SOLVABILITY OF SOME INTEGRO-DIFFERENTIAL EQUATIONS WITH ANOMALOUS DIFFUSION IN TWO DIMENSIONS

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Abstract: The article is devoted to the studies of the existence of solutions of an integro-differential equation in the case of the anomalous diffusion with the negative Laplace operator in a fractional power in two dimensions. The proof of the existence of solutions is based on a fixed point technique. Solvability conditions for non Fredholm elliptic operators in unbounded domains are used.

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1. Introduction

The present article is devoted to the studies of the existence of stationary solutions of the following nonlocal integro-differential equation

$$\frac{\partial u}{\partial t} = -D(-\Delta)^s u + \int_{\mathbb{R}^2} K(x-y)g(u(y,t))dy + f(x), \quad 0 < s < \frac{1}{2}, \tag{1.1}$$

which is relevant to the cell population dynamics. The space variable x here is corresponding to the cell genotype, $u(x,t)$ denotes the cell density as a function of their genotype and time. The right side of this equation describes the evolution of cell density by means of the cell proliferation, mutations and cell influx. The anomalous diffusion term in this context corresponds to the change of genotype via small random mutations, and the integral term describes large mutations. Function $g(u)$ denotes the rate of cell birth depending on u (density dependent proliferation), and the kernel $K(x-y)$ stands for the proportion of newly born cells changing their genotype from y to x. Let us assume here that it depends on the distance between the genotypes. Finally, the last term in the right side of (1.1) designates the influx or efflux of cells for different genotypes.

The operator $(-\Delta)^s$ in equation (1.1) describes a particular case of the anomalous diffusion actively studied in the context of different applications in plasma physics and turbulence [7], [19], surface diffusion [15], [17], semiconductors [18] and so on. Anomalous diffusion can be described as a random process of particle motion characterized by the probability density distribution of jump length. The moments of this density distribution are finite in the case of normal diffusion, but this is not the case for the anomalous diffusion. Asymptotic behavior at infinity of the probability density function determines the value s of the power of our negative Laplace operator (see [16]). The operator $(-\Delta)^s$ is defined by means of the spectral calculus. In the present work we will consider the case of $0 < s < 1/2$. A similar problem in the case of the standard Laplacian in the diffusion term was considered recently in [31].

Let us set $D = 1$ and establish the existence of solutions of the equation

$$-(-\Delta)^s u + \int_{\mathbb{R}^2} K(x-y)g(u(y))dy + f(x) = 0, \quad 0 < s < \frac{1}{2}. \tag{1.2}$$

We will consider the case where the linear part of this operator fails to satisfy the Fredholm property. As a consequence, the conventional methods of nonlinear analysis may not be applicable. Let us use solvability conditions for non Fredholm operators along with the method of contraction mappings.

Consider the equation

$$-\Delta u + V(x)u - au = f, \tag{1.3}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ is either zero identically or tends to 0 at infinity. For $a \geq 0$, the essential spectrum of the operator $A : E \to F$ which corresponds to the left side of problem (1.3) contains the origin. As a consequence, this operator fails to satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present work is devoted to the studies of certain properties of the operators of this kind. Note that elliptic problems with non Fredholm operators were treated actively in recent years. Approaches in weighted Sobolev and Hölder spaces were developed in [2], [3], [4], [5], [6]. The non Fredholm Schrödinger type operators were studied with the methods of the spectral and the scattering theory in [20], [25], [27]. The Laplace operator with drift from the point of view of non Fredholm operators was treated in [26] and linearized Cahn-Hilliard equations in [23] and [29]. Nonlinear non Fredholm elliptic problems were considered in [28] and [30]. Important applications to the theory of reaction-diffusion equations were developed in [9], [10]. Non Fredholm operators arise also when studying wave systems with an infinite number of localized traveling waves (see [1]). Particularly, when $a = 0$ the operator A is Fredholm in some properly chosen weighted spaces (see [2], [3], [4], [5], [6]). However, the case of $a \neq 0$ is significantly different and the approach developed in these articles cannot be applied. Front propagation problems with anomalous diffusion were studied actively in recent years (see e.g. [21], [22]).

We set $K(x) = \varepsilon \mathcal{K}(x)$, where $\varepsilon \geq 0$ and suppose that the assumption below is fulfilled.

Assumption 1. *Consider* $0 < s < \dfrac{1}{2}$. *Let* $f(x) : \mathbb{R}^2 \to \mathbb{R}$ *be nontrivial, such that* $f(x) \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ *and* $(-\Delta)^{1-s} f(x) \in L^2(\mathbb{R}^2)$. *Assume also that* $\mathcal{K}(x) : \mathbb{R}^2 \to \mathbb{R}$ *and* $\mathcal{K}(x) \in L^1(\mathbb{R}^2)$. *Additionally,* $(-\Delta)^{1-s}\mathcal{K}(x) \in L^2(\mathbb{R}^2)$, *such that*

$$Q := \left\|(-\Delta)^{1-s}\mathcal{K}(x)\right\|_{L^2(\mathbb{R}^2)} > 0.$$

Let us choose the space dimension $d = 2$, which is relevant to the solvability conditions for the linear Poisson type equation (4.31) stated in Lemma 6 below. From the point of view of applications, the space dimension is not limited to $d = 2$ since the space variable corresponds to the cell genotype but not to the usual physical space. In $d = 1$ our problem was treated in [34] with $0 < s < \dfrac{1}{4}$ based on the solvability relations for the analog of (4.31) in one dimension. In $d = 3$ our equation was studied in [32] for $\dfrac{1}{4} < s < \dfrac{3}{4}$. As distinct from the situations in $d = 1, 2$, in three dimensions we were able to use the Sobolev inequality for the fractional Laplacian (see Lemma 2.2 of [12], also [13]). For the technical purposes, we use the Sobolev spaces

$$H^{2s}(\mathbb{R}^2) := \left\{u(x) : \mathbb{R}^2 \to \mathbb{R} \mid u(x) \in L^2(\mathbb{R}^2),\ (-\Delta)^s u \in L^2(\mathbb{R}^2)\right\}, \quad 0 < s \leq 1$$

equipped with the norm

$$\|u\|^2_{H^{2s}(\mathbb{R}^2)} := \|u\|^2_{L^2(\mathbb{R}^2)} + \left\|(-\Delta)^s u\right\|^2_{L^2(\mathbb{R}^2)}. \tag{1.4}$$

By means of the standard Sobolev embedding in two dimensions, we have

$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq c_e \|u\|_{H^2(\mathbb{R}^2)}, \tag{1.5}$$

where $c_e > 0$ is the constant of the embedding. When the nonnegative parameter $\varepsilon = 0$, we obtain the linear Poisson type equation (4.31). By virtue of Lemma 6 below along with Assumption 1, equation (4.31) has a unique solution

$$u_0(x) \in H^{2s}(\mathbb{R}^2), \quad 0 < s < \frac{1}{2},$$

such that no orthogonality conditions are required. By means of Lemma 6, when $\frac{1}{2} \leq s < 1$, a certain orthogonality condition (4.33) is required to be able to solve problem (4.31) in $H^{2s}(\mathbb{R}^2)$. Using Assumption 1, since

$$-\Delta u(x) = (-\Delta)^{1-s} f(x) \in L^2(\mathbb{R}^2),$$

we obtain for the unique solution of linear equation (4.31) that $u_0(x) \in H^2(\mathbb{R}^2)$. We seek the resulting solution of nonlinear problem (1.2) as

$$u(x) = u_0(x) + u_p(x). \tag{1.6}$$

Apparently, we arrive at the perturbative equation

$$(-\Delta)^s u_p(x) = \varepsilon \int_{\mathbb{R}^2} \mathcal{K}(x-y) g(u_0(y) + u_p(y)) dy, \quad 0 < s < \frac{1}{2}. \tag{1.7}$$

We designate a closed ball in the Sobolev space as

$$B_\rho := \{u(x) \in H^2(\mathbb{R}^2) \mid \|u\|_{H^2(\mathbb{R}^2)} \leq \rho\}, \quad 0 < \rho \leq 1. \tag{1.8}$$

Let us look for the solution of equation (1.7) as the fixed point of the auxiliary nonlinear problem

$$(-\Delta)^s u(x) = \varepsilon \int_{\mathbb{R}^2} \mathcal{K}(x-y) g(u_0(y) + v(y)) dy, \quad 0 < s < \frac{1}{2} \tag{1.9}$$

in ball (1.8). For a given function $v(y)$ this is an equation with respect to $u(x)$. The left side of (1.9) involves the operator without Fredholm property

$$(-\Delta)^s : H^{2s}(\mathbb{R}^2) \to L^2(\mathbb{R}^2).$$

Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$. Hence, this operator has no bounded inverse. The similar situation appeared in articles [28] and [30] but as distinct from the present situation, the problems studied there required orthogonality relations. The fixed point technique was used in [24] to estimate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger operator involved in the nonlinear problem there had the Fredholm property (see Assumption 1 of [24], also [8]). The existence of pulses for local and nonlocal reaction- diffusion equations was established via the Leray-Schauder method in [11] using the operators which possessed the Fredholm property as well. We define the interval on the real line

$$I := \big[-c_e \|u_0\|_{H^2(\mathbb{R}^2)} - c_e, \ c_e \|u_0\|_{H^2(\mathbb{R}^2)} + c_e \big] \tag{1.10}$$

along with the closed ball in the space of $C_2(I)$ functions, namely

$$D_M := \{g(z) \in C_2(I) \mid \|g\|_{C_2(I)} \leq M\}, \quad M > 0. \tag{1.11}$$

Here the norm

$$\|g\|_{C_2(I)} := \|g\|_{C(I)} + \|g'\|_{C(I)} + \|g''\|_{C(I)}, \tag{1.12}$$

with $\|g\|_{C(I)} := max_{z \in I}|g(z)|$. Let us make the following technical assumption on the nonlinear part of problem (1.2).

Assumption 2. *Let $g(z) : \mathbb{R} \to \mathbb{R}$, such that $g(0) = 0$ and $g'(0) = 0$. We also assume that $g(z) \in D_M$ and it is not equal to zero identically on the interval I.*

We explain why we impose condition $g'(0) = 0$. Let us assume here that the Fourier image of the kernel $\mathcal{K}(x)$ is positive in the whole $\mathbb{R}^2$, which is common in many biological applications. If $g'(0) < 0$, then the essential spectrum of the corresponding operator is in the left-half plane. Such operator satisfies the Fredholm property, and conventional approaches of nonlinear analysis are applicable here. If $g'(0) \geq 0$, then the operator fails to satisfy the Fredholm property, and the goal of this article is to establish the existence of solutions in such case where usual methods are not applicable. The method developed in this work can be used for $g'(0) = 0$ but not for $g'(0) > 0$. This is the reason we impose such condition on the nonlinearity.

Let us introduce the operator T_g, such that $u = T_g v$, where u is a solution of problem (1.9). Our first main proposition is as follows.

Theorem 3. *Let Assumptions 1 and 2 hold. Then problem (1.9) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon < \varepsilon*$ for some $\varepsilon* > 0$. The unique fixed point $u_p(x)$ of this map T_g is the only solution of equation (1.7) in B_ρ.*

Evidently, the resulting solution of problem (1.2) given by (1.6) will be nontrivial since the source term $f(x)$ is nontrivial and $g(0) = 0$ by means of our assumptions. Let us make use of the following trivial statement.

Lemma 4. *For $R \in (0, +\infty)$ consider the function*

$$\varphi(R) := \alpha R^{2-4s} + \frac{1}{R^{4s}}, \quad 0 < s < \frac{1}{2}, \quad \alpha > 0.$$

It achieves the minimal value at $R^* := \sqrt{\dfrac{2s}{\alpha(1-2s)}}$*, which is given by*

$$\varphi(R^*) = \frac{(1-2s)^{2s-1}}{(2s)^{2s}} \alpha^{2s}.$$

Our second main proposition deals with the continuity of the fixed point of the map T_g which existence was proved in Theorem 3 above with respect to the nonlinear function g.

Theorem 5. *Let $j = 1, 2$, the assumptions of Theorem 3 hold, such that $u_{p,j}(x)$ is the unique fixed point of the map $T_{g_j} : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon < \varepsilon_j^*$ and $\delta := min(\varepsilon_1^*, \varepsilon_2^*)$. Then for all $0 < \varepsilon < \delta$ the estimate*

$$\|u_{p,1} - u_{p,2}\|_{H^2(\mathbb{R}^2)} \leq C\|g_1 - g_2\|_{C_2(I)} \tag{1.13}$$

holds, where $C > 0$ is a constant.

We proceed to the proof of our first main statement.

2. The existence of the perturbed solution

Proof of Theorem 3. We choose arbitrarily $v(x) \in B_\rho$ and denote the term involved in the integral expression in the right side of equation (1.9) as

$$G(x) := g(u_0(x) + v(x)).$$

We use the standard Fourier transform

$$\widehat{\phi}(p) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \phi(x)e^{-ipx}dx. \tag{2.14}$$

Clearly, we have the estimate

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{1}{2\pi}\|\phi(x)\|_{L^1(\mathbb{R}^2)}. \tag{2.15}$$

Let us apply (2.14) to both sides of equation (1.9). This yields

$$\widehat{u}(p) = \varepsilon 2\pi \frac{\widehat{\mathcal{K}}(p)\widehat{G}(p)}{|p|^{2s}}.$$

Therefore, for the norm we arrive at

$$\|u\|^2_{L^2(\mathbb{R}^2)} = 4\pi^2\varepsilon^2 \int_{\mathbb{R}^2} \frac{|\widehat{\mathcal{K}}(p)|^2|\widehat{G}(p)|^2}{|p|^{4s}}dp. \tag{2.16}$$

As distinct from articles [28] and [30] involving the standard Laplacian in the diffusion term, here we do not try to control the norm

$$\left\|\frac{\widehat{\mathcal{K}}(p)}{|p|^{2s}}\right\|_{L^\infty(\mathbb{R}^2)}.$$

Instead, we estimate the right side of (2.16) via the analog of bound (2.15) applied to functions $\mathcal{K}$ and G with $R > 0$ as

$$4\pi^2\varepsilon^2 \int_{|p|\leq R} \frac{|\widehat{\mathcal{K}}(p)|^2|\widehat{G}(p)|^2}{|p|^{4s}}dp + 4\pi^2\varepsilon^2 \int_{|p|>R} \frac{|\widehat{\mathcal{K}}(p)|^2|\widehat{G}(p)|^2}{|p|^{4s}}dp \leq$$

$$\leq \varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^2)} \Bigg\{ \frac{1}{4\pi} \|G(x)\|^2_{L^1(\mathbb{R}^2)} \frac{R^{2-4s}}{1-2s} + \frac{1}{R^{4s}} \|G(x)\|^2_{L^2(\mathbb{R}^2)} \Bigg\}. \tag{2.17}$$

Since $v(x) \in B_\rho$, we obtain

$$\|u_0 + v\|_{L^2(\mathbb{R}^2)} \leq \|u_0\|_{H^2(\mathbb{R}^2)} + 1.$$

Sobolev embedding (1.5) gives us

$$|u_0 + v| \leq c_e(\|u_0\|_{H^2(\mathbb{R}^2)} + 1).$$

Identity $G(x) = \displaystyle\int_0^{u_0+v} g'(z)dz$ with the interval I defined in (1.10) yields

$$|G(x)| \leq sup_{z\in I}|g'(z)||u_0 + v| \leq M|u_0 + v|.$$

Therefore,

$$\|G(x)\|_{L^2(\mathbb{R}^2)} \leq M\|u_0 + v\|_{L^2(\mathbb{R}^2)} \leq M(\|u_0\|_{H^2(\mathbb{R}^2)} + 1).$$

Evidently, $G(x) = \displaystyle\int_0^{u_0+v} dy \Big[\int_0^y g''(z)dz \Big]$. This implies

$$|G(x)| \leq \frac{1}{2} sup_{z\in I}|g''(z)||u_0 + v|^2 \leq \frac{M}{2}|u_0 + v|^2,$$

$$\|G(x)\|_{L^1(\mathbb{R}^2)} \leq \frac{M}{2}\|u_0 + v\|^2_{L^2(\mathbb{R}^2)} \leq \frac{M}{2}(\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^2. \tag{2.18}$$

Hence we arrive at the upper bound for the right side of (2.17) as

$$\varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^2)} M^2 (\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^2 \Bigg\{ \frac{(\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^2 R^{2-4s}}{16\pi(1-2s)} + \frac{1}{R^{4s}} \Bigg\},$$

where $R \in (0, +\infty)$. By virtue of Lemma 4, we arrive at the minimal value of the expression above. Therefore,

$$\|u\|^2_{L^2(\mathbb{R}^2)} \leq \varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^2)} (\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^{2+4s} \frac{M^2}{(1-2s)(32\pi s)^{2s}}. \tag{2.19}$$

Evidently, due to (1.9) we have

$$-\Delta u(x) = \varepsilon(-\Delta)^{1-s} \int_{\mathbb{R}^2} \mathcal{K}(x-y)G(y)dy.$$

By means of the analog of bound (2.15) applied to function G along with (2.18) we obtain

$$\|\Delta u\|^2_{L^2(\mathbb{R}^2)} \leq \varepsilon^2 \|G\|^2_{L^1(\mathbb{R}^2)} Q^2 \leq \varepsilon^2 \frac{M^2}{4} (\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^4 Q^2. \tag{2.20}$$

Therefore, by virtue of the definition of the norm (1.4) with $s = 1$ along with bounds (2.19) and (2.20) we arrive at the estimate from above for $\|u\|_{H^2(\mathbb{R}^2)}$ as

$$\varepsilon(\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^2 M \left[\frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R}^2)}(\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^{4s-2}}{(1-2s)(32\pi s)^{2s}} + \frac{Q^2}{4} \right]^{\frac{1}{2}} \leq \rho$$

for all $\varepsilon > 0$ sufficiently small. Therefore, $u(x) \in B_\rho$ as well. If for some $v(x) \in B_\rho$ there exist two solutions $u_{1,2}(x) \in B_\rho$ of equation (1.9), their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^2)$ solves

$$(-\Delta)^s w = 0.$$

Because the operator $(-\Delta)^s, \ 0 < s < \dfrac{1}{2}$ considered on the whole $\mathbb{R}^2$ does not have nontrivial square integrable zero modes, $w(x) = 0$ a.e. on $\mathbb{R}^2$. Thus, equation (1.9) defines a map $T_g : B_\rho \to B_\rho$ for all $\varepsilon > 0$ small enough.

Our goal is to establish that this map is a strict contraction. Let us choose arbitrarily $v_{1,2}(x) \in B_\rho$. The argument above gives us $u_{1,2} := T_g v_{1,2} \in B_\rho$ as well. By means of (1.9) we arrive at

$$(-\Delta)^s u_1(x) = \varepsilon \int_{\mathbb{R}^2} \mathcal{K}(x-y) g(u_0(y) + v_1(y)) dy, \tag{2.21}$$

$$(-\Delta)^s u_2(x) = \varepsilon \int_{\mathbb{R}^2} \mathcal{K}(x-y) g(u_0(y) + v_2(y)) dy, \tag{2.22}$$

$0 < s < \dfrac{1}{2}$. We define

$$G_1(x) := g(u_0(x) + v_1(x)), \quad G_2(x) := g(u_0(x) + v_2(x))$$

and apply the standard Fourier transform (2.14) to both sides of equations (2.21) and (2.22). This yields

$$\widehat{u_1}(p) = \varepsilon 2\pi \frac{\widehat{\mathcal{K}}(p)\widehat{G_1}(p)}{|p|^{2s}}, \quad \widehat{u_2}(p) = \varepsilon 2\pi \frac{\widehat{\mathcal{K}}(p)\widehat{G_2}(p)}{|p|^{2s}}.$$

Clearly,

$$\|u_1 - u_2\|^2_{L^2(\mathbb{R}^2)} = \varepsilon^2 4\pi^2 \int_{\mathbb{R}^2} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G_1}(p) - \widehat{G_2}(p)|^2}{|p|^{4s}} dp.$$

Evidently, it can be estimated from above by virtue of inequality (2.15) by

$$\varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^2)} \left\{ \frac{1}{2\pi} \|G_1(x) - G_2(x)\|^2_{L^1(\mathbb{R}^2)} \frac{R^{2-4s}}{2-4s} + \|G_1(x) - G_2(x)\|^2_{L^2(\mathbb{R}^2)} \frac{1}{R^{4s}} \right\},$$

where $R \in (0, +\infty)$. We use the identity

$$G_1(x) - G_2(x) = \int_{u_0+v_2}^{u_0+v_1} g'(z) dz.$$

Therefore,

$$|G_1(x) - G_2(x)| \leq sup_{z \in I} |g'(z)| |v_1(x) - v_2(x)| \leq M |v_1(x) - v_2(x)|.$$

Hence,

$$\|G_1(x) - G_2(x)\|_{L^2(\mathbb{R}^2)} \leq M \|v_1 - v_2\|_{L^2(\mathbb{R}^2)} \leq M \|v_1 - v_2\|_{H^2(\mathbb{R}^2)}.$$

Evidently,

$$G_1(x) - G_2(x) = \int_{u_0+v_2}^{u_0+v_1} dy \Big[\int_0^y g''(z) dz \Big].$$

We obtain the upper bound for $G_1(x) - G_2(x)$ in the absolute value as

$$\frac{1}{2} sup_{z \in I} |g''(z)| |(v_1 - v_2)(2u_0 + v_1 + v_2)| \leq \frac{M}{2} |(v_1 - v_2)(2u_0 + v_1 + v_2)|.$$

The Schwarz inequality yields the upper bound for the norm $\|G_1(x) - G_2(x)\|_{L^1(\mathbb{R}^2)}$ as

$$\frac{M}{2} \|v_1 - v_2\|_{L^2(\mathbb{R}^2)} \|2u_0 + v_1 + v_2\|_{L^2(\mathbb{R}^2)} \leq$$
$$\leq M \|v_1 - v_2\|_{H^2(\mathbb{R}^2)} (\|u_0\|_{H^2(\mathbb{R}^2)} + 1). \tag{2.23}$$

Therefore, we arrive at the estimate from above for the norm $\|u_1(x) - u_2(x)\|^2_{L^2(\mathbb{R}^2)}$ given by

$$\varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^2)} M^2 \|v_1 - v_2\|^2_{H^2(\mathbb{R}^2)} \Big\{ \frac{1}{2\pi} (\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^2 \frac{R^{2-4s}}{2 - 4s} + \frac{1}{R^{4s}} \Big\}.$$

Lemma 4 allows us to minimize the expression above over $R \in (0, +\infty)$ to obtain the upper bound for $\|u_1(x) - u_2(x)\|^2_{L^2(\mathbb{R}^2)}$ as

$$\varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^2)} M^2 \|v_1 - v_2\|^2_{H^2(\mathbb{R}^2)} \frac{(\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^{4s}}{(1 - 2s)(8\pi s)^{2s}}. \tag{2.24}$$

Formulas (2.21) and (2.22) give us

$$(-\Delta)(u_1 - u_2)(x) = \varepsilon (-\Delta)^{1-s} \int_{\mathbb{R}^2} \mathcal{K}(x - y)[G_1(y) - G_2(y)] dy.$$

By means of inequalities (2.15) and (2.23) we arrive at

$$\|\Delta(u_1 - u_2)\|^2_{L^2(\mathbb{R}^2)} \leq \varepsilon^2 Q^2 \|G_1 - G_2\|^2_{L^1(\mathbb{R}^2)} \leq$$

$$\leq \varepsilon^2 Q^2 M^2 \|v_1 - v_2\|^2_{H^2(\mathbb{R}^2)} (\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^2. \tag{2.25}$$

By virtue of (2.24) and (2.25) the norm $\|u_1 - u_2\|_{H^2(\mathbb{R}^2)}$ can be estimated from above by the expression $\varepsilon M(\|u_0\|_{H^2(\mathbb{R}^2)} + 1)\times$

$$\times \left\{ \frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R}^2)} (\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^{4s-2}}{(1-2s)(8\pi s)^{2s}} + Q^2 \right\}^{\frac{1}{2}} \|v_1 - v_2\|_{H^2(\mathbb{R}^2)}. \tag{2.26}$$

Therefore, our map $T_g : B_\rho \to B_\rho$ defined by equation (1.9) is a strict contraction for all values of $\varepsilon > 0$ small enough. Its unique fixed point $u_p(x)$ is the only solution of problem (1.7) in the ball B_ρ. The resulting $u(x) \in H^2(\mathbb{R}^2)$ given by (1.6) is a solution of equation (1.2). ∎

Then we proceed to the proof of the second main proposition of our article.

3. The continuity of the fixed point of the map T_g

Proof of Theorem 5. Evidently, for all $0 < \varepsilon < \delta$ we have

$$u_{p,1} = T_{g_1} u_{p,1}, \quad u_{p,2} = T_{g_2} u_{p,2}.$$

Hence

$$u_{p,1} - u_{p,2} = T_{g_1} u_{p,1} - T_{g_1} u_{p,2} + T_{g_1} u_{p,2} - T_{g_2} u_{p,2}.$$

Therefore,

$$\|u_{p,1} - u_{p,2}\|_{H^2(\mathbb{R}^2)} \leq \|T_{g_1} u_{p,1} - T_{g_1} u_{p,2}\|_{H^2(\mathbb{R}^2)} + \|T_{g_1} u_{p,2} - T_{g_2} u_{p,2}\|_{H^2(\mathbb{R}^2)}.$$

By means of bound (2.26), we arrive at

$$\|T_{g_1} u_{p,1} - T_{g_1} u_{p,2}\|_{H^2(\mathbb{R}^2)} \leq \varepsilon\sigma \|u_{p,1} - u_{p,2}\|_{H^2(\mathbb{R}^2)}.$$

Here $\varepsilon\sigma < 1$ because the map $T_{g_1} : B_\rho \to B_\rho$ under our assumptions is a strict contraction and the positive constant

$$\sigma := M(\|u_0\|_{H^2(\mathbb{R}^2)} + 1) \left\{ \frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R}^2)} (\|u_0\|_{H^2(\mathbb{R})^2} + 1)^{4s-2}}{(1-2s)(8\pi s)^{2s}} + Q^2 \right\}^{\frac{1}{2}}.$$

Hence, we obtain

$$(1 - \varepsilon\sigma)\|u_{p,1} - u_{p,2}\|_{H^2(\mathbb{R}^2)} \leq \|T_{g_1} u_{p,2} - T_{g_2} u_{p,2}\|_{H^2(\mathbb{R}^2)}. \tag{3.27}$$

Note that for our fixed point $T_{g_2} u_{p,2} = u_{p,2}$ and denote $\xi(x) := T_{g_1} u_{p,2}$. We arrive at

$$(-\Delta)^s \xi(x) = \varepsilon \int_{\mathbb{R}^2} \mathcal{K}(x-y) g_1(u_0(y) + u_{p,2}(y)) dy, \tag{3.28}$$

$$(-\Delta)^s u_{p,2}(x) = \varepsilon \int_{\mathbb{R}^2} \mathcal{K}(x-y) g_2(u_0(y) + u_{p,2}(y)) dy, \tag{3.29}$$

where $0 < s < \frac{1}{2}$. Let $G_{1,2}(x) := g_1(u_0(x) + u_{p,2}(x))$ and $G_{2,2}(x) := g_2(u_0(x) + u_{p,2}(x))$. We apply the standard Fourier transform (2.14) to both sides of equations (3.28) and (3.29). This yields

$$\widehat{\xi}(p) = \varepsilon 2\pi \frac{\widehat{\mathcal{K}}(p)\widehat{G}_{1,2}(p)}{|p|^{2s}}, \quad \widehat{u}_{p,2}(p) = \varepsilon 2\pi \frac{\widehat{\mathcal{K}}(p)\widehat{G}_{2,2}(p)}{|p|^{2s}}.$$

Evidently,

$$\|\xi(x) - u_{p,2}(x)\|^2_{L^2(\mathbb{R}^2)} = \varepsilon^2 4\pi^2 \int_{\mathbb{R}^2} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G}_{1,2}(p) - \widehat{G}_{2,2}(p)|^2}{|p|^{4s}} dp.$$

Apparently, it can be bounded from above by means of (2.15) by

$$\varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^2)} \left\{ \frac{1}{4\pi} \|G_{1,2} - G_{2,2}\|^2_{L^1(\mathbb{R}^2)} \frac{R^{2-4s}}{1-2s} + \|G_{1,2} - G_{2,2}\|^2_{L^2(\mathbb{R}^2)} \frac{1}{R^{4s}} \right\},$$

with $R \in (0, +\infty)$. We use the identity

$$G_{1,2}(x) - G_{2,2}(x) = \int_0^{u_0(x)+u_{p,2}(x)} [g_1'(z) - g_2'(z)] dz.$$

Thus

$$|G_{1,2}(x) - G_{2,2}(x)| \leq sup_{z \in I} |g_1'(z) - g_2'(z)| |u_0(x) + u_{p,2}(x)| \leq$$

$$\leq \|g_1 - g_2\|_{C_2(I)} |u_0(x) + u_{p,2}(x)|.$$

Hence

$$\|G_{1,2} - G_{2,2}\|_{L^2(\mathbb{R}^2)} \leq \|g_1 - g_2\|_{C_2(I)} \|u_0 + u_{p,2}\|_{L^2(\mathbb{R}^2)} \leq$$
$$\leq \|g_1 - g_2\|_{C_2(I)} (\|u_0\|_{H^2(\mathbb{R}^2)} + 1).$$

Another useful representation formula would be

$$G_{1,2}(x) - G_{2,2}(x) = \int_0^{u_0(x)+u_{p,2}(x)} dy \Big[\int_0^y (g_1''(z) - g_2''(z)) dz \Big].$$

Therefore,

$$|G_{1,2}(x) - G_{2,2}(x)| \leq \frac{1}{2} sup_{z \in I} |g_1''(z) - g_2''(z)| |u_0(x) + u_{p,2}(x)|^2 \leq$$

$$\leq \frac{1}{2} \|g_1 - g_2\|_{C_2(I)} |u_0(x) + u_{p,2}(x)|^2.$$

Hence,

$$\|G_{1,2} - G_{2,2}\|_{L^1(\mathbb{R}^2)} \leq \frac{1}{2}\|g_1 - g_2\|_{C_2(I)}\|u_0 + u_{p,2}\|^2_{L^2(\mathbb{R}^2)} \leq$$

$$\leq \frac{1}{2}\|g_1 - g_2\|_{C_2(I)}(\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^2. \tag{3.30}$$

This allows us to obtain the upper bound for the norm $\|\xi(x) - u_{p,2}(x)\|^2_{L^2(\mathbb{R}^2)}$ as

$$\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^2)}(\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^2\|g_1 - g_2\|^2_{C_2(I)}\Big[\frac{1}{16\pi}(\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^2\frac{R^{2-4s}}{1-2s} + \frac{1}{R^{4s}}\Big].$$

This expression can be trivially minimized over $R \in (0, +\infty)$ via Lemma 4. We arrive at the estimate

$$\|\xi(x) - u_{p,2}(x)\|^2_{L^2(\mathbb{R}^2)} \leq \varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^2)}(\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^{2+4s}\frac{\|g_1 - g_2\|^2_{C_2(I)}}{(1-2s)(32\pi s)^{2s}}.$$

Formulas (3.28) and (3.29) yield

$$-\Delta\xi(x) = \varepsilon(-\Delta)^{1-s}\int_{\mathbb{R}^2}\mathcal{K}(x-y)G_{1,2}(y)dy,$$

$$-\Delta u_{p,2}(x) = \varepsilon(-\Delta)^{1-s}\int_{\mathbb{R}^2}\mathcal{K}(x-y)G_{2,2}(y)dy.$$

Then by means of (2.15) and (3.30), the norm $\|\Delta[\xi(x) - u_{p,2}(x)]\|^2_{L^2(\mathbb{R}^2)}$ can be estimated from above by

$$\varepsilon^2\|G_{1,2} - G_{2,2}\|^2_{L^1(\mathbb{R}^2)}Q^2 \leq \frac{\varepsilon^2 Q^2}{4}(\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^4\|g_1 - g_2\|^2_{C_2(I)}.$$

Thus, $\|\xi(x) - u_{p,2}(x)\|_{H^2(\mathbb{R}^2)} \leq$

$$\leq \varepsilon\|g_1 - g_2\|_{C_2(I)}(\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^2\left[\frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R}^2)}(\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^{4s-2}}{(1-2s)(32\pi s)^{2s}} + \frac{Q^2}{4}\right]^{\frac{1}{2}}.$$

By virtue of inequality (3.27), the norm $\|u_{p,1} - u_{p,2}\|_{H^2(\mathbb{R}^2)}$ can be bounded from above by

$$\frac{\varepsilon}{1-\varepsilon\sigma}(\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^2\left[\frac{\|\mathcal{K}\|^2_{L^1(\mathbb{R}^2)}(\|u_0\|_{H^2(\mathbb{R}^2)} + 1)^{4s-2}}{(1-2s)(32\pi s)^{2s}} + \frac{Q^2}{4}\right]^{\frac{1}{2}}\|g_1 - g_2\|_{C_2(I)},$$

which completes the proof of our theorem. ■

4. Auxiliary results

Below we state the solvability conditions for the linear Poisson type equation with a square integrable right side

$$(-\Delta)^s u = f(x), \quad x \in \mathbb{R}^2, \quad 0 < s < 1. \tag{4.31}$$

We designate the inner product as

$$(f(x), g(x))_{L^2(\mathbb{R}^2)} := \int_{\mathbb{R}^2} f(x)\bar{g}(x)dx, \tag{4.32}$$

with a slight abuse of notations when the functions involved in (4.32) are not square integrable, like for example the one present in orthogonality condition (4.33) of Lemma 6 below. Indeed, if $f(x) \in L^1(\mathbb{R}^2)$ and $g(x) \in L^\infty(\mathbb{R}^2)$, then the integral in the right side of (4.32) makes sense. We have the following technical statement, which can be easily proved by applying the standard Fourier transform (2.14) to both sides of equation (4.31) (see the part b) of the first theorem of [35] and for $s = \dfrac{1}{2}$ the part 2) of Lemma 3.1 of [33]).

Lemma 6. *Let* $f(x) : \mathbb{R}^2 \to \mathbb{R}$ *and* $f(x) \in L^2(\mathbb{R}^2)$.

1) When $0 < s < \frac{1}{2}$ *and in addition* $f(x) \in L^1(\mathbb{R}^2)$*, problem (4.31) has a unique solution* $u(x) \in H^{2s}(\mathbb{R}^2)$.

2) When $\frac{1}{2} \leq s < 1$ *and additionally* $|x|f(x) \in L^1(\mathbb{R}^2)$*, equation (4.31) admits a unique solution* $u(x) \in H^{2s}(\mathbb{R}^2)$ *if and only if the orthogonality condition*

$$(f(x), 1)_{L^2(\mathbb{R}^2)} = 0 \tag{4.33}$$

holds.

Note that for the lower values of the power of the negative Laplace operator $0 < s < \dfrac{1}{2}$ under the conditions given above no orthogonality relations are required to solve the linear Poisson type problem (4.31) in $H^{2s}(\mathbb{R}^2)$.

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