 0.$$

It remains to notice that the form $\mathfrak{F}_{m1}^{\mathcal{S}}$ is unitarily equivalent via

(3.3) $$\mathsf{U}_1 : L^2(\mathcal{G}_1; r\mathrm{d}r\mathrm{d}z) \to L^2(\mathcal{G}_1), \qquad \mathsf{U}_1\psi := \sqrt{r}\psi,$$

to the form $\mathfrak{f}_{m1}^{\mathrm{D}}$ in (3.1b), and therefore the desired inequality in (i) holds.

(ii) Let $\chi_0 : \overline{\mathbb{R}}_+ \to [0,1]$ be a $\mathcal{C}^\infty$-smooth function satisfying $\chi_0(s) = 1$ for all $s < \frac{1}{2}$ and $\chi_0(s) = 0$ for all $s > 1$. Define the function $\chi_1 : \overline{\mathbb{R}}_+ \to [0,1]$ through the identity $\chi_0^2 + \chi_1^2 \equiv 1$ on $\overline{\mathbb{R}}_+$. Employing the curvilinear coordinates (s, u) on the parabolic strip $\mathcal{G}$ we introduce the cut-off functions $\mathcal{X}_{j,p} : \mathcal{G} \to [0,1]$, $j = 0, 1$, by

$$\mathcal{X}_{0,p}(s, u) := \chi_0(p^{-1}s) \qquad \text{and} \qquad \mathcal{X}_{1,p}(s, u) := \chi_1(p^{-1}s), \qquad p > 0.$$

These cut-off functions can also be viewed as functions of the arguments (r, z). Define the expression $X_p \colon \mathcal{G} \to \mathbb{R}_+$ by

$$(3.4) \qquad\qquad X_p := |\nabla \mathcal{X}_{0,p}|^2 + |\nabla \mathcal{X}_{1,p}|^2.$$

Set $\Omega'_0 := (0, p) \times \mathbb{I}_a$ and $\Omega'_1 := (\frac{p}{2}, \infty) \times \mathbb{I}_a$. The parabolic strip $\mathcal{G}$ can be represented as the union of the domains $\mathcal{G}'_0 := \tau(\Omega'_0)$ and $\mathcal{G}'_1 := \tau(\Omega'_1)$, having a non-empty bounded intersection $\mathcal{G}'_{01} := \mathcal{G}'_0 \cap \mathcal{G}'_1$. The expression X_p in (3.4) can be pointwise estimated from above as $X_p \leq 2C_{\mathcal{X}} p^{-2} \mathbb{1}_{\mathcal{G}'_{01}}$ with some constant $C_{\mathcal{X}} > 0$, here $\mathbb{1}_{\mathcal{G}'_{01}}$ is the characteristic function of $\mathcal{G}'_{01}$. Next we define a family of quadratic forms parametrized by $m \in \mathbb{Z}$ and $j = 0, 1$ through

$$\mathfrak{E}^{\mathrm{D}}_{mj}[\psi] := \int_{\mathcal{G}'_j} \mathcal{E}_m[\psi] r \mathrm{d}r \mathrm{d}z - C_{\mathcal{X}} p^{-2} \int_{\mathcal{G}'_{01}} |\psi|^2 r \mathrm{d}r \mathrm{d}z,$$

$$\mathrm{dom}\, \mathfrak{E}^{\mathrm{D}}_{mj} := \big\{ \Psi|_{\mathcal{G}'_j} \colon \Psi \in \mathrm{dom}\, \mathfrak{F}_m, \mathrm{supp}\, \Psi \subset \overline{\mathcal{G}'_j} \big\},$$

where the integrand $\mathcal{E}_m[\psi]$ is as in (2.10). Further, applying the IMS formula [CFKS, §3.1] to the quadratic form $\mathfrak{F}_m$, we obtain that $(\mathcal{X}_{j,p}\psi)|_{\mathcal{G}'_j} \in \mathrm{dom}\, \mathfrak{E}^{\mathrm{D}}_{mj}$, $j = 0, 1$ and with a slight abuse of notation we have

$$\mathfrak{F}_m[\psi] \geq \mathfrak{E}^{\mathrm{D}}_{m0}[\mathcal{X}_{0,p}\psi] + \mathfrak{E}^{\mathrm{D}}_{m1}[\mathcal{X}_{1,p}\psi], \qquad \forall\, \psi \in \mathrm{dom}\, \mathfrak{F}_m.$$

Hence, by [DLR12, Lem. 5.2] we get

$$\mathcal{N}_{E_\infty - E}(\mathfrak{F}_m) \leq \mathcal{N}_{E_\infty - E}(\mathfrak{E}^{\mathrm{D}}_{m0}) + \mathcal{N}_{E_\infty - E}(\mathfrak{E}^{\mathrm{D}}_{m1}), \qquad \forall\, E > 0.$$

Notice that the quadratic form $\mathfrak{E}^{\mathrm{D}}_{m0}$ corresponds to a self-adjoint operator with a compact resolvent, while the form $\mathfrak{E}^{\mathrm{D}}_{m1}$ is unitarily equivalent via $\mathsf{U}'_1 \colon L^2(\mathcal{G}'_1; r\mathrm{d}r\mathrm{d}z) \to L^2(\mathcal{G}'_1)$, $\mathsf{U}'_1 \psi := \sqrt{r}\psi$, to the following form in the Hilbert space $L^2(\mathcal{G}'_1)$,

$$H^1_0(\mathcal{G}'_1) \ni \psi \mapsto \int_{\mathcal{G}'_1} \mathcal{F}_m[\psi] \mathrm{d}r\mathrm{d}z - C_{\mathcal{X}} p^{-2} \int_{\mathcal{G}'_{01}} |\psi|^2 \mathrm{d}r\mathrm{d}z,$$

which is larger in the sense of ordering than the orthogonal sum $\mathfrak{f} \oplus \mathfrak{f}^{\mathrm{N}}_{m1}$ of the form $\mathfrak{f}^{\mathrm{N}}_{m1}$ in the Hilbert space $L^2(\mathcal{G}_1)$ defined in (3.1a) and the form in the Hilbert space $L^2(\mathcal{G}'_{01})$,

$$\mathfrak{f}[\psi] := \int_{\mathcal{G}'_{01}} \big(\mathcal{F}_m[\psi] - C_{\mathcal{X}} p^{-2} |\psi|^2 \big) \mathrm{d}r\mathrm{d}z, \qquad \mathrm{dom}\, \mathfrak{f} := \big\{ \Psi|_{\mathcal{G}'_{01}} \colon \Psi \in H^1_0(\mathcal{G}'_1) \big\}.$$

It only remains to notice that the last form also corresponds to a self-adjoint operator in the Hilbert space $L^2(\mathcal{G}'_{01})$ with a compact resolvent. Thus, the desired inequality holds with the constant $C^{\mathrm{N}}_m = \mathcal{N}_{E_\infty}(\mathfrak{E}^{\mathrm{D}}_{m0}) + \mathcal{N}_{E_\infty}(\mathfrak{f})$. □

4. Straightening of the meridian domain

Recall that the curve Γ in (2.2) is parametrized via the mapping $\overline{\mathbb{R}}_+ \ni s \mapsto (\phi(s), f(\phi(s)))$ with ϕ satisfying the differential equation (2.11). Furthermore, recall that the quadratic

form $\mathfrak{F}_m$ in (2.7) is unitarily equivalent through the unitary transformation U to the form $\mathfrak{f}_m$ in (2.8) expressed in the flat metric.

Let us introduce auxiliary potentials $V_m^{\mathrm{C}}, V_m \colon \Omega \to \mathbb{R}$, $m \in \mathbb{Z}$, on the half-strip $\Omega = \mathbb{R}_+ \times \mathbb{I}_a$ by the formulæ

$$
(4.1) \qquad V_m^{\mathrm{C}}(s, u) := \frac{m^2 - \frac{1}{4}}{\left(\phi(s) - u\dot{f}(\phi(s))\dot{\phi}(s)\right)^2},
$$

$$
V_m(s, u) := \frac{u\ddot{\gamma}(s)}{2g^{3/2}(s, u)} - \frac{\gamma^2(s)}{4g(s, u)} - \frac{5}{4}\frac{u^2\dot{\gamma}^2(s)}{g^2(s, u)} + V_m^{\mathrm{C}}(s, u).
$$

The first three terms in the definition of V_m correspond to the curvature-induced potential arising while straightening the curved two-dimensional half-strip $\mathcal{G}$; *cf.* [EK, Sec. 1.1]. The fourth term V_m^{C} corresponds to the centrifugal potential written in the (s, u)-coordinates.

For the sake of brevity, we introduce shorthand notations for the integrands,

$$
\mathcal{T}_m[\psi](s, u) := \frac{|\partial_s \psi|^2}{g(s, u)} + |\partial_u \psi|^2 + V_m |\psi|^2, \qquad m \in \mathbb{Z},
$$

$$
(4.2) \qquad \mathcal{B}_p[\psi](u) := \frac{\gamma'(p)u|\psi(p, u)|^2}{2g^{3/2}(p, u)}, \qquad p \geq 0.
$$

Next, we define the unitary operator $\mathsf{V} \colon L^2(\mathcal{G}) \to L^2(\Omega)$ by

$$
(\mathsf{V}\psi)(s, u) := g(s, u)^{1/4}\,\psi\big(\tau(s, u)\big),
$$

where the map τ is as in (2.15), its Jacobian $\mathcal{J}$ is given by (2.16) and $g = \mathcal{J}^2$ as in (2.17). The quadratic form $\mathfrak{f}_m$ is unitarily equivalent via the mapping V to the quadratic form

$$
(4.3) \qquad \mathfrak{t}_m[\psi] := \mathfrak{f}_m[\mathsf{V}^{-1}\psi], \qquad \operatorname{dom}\mathfrak{t}_m := \mathsf{V}(\operatorname{dom}\mathfrak{f}_m),
$$

on the Hilbert space $L^2(\Omega)$. Furthermore, recall that $\Omega_1 = (p, \infty) \times \mathbb{I}_a \subset \Omega$ and introduce the unitary operator $\mathsf{V}_1 \colon L^2(\mathcal{G}_1) \to L^2(\Omega_1)$ by

$$
(\mathsf{V}_1\psi)(s, u) := g(s, u)^{1/4}\,\psi\big(\tau(s, u)\big).
$$

The quadratic forms $\mathfrak{f}_{m1}^{\mathrm{D}}$ and $\mathfrak{f}_{m1}^{\mathrm{N}}$ in (3.1) can be further unitarily transformed into the forms

$$
(4.4) \qquad \mathfrak{t}_{m1}^{\mathrm{D}}[\psi] := \mathfrak{f}_{m1}^{\mathrm{D}}[\mathsf{V}_1^{-1}\psi], \qquad\qquad \operatorname{dom}\mathfrak{t}_{m1}^{\mathrm{D}} := \mathsf{V}_1(\operatorname{dom}\mathfrak{f}_{m1}^{\mathrm{D}}),
$$

$$
(4.5) \qquad \mathfrak{t}_{m1}^{\mathrm{N}}[\psi] := \mathfrak{f}_{m1}^{\mathrm{N}}[\mathsf{V}_1^{-1}\psi], \qquad\qquad \operatorname{dom}\mathfrak{t}_{m1}^{\mathrm{N}} := \mathsf{V}_1(\operatorname{dom}\mathfrak{f}_{m1}^{\mathrm{N}}),
$$

on the Hilbert space $L^2(\Omega_1)$.

In the remaining part of this section we will get more explicit expressions for the forms $\mathfrak{t}_m$, $\mathfrak{t}_{m1}^{\mathrm{D}}$ and $\mathfrak{t}_{m1}^{\mathrm{N}}$. Let $\Psi \in H^1(\mathcal{G})$ and $\psi \in H^1(\Omega)$ be connected through the relation $\Psi \circ \tau = \mathcal{J}^{-1/2}\psi$. Let also $(t_1, t_2)^\top$ and $(n_1, n_2)^\top$ be, respectively, the unit tangential and normal vectors to Γ. Using the Frenet formula we find

$$
\partial_s(\mathcal{J}^{-1/2}\psi) = (\partial_r\Psi \circ \tau)\partial_s\tau_1 + (\partial_z\Psi \circ \tau)\partial_s\tau_2
$$

$$
= (\partial_r\Psi \circ \tau)t_1(1 - u\gamma) + (\partial_z\Psi \circ \tau)t_2(1 - u\gamma) = \mathcal{J}\big((\partial_r\Psi \circ \tau)t_1 + (\partial_z\Psi \circ \tau)t_2\big),
$$

$$
\partial_u(\mathcal{J}^{-1/2}\psi) = (\partial_r\Psi \circ \tau)\partial_u\tau_1 + (\partial_z\Psi \circ \tau)\partial_u\tau_2 = (\partial_r\Psi \circ \tau)n_1 + (\partial_z\Psi \circ \tau)n_2,
$$

Hence, we obtain

$$
(4.6) \qquad |(\nabla\Psi)\circ\tau|^2 = \frac{|\partial_s(\mathcal{J}^{-1/2}\psi)|^2}{\mathcal{J}^2} + |\partial_u(\mathcal{J}^{-1/2}\psi)|^2.
$$

In the spirit of [EK, Sec. 1.1], we find that $\operatorname{dom}\mathfrak{t}_m = H_0^1(\Omega)$ for all $m \in \mathbb{Z}\setminus\{0\}$ and by performing elementary computations relying on (4.6) we get

$$
\begin{aligned}
\mathfrak{t}_m[\psi] &= \int_\Omega \frac{|\partial_s(\mathcal{J}^{-1/2}\psi)|^2}{\mathcal{J}} + |\partial_u(\mathcal{J}^{-1/2}\psi)|^2\mathcal{J} + V_m^{\mathrm{C}}|\mathcal{J}^{-1/2}\psi|^2\mathcal{J} \\
&= \int_\Omega \frac{|\partial_s\psi|^2}{\mathcal{J}^2} - \frac{u\dot\gamma\mathrm{Re}\,(\overline\psi\partial_s\psi)}{\mathcal{J}^3} + \frac{u^2\dot\gamma^2|\psi|^2}{4\mathcal{J}^4} + |\partial_u\psi|^2 - \frac{\gamma\mathrm{Re}\,(\overline\psi\partial_u\psi)}{\mathcal{J}} + \frac{\gamma^2|\psi|^2}{4\mathcal{J}^2} + V_m^{\mathrm{C}}|\psi|^2 \\
&= \int_\Omega \frac{|\partial_s\psi|^2}{\mathcal{J}^2} - \frac{u\dot\gamma\partial_s(|\psi|^2)}{2\mathcal{J}^3} + \frac{u^2\dot\gamma^2|\psi|^2}{4\mathcal{J}^4} + |\partial_u\psi|^2 - \frac{\gamma\partial_u(|\psi|^2)}{2\mathcal{J}} + \frac{\gamma^2|\psi|^2}{4\mathcal{J}^2} + V_m^{\mathrm{C}}|\psi|^2.
\end{aligned}
$$

Integrating by parts in the above formula, we obtain

$$
\begin{aligned}
\mathfrak{t}_m[\psi] &= \int_\Omega \frac{|\partial_s\psi|^2}{\mathcal{J}^2} + |\partial_u\psi|^2 + \left[\frac{u^2\dot\gamma^2}{4\mathcal{J}^4} + \partial_s\left(\frac{u\dot\gamma}{2\mathcal{J}^3}\right) + \partial_u\left(\frac{\gamma}{2\mathcal{J}}\right) + \frac{\gamma^2}{4\mathcal{J}^2} + V_m^{\mathrm{C}}\right]|\psi|^2 \\
&= \int_\Omega \frac{|\partial_s\psi|^2}{\mathcal{J}^2} + |\partial_u\psi|^2 + \left[\frac{u^2\dot\gamma^2}{4\mathcal{J}^4} + \frac{u\ddot\gamma}{2\mathcal{J}^3} - \frac{3u^2\dot\gamma^2}{2\mathcal{J}^4} - \frac{\gamma^2}{2\mathcal{J}^2} + \frac{\gamma^2}{4\mathcal{J}^2} + V_m^{\mathrm{C}}\right]|\psi|^2 \\
(4.7) \qquad\qquad &= \int_\Omega \frac{|\partial_s\psi|^2}{\mathcal{J}^2} + |\partial_u\psi|^2 + \left[\frac{u\ddot\gamma}{2\mathcal{J}^3} - \frac{5u^2\dot\gamma^2}{4\mathcal{J}^4} - \frac{\gamma^2}{4\mathcal{J}^2} + V_m^{\mathrm{C}}\right]|\psi|^2 \\
&= \int_\Omega \mathcal{T}_m[\psi],
\end{aligned}
$$

the boundary terms vanished thanks to the Dirichlet boundary condition.

Analogously, we get for all $m \in \mathbb{Z}$

$$
\mathfrak{t}_{m1}^{\mathrm{D}}[\psi] := \int_{\Omega_1} \mathcal{T}_m[\psi]\,\mathsf{d}s\mathsf{d}u, \qquad \operatorname{dom}\mathfrak{t}_{m1}^{\mathrm{D}} := H_0^1(\Omega_1),
$$

Mimicking the above computation for the form $\mathfrak{t}_{m1}^{\mathrm{N}}$, $m \in \mathbb{Z}$, we arrive at $\operatorname{dom}\mathfrak{t}_{m1}^{\mathrm{N}} = \{\psi|_{\Omega_1} : \psi \in H_0^1(\Omega)\}$ and obtain that

$$
\mathfrak{t}_{m1}^{\mathrm{N}}[\psi] := \int_{\Omega_1} \frac{|\partial_s\psi|^2}{\mathcal{J}^2} - \frac{u\dot\gamma\partial_s(|\psi|^2)}{2\mathcal{J}^3} + \frac{u^2\dot\gamma^2|\psi|^2}{4\mathcal{J}^4} + |\partial_u\psi|^2 - \frac{\gamma\partial_u(|\psi|^2)}{2\mathcal{J}} + \frac{\gamma^2|\psi|^2}{4\mathcal{J}^2} + V_m^{\mathrm{C}}|\psi|^2.
$$

Again integrating by parts, we finally get

$$
\begin{aligned}
\mathfrak{t}_{m1}^{\mathrm{N}}[\psi] &= \int_{\Omega_1} \left(\frac{|\partial_s\psi|^2}{\mathcal{J}^2} + |\partial_u\psi|^2 + V_m|\psi|^2\right)\mathsf{d}s\mathsf{d}u + \int_{\mathbb{I}_a} \frac{\dot\gamma(p)u|\psi(p,u)|^2}{2\mathcal{J}^3(p,u)}\mathsf{d}u \\
&= \int_{\Omega_1} \mathcal{T}_m[\psi]\mathsf{d}s\mathsf{d}u + \int_{\mathbb{I}_a} \mathcal{B}_p[\psi]\mathsf{d}u.
\end{aligned}
$$

While the Dirichlet boundary condition on $\partial\mathcal{G}_1 \setminus \mathcal{S}$ is preserved upon straightening of $\mathcal{G}_1$, the Neumann boundary condition on the line segment $\mathcal{S}$ transforms into the Robin boundary condition with the coupling function $\mathbb{I}_a \ni u \mapsto \frac{\dot\gamma(p)u}{2\mathcal{J}^3(p,u)}$. This peculiarity manifests in the presence of the boundary in the expression for $\mathfrak{t}_{m1}^{\mathrm{N}}$.

5. **Properties of $\mathcal{N}_{E_\infty - E}(\mathsf{F}_m)$ for $m \in \mathbb{Z} \setminus \{0\}$**

In this section, we investigate the spectral counting function of the fiber operators F_m for all $m \in \mathbb{Z} \setminus \{0\}$. First, we show that for all $m \neq 0$ the discrete spectrum of the fiber operators below the threshold E_∞ is at most finite, and secondly, that this discrete spectrum is empty for all but finitely many such fiber operators. This informal explanation is precisely formulated in the proposition below, the proof of which relies on the min-max principle and on the finiteness of the discrete spectrum for a class of one-dimensional Schrödinger operators stated in Proposition 2.1 (ii).

Proposition 5.1. *Let F_m, $m \in \mathbb{Z} \setminus \{0\}$, be the self-adjoint fiber operators in the Hilbert space $L^2(\mathcal{G}; r\mathrm{d}r\mathrm{d}z)$ associated with the quadratic forms in (2.7). Then there exists an $M = M(f, a) \in \mathbb{N}_0$ such that:*

(i) $1 \leq \mathcal{N}_{E_\infty}(\mathsf{F}_m) < \infty$ *for all $m = 1, 2, \ldots, M$;*

(ii) $\mathcal{N}_{E_\infty}(\mathsf{F}_m) = 0$ *for all $m > M$.*

Proof. First of all, recall that the form $\mathfrak{t}_m$ in (4.3) is unitarily equivalent to the form $\mathfrak{F}_m$ which is represented by the operator F_m. We organize the argument into three steps.

Step 1: bracketing. We infer that the ordering $\mathfrak{s}_m \prec \mathfrak{t}_m$ holds, where the form $\mathfrak{s}_m$ is given by

$$(5.1) \qquad \mathfrak{s}_m[\psi] = \int_\Omega \left(\frac{|\partial_s \psi|^2}{(1 + \xi_0)^2} + |\partial_u \psi|^2 + U_m |\psi|^2 \right) \mathrm{d}u\mathrm{d}s, \qquad \mathrm{dom}\,\mathfrak{s}_m = H_0^1(\Omega),$$

with ξ_0 as in (2.25) for $p = 0$ and the potential U_m defined by

$$(5.2) \qquad U_m(s) = \frac{1}{\phi^2(s)} \left(\frac{m^2 - \frac{1}{4}}{(1 + \xi_0)^2} - \frac{\phi^2(s)\gamma^2(s)}{4(1 - \xi_0)^2} - \frac{a\phi^2(s)|\ddot{\gamma}(s)|}{2(1 - \xi_0)^3} - \frac{5}{4} \frac{a^2 \phi^2(s)\dot{\gamma}^2(s)}{(1 - \xi_0)^4} \right).$$

Step 2: large $|m| \in \mathbb{N}$. In view of Propositions A.2 (i) and A.3 (i)-(iii), the functions

$$s \mapsto \phi^2(s)\gamma^2(s), \qquad s \mapsto \phi^2(s)|\ddot{\gamma}(s)| \qquad \text{and} \qquad s \mapsto \phi^2(s)\dot{\gamma}^2(s)$$

are all bounded on $\overline{\mathbb{R}}_+$. Hence, for any $|m| \geq M$ with $M \in \mathbb{N}$ large enough, the potential U_m is pointwise positive. Fur such m's we obtain

$$\mathcal{N}_{E_\infty}(\mathsf{F}_m) = \mathcal{N}_{E_\infty}(\mathfrak{f}_m) = \mathcal{N}_{E_\infty}(\mathfrak{t}_m) \leq \mathcal{N}_{E_\infty}(\mathfrak{s}_m) = 0.$$

Step 3: small $|m| \in \mathbb{N}$. In view of the asymptotics of $\gamma(s)$, $\dot{\gamma}(s)$ and $\ddot{\gamma}(s)$ as $s \to \infty$ shown in Proposition A.3, the potential $q \colon \mathbb{R}_+ \to \mathbb{R}$ defined by

$$(5.3) \qquad q(s) := -\frac{1}{\zeta_0} \left(\frac{\gamma^2(s)}{4} + \frac{a|\ddot{\gamma}(s)|}{2(1 - \xi_0)} + \frac{5}{4} \frac{a^2 \dot{\gamma}^2(s)}{(1 - \xi_0)^2} \right),$$

where ξ_0 and ζ_0 are as in (2.25) with $p = 0$, satisfies the condition (2.27) with $\beta = 2$ and $c = 0$ and therefore by Proposition 2.1 the negative discrete spectrum of the self-adjoint operator h_q in (2.26) is finite.

Next, we define the auxiliary quadratic form

$$(5.4) \qquad H_0^1(\mathbb{I}_a) \ni \psi \mapsto \mathfrak{h}_a^{\mathrm{D}}[\psi] := \|\psi'\|_{L^2(\mathbb{I}_a)}^2$$

in the Hilbert space $L^2(\mathbb{I}_a)$ and consider the closed, densely defined, symmetric and semi-bounded quadratic form in $L^2(\Omega) = L^2(\mathbb{R}_+) \otimes L^2(\mathbb{I}_a)$ having the tensor product structure $\mathfrak{h} := (1 + \xi_0)^{-2}\mathfrak{h}_q \otimes \mathfrak{i} + \mathfrak{i} \otimes \mathfrak{h}_a^{\mathrm{D}}$, where $\mathfrak{h}_q$ is the form represented by h_q in (2.26) and $\mathfrak{i}$ stands for the quadratic form of the identity operator on a generic Hilbert space. Comparing the definition (5.1) of the form $\mathfrak{s}_m$, the expression (5.2) for the potential U_m, and the expression (5.3) for the potential q, we obtain that the ordering $\mathfrak{h} \prec \mathfrak{s}_m$ holds for all $m \neq 0$. Furthermore, let $\lambda_n^{\mathrm{D}}(a)$, $n \in \mathbb{N}$, be the eigenvalues of $\mathfrak{h}_a^{\mathrm{D}}$ enumerated in a non-decreasing way. Note also that we have $E_\infty = \lambda_1^{\mathrm{D}}$. Since $\mathfrak{h}_q$ is semi-bounded, there exists $N = N(f, a)$ such that $(1 + \xi_0)^{-2}\mathfrak{h}_q \geq E_\infty - \lambda_N^{\mathrm{D}}$. Consequently, for any $|m| < M$ we get

$$\mathcal{N}_{E_\infty}(\mathsf{F}_m) = \mathcal{N}_{E_\infty}(\mathfrak{f}_m) = \mathcal{N}_{E_\infty}(\mathfrak{t}_m) \leq \mathcal{N}_{E_\infty}(\mathfrak{s}_m)$$

$$\leq \sum_{n=1}^{\infty} \mathcal{N}_{E_\infty - \lambda_n^{\mathrm{D}}}\big((1 + \xi_0)^{-2}\mathfrak{h}_q\big) \leq \sum_{n=1}^{N} \mathcal{N}_0\big((1 + \xi_0)^{-2}\mathfrak{h}_q\big) = N \cdot \mathcal{N}_0(\mathfrak{h}_q) < \infty,$$

which concludes the proof. $\qquad\square$

6. Asymptotics of $\mathcal{N}_{E_\infty - E}(\mathsf{F}_0)$

The fiber operator F_0 corresponding to $m = 0$ requires a separate consideration. Recall that this fiber operator is associated with the quadratic form $\mathfrak{F}_0$ on the Hilbert space $L^2(\mathcal{G}; r\mathrm{d}r\mathrm{d}z)$ defined as

$$(6.1) \qquad \mathfrak{F}_0[\psi] = \int_{\mathcal{G}} \mathcal{E}_0[\psi]r\mathrm{d}r\mathrm{d}z, \qquad \mathrm{dom}\,\mathfrak{F}_0 = \Pi_0(H_0^1(\mathcal{P})),$$

where $\mathcal{E}_0$ is as in (2.10). Throughout this section we use the shorthand notation

$$(6.2) \qquad \mathsf{g}_{\alpha,k}(E) := \frac{1}{2\pi} \frac{\alpha k}{2^\alpha} \frac{\mathsf{B}\left(\frac{3}{2}, \frac{\alpha}{2} - \frac{1}{2}\right)}{E^{\frac{\alpha}{2} - \frac{1}{2}}}.$$

6.1. A lower bound on $\mathcal{N}_{E_\infty - E}(\mathsf{F}_0)$

In this subsection we obtain an asymptotic lower bound on the counting function $\mathcal{N}_{E_\infty - E}(\mathsf{F}_0)$ in the limit $E \to 0^+$. To this aim we modify the strategy used in the proof of Proposition 5.1 by additionally involving Lemma 3.1 (i). At the end we reduce the problem to the spectral asymptotics of a one-dimensional Schrödinger operator covered by Proposition 2.1 (iii).

Proposition 6.1. *Let the self-adjoint fiber operator F_0 in the Hilbert space $L^2(\mathcal{G}; r\mathrm{d}r\mathrm{d}z)$ be associated with the form* (6.1) *and let the function $\mathsf{g}_{\alpha,k}(\cdot)$ be as in* (6.2). *Then*

$$\liminf_{E \to 0^+} \frac{\mathcal{N}_{E_\infty - E}(\mathsf{F}_0)}{\mathsf{g}_{\alpha,k}(E)} \geq 1.$$

Proof. We reduce finding the asymptotic lower bound to the analysis of one-dimensional Schrödinger operators. Recall first that Lemma 3.1 (i) and the construction described in Section 4 imply that

$$\mathcal{N}_{E_\infty - E}(\mathsf{F}_0) \ge \mathcal{N}_{E_\infty - E}(\mathsf{t}_{01}^{\mathrm{D}}), \qquad \forall\, E > 0.$$

Let the auxiliary functions ξ_p and ζ_p be as in (2.25). Next we introduce the quadratic form

$$(6.3) \qquad \mathfrak{u}_{01}^{\mathrm{D}}[\psi] = \int_{\Omega_1} \left(\frac{|\partial_s \psi|^2}{(1 - \xi_p)^2} + |\partial_u \psi|^2 + W_p(s)|\psi|^2 \right) \mathrm{d}u\,\mathrm{d}s, \qquad \mathrm{dom}\, \mathfrak{u}_{01}^{\mathrm{D}} = H_0^1(\Omega_1),$$

where $\Omega_1 = (p, \infty) \times \mathbb{I}_a$ and the potential $W_p(s)$ is given by

$$(6.4) \qquad W_p(s) = \frac{a|\ddot{\gamma}(s)|}{2(1 - \xi_p)^3} - \frac{1}{4(1 + \xi_p)^2} \frac{1}{\phi^2(s)}.$$

For g in (2.17) and V_0 in (4.1) with $m = 0$, we have $(1 - \xi_p)^2 \le g(s, u) \le (1 + \xi_p)^2$ and $V_0(s, u) \le W_p(s)$ for all $(s, u) \in \Omega_1$, hence the ordering of the forms $\mathsf{t}_{01}^{\mathrm{D}} \prec \mathfrak{u}_{01}^{\mathrm{D}}$ holds.

Now we can reduce the asymptotics analysis to investigation of one-dimensional Schrödinger operators. Define the potential $q \colon \mathbb{R}_+ \to \mathbb{R}$ as

$$(6.5) \qquad q(s) := W_p(s - p)\,(1 - \xi_p)^2.$$

Proposition A.2 (i) and Proposition A.3 (ii) imply that the potential q in (6.5) satisfies the condition (2.27) in Proposition 2.1 with $\beta = \frac{2}{\alpha} \in (0, 2)$ and $c = \frac{\zeta_p}{4} k^{\frac{2}{\alpha}}$. Consequently, using the decomposition

$$\mathfrak{u}_{01}^{\mathrm{D}} = (1 - \xi_p)^{-2}\mathfrak{h}_q \otimes \mathfrak{i} + \mathfrak{i} \otimes \mathfrak{h}_a^{\mathrm{D}} \cong \bigoplus_{n \in \mathbb{N}} \left(\lambda_n^{\mathrm{D}}(a) + (1 - \xi_p)^{-2}\mathfrak{h}_q \right)$$

with $\mathfrak{h}_a^{\mathrm{D}}$ as in 5.4, and applying Proposition 2.1 (iii), we obtain that

$$
\begin{aligned}
\liminf_{E \to 0^+} \frac{\mathcal{N}_{E_\infty - E}(\mathsf{F}_0)}{\mathsf{g}_{\alpha, k}(E)} &\ge \liminf_{E \to 0^+} \frac{\mathcal{N}_{E_\infty - E}(\mathsf{t}_{01}^{\mathrm{D}})}{\mathsf{g}_{\alpha, k}(E)} \\
&\ge \liminf_{E \to 0^+} \frac{\mathcal{N}_{E_\infty - E}(\mathfrak{u}_{01}^{\mathrm{D}})}{\mathsf{g}_{\alpha, k}(E)} \ge \liminf_{E \to 0^+} \frac{\mathcal{N}_{-E(1 - \xi_p)^2}(\mathfrak{h}_q)}{\mathsf{g}_{\alpha, k}(E)} \\
&= \liminf_{E \to 0^+} \frac{1}{2\pi} \frac{\alpha k}{2^\alpha} \frac{\mathrm{B}\left(\frac{3}{2}, \frac{\alpha}{2} - \frac{1}{2}\right)}{E^{\frac{\alpha}{2} - \frac{1}{2}}} \frac{1}{\mathsf{g}_{\alpha, k}(E)} \zeta_p^{\frac{\alpha}{2}}(1 - \xi_p)^{1 - \alpha} = \zeta_p^{\frac{\alpha}{2}}(1 - \xi_p)^{1 - \alpha}.
\end{aligned}
$$

Eventually, passing to the limit $p \to \infty$ in the above inequality and making use of $\xi_p \to 0$, $\zeta_p \to 1$ as $p \to \infty$ we obtain the claim. $\qquad\square$

6.2. **An upper bound on $\mathcal{N}_{E_\infty - E}(\mathsf{F}_0)$**

Finally, we derive an asymptotic upper bound on $\mathcal{N}_{E_\infty - E}(\mathsf{F}_0)$. To this aim we combine the strategy used in the proof of Proposition 5.1 with Lemma 3.1 (ii).

Proposition 6.2. *Let the self-adjoint fiber operator F_0 in the Hilbert space $L^2(\mathcal{G}; r\mathrm{d}r\mathrm{d}z)$ be associated with the form (6.1) and let the function $\mathsf{g}_{\alpha, k}(\cdot)$ be as in (6.2). Then*

$$\limsup_{E \to 0^+} \frac{\mathcal{N}_{E_\infty - E}(\mathsf{F}_0)}{\mathsf{g}_{\alpha, k}(E)} \le 1.$$

Proof. We reduce finding the asymptotic upper bound again to the analysis of one-dimensional Schrödinger operators. Recall that Lemma 3.1 (ii) and the construction described in Section 4 imply that

$$\mathcal{N}_{E_\infty - E}(\mathsf{F}_0) \leq \mathcal{N}_{E_\infty - E}(\mathfrak{t}_{01}^{\mathrm{N}}) + C_0^{\mathrm{N}}, \qquad \forall\, E > 0.$$

Let the auxiliary functions ξ_p and ζ_p be as in (2.25). Introduce the quadratic form

(6.6)
$$\mathfrak{s}_{01}^{\mathrm{N}}[\psi] = \int_{\Omega_1} \left(\frac{|\partial_s \psi|^2}{(1+\xi_p)^2} + |\partial_u \psi|^2 + U_p(s)|\psi|^2 \right) \mathrm{d}s\mathrm{d}u - \int_{\mathbb{I}_a} \frac{a|\dot{\gamma}(p)||\psi(p,u)|^2}{2(1-\xi_p)^3}\mathrm{d}u,$$
$$\mathrm{dom}\,\mathfrak{s}_{01}^{\mathrm{N}} := \left\{ \Psi|_{\Omega_1} : \Psi \in H_0^1(\Omega) \right\},$$

where $\Omega_1 = (p, \infty) \times \mathbb{I}_a$ and the potential U_p is given by

(6.7)
$$U_p(s) := -\frac{\gamma^2(s)}{4(1-\xi_p)^2} - \frac{5a^2\dot{\gamma}^2(s)}{4(1-\xi_p)^4} - \frac{a|\ddot{\gamma}(s)|}{2(1-\xi_p)^3} - \frac{1}{4(1-\xi_p)^2}\frac{1}{\phi^2(s)}.$$

For g in (2.17) and V_0 in (4.1) with $m = 0$, we have $(1 - \xi_p)^2 \leq g(s,u) \leq (1+\xi_p)^2$ and $V_0(s,u) \geq U_p(s)$ for all $(s,u) \in \Omega_1$, hence the ordering of the forms $\mathfrak{s}_{01}^{\mathrm{N}} \prec \mathfrak{t}_{01}^{\mathrm{N}}$ holds.

Define the potential $q \colon \mathbb{R}_+ \to \mathbb{R}$ as

(6.8)
$$q(s) := U_p(s - p)(1 + \xi_p)^2.$$

Proposition A.2 (i) and Proposition A.3 (i)-(iii) imply that q satisfies the condition (2.27) with $\beta = \frac{2}{\alpha} \in (0,2)$ and $c = \frac{k^{\frac{2}{\alpha}}}{4\zeta_p}$. Next we define the quadratic form

$$H^1(\mathbb{R}_+) \ni \psi \mapsto \mathfrak{h}_q'[\psi] := \int_{\mathbb{R}_+} \left(|\psi'(s)|^2 + q(s)|\psi(s)|^2 \right) \mathrm{d}s - \frac{a|\dot{\gamma}(p)||\psi(0)|^2}{2(1-\xi_p)\zeta_p},$$

noting that the self-adjoint operator $\mathfrak{h}_q'$ corresponding to the quadratic form $\mathfrak{h}_q'$ is a rank-one perturbation of the operator $\mathfrak{h}_q$ defined in (2.26) with the potential as in (6.8). Hence by [BS, §9.3, Thm. 3] we have

(6.9)
$$|\mathcal{N}_{-E}(\mathfrak{h}_q') - \mathcal{N}_{-E}(\mathfrak{h}_q)| \leq 1, \qquad \forall\, E > 0.$$

In the tensor-product decomposition of $\mathfrak{s}_{01}^{\mathrm{N}}$ into the orthogonal sum,

$$\mathfrak{s}_{01}^{\mathrm{N}} = (1+\xi_p)^{-2}\mathfrak{h}_q' \otimes \mathfrak{i} + \mathfrak{i} \otimes \mathfrak{h}_a^{\mathrm{D}} \cong \bigoplus_{n \in \mathbb{N}} \left(\lambda_n^{\mathrm{D}}(a) + (1+\xi_p)^{-2}\mathfrak{h}_q' \right),$$

only finitely many summands have non-empty discrete spectrum below the threshold $\lambda_1^{\mathrm{D}}(a) = E_\infty$ and for all of them except for the lowest one the discrete spectrum below E_∞ is finite. Making now use of Proposition 2.1 (iii) with $c = \frac{k^{\frac{2}{\alpha}}}{4\zeta_p}$ and $\beta = \frac{2}{\alpha}$ we obtain

$$\limsup_{E \to 0^+} \frac{\mathcal{N}_{E_\infty - E}(\mathsf{F}_0)}{\mathsf{g}_{\alpha,k}(E)} \leq \limsup_{E \to 0^+} \frac{\mathcal{N}_{E_\infty - E}(\mathfrak{t}_{01}^{\mathrm{N}})}{\mathsf{g}_{\alpha,k}(E)} \leq \limsup_{E \to 0^+} \frac{\mathcal{N}_{E_\infty - E}(\mathfrak{s}_{01}^{\mathrm{N}})}{\mathsf{g}_{\alpha,k}(E)}$$
$$\leq \limsup_{E \to 0^+} \frac{\mathcal{N}_{-E(1+\xi_p)^2}(\mathfrak{h}_q')}{\mathsf{g}_{\alpha,k}(E)} = \limsup_{E \to 0^+} \frac{\mathcal{N}_{-E(1+\xi_p)^2}(\mathfrak{h}_q)}{\mathsf{g}_{\alpha,k}(E)}$$
$$= \limsup_{E \to 0^+} \frac{1}{2\pi} \frac{\alpha k}{2^\alpha} \frac{\mathsf{B}\left(\frac{3}{2}, \frac{\alpha}{2} - \frac{1}{2}\right)}{E^{\frac{\alpha}{2} - \frac{1}{2}}} \frac{1}{\mathsf{g}_{\alpha,k}(E)} \zeta_p^{-\frac{\alpha}{2}}(1+\xi_p)^{1-\alpha} = \zeta_p^{-\frac{\alpha}{2}}(1+\xi_p)^{1-\alpha}.$$

Passing to the limit $p \to \infty$ in the above inequality we arrive at the sought claim.   □

7. **Proof of Theorem 1.2**

With all the preparations made above, the proof of the main result turns out to be very short. By Proposition 5.1 we get

$$\liminf_{E \to 0^+} \frac{\mathcal{N}_{E_\infty - E}(\mathsf{H})}{\mathsf{g}_{\alpha,k}(E)} = \liminf_{E \to 0^+} \frac{\mathcal{N}_{E_\infty - E}(\mathsf{F}_0)}{\mathsf{g}_{\alpha,k}(E)},$$

$$\limsup_{E \to 0^+} \frac{\mathcal{N}_{E_\infty - E}(\mathsf{H})}{\mathsf{g}_{\alpha,k}(E)} = \limsup_{E \to 0^+} \frac{\mathcal{N}_{E_\infty - E}(\mathsf{F}_0)}{\mathsf{g}_{\alpha,k}(E)}.$$

and furthermore, applying Propositions 6.1 and 6.2 we conclude that

$$\lim_{E \to 0^+} \frac{\mathcal{N}_{E_\infty - E}(\mathsf{H})}{\mathsf{g}_{\alpha,k}(E)} = 1,$$

which is our main result. □

8. **Discussion**

Let us briefly outline a few possible extensions of the results obtained here. Recall first that a domain $\Omega_1 \subset \mathbb{R}^3$ is said to be a *local perturbation* of the domain $\Omega_2 \subset \mathbb{R}^3$, if $\Omega_1 \setminus K = \Omega_2 \setminus K$ for a compact set $K \subset \mathbb{R}^3$. Without much effort, our main result in Theorem 1.2 allows for an extension to all local perturbations of the generalized, radially symmetric parabolic layers. It is worth to stress that such perturbed domains need not necessarily be radially symmetric themselves.

Furthermore, one can introduce into the present model an *Aharonov-Bohm-type magnetic field* along the axis of the layer in the spirit of [EK18, KLO17]. In our geometric setting, we expect that such a singular magnetic field can neither 'switch off' the infiniteness of the discrete spectrum nor will it modify the principal term in the spectral asymptotics. Note that this conjectured behaviour would be in sheer contrast to that of the conical layers [KLO17].

Apparently, an analysis similar to the present one can also be performed for the self-adjoint three-dimensional *Schrödinger operator with an attractive δ-interaction* of constant strength supported on the (generalized) paraboloid Σ in (1.1). Taking the results of [BEL14, LO16, OP17] into account, one may conjecture that the spectral asymptotics will be the same as in Theorem 1.2. However, proof of such a claim might be technically more involved than for the Dirichlet layers.

Finally, it is worth noting that the analogous spectral problem can be also considered for the self-adjoint *Robin Laplacian* in the unbounded domain $\{(\boldsymbol{x}, x_3) \in \mathbb{R}^3 \colon x_3 > f(|\boldsymbol{x}|)\}$, lying above the surface Σ. With a reference to [BPP18, P16] we expect the infiniteness of the discrete spectrum for the Robin boundary condition with a negative parameter, in other words, for an attractive boundary interaction. In view of [BPP18], however, the principal term in the spectral asymptotics might not be the same as in Theorem 1.2.

Appendix A. The signed curvature of Γ

In this appendix, we analyze properties of the signed curvature γ of the curve Γ defined in (2.13). This material can be seen as an exercise in the differential geometry, however, the claims we need are scattered and not easy to find in textbooks. Recall that the curve Γ is parametrized by $\overline{\mathbb{R}}_+ \ni s \to (\phi(s), f(\phi(s)))$, where the increasing function $\phi \colon \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ fulfils $\phi(0) = 0$ and satisfies the ordinary differential equation (2.11). In the remaining part of this appendix, all the functions depend on s and their derivatives are taken with respect to that variable. For the sake of brevity, the indication of the dependence on s is occasionally dropped.

First, we formulate and prove an auxiliary lemma on the second principal curvature κ_2 of Σ, explicitly given in (2.19).

Lemma A.1. *Let* $\phi \colon \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ *be the solution of* (2.11) *with* $\phi(0) = 0$*. Then the function*

$$\overline{\mathbb{R}}_+ \ni p \mapsto \sup_{s \in [p, \infty)} \frac{\dot{f}(\phi(s))\dot{\phi}(s)}{\phi(s)}$$

is bounded and vanishes as $p \to \infty$*.*

Proof. First, we observe that the function $\mathbb{R}_+ \ni s \mapsto \frac{\dot{f}(\phi(s))\dot{\phi}(s)}{\phi(s)}$ is $\mathcal{C}^\infty$-smooth on $\mathbb{R}_+$. Moreover, using that $\phi(0) = 0$, $\dot{\phi}(0) = 1$ and the Taylor expansion $\dot{f}(x) = \ddot{f}(0)x + o(x)$ as $x \to 0^+$ we get

$$\lim_{s \to 0^+} \frac{\dot{f}(\phi)\dot{\phi}}{\phi} = \ddot{f}(0) \qquad \text{and} \qquad \lim_{s \to \infty} \frac{\dot{f}(\phi)\dot{\phi}}{\phi} = \lim_{s \to \infty} \frac{1}{\phi}\left(\frac{\dot{f}^2(\phi)}{1 + \dot{f}^2(\phi)}\right)^{\frac{1}{2}} = \lim_{s \to \infty} \frac{1}{\phi} = 0.$$

The above two limits and smoothness of $\frac{\dot{f}(\phi)\dot{\phi}}{\phi}$ yield the claims. $\qquad\square$

Next, we prove a proposition, on the asymptotic behaviour of ϕ and its derivatives up to the third, in the limit $s \to \infty$.

Proposition A.2. *The solution* $\phi \colon \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ *of* (2.11) *with* $\phi(0) = 0$ *has the following properties.*

(i) $\lim_{s \to \infty} s^{-\frac{1}{\alpha}}\phi = k^{-\frac{1}{\alpha}}$.

(ii) $\lim_{s \to \infty} s^{\frac{\alpha-1}{\alpha}}\dot{\phi} = \frac{k^{-\frac{1}{\alpha}}}{\alpha}$.

(iii) $\lim_{s \to \infty} s^{\frac{2\alpha-1}{\alpha}}\ddot{\phi} = -\frac{\alpha-1}{\alpha^2}k^{-\frac{1}{\alpha}}$.

(iv) $\lim_{s \to \infty} s^{\frac{3\alpha-1}{\alpha}}\dddot{\phi} = \frac{(\alpha-1)(2\alpha-1)}{\alpha^3}k^{-\frac{1}{\alpha}}$.

Proof. Notice that we obviously have $\lim_{s \to \infty} \phi(s) = \infty$, because the map $\overline{\mathbb{R}}_+ \ni s \mapsto (\phi(s), f(\phi(s))$ is a re-parametrization of the curve $\overline{\mathbb{R}}_+ \ni x \mapsto (x, f(x))$. The differential equation (2.11) implies that on the interval $[R, \infty)$ the following estimates hold:

$$\alpha k \phi^{\alpha-1}\dot{\phi} \leq 1 \qquad \text{and} \qquad \left(1 + \alpha k \phi^{\alpha-1}\right)\dot{\phi} \geq 1.$$

Integrating the above inequalities on the interval $[R, s]$ we get

$$(A.1) \qquad k\phi^\alpha(s) \leq s + \mathcal{O}(1) \qquad \text{and} \qquad \phi(s) + k\phi^\alpha(s) \geq s + \mathcal{O}(1), \qquad \forall s \geq R.$$

Plugging the first inequality in (A.1) into the second, we obtain

$$
(A.2) \qquad k\phi^{\alpha}(s) \geq s - \left(\frac{s}{k}\right)^{\frac{1}{\alpha}} + \mathcal{O}(1), \qquad \forall\, s \geq R.
$$

Combining the first inequality in (A.1) with (A.2) and using the assumption $\alpha > 1$ we get the limit in (i). Furthermore, the limit in (ii) can be shown as follows,

$$
\lim_{s\to\infty} s^{\frac{\alpha-1}{\alpha}} \dot{\phi} = \lim_{s\to\infty} s^{\frac{\alpha-1}{\alpha}} \left(1 + \alpha^2 k^2 \phi^{2\alpha-2}\right)^{-\frac{1}{2}} = \lim_{s\to\infty} \left(s^{-\frac{2(\alpha-1)}{\alpha}} + \alpha^2 k^2 s^{-\frac{2(\alpha-1)}{\alpha}} \phi^{2\alpha-2}\right)^{-\frac{1}{2}}
$$
$$
= \lim_{s\to\infty} \frac{1}{\alpha k s^{-\frac{\alpha-1}{\alpha}} \phi^{\alpha-1}} = \frac{k^{-\frac{1}{\alpha}}}{\alpha}.
$$

The differential equation (2.11) can be alternatively written as

$$
(A.3) \qquad \dot{\phi}(s) = \frac{1}{\left(1 + \dot{f}^2(\phi(s))\right)^{\frac{1}{2}}}.
$$

Differentiating the left and right hand sides of equation (A.3), we express $\ddot{\phi}$ as follows,

$$
(A.4) \qquad \ddot{\phi} = -\frac{\dot{f}(\phi)\ddot{f}(\phi)\dot{\phi}}{(1 + \dot{f}^2(\phi))^{\frac{3}{2}}} = -\frac{\dot{f}(\phi)\ddot{f}(\phi)}{(1 + \dot{f}^2(\phi))^2}.
$$

Hence, on the interval $[R, \infty)$, we have

$$
\ddot{\phi} = -\frac{\alpha^2 k^2 (\alpha-1)\phi^{2\alpha-3}}{\left(1 + \alpha^2 k^2 \phi^{2\alpha-2}\right)^2}.
$$

Eventually, using (i) we get

$$
\lim_{s\to\infty} s^{\frac{2\alpha-1}{\alpha}} \ddot{\phi} = -\lim_{s\to\infty} \frac{\alpha-1}{\alpha^2 k^2 s^{-\frac{2\alpha-1}{\alpha}} \phi^{2\alpha-1}} = \frac{\alpha-1}{\alpha^2 k^2 k^{-\frac{2\alpha-1}{\alpha}}} = -\frac{\alpha-1}{\alpha^2} k^{-\frac{1}{\alpha}},
$$

and in this way the limit in (iii) is also obtained.

Differentiating the left and the right hand sides of (A.4), we express $\dddot{\phi}$ as follows,

$$
\dddot{\phi} = -\frac{(\ddot{f}^2(\phi) + \dot{f}(\phi)\dddot{f}(\phi))(1 + \dot{f}^2(\phi))\dot{\phi} - 4\dot{f}^2(\phi)\ddot{f}^2(\phi)\dot{\phi}}{(1 + \dot{f}^2(\phi))^3}
$$
$$
= \frac{3\dot{f}^2(\phi)\ddot{f}^2(\phi) - \ddot{f}^2(\phi) - \dot{f}(\phi)\dddot{f}(\phi) - \dot{f}^3(\phi)\dddot{f}(\phi)}{(1 + \dot{f}^2(\phi))^{\frac{7}{2}}}.
$$

The latter yields that on the interval $[R, \infty)$

$$
\dddot{\phi} = \frac{3\alpha^4(\alpha-1)^2 k^4 \phi^{4\alpha-6} - \alpha^2(\alpha-1)(2\alpha-3)k^2\phi^{2\alpha-4} - \alpha^4(\alpha-1)(\alpha-2)k^4\phi^{4\alpha-6}}{\left(1 + \alpha^2 k^2 \phi^{2\alpha-2}\right)^{\frac{7}{2}}}
$$
$$
= \frac{\alpha^4(\alpha-1)(2\alpha-1)k^4\phi^{4\alpha-6} - \alpha^2(\alpha-1)(2\alpha-3)k^2\phi^{2\alpha-4}}{\left(1 + \alpha^2 k^2 \phi^{2\alpha-2}\right)^{\frac{7}{2}}}.
$$

Again using (i) we obtain

$$\lim_{s\to\infty} s^{\frac{3\alpha-1}{\alpha}}\dddot{\phi} = \lim_{s\to\infty} s^{\frac{3\alpha-1}{\alpha}}\frac{\alpha^4(\alpha-1)(2\alpha-1)k^4\phi^{4\alpha-6} - \alpha^2(\alpha-1)(2\alpha-3)k^2\phi^{2\alpha-4}}{\left(1+\alpha^2k^2\phi^{2\alpha-2}\right)^{\frac{7}{2}}}$$

$$= \lim_{s\to\infty} s^{\frac{3\alpha-1}{\alpha}}\frac{\alpha^4(\alpha-1)(2\alpha-1)k^4\phi^{4\alpha-6}}{\alpha^7k^7\phi^{7\alpha-7}} = \lim_{s\to\infty}\frac{(\alpha-1)(2\alpha-1)}{\alpha^3k^3s^{-\frac{3\alpha-1}{\alpha}}\phi^{3\alpha-1}}$$

$$= \frac{(\alpha-1)(2\alpha-1)}{\alpha^3k^3k^{-\frac{3\alpha-1}{\alpha}}} = \frac{(\alpha-1)(2\alpha-1)}{\alpha^3}k^{-\frac{1}{\alpha}},$$

by which the limit in (iv) is also shown. □

Recall that the signed curvature of the curve Γ is given by the formula

(A.5) $$\gamma = \ddot{f}(\phi)\dot{\phi}^3.$$

Finally, we prove a claim about the asymptotic behaviour of γ and its derivatives up to the second order, in the limit $s \to \infty$.

Proposition A.3. *Let the signed curvature $\gamma\colon \overline{\mathbb{R}}_+ \to \mathbb{R}$ be as in* (A.5). *Then there exist $g_j \in \mathbb{R}$, $j = 0, 1, 2$, such that:*

(i) $\lim_{s\to\infty} s^{\frac{2\alpha-1}{\alpha}}\gamma = g_0,$

(ii) $\lim_{s\to\infty} s^{\frac{3\alpha-1}{\alpha}}\dot{\gamma} = g_1,$

(iii) $\lim_{s\to\infty} s^{\frac{4\alpha-1}{\alpha}}\ddot{\gamma} = g_2.$

Proof. The first and the second derivatives of the signed curvature γ are given by

$$\dot{\gamma} = 3\dot{\phi}^2\ddot{\phi}\ddot{f}(\phi) + \dot{\phi}^4\dddot{f}(\phi),$$
$$\ddot{\gamma} = 6\dot{\phi}\ddot{\phi}^2\ddot{f}(\phi) + 3\dot{\phi}^2\dddot{\phi}\ddot{f}(\phi) + 7\dot{\phi}^3\ddot{\phi}\dddot{f}(\phi) + \dot{\phi}^5 f^{(4)}(\phi).$$

Hence, using the notation $\kappa := \alpha(\alpha-1)k$ we infer that on the interval $[R, \infty)$ the following relations hold:

$$\gamma = \kappa\phi^{\alpha-2}\dot{\phi}^3,$$
$$\dot{\gamma} = \kappa\left[3\phi^{\alpha-2}\dot{\phi}^2\ddot{\phi} + (\alpha-2)\phi^{\alpha-3}\dot{\phi}^4\right],$$
$$\ddot{\gamma} = \kappa\left[\phi^{\alpha-2}\left(6\dot{\phi}\ddot{\phi}^2 + 3\dot{\phi}^2\dddot{\phi}\right) + 7(\alpha-2)\phi^{\alpha-3}\dot{\phi}^3\ddot{\phi} + (\alpha-2)(\alpha-3)\phi^{\alpha-4}\dot{\phi}^5\right].$$

Eventually, existence of finite limits in (i)-(iii) directly follows from Proposition A.2 (i)-(iv). □

Acknowledgment

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