EIGENVALUE BOUNDS FOR STARK OPERATORS WITH COMPLEX POTENTIALS

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Abstract. We consider the 3-dimensional Stark operator perturbed by a complex-valued potential. We obtain an estimate for the number of eigenvalues of this operator as well as for the sum of imaginary parts of eigenvalues situated in the upper half-plane.

1. Introduction and main results

Let H_0 be the free Stark operator

$$H_0 = -\Delta + x_1, \tag{1.1}$$

acting in the space $L^2(\mathbb{R}^3)$. In the formula above, x_1 denotes the function whose value at a point $x \in \mathbb{R}^3$ coinsides with the first coordinate of x. Since H_0 is an unbounded operator, one has to specify its domain of definition. For this purpose we simply mention that H_0 is essentially selfadjoint on $C_0^\infty(\mathbb{R}^3)$. We study the spectral properties of the operator

$$H = H_0 + V,$$

where the potential V is a bounded complex-valued function, satisfying

$$\int_{\mathbb{R}^3} (1 + |x_1|)^{3/2} |V(x)| dx < \infty. \tag{1.2}$$

While the interest of mathematicians in the theory of non-selfadjoint operators of this type is quite new, Stark operators with real potentials have been studied thoroughly in mathematical physics for a long time. Among the classical results applicable to the self-adjoint case are the theorems of Avron and Herbst [2] who considered scattering for the pair of operators H and H_0 in the case where V is a short-range potential. In particular, it was established that the spectrum of H is purely absolutely continuous and covers the real line $\mathbb{R}$ (besides [2], see Herbst [8]). It was proved in [2], [8] and [33] that for a short-range potential V, the wave operators

$$\Omega_\pm = s - \lim e^{itH} e^{-itH_0}, \qquad \text{as} \qquad t \to \pm\infty,$$

exist and are unitary. Further development of the methods used to study Stark operators led to the theory of scattering of several particles in an external constant electric field (see the papers [11] and [19]).

Besides the results related to the scattering theory, the mathematical literature on Stark operators contains numerous statements about the distribution of resonances in the models involving a constant electric field. Here, we only mention the article [9] and the recent paper [20], which can be also used for finding other relevant references.

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Let us now describe the main results of the present paper devoted to the non-selfadjoint case. Under the condition (1.2), the spectrum $\sigma(H)$ of H coincides with the union of the line $\mathbb{R}$ and the discrete set of complex eigenvalues that might accumulate only to real points. We denote by $\{\lambda_j\}_{j=1}^\infty$ the sequence of eigenvalues of H in $\mathbb{C}\setminus\mathbb{R}$ enumerated in an arbitrary order. The number of times an eigenvalue appears in the sequence $\{\lambda_j\}_{j=1}^\infty$ coincides with the algebraic multiplicity of the eigenvalue. We will show that the condition (1.2) guarantees that

$$\sum_j |\operatorname{Im}\lambda_j| < \infty.$$

Theorem 1.1. *Let V be a bounded complex-valued function satisfying the condition* (1.2). *Then the eigenvalues λ_j of the operator H obey the estimate*

$$\sum_j |\operatorname{Im}\lambda_j| \leqslant C\Big[\Big(\int_{\mathbb{R}^3}|V|^2dx\Big)^{1/2} + \int_{\mathbb{R}^3}(1+|x_1|)^{3/2}|V|dx\Big]^5.$$

The constant C in this inequality is independent of V.

As a consequence of the method used in the proof of Theorem 1.1 we will obtain the following statement, where V might decay slower than a potential satisfying (1.2).

Theorem 1.2. *Let $\mathcal{P}_q$ be the function defined on $\mathbb{R}^3$ by*

$$\mathcal{P}_q(x) = (1+|x|)^q(1+|x_1|)^{15/2}, \qquad q > 12.$$

Let V be a bounded complex-valued potential such that

$$\int_{\mathbb{R}^3}\mathcal{P}_q(x)|V(x)|^{5+(p-1)/2}\,dx < \infty, \qquad p > 1.$$

Then the spectrum of H is discrete in $\mathbb{C}\setminus\mathbb{R}$. Moreover, the eigenvalues λ_j of the operator H satisfy the estimate

$$\sum_j |\operatorname{Im}\lambda_j|^p \leqslant C_{p,q}||V||_{L^\infty}^{(p-1)/2}\int_{\mathbb{R}^3}\mathcal{P}_q(x)|V(x)|^{5+(p-1)/2}\,dx \tag{1.3}$$

with a positive constant $C_{p,q} > 0$ depending only on $p > 1$ and $q > 12$.

The next theorem gives a bound on the number of eigenvalues of H in the half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < \alpha\}$ under the condition

$$\int_{\mathbb{R}^3}\mathcal{P}(x_1)|V(x)|dx + \int_{\mathbb{R}^3}\mathcal{P}^2(x_1)|V(x)|^2dx < \infty \tag{1.4}$$

where

$$\mathcal{P}(x_1) = (1+|x_1|)^2(1+e^{-x_1}). \tag{1.5}$$

Functions V satisfying the condition (1.4) decay exponentially fast in some integral sense in the direction of the negative x_1-axis. However, such potentials do not have to decay exponentially fast in other directions. For instance,

$$V(x) = \frac{v_0}{(1+|x_1|)^2(1+e^{-x_1})(1+|x|)^p}, \qquad p > 3,\ v_0 \in \mathbb{C}, \qquad \text{is fine.}$$

Theorem 1.3. *Let V be a bounded complex-valued function satisfying* (1.4) *with $\mathcal{P}(x_1)$ defined by* (1.5) *and let $\alpha > 0$. Then the number $N(\alpha)$ of non-real eigenvalues of the operator H situated in the half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < \alpha\}$ obeys the estimate*

$$N(\alpha) \leqslant C_\alpha \Big[\int_{\mathbb{R}^3} \mathcal{P}(x_1)|V(x)|dx + \Big(\int_{\mathbb{R}^3} \mathcal{P}^2(x_1)|V(x)|^2 dx \Big)^{1/2} \Big]^{12}, \tag{1.6}$$

where

$$C_\alpha = C \min_{\varepsilon>0} \Big[\varepsilon^{-2} e^{12(\alpha+\varepsilon)} \Big(\frac{\alpha+\varepsilon}{\varepsilon^{7/5}} + (1+\varepsilon^2)e^{12\varepsilon^2} + \frac{1}{(\alpha+\varepsilon)^{2/5}} \Big) \Big],$$

and $C > 0$ is a universal constant.

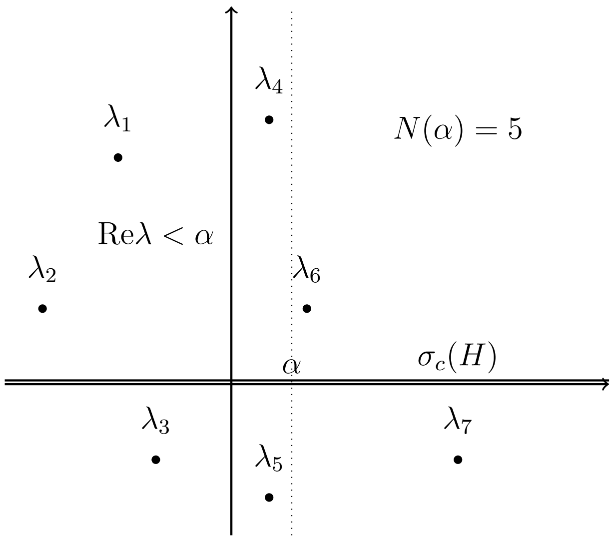

Fig. 1. Eigenvalues of H

One can combine this theorem with the fact that all eigenvalues are situated in a disk of a finite radius, to obtain an estimate for the total number of non-real eigenvalues.

Theorem 1.4. *Let $V \in L^\infty(\mathbb{R}^3)$. There exists a universal constant $C > 0$, such that all eigenvalues $\lambda \in \mathbb{C} \setminus \mathbb{R}$ of the operator H are situated in the disk*

$$|\lambda| \leqslant C \Big(\int_{\mathbb{R}^3} (1+|x|)^4 |V(x)|\, dx + \Big(\int_{\mathbb{R}^3} |V|^2 dx \Big)^{1/2} \Big)^4. \tag{1.7}$$

In particular, the condition

$$\int_{\mathbb{R}^3} (1+|x|)^4 (1+e^{-x_1}) |V(x)| dx + \int_{\mathbb{R}^3} \mathcal{P}^2(x_1)|V(x)|^2 dx < \infty, \tag{1.8}$$

with $\mathcal{P}$ as in (1.5)*, implies that the total number N of non-real eigenvalues of the operator H is finite*

$$(1.8) \implies N < \infty$$

and coincides with $N(\alpha)$, where α equals the right hand side of (1.7)*.*

Let us say a couple of words about our approach to the problem. It is well known that eigenvalues of most important differential operators can be described as zeros of the corresponding perturbation determinants, which depend analytically on the spectral parameter. The latter observation allows one to turn the analysis of eigenvalues into the study of zeros of analytic functions. Similar ideas were successfully used in the paper [6] by Frank and Sabin for the study of the eigenvalues of the Schrödinger operator perturbed by a decaying potential. Among other related papers are the articles [5], [7]. The problem pertaining to the Stark operator is however more complicated compared to the one involving the usual Schrödinger equation, simply because the free Stark operator is not diagonalized by the Fourier transformation.

Notations. We denote by C various possibly different constants whose values are irrelevant. The upper half-plane $\{\lambda \in \mathbb{C} : \mathrm{Im}\lambda > 0\}$ will be denoted by the symbol $\mathbb{C}_+$. By $\mathcal{B}$ and $\mathfrak{S}_\infty$ we denote the classes of bounded and compact operators, respectively. The symbols $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are used to denote the trace class and the Hilbert-Schmidt class equipped with the norms $\|\cdot\|_{\mathfrak{S}_1}$ and $\|\cdot\|_{\mathfrak{S}_2}$, respectively. More generally, $\mathfrak{S}_p$ denotes the class of compact operators K obeying

$$||K||^p_{\mathfrak{S}_p} = \mathrm{tr}\Big(K^*K\Big)^{p/2} < \infty, \qquad p \geqslant 1.$$

Note that if $K \in \mathfrak{S}_p$ for some $p \geqslant 1$, then $K \in \mathfrak{S}_q$ for $q > p$ and

$$||K||_{\mathfrak{S}_q} \leqslant ||K||_{\mathfrak{S}_p}.$$

For a self-adjoint operator $T = T^*$ the symbol $E_T(\cdot)$ denotes its (operator-valued) spectral measure.

2. Preliminaries

Very often, eigenvalues of closed operators can be described as zeros of analytic functions. The latter circumstance allows one to use known results on the distribution of zeros of holomorphic functions to obtain bounds on the eigenvalues of a given operator. In particular, the eigenvalues of H coincide with zeros of the so called perturbation determinant $D_n(\lambda)$, which depends analytically on λ.

The definition of $D_n(\lambda)$ requires that we find two functions W_1 and W_2 having the properties

$$V = W_2W_1, \qquad |W_1| = |W_2|,$$

and set

$$Y_0(\lambda) = W_1\, R_0(\lambda)\, W_2, \qquad R_0(\lambda) = (H_0 - \lambda)^{-1}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{2.9}$$

The condition (1.2) implies that $Y_0(\lambda)$ is a Hilbert-Schmidt operator whenever $\mathrm{Im}\lambda \neq 0$. Therefore, we can define the determinants

$$D_n(\lambda) = \mathrm{det}_n(I + Y_0(\lambda)), \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{2.10}$$

for $n \geqslant 2$. The standard way to describe $\mathrm{det}_n(I + K)$ in terms of eigenvalues z_j of a compact operator $K \in \mathfrak{S}_n$ is to define it as

$$\mathrm{det}_n(I + K) = \prod_j (1 + z_j) \exp\Big(\sum_{m=1}^{n-1} \frac{(-1)^m z_j^m}{m}\Big), \qquad n \geqslant 2;$$

$$\mathrm{det}(I + K) = \prod_j (1 + z_j), \qquad n = 1.$$

The following relations can be found in Section 3 of the book [31]. If $X, Y \in \mathcal{B}$ and both products XY, YX belong to $\mathfrak{S}_n$, then

$$\mathrm{det}_n(I + XY) = \mathrm{det}_n(I + YX). \tag{2.11}$$

The mapping $X \to \det(I+X)$ is continuous on $\mathfrak{S}_1$, which is guaranteed by the inequality

$$|\det(I+X) - \det(I+Y)| \leqslant \|X-Y\|_{\mathfrak{S}_1} e^{1+\|X\|_{\mathfrak{S}_1}+\|Y\|_{\mathfrak{S}_1}}. \tag{2.12}$$

Moreover, there exists a constant $C_n > 0$ depending only on n such that

$$|\det{}_n(I+X)| \leqslant e^{C_n\|X\|^n_{\mathfrak{S}_n}}, \qquad \forall X \in \mathfrak{S}_n. \tag{2.13}$$

If an operator-valued function $X : \Omega \to \mathfrak{S}_1$ is analytic on a domain $\Omega \subset \mathbb{C}$ and $(I+X(z))^{-1} \in \mathcal{B}$ for all $z \in \Omega$, then the function $F(z) = \det(I+X(z))$ is also analytic and its derivative satisfies the relation

$$F'(z) = F(z)\,\mathrm{Tr}\Big((I+X(z))^{-1}X'(z)\Big), \qquad z \in \Omega. \tag{2.14}$$

Similarly, if an operator-valued function $X : \Omega \to \mathfrak{S}_n$, (here, $n \geqslant 2$) is analytic on a domain $\Omega \subset \mathbb{C}$ and $(I+X(z))^{-1} \in \mathcal{B}$ for all $z \in \Omega$, then the function $F(z) = \det_n(I+X(z))$ is analytic and its derivative equals

$$F'(z) = F(z)\,\mathrm{Tr}\Big(\big((I+X(z))^{-1} - \sum_{j=0}^{n-2}(-1)^j X^j\big)X'(z)\Big), \qquad z \in \Omega. \tag{2.15}$$

Let $n \geqslant 2$ be integer. We will show that if V is a bounded function satisfying (1.2), then $Y_0(\lambda)$ is a Hilbert-Schmidt operator for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$. The latter condition implies that the function $D_n(\lambda)$ is analytic on the open domain $\mathbb{C}\setminus\mathbb{R}$. The following statement is known as the Birman-Schwinger principle (for more detailed description, see [7]).

Lemma 2.1. *Let $n \geqslant 2$ be integer. Let $V \in L^\infty(\mathbb{R}^3)$ satisfy* (1.2). *The point $\lambda \in \mathbb{C}\setminus\mathbb{R}$ is an eigenvalue of the operator H if and only if λ is a zero of $D_n(\lambda)$. The algebraic multiplicity of each eigenvalue $\lambda \in \sigma(H)\setminus\mathbb{R}$ coincides with the multiplicity of the corresponding zero of the function $D_n(\cdot)$.*

3. Estimates of the norms of the Birman-Schwinger operator

Recall that the integral kernels of the operators $(-\Delta-\lambda)^{-1}$ and $e^{it\Delta}$ on $L^2(\mathbb{R}^3)$ are functions of the form

$$(-\Delta-\lambda)^{-1}(x,y) = \frac{1}{4\pi|x-y|}e^{i\sqrt{\lambda}|x-y|}, \qquad \lambda \in \mathbb{C}\setminus\overline{\mathbb{R}}_+, \quad \sqrt{\lambda} \in \mathbb{C}_+, \tag{3.1}$$

$$(e^{it\Delta})(x,y) = \frac{e^{-i3\pi/4}}{(4\pi t)^{3/2}}e^{i|x-y|^2/4t}, \qquad t > 0, \tag{3.2}$$

$x, y \in \mathbb{R}^3$. Let $H_0 = -\Delta + x_1$ be the Stark operator. We are going to use the representation of $\exp(-itH_0)$ as a product of different factors, one of which is $\exp(it\Delta)$. This formula was discovered in [2] and is given by

$$e^{-itH_0} = e^{-itx_1}e^{it\Delta}e^{t^2\frac{\partial}{\partial x_1}}e^{-i\frac{t^3}{3}}, \qquad \forall\, t \in \mathbb{R}. \tag{3.3}$$

where $x_1 = (x, e)$. Consequently, the resolvent operator $R_0(\lambda) = (H_0-\lambda)^{-1}$ can be formally written as the integral

$$R_0(\lambda) = i\int_0^\infty e^{-it(H_0-\lambda)}dt = i\int_0^\infty e^{-itx_1}e^{t^2\frac{\partial}{\partial x_1}}e^{it\Delta}e^{it\lambda-i\frac{t^3}{3}}dt. \tag{3.4}$$

If $\operatorname{Im}\lambda > 0$, then the integrals in (3.4) converge (absolutely) in the operator-norm topology. Throughout below, we use the following convenient notation

$$\Lambda = \lambda - 2^{-1}(x_1 + y_1).$$

Proposition 3.1. *The integral kernel $r_0(x, y, \lambda)$ of the operator $R_0(\lambda)$ equals*

$$r_0(x, y, \lambda) = \frac{e^{i\sqrt{\Lambda}|x-y|}}{4\pi|x-y|} + \frac{e^{-i\frac{\pi}{4}}}{\sqrt{(4\pi)^3}} \int_0^\infty e^{\frac{i}{4t}|x-y|^2}\Big(e^{-i\frac{t^3}{12}} - 1\Big)e^{it\Lambda}\frac{dt}{t^{3/2}}, \tag{3.5}$$

for $x, y \in \mathbb{R}^3$ and $\lambda \in \mathbb{C}_+$.

Proof. Indeed, according to (3.4),

$$r_0(x, y, \lambda) = i\int_0^\infty e^{-itx_1}e^{it\lambda - i\frac{t^3}{3}}\Big((2\pi)^{-3}\int_{\mathbb{R}^3} e^{ip(x-y)}e^{it^2p_1}e^{-it|p|^2}dp\Big)\,dt.$$

The substitution $\tilde{p} = p - 2^{-1}te_1$ turns this integral into

$$r_0(x, y, \lambda) = i\int_0^\infty e^{-it(x_1+y_1)/2}e^{it\lambda - i\frac{t^3}{12}}\Big((2\pi)^{-3}\int_{\mathbb{R}^3} e^{i\tilde{p}(x-y)}e^{-it|\tilde{p}|^2}d\tilde{p}\Big)\,dt.$$

The formula (3.5) follows from the fact that

$$i(2\pi)^{-3}\int_0^\infty e^{-it(x_1+y_1)/2}e^{it\lambda}\Big(\int_{\mathbb{R}^3} e^{i\tilde{p}(x-y)}e^{-it|\tilde{p}|^2}d\tilde{p}\Big)\,dt = \frac{e^{i\sqrt{\Lambda}|x-y|}}{4\pi|x-y|},$$

combined with the observation that

$$(2\pi)^{-3}\int_{\mathbb{R}^3} e^{i\tilde{p}(x-y)}e^{-it|\tilde{p}|^2}d\tilde{p} = \frac{e^{-i3\pi/4}}{(4\pi t)^{3/2}}e^{i|x-y|^2/4t}.$$

The proof is completed. ■

The following estimate plays a key role in our arguments.

Lemma 3.2. *Let W_1 and W_2 be two functions from the space $L^3(\mathbb{R}^3)$. Then*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|W_1(x)|^2|W_2(y)|^2}{|x-y|^2}dxdy \leqslant C||W_1||_{L^3}^2||W_2||_{L^3}^2. \tag{3.6}$$

where $C > 0$ is independent of W_1 and W_2.

Proof. Note that the function $W_1(x)W_2(y)/(4\pi|x-y|)$ is the integral kernel of the operator $T = W_1(-\Delta)^{-1}W_2$. According to the Cwikel-Lieb-Rozenblum inequality (see [4],[24] and [30]), the number $n(s, T)$ of singular values of T lying to the right of $s > 0$ satisfies the relation

$$n(s, T) \leqslant Cs^{-3/2}||W_1||_{L^3}^{3/2}||W_2||_{L^3}^{3/2}$$

with a constant $C > 0$ independent of W_1 and W_2. In particular, it implies the bound $||T|| \leqslant C^{2/3}||W_1||_{L^3}||W_2||_{L^3}$. It remains to note that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|W_1(x)|^2|W_2(y)|^2}{|x-y|^2}dxdy = ||T||_{\mathfrak{S}_2}^2 = 2\int_0^{||T||} n(s, T)s\,ds.$$

■

Corollary 3.3. *Let W_1 and W_2 be two functions from the space $L^3(\mathbb{R}^3)$. Then*

$$||W_1(-\Delta-\lambda)^{-1}W_2||_{\mathfrak{S}_2} \leqslant C||W_1||^2_{L^3}||W_2||^2_{L^3}, \qquad \forall\lambda\in\overline{\mathbb{C}}_+,$$

where C is the same as in (3.6).

Proof. The function $e^{i\sqrt{\lambda}|x-y|}/(4\pi|x-y|)$ is the kernel of the operator $(-\Delta-\lambda)^{-1}$. Consequently,

$$||W_1(-\Delta-\lambda)^{-1}W_2||^2_{\mathfrak{S}_2} \leqslant \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|W_1(x)|^2|W_2(y)|^2}{|x-y|^2}dxdy.$$

■

Let us now introduce the following convenient notations

$$\rho(\lambda,x,y)=\frac{e^{-i\frac{\pi}{4}}}{\sqrt{(4\pi)^3}}\int_0^\infty e^{\frac{i}{4t}|x-y|^2}\left(e^{-i\frac{t^3}{12}}-1\right)e^{it\Lambda}\frac{dt}{t^{3/2}},$$

and

$$\rho_1(\lambda,x,y)=\rho_0(\lambda-2^{-1}(x_1+y_1),x,y)-\rho_0(\lambda,x,y),$$

where

$$\rho_0(\lambda,x,y):=\frac{1}{4\pi|x-y|}e^{i\sqrt{\lambda}|x-y|}.$$

In these notations,

$$r_0(x,y,\lambda)=\rho_0(\lambda,x,y)+\rho_1(\lambda,x,y)+\rho(\lambda,x,y).$$

This representation of the integral kernel leads to the corresponding decomposition of the resolvent operator

$$R_0(\lambda)=\mathfrak{R}_0(\lambda)+\mathfrak{R}_1(\lambda)+\mathfrak{R}(\lambda).$$

We also need to introduce the characteristic function $\chi_\lambda(x)$ of the set $\{x\in\mathbb{R}^3:\ |x_1|<|\lambda|/2\}$.

Lemma 3.4. *Let V, W_1 and W_3 be three functions on $\mathbb{R}^3$ such that $V=W_2W_1$, and $|W_1|=|W_2|$. Let $\mathfrak{R}_1(\lambda)$ be the integral operator on $L^2(\mathbb{R}^3)$ with the kernel $\rho_1(\lambda,x,y)$. Let C_0 be the best constant in* (3.6). *Then*

$$||W_1\mathfrak{R}_1(\lambda)W_2-\chi_\lambda W_1\mathfrak{R}_1(\lambda)W_2\chi_\lambda||_{\mathfrak{S}_2}\leqslant\frac{2^{5/4}C_0}{(2+|\lambda|)^{1/4}}\Bigl(\int_{\mathbb{R}^3}(1+|x_1|)^{3/4}|V|^{3/2}dx\Bigr)^{2/3}.$$

Proof. Note that due to the fact that square of the Hilbert-Schmidt norm of an operator is the integral of the square of its kernel, we have

$$||W_1\mathfrak{R}_1(\lambda)W_2||_{\mathfrak{S}_2}\leqslant||W_1\mathfrak{R}_0(0)W_2||_{\mathfrak{S}_2}\leqslant C_0||W_1||_{L^3}||W_2||_{L^3}.$$

The statement of the lemma follows from the simple fact that

$$||W_1(1-\chi_\lambda)||_{L^3}+||W_2(1-\chi_\lambda)||_{L^3}\leqslant\frac{2^{5/4}}{(2+|\lambda|)^{1/4}}\Bigl(\int_{\mathbb{R}^3}(1+|x_1|)^{3/4}|V|^{3/2}dx\Bigr)^{1/3}.$$

The proof is completed. ■

Lemma 3.5. *Let $\mathfrak{R}_1(\lambda)$ be the operator with the kernel $\rho_1(\lambda,x,y)$. Let $\lambda\in\overline{\mathbb{C}}_+\setminus\{0\}$. Then*

$$||\chi_\lambda W_1\mathfrak{R}_1(\lambda)W_2\chi_\lambda||_{\mathfrak{S}_2}\leqslant\frac{1}{4\pi(1+|\lambda|)^{1/4}}\Bigl(\int_{\mathbb{R}^3}(1+|x_1|)^{3/2}|V|dx\Bigr).$$

Proof. It is easy to see that the kernel ρ_1 satisfies the estimate

$$|\chi_\lambda(x)\rho_1(\lambda,x,y)\chi_\lambda(y)| \leqslant \frac{1}{4\pi(1+|\lambda|)^{1/4}}\big((1+|x_1|)(1+|y_1|)\big)^{3/4}. \tag{3.7}$$

Indeed, since

$$\rho_1(\lambda,x,y) = \frac{e^{i\sqrt{\Lambda}|x-y|}}{4\pi|x-y|} - \frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|},$$

we obtain that

$$|\chi_\lambda(x)\rho_1(\lambda,x,y))\chi_\lambda(y)| \leqslant \frac{\left|\sqrt{\Lambda}-\sqrt{\lambda}\right|\cdot|x-y|}{4\pi|x-y|} \leqslant \frac{\sqrt{2}}{16\pi|\lambda|^{1/2}}|x_1+y_1|,$$

for $(|x_1|+|y_1|)<|\lambda|$. The latter implies that

$$|\chi_\lambda(x)\rho_1(\lambda,x,y))\chi_\lambda(y)| \leqslant \frac{1}{8\pi}, \qquad \text{for} \quad |\lambda|\leqslant 1.$$

That proves (3.7) for $|\lambda|\leqslant 1$. The estimate (3.7) in the case $|\lambda|>1$ follows from the inequality

$$\frac{\sqrt{2}}{16\pi|\lambda|^{1/2}}|x_1+y_1| \leqslant \frac{1}{4\pi(1+|\lambda|)^{1/4}}|x_1+y_1|^{3/4}, \qquad \text{for } |\lambda|>1.$$

The statement of the lemma immediately follows from (3.7) and the definition of the Hilbert-Schmidt norm. ∎

Corollary 3.6. *Let $\mathfrak{R}_1(\lambda)$ be the operator with the kernel $\rho_1(\lambda,x,y)$. Let C_0 be the best constant in* (3.6)*. Let $\lambda\in\overline{\mathbb{C}}_+\setminus\{0\}$. Then*

$$||W_1\mathfrak{R}_1(\lambda)W_2||_{\mathfrak{S}_2} \leqslant \frac{2^{5/4}C_0}{(2+|\lambda|)^{1/4}}\Big(\int_{\mathbb{R}^3}(1+|x_1|)^{3/4}|V|^{3/2}dx\Big)^{2/3} + \frac{1}{4\pi(1+|\lambda|)^{1/4}}\Big(\int_{\mathbb{R}^3}(1+|x_1|)^{3/2}|V|dx\Big).$$

Let us now consider the operator $W_1\mathfrak{R}_0(\lambda)W_2$, where $\mathfrak{R}_0(\lambda)$ is the operator with the integral kernel $\rho_0(\lambda,x,y)$. According to Lemma 12 of the paper [6], we can state the following:

Theorem 3.7. *The $\mathfrak{S}_p$-norms of the operator $W_1\mathfrak{R}_0(\lambda)W_2$ satisfy the estimates*

$$||W_1\mathfrak{R}_0(\lambda)W_2||_{\mathfrak{S}_p} \leqslant C_q|\lambda|^{-1+3/(2q)}||W_1||_{L^{2q}}||W_1||_{L^{2q}},$$

with $3/2<q\leqslant 2$ and $p=2q/(3-q)$. In particular,

$$||W_1\mathfrak{R}_0(\lambda)W_2||_{\mathfrak{S}_4} \leqslant C|\lambda|^{-1/4}\Big(\int_{\mathbb{R}^3}|V|^2dx\Big)^{1/2}, \tag{3.8}$$

where the positive constant C is independent of V.

Finally, we obtain an estimate for the integral of the Hilbert-Schmidt norms of the operators $W_1\mathfrak{R}(\lambda+i\varepsilon)W_2$, where $\varepsilon\geqslant 0$ and $\lambda\in\mathbb{R}$.

Proposition 3.8. *There is a universal positive constant C such that*

$$\int_{\mathbb{R}}||W_1\mathfrak{R}(\lambda+i\varepsilon)W_2||^5_{\mathfrak{S}_2}d\lambda \leqslant C\Big(\int|V|dx\Big)^5, \qquad \forall\varepsilon\geqslant 0. \tag{3.9}$$

Proof. Let us study the properties of $\rho(\lambda + i\varepsilon, x, y)$ for fixed ε, x and y. It is easy to see that, as a function of $\lambda \in \mathbb{R}$, the integral kernel $\rho(\lambda + i\varepsilon, x, y)$ of the operator $\mathfrak{R}(\lambda + i\varepsilon)$ is the Fourier transform of a function $\psi(t)$ satisfying

$$|\psi(t)| \leqslant C|\sin(t^3/24)|t^{-3/2}.$$

The function in the right hand side belongs simultaneously to $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$. Consequently,

$$\sup_{x,y,\lambda,\varepsilon} |\rho(\lambda + i\varepsilon, x, y)| + \sup_{x,y,\varepsilon} \int_{\mathbb{R}} |\rho(\lambda + i\varepsilon, x, y)|^2 d\lambda \leqslant C < \infty,$$

which implies that

$$\sup_{\lambda} ||W_1\mathfrak{R}(\lambda + i\varepsilon)W_2||^2_{\mathfrak{S}_2} + \int_{\mathbb{R}} ||W_1\mathfrak{R}(\lambda + i\varepsilon)W_2||^2_{\mathfrak{S}_2} d\lambda \leqslant C\Big(\int |V|dx\Big)^2,$$

where C does not depend on ε. Now (3.9) follows from the trivial estimate

$$\int_{\mathbb{R}} ||W_1\mathfrak{R}(\lambda + i\varepsilon)W_2||^5_{\mathfrak{S}_2} d\lambda \leqslant \Big(\sup_{\lambda} ||W_1\mathfrak{R}(\lambda + i\varepsilon)W_2||^3_{\mathfrak{S}_2}\Big) \int_{\mathbb{R}} ||W_1\mathfrak{R}(\lambda + i\varepsilon)W_2||^2_{\mathfrak{S}_2} d\lambda.$$

The proof is completed. ∎

Proposition 3.9. *Let $a(\cdot)$ be an analytic function on $\mathbb{C}_+ = \{\operatorname{Im}\lambda > 0\}$ satisfying*

$$a(\lambda) = 1 + o(|\lambda|^{-1}) \qquad \text{as } |\lambda| \to \infty \text{ in } \mathbb{C}_+. \tag{3.10}$$

Assume that there is a family of positive functions $f_\varepsilon \in L^1(\mathbb{R})$ and a number $\varepsilon_0 > 0$ such that

$$\ln|a(\lambda + i\varepsilon)| \leqslant f_\varepsilon(\lambda), \qquad \forall\lambda \in \mathbb{R}, \qquad \forall\varepsilon \in (0, \varepsilon_0). \tag{3.11}$$

Then the zeroes λ_j of a in $\mathbb{C}_+$, repeated according to multiplicities, satisfy

$$\sum_j \operatorname{Im}\lambda_j \leqslant \frac{1}{2\pi} \sup_{0<\varepsilon<\varepsilon_0} \int_{\mathbb{R}} f_\varepsilon(\lambda)\, d\lambda. \tag{3.12}$$

Proof. Consider first the function $a_\varepsilon(\lambda) = a(\lambda + i\varepsilon)$ for $0 < \varepsilon < \varepsilon_0$. Recall that the Blaschke product for a_ε is

$$B_\varepsilon(\lambda) = \prod_{\operatorname{Im}\lambda_j > \varepsilon} \frac{\lambda + i\varepsilon - \lambda_j}{\lambda - i\varepsilon - \overline{\lambda_j}},$$

Since $a_\varepsilon(\lambda)/B_\varepsilon(\lambda)$ is analytic and non-zero in $\mathbb{C}_+ = \{\operatorname{Im}\lambda > 0\}$, the function $\log(a_\varepsilon(\lambda)/B_\varepsilon(\lambda))$ exists and is analytic there. For $R > 0$ we denote by C_R the contour which consists of the interval $[-R, R]$, traversed from left to right, and the circular part $\Gamma_R := \{\lambda \in \mathbb{C} : |\lambda| = R, \operatorname{Im}\lambda > 0\}$, traversed counterclockwise. Then

$$\int_{C_R} \log\frac{a_\varepsilon(\lambda)}{B_\varepsilon(\lambda)}\, d\lambda = 0,$$

and, therefore,

$$\operatorname{Re}\int_{-R}^{R} \log\frac{a_\varepsilon(x)}{B_\varepsilon(x)}\, dx + \operatorname{Re}\int_{\Gamma_R} \log\frac{a_\varepsilon(\lambda)}{B_\varepsilon(\lambda)}\, d\lambda = 0. \tag{3.13}$$

We note that $|B_\varepsilon(x)| = 1$ if $x \in \mathbb{R}$ and, therefore,

$$\mathrm{Re} \int_{-R}^{R} \log \frac{a_\varepsilon(x)}{B_\varepsilon(x)}\, dx = \int_{-R}^{R} \ln \left| \frac{a_\varepsilon(x)}{B_\varepsilon(x)} \right| dx = \int_{-R}^{R} \ln |a_\varepsilon(x)|\, dx\,.$$

(We denote by ln the natural logarithm to distinguish it from the particular branch of the complex logarithm log chosen before.) On the other hand, by (3.10) and $B_\varepsilon(\lambda) = 1 + O(|\lambda|^{-1})$ (note that the zeros λ_j are contained in a bounded set as a consequence of (3.10)), both $\log a_\varepsilon(\lambda)$ and $\log B_\varepsilon(\lambda)$ are well-defined for all sufficiently large $|\lambda|$ and we have, for all sufficiently large R,

$$\mathrm{Re} \int_{\Gamma_R} \log \frac{a_\varepsilon(\lambda)}{B_\varepsilon(\lambda)}\, d\lambda = \mathrm{Re} \int_{\Gamma_R} \log a_\varepsilon(\lambda)\, d\lambda - \mathrm{Re} \int_{\Gamma_R} \log B_\varepsilon(\lambda)\, d\lambda\,. \tag{3.14}$$

We conclude from (3.13), (3.14) and (3.14) that

$$\mathrm{Re} \int_{\Gamma_R} \log B_\varepsilon(\lambda)\, d\lambda = \int_{-R}^{R} \ln |a_\varepsilon(x)|\, dx + \mathrm{Re} \int_{\Gamma_R} \log a_\varepsilon(\lambda)\, d\lambda \tag{3.15}$$

for all sufficiently large R. We assume that $|\lambda_j - i\varepsilon| < R$ for all j. Since

$$\log B_\varepsilon(\lambda) = 2i \sum_{\mathrm{Im}\,\lambda_j > \varepsilon} \frac{\varepsilon - \mathrm{Im}\,\lambda_j}{\lambda} + O((\lambda)^{-2})\,,$$

we get

$$\int_{\Gamma_R} \log B_\varepsilon(\lambda)\, d\lambda = = -2\pi \sum_{\mathrm{Im}\,\lambda_j > \varepsilon} (\varepsilon - \mathrm{Im}\,\lambda_j) + O(R^{-1}) \qquad \text{as } R \to \infty. \tag{3.16}$$

On the other hand, by (3.10),

$$\mathrm{Re} \int_{\Gamma_R} \log a_\varepsilon(\lambda)\, d\lambda = o(1) \qquad \text{as } R \to \infty\,. \tag{3.17}$$

Finally, by (3.11),

$$\int_{-R}^{R} \ln |a_\varepsilon(x)|\, dx \leqslant \int_{-R}^{R} f_\varepsilon(\lambda) d\lambda \leqslant \int_{-\infty}^{\infty} f_\varepsilon(\lambda) d\lambda\,. \tag{3.18}$$

Relations (3.15), (3.16), (3.17) and (3.18) imply

$$\sum_j (\mathrm{Im}\,\lambda_j - \varepsilon)_+ \leqslant \frac{1}{2\pi} \int_{\mathbb{R}} f_\varepsilon(\lambda)\, d\lambda \leqslant \frac{1}{2\pi} \sup_{0<\varepsilon<\varepsilon_0} \int_{\mathbb{R}} f_\varepsilon(\lambda)\, d\lambda. \tag{3.19}$$

Inequality (3.12) now follows from (3.19) by the monotone convergence theorem. ∎

4. Pointwise estimate of the kernel $\rho(\lambda, x, y)$ and related eigenvalue bounds

Here we obtain an estimate of the Hilbert-Schmidt norm of the Birman-Schwinger operator that allow us to say something about the location of eigenvalues of H in the complex plane. We remind the reader that

$$\Lambda = \lambda - 2^{-1}(x_1 + y_1).$$

Let ρ be the function

$$\rho(\lambda, x, y) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{(4\pi)^3}} \int_0^\infty e^{\frac{i}{4t}|x-y|^2}\Big(e^{-i\frac{t^3}{12}} - 1\Big)e^{it\Lambda}\frac{dt}{t^{3/2}},$$

where $x, y \in \mathbb{R}^3$ and $\lambda \in \overline{\mathbb{C}}_+$.

Proposition 4.1. *There exists a universal constant $C > 0$ such that*

$$|\rho(\lambda, x, y)| \leqslant C\frac{(1 + |x|)^2(1 + |y|)^2}{(1 + |\lambda|)^{1/4}}, \qquad \textit{for all } x, y \in \mathbb{R}^3 \textit{ and } \lambda \in \overline{\mathbb{C}}_+. \tag{4.20}$$

Proof. First, note that the function $\rho(\lambda, x, y)$ is bounded by a constant independent of the variables λ, x and y. The latter impplies that one needs to prove (4.20) only for $|\lambda| > 1$. For this purpose we set $\beta = |\lambda|^{1/2}$ and write $\rho(\lambda, x, y)$ as the sum of two integrals

$$\begin{aligned}\rho(\lambda, x, y) = &\frac{e^{-i\frac{\pi}{4}}}{\sqrt{(4\pi)^3}} \int_0^\beta e^{\frac{i}{4t}|x-y|^2}\Big(e^{-i\frac{t^3}{12}} - 1\Big)e^{it\Lambda}\frac{dt}{t^{3/2}} + \\ &\frac{e^{-i\frac{\pi}{4}}}{\sqrt{(4\pi)^3}} \int_\beta^\infty e^{\frac{i}{4t}|x-y|^2}\Big(e^{-i\frac{t^3}{12}} - 1\Big)e^{it\Lambda}\frac{dt}{t^{3/2}}.\end{aligned} \tag{4.21}$$

The second integral can be easily estimated by $C/|\beta|^{1/2}$, while the first integral equals

$$\frac{ie^{-i\frac{\pi}{4}}}{\lambda\sqrt{(4\pi)^3}} \int_0^\beta e^{it\lambda}\frac{d}{dt}\Big[\frac{e^{\frac{i}{4t}|x-y|^2}\Big(e^{-i\frac{t^3}{12}} - 1\Big)e^{-it(x_1+y_1)/2}}{t^{3/2}}\Big]dt + O(1/|\lambda|^{7/4}),$$

as $|\lambda| \to \infty$. Now we observe that

$$\Big|\frac{d}{dt}\Big[\frac{e^{\frac{i}{4t}|x-y|^2}\Big(e^{-i\frac{t^3}{12}} - 1\Big)e^{-it(x_1+y_1)/2}}{t^{3/2}}\Big]\Big| \leqslant C(t^{-1/2}|x-y|^2 + |x_1| + |y_1| + 1 + t^{1/2})$$

and integrate the expression in the right hand side with respect to t from 0 to β. We will obtain that

$$|\rho(\lambda, x, y)| \leqslant C(|\lambda|^{-3/4}|x-y|^2 + (|x_1| + |y_1| + 1)|\lambda|^{-1/2} + |\lambda|^{-1/4}) \qquad \text{for } |\lambda| > 1.$$

It remains to note that $|x-y|^2 + |x_1| + |y_1| + 1 \leqslant 2(1+|x|)^2(1+|y|)^2$. The proof is completed. ∎

Corollary 4.2. *Let $\mathfrak{R}(\lambda)$ be the operator on $L^2(\mathbb{R}^3)$ with the integral kernel $\rho(\lambda, x, y)$. Let W_1, $W_2 \in C_0^\infty(\mathbb{R}^3)$ be two functions such that $|W_1| = |W_2|$ and let $V = W_1W_2$. Then*

$$||W_1\mathfrak{R}(\lambda)W_2||_{\mathfrak{S}_2} \leqslant \frac{C}{(1 + |\lambda|)^{1/4}} \int_{\mathbb{R}^3} (1 + |x|)^4|V(x)|\, dx, \tag{4.22}$$

where C is the same as in (4.20)

Corollary 4.3. *The exists a universal constant $C>0$, such that all eigenvalues $\lambda\in\mathbb{C}_+$ of the operator H are situated in the disk*

$$|\lambda|^{1/4}\leqslant C\Big(\int_{\mathbb{R}^3}(1+|x|)^4|V(x)|\,dx+\Big(\int_{\mathbb{R}^3}|V|^2dx\Big)^{1/2}\Big).$$

Moreover, there is a universal constant $C_1>0$ such that the condition

$$\int_{\mathbb{R}^3}(1+|x|)^2|V(x)|\,dx+\Big(\int_{\mathbb{R}^3}|V|^2dx\Big)^{1/2}<C_1$$

implies that the spectrum of H coincides with the real line $\mathbb{R}$.

Proof. It is very well known that all non-real eigenvalues of H are situated in the set

$$\{\lambda\in\mathbb{C}_+:\ \ ||Y_0(\lambda)||\geqslant 1)\}.$$

Obviously,

$$||Y_0(\lambda)||=||W_1R_0(\lambda)W_2||\leqslant||W_1\mathfrak{R}_0(\lambda)W_2||_{\mathfrak{S}_4}+||W_1\mathfrak{R}_1(\lambda)W_2||_{\mathfrak{S}_2}+||W_1\mathfrak{R}(\lambda)W_2||_{\mathfrak{S}_2}.$$

On the other hand, according to Theorem 3.7 combined with Corollary 3.3,

$$||W_1\mathfrak{R}_0(\lambda)W_2||_{\mathfrak{S}_4}\leqslant C\frac{1}{1+|\lambda|^{1/4}}\Big[\Big(\int_{\mathbb{R}^3}|V|^2dx\Big)^{1/2}+\Big(\int_{\mathbb{R}^3}(1+|x_1|)^{3/4}|V|^{3/2}dx\Big)^{2/3}\Big].$$

Due to Corollary 3.6, we also have

$$||W_1\mathfrak{R}_1(\lambda)W_2||_{\mathfrak{S}_2}\leqslant C\frac{1}{1+|\lambda|^{1/4}}\Big[\Big(\int_{\mathbb{R}^3}(1+|x_1|)^{3/2}|V|dx\Big)+\Big(\int_{\mathbb{R}^3}(1+|x_1|)^{3/4}|V|^{3/2}dx\Big)^{2/3}\Big].$$

Finally, Corollary 4.2 gives us the estimate

$$||W_1\mathfrak{R}(\lambda)W_2||_{\mathfrak{S}_2}\leqslant\frac{C}{(1+|\lambda|)^{1/4}}\int_{\mathbb{R}^3}(1+|x|)^4|V(x)|\,dx.$$

Consequently,

$$||Y_0(\lambda)||\leqslant\frac{C}{(1+|\lambda|)^{1/4}}\Big[\int_{\mathbb{R}^3}(1+|x|)^4|V(x)|\,dx+\Big(\int_{\mathbb{R}^3}|V|^2dx\Big)^{1/2}\Big]$$

The latter implies both statements of Corollary 4.3. ■

As a consequence of the method, we obtain the following estimate with a very short expression in the right hand side:

Theorem 4.4. *Let* $\mathrm{Im}\lambda\geqslant 0$. *Then for any $p>11$, there exists a positive constant $C_p>0$ depending only on p such that*

$$||Y_0(\lambda)||_{\mathfrak{S}_4}\leqslant\frac{C_p}{1+|\lambda|^{1/4}}\Big(\int_{\mathbb{R}^3}(1+|x|)^p|V|^2dx\Big)^{1/2}.\qquad(4.23)$$

Proof. It is enough to note that

$$\int_{\mathbb{R}^3}(1+|x|)^4|V(x)|\,dx+\Big(\int_{\mathbb{R}^3}|V|^2dx\Big)^{1/2}\leqslant C_p\Big(\int_{\mathbb{R}^3}(1+|x|)^p|V|^2dx\Big)^{1/2}.$$

■

5. Proof of Theorem 1.1

We apply Proposition 3.9 with $a(\lambda) = \det_5(I + W_1 R_0(\lambda)W_2)$ and $f_\varepsilon(\lambda) = C||W_1R_0(\lambda + i\varepsilon)W_2||^5_{\mathfrak{S}_5}$ where C is the constant from (2.13) with $n = 5$. Note that the zeros of $a(\lambda)$ are eigenvalues of H.

Obviously,

$$||W_1R_0(\lambda)W_2||_{\mathfrak{S}_5} \leqslant ||W_1\mathfrak{R}_0(\lambda)W_2||_{\mathfrak{S}_5} + ||W_1\mathfrak{R}_1(\lambda)W_2||_{\mathfrak{S}_5} + ||W_{,1}\mathfrak{R}(\lambda)W_2||_{\mathfrak{S}_5}.$$

On the other hand, according to Theorem 3.7, Corollaries 3.3, 3.6 and Proposition 3.8,

$$||W_1\mathfrak{R}_0(\lambda)W_2||_{\mathfrak{S}_5} \leqslant C\frac{1}{(1+|\lambda|)^{1/4}}\Big[\Big(\int_{\mathbb{R}^3}|V|^2dx\Big)^{1/2} + \Big(\int_{\mathbb{R}^3}(1+|x_1|)^{3/4}|V|^{3/2}dx\Big)^{2/3}\Big],$$

$$||W_1\mathfrak{R}_1(\lambda)W_2||_{\mathfrak{S}_5} \leqslant C\frac{1}{(1+|\lambda|)^{1/4}}\Big[\Big(\int_{\mathbb{R}^3}(1+|x_1|)^{3/2}|V|dx\Big) + \Big(\int_{\mathbb{R}^3}(1+|x_1|)^{3/4}|V|^{3/2}dx\Big)^{2/3}\Big],$$

and

$$\int_{\mathbb{R}}||W_1\mathfrak{R}(\lambda+i\varepsilon)W_2||^5_{\mathfrak{S}_5}d\lambda \leqslant C\Big(\int_{\mathbb{R}^3}|V|dx\Big)^5.$$

Therefore, Theorem 1.1 follows indeed from Proposition 3.9, because

$$\int_{\mathbb{R}} f_\varepsilon(\lambda)d\lambda \leqslant C\Big[\Big(\int_{\mathbb{R}^3}|V|^2dx\Big)^{1/2} + \int_{\mathbb{R}^3}(1+|x_1|)^{3/2}|V|dx\Big]^5.$$

It is also clear that $a(\lambda) = 1 + O(|\lambda|^{-5/4})$, as $|\lambda| \to \infty$. So, all conditions of Proposition 3.9 are fulfilled.

6. Resolvent operator. Revised

Recall again that the integral kernels of the operators $(-\Delta - \lambda)^{-1}$, $e^{it\Delta}$ and $e^{t\Delta}$ on $L^2(\mathbb{R}^3)$ are functions of the form

$$(-\Delta - \lambda)^{-1}(x,y) = \frac{1}{4\pi|x-y|}e^{i\sqrt{\lambda}|x-y|}, \qquad \lambda \in \mathbb{C}\setminus\overline{\mathbb{R}}_+, \quad \sqrt{\lambda}\in\mathbb{C}_+, \tag{6.24}$$

$$(e^{it\Delta})(x,y) = \frac{e^{-i3\pi/4}}{(4\pi t)^{3/2}}e^{i|x-y|^2/4t}, \qquad t > 0, \tag{6.25}$$

$$(e^{t\Delta})(x,y) = \frac{1}{(4\pi t)^{3/2}}e^{-|x-y|^2/4t}, \qquad t > 0, \tag{6.26}$$

where $x, y \in \mathbb{R}^3$. Let $H_0 = -\Delta + (x,e)$ be the free Stark operator and let $x_1 := (x,e)$. The representation

$$e^{-itH_0} = e^{-itx_1}e^{it\Delta}e^{t^2\frac{\partial}{\partial x_1}}e^{-i\frac{t^3}{3}}, \qquad \forall\, t\in\mathbb{R}, \tag{6.27}$$

discovered in [2] implies that

$$e^{-tH_0} = e^{-tx_1}e^{t\Delta}e^{-t^2\frac{\partial}{\partial x_1}}e^{\frac{t^3}{3}}, \qquad \forall\, t \geqslant 0. \tag{6.28}$$

The formula (6.28) follows from the observation that the quantity $\big(e^{-zH_0}E_{H_0}(a,b)f,g\big)$ depends analytically on z for any $-\infty < a < b < \infty$, $f \in L^2(\mathbb{R}^3)$ and $g \in C_0^\infty(\mathbb{R}^3)$. On the other hand, for the same a, b, f and g, the quantity

$$\big(e^{-zx_1}e^{z\Delta}e^{-z^2\frac{\partial}{\partial x_1}}e^{\frac{z^3}{3}}E_{H_0}(a,b)f,g\big)$$

depends analytically on z in the right half-plane $\{z:\ \mathrm{Re}\, z>0\}$. Due to the fact that it is also continuous up to the boundary of the half-plane, we obtain from (6.27) that

$$\big(e^{-zH_0}E_{H_0}(a,b)f,g\big)=\big(e^{-zx_1}e^{z\Delta}e^{-z^2\frac{\partial}{\partial x_1}}e^{\frac{z^3}{3}}E_{H_0}(a,b)f,g\big),\qquad \mathrm{Re}\, z\geqslant 0.$$

The latter relation implies (6.28).

As we see, e^{-tH_0} is not a continuous operator. However, the product $e^{tx_1}e^{-tH_0}$ is bounded for all $t\geqslant 0$, due to the fact that Δ is a negative operator.

Observe now that for any $-\infty<a<b<\infty$, the product of resolvent operator $R_0(\lambda)=(H_0-\lambda)^{-1}$ and the spectral projection $E_{H_0}(a,b)$ can be written as the sum of two integrals

$$R_0(\lambda)E_{H_0}(a,b)=\int_0^1 e^{-t(H_0-\lambda)}E_{H_0}(a,b)dt+i\int_0^\infty e^{-i(t-i)(H_0-\lambda)}E_{H_0}(a,b)dt. \tag{6.29}$$

While the first integral converges for all λ, the second integral in the right hand side of (6.29) converges (absolutely) in the operator-norm topology only for $\mathrm{Im}\lambda>0$. We will often drop the projection $E_{H_0}(a,b)$ and write formally that

$$R_0(\lambda)=\int_0^1 e^{-t(H_0-\lambda)}dt+i\int_0^\infty e^{-i(t-i)(H_0-\lambda)}dt. \tag{6.30}$$

Now we are going to obtain a useful representation for the integral kernal of one of the operators in the right hand side of (6.30). In particular, we will show that the quadratic form of this operator is well-defined on $C_0^\infty(\mathbb{R}^3)$.

Proposition 6.1. *The integral kernel of the operator* $\int_0^1 e^{-t(H_0-\lambda)}dt$ *equals*

$$\kappa_1(x,y,\lambda)=\frac{1}{\sqrt{(4\pi)^3}}\int_0^1 e^{\frac{-1}{4t}|x-y|^2}e^{\frac{t^3}{12}+t\Lambda}\frac{dt}{t^{3/2}},\qquad \Lambda=\lambda-2^{-1}(x_1+y_1), \tag{6.31}$$

for $x,y\in\mathbb{R}^3$ *and* $\lambda\in\mathbb{C}_+$.

Proof. Indeed, using (6.28), we obtain

$$\kappa_1(x,y,\lambda)=(2\pi)^{-3}\int_0^1 e^{-tx_1}e^{t\lambda+\frac{t^3}{3}}\Big(\int_{\mathbb{R}^3}e^{ip(x-y)}e^{-it^2p_1}e^{-t|p|^2}dp\Big)\,dt. \tag{6.32}$$

The substitution $\tilde p=p+2^{-1}ite_1$ turns this integral into

$$\kappa_1(x,y,\lambda)=(2\pi)^{-3}\int_0^1 e^{-t(x_1+y_1)/2}e^{t\lambda+\frac{t^3}{12}}\Big(\int_{\mathbb{R}^3}e^{i\tilde p(x-y)}e^{-t|\tilde p|^2}d\tilde p\Big)\,dt. \tag{6.33}$$

Now, the formula (6.31) follows from the fact that

$$(2\pi)^{-3}\int_{\mathbb{R}^3}e^{i\tilde p(x-y)}e^{-t|\tilde p|^2}d\tilde p=\frac{1}{\sqrt{(4\pi)^3}}\frac{e^{\frac{-1}{4t}|x-y|^2}}{t^{3/2}}. \tag{6.34}$$

■

Similarly, one can show that the following statement holds true.

Proposition 6.2. *The integral kernel of the operator* $i\int_0^\infty e^{-i(t-i)(H_0-\lambda)}dt$ *is the function*

$$\kappa_2(\lambda,x,y)=\frac{e^{-i\frac{\pi}{4}}}{\sqrt{(4\pi)^3}}\int_0^\infty e^{\frac{i}{4(t-i)}|x-y|^2}e^{-i\frac{(t-i)^3}{12}}e^{i(t-i)\Lambda}\frac{dt}{(t-i)^{3/2}},$$

where the agreement about the choice of the branch of $(t-i)^{3/2}$ *is that* $(t-i)^{3/2}\big|_{t=0}=e^{-i3\pi/4}$.

Proof. Instead of using (6.28), we use (6.27) to obtain

$$\kappa_2(x,y,\lambda) = (2\pi)^{-3} i \int_0^\infty e^{-i(t-i)x_1} e^{i(t-i)\lambda - \frac{i(t-i)^3}{3}} \Big(\int_{\mathbb{R}^3} e^{ip(x-y)} e^{i(t-i)^2 p_1} e^{-i(t-i)|p|^2} dp \Big)\, dt. \quad (6.35)$$

The substitution $\tilde{p} = p - 2^{-1}(t-i)e_1$ turns this integral into

$$\kappa_2(x,y,\lambda) = (2\pi)^{-3} i \int_0^\infty e^{-i(t-i)(x_1+y_1)/2} e^{i(t-i)\lambda - \frac{i(t-i)^3}{12}} \Big(\int_{\mathbb{R}^3} e^{i\tilde{p}(x-y)} e^{-i(t-i)|\tilde{p}|^2} d\tilde{p} \Big)\, dt. \quad (6.36)$$

Now, the statement of the proposition follows from the fact that

$$(2\pi)^{-3} \int_{\mathbb{R}^3} e^{i\tilde{p}(x-y)} e^{-i(t-i)|\tilde{p}|^2} d\tilde{p} = \frac{e^{-i3\pi/4}}{\sqrt{(4\pi)^3}} \frac{e^{\frac{i}{4(t-i)}|x-y|^2}}{(t-i)^{3/2}}. \quad (6.37)$$

■

Lemma 6.3. *There exists a universal positive constant $C > 0$ such that*

$$|\kappa_1(x,y,\lambda)| \leqslant C(1+e^{\mathrm{Re}\lambda}) \frac{1+e^{-(x_1+y_1)/2}}{|x-y|} \quad (6.38)$$

Proof. The statement is almost obvious. One only needs to realize that

$$\int_0^\infty \frac{e^{\frac{-1}{4t}|x-y|^2}}{t^{3/2}} dt = |x-y|^{-1} \int_0^\infty \frac{e^{\frac{-1}{4t}}}{t^{3/2}} dt. \qquad ■$$

The main disadvantage of the estimate (6.38) is that it does not establish any decay of k_1 as $|\lambda| \to \infty$. It turns out that k_1 does decay in the half-plane $\{\lambda :\ \mathrm{Re}\,\lambda < \alpha\}$ for any $\alpha \in \mathbb{R}$. In order to obtain an estimate that shows such a behavior of k_1, we write the relation (6.31) in the form

$$\kappa_1(x,y,\lambda) = \frac{1}{\lambda\sqrt{(4\pi)^3}} \int_0^1 e^{\frac{-1}{4t}|x-y|^2} e^{\frac{t^3}{12} - 2^{-1}t(x_1+y_1))} \big(\frac{de^{\lambda t}}{dt}\big) \frac{dt}{t^{3/2}}, \quad (6.39)$$

for $x, y \in \mathbb{R}^3$ and $\lambda \in \mathbb{C}_+ \setminus \{0\}$. Integrating by parts in (6.39), we obtain

$$\kappa_1(x,y,\lambda) = \frac{1}{\lambda\sqrt{(4\pi)^3}} \int_0^1 e^{\frac{-1}{4t}|x-y|^2 + \frac{t^3}{12} + t\Lambda} \Big(2^{-1}(x_1+y_1) + \frac{3}{2t} - \frac{|x-y|^2 + t^4}{4t^2} \Big) \frac{dt}{t^{3/2}} + + \frac{1}{\lambda\sqrt{(4\pi)^3}} e^{\frac{-1}{4}|x-y|^2 + \frac{1}{12} + \Lambda} \quad (6.40)$$

for $x, y \in \mathbb{R}^3$ and $\lambda \in \mathbb{C}_+$.The formula (6.40) leads to the estimate

Lemma 6.4. *There exists a universal positive constant $C > 0$ such that*

$$|\kappa_1(x,y,\lambda)| \leqslant C \frac{(1+e^{\mathrm{Re}\lambda})}{|\lambda|} (1+e^{-(x_1+y_1)/2}) \frac{(1+|x-y|^2(|x_1|+|y_1|))}{|x-y|^3} \quad (6.41)$$

Proof. The estimate is quite obvious. Again, one needs to use the observe that

$$\int_0^\infty \frac{e^{\frac{-1}{4t}|x-y|^2}}{t^{3/2+p}}dt = |x-y|^{-1-2p}\int_0^\infty \frac{e^{\frac{-1}{4t}}}{t^{3/2+p}}dt, \qquad p \geqslant 0.$$

Besides this fact, one should also use the inequality $t^4 < t$ for $0 < t < 1$ and the estimate $e^{\frac{-1}{4}|x-y|^2} \leqslant C|x-y|^{-3}$. ■

Let now $X_1(\lambda)$ be the integral operator with the kernel

$$W_1(x)\kappa_1(x,y,\lambda)W_2(y)$$

Lemma 6.5. *Let $X_1(\lambda)$ be analytically continued into the lower half-plane $\mathbb{C}_-$. There exists a universal positive constant $C > 0$ such that*

$$||X_1(\lambda)||_{\mathfrak{S}_2} \leqslant C\frac{(1+e^{\mathrm{Re}\lambda})}{|\lambda|^{1/5}}\Big(\Big(\int_{\mathbb{R}^3}(1+|x_1|)^3(1+e^{-3x_1/2})|V|^{3/2}(x)dx\Big)^{2/3} + \Big(\int_{\mathbb{R}^3}(1+e^{-2x_1})|V(x)|^2dx\Big)^{1/2}\Big), \tag{6.42}$$

for all $\lambda \in \mathbb{C}$.

Proof. Note that due to the estimate (6.38) and the inequality (3.6), one can always find a universal constant C such that

$$||X_1(\lambda)||_{\mathfrak{S}_2} \leqslant C(1+e^{\mathrm{Re}\lambda})\Big(\int_{\mathbb{R}^3}(1+e^{-3x_1/2})|V|^{3/2}(x)dx\Big)^{2/3}. \tag{6.43}$$

The difficulty of the proof of (6.42) is related to the circumstance that we need $|\lambda|^{1/5}$ in the denominator. Due to (6.43), it is sufficient to prove (6.42) for $|\lambda| > 1$. Let χ_λ be the characteristic function of the ball $\{x \in \mathbb{R}^3 : \quad |x| < |\lambda|^{-2/5}\}$. We can represent the operator $X_1(\lambda)$ as the sum $X_1(\lambda) = \mathfrak{X}(\lambda) + \mathfrak{Y}(\lambda)$ where the operator $\mathfrak{X}(\lambda)$ is the operator with the integral kernel

$$W_1(x)\kappa_1(x,y,\lambda)\chi_\lambda(x-y)W_2(y).$$

Using the estimate (6.38), we obtain

$$||\mathfrak{X}(\lambda)||^2_{\mathfrak{S}_2} \leqslant C(1+e^{\mathrm{Re}\lambda})^2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\chi_\lambda(x-y)}{|x-y|^2}\big(1+e^{-x_1/2}\big)^2|V(x)|\big(1+e^{-y_1/2}\big)^2|V(y)|\,dxdy \leqslant$$

$$C(1+e^{\mathrm{Re}\lambda})^2|\lambda|^{-2/5}\int_{\mathbb{R}^3}\big(1+e^{-x_1/2}\big)^4|V(x)|^2\,dx \leqslant \tilde{C}(1+e^{\mathrm{Re}\lambda})^2|\lambda|^{-2/5}\int_{\mathbb{R}^3}\big(1+e^{-2x_1}\big)|V(x)|^2\,dx.$$

Thus,

$$||\mathfrak{X}(\lambda)||_{\mathfrak{S}_2} \leqslant \tilde{C}\frac{(1+e^{\mathrm{Re}\lambda})}{|\lambda|^{1/5}}\Big(\int_{\mathbb{R}^3}\big(1+e^{-2x_1}\big)|V(x)|^2\,dx\Big)^{1/2}. \tag{6.44}$$

On the other hand, if $|x-y| > |\lambda|^{-2/5}$, the estimate (6.41) leads to the inequality

$$|\kappa_1(x,y,\lambda)| \leqslant C\frac{(1+e^{\mathrm{Re}\lambda})}{\lambda}(1+e^{-(x_1+y_1)/2})\frac{(|\lambda|^{4/5}+(|x_1|+|y_1|))}{|x-y|} \tag{6.45}$$

The latter implies that

$$||\mathfrak{Y}(\lambda)||^2_{\mathfrak{S}_2} \leqslant$$

$$C\frac{(1+e^{\mathrm{Re}\lambda})^2}{|\lambda|^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(1+e^{-x_1/2})^2|V(x)|(1+|x_1|)^2\frac{(|\lambda|^{4/5}+1)^2}{|x-y|^2}(1+e^{-y_1/2})^2|V(y)|(1+|y_1|)^2\,dxdy,$$

which, according to (3.6), immediately leads to

$$||\mathfrak{Y}(\lambda)||_{\mathfrak{S}_2} \leqslant \tilde{C}\frac{(1+e^{\mathrm{Re}\lambda})}{|\lambda|^{1/5}}\Big(\int_{\mathbb{R}^3}(1+e^{-x_1/2})^3|V(x)|^{3/2}(1+|x_1|)^3\,dx\Big)^{2/3} \tag{6.46}$$

for $|\lambda|>1$. It remains to combine (6.44) with (6.46) and (6.43). ■

Corollary 6.6. *Let $X_1(\lambda)$ be analytically continued into the lower half-plane $\mathbb{C}_-$. Let $\mathcal{P}(x_1)=(1+|x_1|)^2(1+e^{-x_1})$. Then there exists a positive constant $C>0$ such that*

$$||X_1(\lambda)||_{\mathfrak{S}_2} \leqslant C\frac{(1+e^{\mathrm{Re}\lambda})}{|\lambda|^{1/5}}\Big[\int_{\mathbb{R}^3}\mathcal{P}(x_1)|V(x)|dx+\Big(\int_{\mathbb{R}^3}\mathcal{P}^2(x_1)|V(x)|^2dx\Big)^{1/2}\Big], \qquad \forall\lambda\in\mathbb{C}. \tag{6.47}$$

Proof. The relation (6.47) is obtained from (6.42) by an application of Hölder's inequality to the first integral in the right hand side. ■

Let us now deal with the integral kernel of the operator $i\int_0^\infty e^{-i(t-i)(H_0-\lambda)}dt$, which is the function

$$\kappa_2(\lambda,x,y)=\frac{e^{-i\frac{\pi}{4}}}{\sqrt{(4\pi)^3}}\int_0^\infty e^{\frac{i}{4(t-i)}|x-y|^2}e^{-i\frac{(t-i)^3}{12}}e^{i(t-i)\,\Lambda}\frac{dt}{(t-i)^{3/2}}.$$

Obviously,

$$|\kappa_2(\lambda,x,y)|\leqslant\frac{e^{-2^{-1}(x_1+y_1)}}{\sqrt{(4\pi)^3}}\int_0^\infty e^{\frac{-3t^2+1}{12}}e^{-t\mathrm{Im}\lambda}e^{\mathrm{Re}\lambda}dt\leqslant Ce^{-2^{-1}(x_1+y_1)}e^{(\mathrm{Im}\lambda)^2}e^{\mathrm{Re}\lambda}.$$

Lemma 6.7. *Let $X_2(\lambda)$ be the analytic continuation of the operator-valued function $i\int_0^\infty W_1e^{-i(t-i)(H_0-\lambda)}W_2dt$ into the lower half-plane $\mathbb{C}_-$. Then there exists a positive constant $C>0$ such that*

$$||X_2(\lambda)||_{\mathfrak{S}_2}\leqslant Ce^{(\mathrm{Im}\lambda)^2}e^{\mathrm{Re}\lambda}\int_{\mathbb{R}^3}e^{-x_1}|V(x)|\,dx, \qquad \lambda\in\mathbb{C}_-. \tag{6.48}$$

The estimate (6.48) is however not suitable for our purposes in the case $\lambda\in\mathbb{C}_+$. Therefore, we formulate and prove the following statement.

Lemma 6.8. *Let $\lambda\in\mathbb{C}_+$. Then there exists a positive constant $C>0$ such that*

$$||X_2(\lambda)||_{\mathfrak{S}_2}\leqslant\frac{Ce^{\mathrm{Re}\lambda}}{1+\mathrm{Im}\lambda}\int_{\mathbb{R}^3}e^{-x_1}|V(x)|\,dx, \qquad \lambda\in\mathbb{C}_+. \tag{6.49}$$

Proof. It is enough to consider the case $\mathrm{Im}\lambda>1$, where , (6.49) follows from the inequality

$$|\kappa_2(\lambda,x,y)|\leqslant\frac{e^{-2^{-1}(x_1+y_1)}}{\sqrt{(4\pi)^3}}e^{\mathrm{Re}\lambda}\int_0^\infty e^{\frac{-3t^2+1}{12}}e^{-t\mathrm{Im}\lambda}dt\leqslant\frac{Ce^{-2^{-1}(x_1+y_1)}}{\mathrm{Im}\lambda}e^{\mathrm{Re}\lambda}\Big(1+\int_0^\infty te^{\frac{-t^2}{4}}e^{-t\mathrm{Im}\lambda}dt\Big).$$

■

We remind the reader that $Y_0(\lambda) = W_1(H_0-\lambda)^{-1}W_2 = X_1(\lambda)+X_2(\lambda)$. Combining the estimates (6.47), (6.48) and (6.49), we obtain the following statement, where $(|\mathrm{Im}\lambda| \pm \mathrm{Im}\lambda)/2$ is denoted by $(\mathrm{Im}\lambda)_\pm$:

Theorem 6.9. *Let $\mathcal{P}(x_1) = (1+|x_1|)^2(1+e^{-x_1})$. There exists a universal positive constant $C>0$ such that*

$$||Y_0(\lambda)||_{\mathfrak{S}_2} \leqslant$$
$$C\Big(\frac{(1+e^{\mathrm{Re}\lambda})}{|\lambda|^{1/5}} + \frac{e^{(\mathrm{Im}\lambda)_-^2+\mathrm{Re}\lambda}}{1+(\mathrm{Im}\lambda)_+}\Big)\Big[\int_{\mathbb{R}^3}\mathcal{P}(x_1)|V(x)|dx \quad +\Big(\int_{\mathbb{R}^3}\mathcal{P}^2(x_1)|V(x)|^2dx\Big)^{1/2}\Big]. \tag{6.50}$$

7. Jensen's inequality for a function analytic in a corner and its applications

Here we prove the following result about zeros of an analytic function.

Proposition 7.1. *Let $\varepsilon>0$ and $\alpha>0$ be two positive numbers. Let $a(z)$ be an analytic function on the domain $\Omega = \{z\in\mathbb{C}:\ \mathrm{Re}\, z\leqslant \alpha+\varepsilon,\ \mathrm{Im}\, z\geqslant -\varepsilon\}$, having the asymptotics $a(z) = 1+o(|z|^{-2})$ as $|z|\to\infty$ in Ω. Assume also that*

$$\ln|a(z)| \leqslant \Big(\frac{(1+e^{\mathrm{Re}z})}{|z|^{1/5}} + \frac{e^{(\mathrm{Im}z)_-^2+\mathrm{Re}z}}{1+(\mathrm{Im}z)_+}\Big)^{12}\cdot M, \qquad \text{if}\quad z\in\Omega, \tag{7.51}$$

for some $M>0$. Then the number N of zeros of $a(z)$ in the domain $\{z\in\mathbb{C}:\ \mathrm{Re}\, z\leqslant \alpha,\ \mathrm{Im}\, z\geqslant 0\}$ satisfies

$$N\leqslant \varepsilon^{-2}C\cdot M\Big[e^{12(\alpha+\varepsilon)}\Big(\frac{\alpha+\varepsilon}{\varepsilon^{7/5}} + (1+\varepsilon^2)e^{12\varepsilon^2} + \frac{1}{(\alpha+\varepsilon)^{2/5}}\Big)\Big], \tag{7.52}$$

where $C>0$ is independent of α, ε and M.

Proof. The function $\log(a(z))$ it is not analytic in Ω, due to the possibility of having zeros of $a(z)$ in Ω. To get rid of the zeros, we introduce the following Blaschke product:

$$B(z) = \prod_j \frac{(z-\alpha+(i-1)\varepsilon)^2 - (z_j-\alpha+(i-1)\varepsilon)^2}{(z-\alpha+(i-1)\varepsilon)^2-(\bar z_j-\alpha-(1+i)\varepsilon)^2},$$

where z_j are zeros of $a(z)$. It is easy to see that the function $\log[a(z)/B(z)]$ is analytic on Ω, because $B(z)$ vanishes exactly at the points $z=z_j$. On the other hand, $|B(z)|=1$ for all z that belong to the boundary of Ω.

Let $C_R = \{z\in\Omega:\ |z-\alpha| = R\}$, let I_R be the interval $\{z\in\Omega:\ -\sqrt{R^2-\varepsilon^2}\leqslant \mathrm{Re}\, z-\alpha\leqslant \varepsilon$, and $\mathrm{Im}z=-\varepsilon\}$, and let J_R be the interval $\{z\in\Omega:\ \mathrm{Re}\, z = \alpha+\varepsilon$, and $-\varepsilon\leqslant \mathrm{Im}\, z\leqslant \sqrt{R^2-\varepsilon^2}\}$. Define $\Gamma_R = C_R\cup I_R\cup J_R$ as a traversed counterclockwise contour. Then

$$\int_{\Gamma_R}\log[a(z)/B(z)](z-\alpha+(i-1)\varepsilon)dz = 0.$$

Consequently,

$$\lim_{R\to\infty}\mathrm{Re}\int_{C_R}\log[B(z)/a(z)](z-\alpha+(i-1)\varepsilon)dz = \lim_{R\to\infty}\int_{I_R}\log|a(z)|(z-\alpha+(i-1)\varepsilon)dz+$$
$$\lim_{R\to\infty}\int_{J_R}\log|a(z)|(z-\alpha+(i-1)\varepsilon)dz,$$

which implies that

$$\lim_{R\to\infty} \mathrm{Re} \int_{C_R} \log[B(z)](z-\alpha+(i-1)\varepsilon)dz = \lim_{R\to\infty} \int_{I_R} \log|a(z)|(z-\alpha+(i-1)\varepsilon)dz + \lim_{R\to\infty} \int_{J_R} \log|a(z)|(z-\alpha+(i-1)\varepsilon)dz. \tag{7.53}$$

On the other hand, due to the expansion

$$\log[B(z)](z-\alpha+(i-1)\varepsilon) = -\frac{2i}{z}\sum_j \mathrm{Im}\,(z_j-\alpha+(i-1)\varepsilon)^2 + O(1/|z|^2), \qquad \text{as} \quad |z|\to\infty,$$

the limit of the integral in the left hand side can be easily computed. Namely,

$$\lim_{R\to\infty} \mathrm{Re} \int_{C_R} \log[B(z)](z-\alpha+(i-1)\varepsilon)dz = \pi \sum_j \mathrm{Im}\,(z_j-\alpha+(i-1)\varepsilon)^2.$$

Therefore,

$$\lim_{R\to\infty} \mathrm{Re} \int_{C_R} \log[B(z)](z-\alpha+(i-1)\varepsilon)dz = 2\pi \sum_j (\mathrm{Im}\, z_j+\varepsilon)(\mathrm{Re} z_j-(\alpha+\varepsilon)) \leqslant -2\pi\varepsilon^2 N.$$

Taking into account the condition (7.51), we obtain from (7.53) that

$$2\pi\varepsilon^2 N \leqslant M \int_{-\infty}^{\alpha+\varepsilon} \Big(\frac{(1+e^t)}{|t-i\varepsilon|^{1/5}} + e^{\varepsilon^2}e^t\Big)^{12} |t-\alpha-\varepsilon|dt + M\int_{-\varepsilon}^{\infty}\Big(\frac{1+e^{\alpha+\varepsilon}}{|\alpha+\varepsilon+it|^{1/5}} + \frac{e^{\alpha+\varepsilon}e^{t_-^2}}{(1+t_+)}\Big)^{12}(t+\varepsilon)dt. \tag{7.54}$$

Note that

$$\int_{-\varepsilon}^{\infty}\Big(\frac{1+e^{\alpha+\varepsilon}}{|\alpha+\varepsilon+it|^{1/5}} + \frac{e^{\alpha+\varepsilon}e^{t_-^2}}{(1+t_+)}\Big)^{12}(t+\varepsilon)dt \leqslant C\Big[\frac{1+e^{12(\alpha+\varepsilon)}}{(\alpha+\varepsilon)^{2/5}} + e^{12(\alpha+\varepsilon)}(1+\varepsilon+e^{12\varepsilon^2}\varepsilon^2)\Big]$$

and

$$\int_{-\infty}^{\alpha+\varepsilon} \Big(\frac{(1+e^t)}{|t-i\varepsilon|^{1/5}} + e^{\varepsilon^2}e^t\Big)^{12} |t-\alpha-\varepsilon|dt \leqslant C\Big[\frac{(1+e^{\alpha+\varepsilon})^{12}(\alpha+\varepsilon)}{\varepsilon^{7/5}} + e^{12\varepsilon^2}e^{12(\alpha+\varepsilon)}\Big].$$

Consequently, (7.54) can be written in the form

$$2\pi\varepsilon^2 N \leqslant C\cdot M\Big[e^{12(\alpha+\varepsilon)}\Big(\frac{\alpha+\varepsilon}{\varepsilon^{7/5}} + (1+\varepsilon^2)e^{12\varepsilon^2} + \frac{1}{(\alpha+\varepsilon)^{2/5}}\Big)\Big].$$

The proof is completed. ■

We now can apply this proposition to the function

$$a(z) = \det_{12}(I+Y_0(z)).$$

Let's remind the reader that according to Theorem 6.9 combined with the inequality (2.13), there exists a universal positive constant $C_0 > 0$ such that (7.51) holds with

$$M = C_0\Big[\int_{\mathbb{R}^3} \mathcal{P}(x_1)|V(x)|dx + \Big(\int_{\mathbb{R}^3} \mathcal{P}^2(x_1)|V(x)|^2dx\Big)^{1/2}\Big]^{12}.$$

Thus, Theorem 1.3 follows from Proposition 7.1.

8. Proof of Theorem 1.2

In this section, we establish some bounds on the sums of the powers of imaginary parts of the eigenvalues of the operator H. First, note that a simple application of Hölder's inequality leads us to the following consequence of Theorem 1.1.

Corollary 8.1. *Let $q > 12$ and let $\mathcal{P}_q$ be the positive weight-function defined by*

$$\mathcal{P}_q(x) = (1 + |x|)^q (1 + |x_1|)^{15/2} \tag{8.55}$$

for all $x \in \mathbb{R}^3$. Assume that V is a bounded complex-valued potential obeying the condition

$$\int_{\mathbb{R}^3} \mathcal{P}_q(x)|V(x)|^5\, dx < \infty. \tag{8.56}$$

Then there exists a positive constant $C_q > 0$ depending only on $q > 12$ such that eigenvalues λ_j of the operator H satisfy the estimate

$$\sum_j |\mathrm{Im}\, \lambda_j| \leqslant C_q \int_{\mathbb{R}^3} \mathcal{P}_q(x)|V(x)|^5\, dx. \tag{8.57}$$

Moreover, the following statement holds true.

Theorem 8.2. *Let V be a bounded complex-valued potential obeying the condition* (8.56) *with $\mathcal{P}_q$ defined by* (8.55)*. Then there exists a positive constant $C_q > 0$ depending only on $q > 12$, such that for any $\gamma > 0$, the eigenvalues λ_j of the operator H satisfy the estimate*

$$\sum_j (\mathrm{Im}\, \lambda_j - \gamma_0)_+ \leqslant C_q \int_{\mathbb{R}^3} \mathcal{P}_q(x)\big(|V(x)|^{1/2} - \gamma\big)_+^{10}\, dx, \tag{8.58}$$

where $\gamma_0 = 4\gamma||V||_{L^\infty}^{1/2}$.

Proof. In order to prove this result, we need to specify our choice of the functions W_1 and W_2. Let us agree that

$$W_2 = |V|^{1/2}, \qquad \text{and} \qquad W_1 W_2 = V.$$

Then there exists a unitary operator S such that $W_1 = SW_2$. For any positive $\gamma > 0$ we introduce $W_{1,\gamma}$ and $W_{2,\gamma}$, setting

$$W_{2,\gamma} = (W_2 - \gamma)_+ \qquad \text{and} \qquad W_{1,\gamma} = SW_{2,\gamma}.$$

Now, we decompose the operator $Y_0(\lambda) = W_1 R_0(\lambda) W_2$ into the sum

$$Y_0(\lambda) = Y_\gamma(\lambda) + \tilde{Y}_\gamma(\lambda), \qquad \text{where} \qquad Y_\gamma(\lambda) = W_{1,\gamma} R_0(\lambda) W_{2,\gamma}. \tag{8.59}$$

Obviously, since $||W_j - W_{j,\gamma}||_{L^\infty} \leqslant \gamma$ and $||R_0(\lambda)|| \leqslant |\mathrm{Im}\,\lambda|^{-1}$, we have the estimate

$$||\tilde{Y}_\gamma(\lambda)|| \leqslant \frac{2\gamma||V||_{L^\infty}^{1/2}}{|\mathrm{Im}\lambda|}, \qquad \forall \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Consequently, $||\tilde{Y}_\gamma(\lambda)|| \leqslant \frac{1}{2}$ in the half-plane $\{\lambda \in \mathbb{C}_+ :\ \mathrm{Im}\,\lambda > 4\gamma||V||_{L^\infty}^{1/2}\}$. Therefore, the operator $I + \tilde{Y}_\gamma(\lambda)$ is invertible for all λ that belong to this region and

$$||(I + \tilde{Y}_\gamma(\lambda))^{-1}|| \leqslant 2, \qquad \forall \lambda \in \{\lambda \in \mathbb{C}_+ :\ \mathrm{Im}\,\lambda > 4\gamma||V||_{L^\infty}^{1/2}\}.$$

Let $\gamma_0 = 4\gamma||V||_{L^\infty}^{1/2}$. We apply Proposition 3.9 with

$$a(\lambda) = \det_5\Big(I + \big(I + \tilde{Y}_\gamma(\lambda + i\gamma_0)\big)^{-1} Y_\gamma(\lambda + i\gamma_0)\Big)$$

and $f_\varepsilon(\lambda) = 2^5 C||Y_\gamma(\lambda + i(\gamma_0 + \varepsilon))||_{\mathfrak{S}_5}^5$ where C is the constant from (2.13) with $n = 5$. Note that a point $\lambda \in \mathbb{C}_+$ is a zero of $a(\lambda)$ if and only if $\lambda + i\gamma_0$ is an eigenvalue of H.

It is also clear that

$$||W_{1,\gamma} R_0(\lambda) W_{2,\gamma}||_{\mathfrak{S}_5} \leqslant ||W_{1,\gamma}\mathfrak{R}_0(\lambda)W_{2,\gamma}||_{\mathfrak{S}_5} + ||W_{1,\gamma}\mathfrak{R}_1(\lambda)W_{2,\gamma}||_{\mathfrak{S}_5} + ||W_{1,\gamma}\mathfrak{R}(\lambda)W_{2,\gamma}||_{\mathfrak{S}_5}.$$

On the other hand, according to Theorem 3.7, Corollaries 3.3, 3.6 and Proposition 3.8,

$$||W_{1,\gamma}\mathfrak{R}_0(\lambda)W_{2,\gamma}||_{\mathfrak{S}_5} \leqslant C\frac{1}{(1+|\lambda|)^{1/4}}\Big[\Big(\int_{\mathbb{R}^3}(|V|^{1/2}-\gamma)_+^4 dx\Big)^{1/2} + \Big(\int_{\mathbb{R}^3}(1+|x_1|)^{3/4}(|V|^{1/2}-\gamma)_+^3 dx\Big)^{2/3}\Big],$$

$$||W_{1,\gamma}\mathfrak{R}_1(\lambda)W_{2,\gamma}||_{\mathfrak{S}_5} \leqslant C\frac{1}{(1+|\lambda|)^{1/4}}\Big[\Big(\int_{\mathbb{R}^3}(1+|x_1|)^{3/2}(|V|^{1/2}-\gamma)_+^2 dx\Big) + \Big(\int_{\mathbb{R}^3}(1+|x_1|)^{3/4}(|V|^{1/2}-\gamma)_+^3 dx\Big)^{2/3}\Big],$$

and

$$\int_{\mathbb{R}} ||W_{1,\gamma}\mathfrak{R}(\lambda + i(\gamma_0+\varepsilon))W_{2,\gamma}||_{\mathfrak{S}_5}^5 d\lambda \leqslant C\Big(\int_{\mathbb{R}^3}(|V|^{1/2}-\gamma)_+^2 dx\Big)^5, \quad \forall\varepsilon \geqslant 0.$$

Therefore, Theorem 8.2 follows indeed from Proposition 3.9, because

$$\int_{\mathbb{R}} f_\varepsilon(\lambda) d\lambda \leqslant \tilde{C}_p \int_{\mathbb{R}^3} \mathcal{P}_q(x)\Big(|V|^{1/2}-\gamma\Big)_+^{10} dx.$$

It is also clear that $a(\lambda) = 1 + O(|\lambda|^{-5/4})$, as $|\lambda| \to \infty$. So, all conditions of Proposition 3.9 are fulfilled. ■

Proof of Theorem 1.2. One only needs to multiply (8.58) by γ^{p-2} and integrate the resulting inequality with respect to γ from 0 to ∞.

$$\sum_j \int_0^\infty (\operatorname{Im}\lambda_j - 4\gamma||V||_{L^\infty}^{1/2})_+\gamma^{p-2}d\gamma \leqslant C_q \int_{\mathbb{R}^3}\mathcal{P}_q(x)\int_0^\infty \big(|V(x)|^{1/2}-\gamma\big)_+^{10}\gamma^{p-2}d\gamma\, dx,$$

A simple change of the variable in the corresponding integral leads to the equality

$$\int_0^\infty (\operatorname{Im}\lambda_j - 4\gamma||V||_{L^\infty}^{1/2})_+\gamma^{p-2}d\gamma = \frac{|\operatorname{Im}\lambda_j|^p}{||V||^{(p-1)/2}}\int_0^\infty (1-4\gamma)_+\gamma^{p-2}d\gamma.$$

In the same way one obtains that

$$\int_0^\infty \big(|V(x)|^{1/2}-\gamma\big)_+^{10}\gamma^{p-2}d\gamma = |V(x)|^{(p+9)/2}\int_0^\infty \big(1-\gamma\big)_+^{10}\gamma^{p-2}d\gamma.$$

Thus, Theorem 1.2 follows. ■

We provide an extensive list of mathematical articles [1]-[3], [8]-[23], [25]- [27], [29], [32], [33] containing the important work on Stark operators, which are operators with the potential corresponding to a constant electric field, and the work related to the study of the Stark effect. Our list includes the titles of the books [28] and [31] containing the relevant theory of Schrödinger operators and perturbation determinants.

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