Non-deterministic Quantum Mechanics

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The paper presents an interpretation of quantum mechanics in which a particle's motion is inherently non-deterministic while it has definite unambiguous momentum and position whose existence is independent of measurement. It is therefore distinct from the Copenhagen Interpretation while it is also distinct from the pilot wave theory in that this interpretation is inherently non-deterministic. This paper paves the way for rigorous investigation into the spins of particles in the succeeding paper.

1. Introduction

This paper seeks to identify the non-deterministic component and the deterministic component of the motion of a particle. The Copenhagen interpretation of quantum mechanics has a very open non-deterministic view of the position and momentum of a particle while the pilot wave interpretation has a closed deterministic view of a particle's position and momentum. Here, it is proposed that a particle's motion consists of a deterministic component and a non-deterministic component, and the unambiguous position and momentum of a particle at any instant exist independently of any observation or measurement. In this sense, the proposed interpretation has some flavour of the pilot wave interpretation and some flavour of the Copenhagen interpretation (because of the indeterminacy), and yet is fundamentally distinct from them.

2. The Non-Deterministic Component and the Deterministic Component of a Particle's Motion

Our investigation begins with writing the wave function in the following form:

$$\psi = Re^{iS}$$

where S is the wave phase, R and S are non-dimensional real functions of time and space.[1] The Schrödinger equation is

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + U\psi \tag{1}$$

where $\hbar$ is the reduced Planck constant, m is the mass of the particle and U is the potential 'experienced' by the particle. The Schrödinger equation can be exactly re-written as the

[†]This paper is dedicated to my father, Mr. Chun Loy So, who worked in the UK as a migrant worker from the late 1950s to support his family.

[1] In other treatments, the non-dimensional S here is replaced by $S/\hbar$.

following two equations by considering its real and imaginary parts:

$$\hbar \frac{\partial S}{\partial t} + \frac{\hbar^2}{2m} (\nabla S)^2 - \frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} + U = 0 \tag{2}$$

$$\frac{\partial R^2}{\partial t} + Div\left(\frac{\hbar}{m} R^2 \nabla S \right) = 0 \ . \tag{3}$$

Equation (2) is in a modified form of the Hamilton-Jacobi equation. The term, $-\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$, is called the quantum potential.[2] Equation (3) in this paper is called the pseudo continuity equation for reasons which will become clear. To these two equations derived from the Schrödinger equation, we add the following equation which should also be satisfied,

$$\frac{\partial \rho}{\partial t} + Div(\rho \underset{\sim}{v}) = 0 \tag{4}$$

where $\rho \equiv R^2$ is the probability density, $\underset{\sim}{v}$ is the particle's velocity in three dimensions and $\rho \underset{\sim}{v}$ is the flux of probability.[3] This equation is called the generic continuity equation and its counterparts can be found in other branches of physics, e.g., fluid mechanics.

In the usual pilot wave formulation of quantum mechanics, the velocity of the particle is unambiguously described by a single component evaluated at the particle's unambiguous position,

$$\underset{\sim}{v} = \underset{\sim}{v}_1 \equiv \frac{\hbar}{m} \nabla S \ .$$

This is the usual guiding equation in the deterministic pilot wave theory (see Bohm [1]). Since equation (3) must hold, in this case it can be re-written as

$$\frac{\partial \rho}{\partial t} + Div(\rho \underset{\sim}{v}_1) = 0 \ . \tag{5}$$

With $\underset{\sim}{v}$ prescribed as $\underset{\sim}{v}_1$, the generic continuity equation (4) is also satisfied, as required. Since the time evolution of S and R can be determined by equations (2) and (3), and $\underset{\sim}{v}_1$ depends only on S, the time evolution of $\underset{\sim}{v}_1$ is also determined. Attempts have been made to find an additional component of the velocity such that the generic continuity equation (4) is also satisfied by the sum of $\underset{\sim}{v}_1$ and this additional velocity component. These attempts were also made in a deterministic framework so that inherently they will not be able to identify the non-deterministic component of the velocity of a particle. However, the aim of this section is to locate the non-deterministic component. Inevitably, it will look at these attempts and see how progress can be made from them. For this purpose, it will be sufficient to consider Esposito's paper [2] which is a further derivation of the papers by Salesi [3], Recami and Salesi [4].

[2] The quantum potential is like the pressure term in the Navier-Stokes equation for fluid dynamics where the pressure can also be considered as a kind of energy potential.

[3] In many papers dealing with this subject, this generic continuity equation is not explicitly given but giving it here explicitly will help our discussion later.

Esposito introduced a second velocity component, $\underset{\sim}{v}_2$, in the following way.[4] Defining $\vec{u}_s \equiv \frac{\hbar}{2m}\frac{\nabla\rho}{\rho}$, he further defined

$$\underset{\sim}{v}_2 \equiv \vec{u}_s \wedge \vec{s} = \frac{\hbar}{2m}\frac{\nabla\rho}{\rho} \wedge \vec{s} \tag{6}$$

where $\vec{s}$ is a unit vector constant over space, defined at the point where the particle is (which is called the reference point in this paper). $\vec{s}$ at the reference point is perpendicular to $\nabla\rho$ (and thus $\vec{u}_s$) and *lies on the plane* formed by $\underset{\sim}{v}_1$ and $\vec{u}_s$, or equivalently the plane formed by ∇S and $\nabla\rho$.[5] The following diagram illustrates the relationships between some of the vectors at the reference point where the particle is.

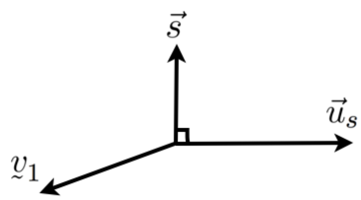

Fig. 1 Three vectors on the same plane

$\underset{\sim}{v}_2$ at the reference point is perpendicular to the plane containing the following three vectors: $\underset{\sim}{v}_1$ (and thus ∇S), $\vec{u}_s$ (and thus $\nabla\rho$) and $\vec{s}$. Visually, it is coming out or into the page of Figure 1. Since $\rho\underset{\sim}{v}_2$ can be expressed as

$$Curl\left(\frac{\hbar}{2m}\rho\vec{s}\right)$$

and *Div Curl* of a vector is 0,

$$Div\big(\rho\underset{\sim}{v}_2\big) = 0\ .$$

Combining this with the condition in (5) which comes from the pseudo continuity equation (3), we have

$$\frac{\partial\rho}{\partial t} + Div\big(\rho(\underset{\sim}{v}_1 + \underset{\sim}{v}_2)\big) = 0\ .$$

And if we prescribe $\underset{\sim}{v}$ in the following way,

$$\underset{\sim}{v} = \underset{\sim}{v}_1 + \underset{\sim}{v}_2\ ,$$

then the generic continuity equation (4) is satisfied as required. That is, this continuity equation can be satisfied in two different ways with two somewhat different prescriptions for the velocity, $\underset{\sim}{v}$. The addition of another velocity component, $\underset{\sim}{v}_2$, to the usual prescribed velocity, $\underset{\sim}{v}_1$, takes the pilot wave theory one step further.

This much has been established by Esposito [2], Salesi and Recami [3] [4], though

[4] Esposito chose to use 'natural units' in his paper and set $\hbar$ to 1. However, $\hbar$ is given explicitly and made visible in this paper instead of the invisible 1. His definition of terms will be slightly adjusted by this factor of $\hbar$ and his mathematical formulation is virtually reproduced here.

[5] Note the scalar quantity S in ∇S is different from the unit vector $\vec{s}$.

their treatments have been unpacked somewhat in the above by stating explicitly that the additional generic continuity equation must also be satisfied. Also, it needs to be noted that $\underset{\sim}{v}_2$ is perpendicular to $\underset{\sim}{v}_1$ (and thus ∇S) at the position where the particle is at, i.e., where the constant unit vector $\vec{s}$ is defined at the reference point. At other points, this perpendicular relationship cannot be guaranteed since at those points ∇S, $\nabla \rho$ and $\vec{s}$ may not lie on the same plane; nevertheless, $Div\big(\rho \underset{\sim}{v}_2\big) = 0$ at all points and the generic continuity equation (4) is satisfied at all points. Furthermore, it has to be remembered that their treatments were carried out in a deterministic framework; this paper will move beyond such a deterministic framework which may be considered to be more rigid compared to the more open Copenhagen interpretation of quantum mechanics.

Since in general there can be three components of a velocity in three dimensions with the components orthogonal to each other, and we have two such orthogonal components in $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ at the reference point where the particle is, it is reasonable to postulate that a third velocity $\underset{\sim}{v}_3$, orthogonal to $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ at the reference point, exists such that $\underset{\sim}{v} \equiv \underset{\sim}{v}_1 + \underset{\sim}{v}_2 + \underset{\sim}{v}_3$ satisfies the generic continuity equation (4) at all points in space. Clearly, this requires $Div(\rho \underset{\sim}{v}_3) = 0$. If $\rho \underset{\sim}{v}_3$ is a rotor, i.e., if it can be expressed as the $Curl$ of a vector, then $Div(\rho \underset{\sim}{v}_3) = 0$ is automatically satisfied since $Div\, Curl$ operating on any vector is 0. Consider the vector $S\vec{a}$, where S is the phase in equation (2) and equation (3) and $\vec{a}$ is the unit vector constant over space in the direction of $\underset{\sim}{v}_2$ defined at the reference point where the particle is,

$$Curl\,(S\vec{a}) \equiv S\, Curl\, \vec{a} + \nabla S \wedge \vec{a} = \nabla S \wedge \vec{a}\ .$$

If $\underset{\sim}{v}_3$ is defined as follows:

$$\underset{\sim}{v}_3 \equiv \frac{\hbar}{m}\frac{\nabla S}{\rho} \wedge \vec{a}\ ,$$

then

$$\rho \underset{\sim}{v}_3 = \frac{\hbar}{m}\nabla S \wedge \vec{a} = \frac{\hbar}{m} Curl\,(S\vec{a})$$

so that indeed $Div(\rho \underset{\sim}{v}_3) = 0$.[6] Notice that $\underset{\sim}{v}_3$ at the reference point is orthogonal to $\underset{\sim}{v}_1$ since it is perpendicular to ∇S, and $\underset{\sim}{v}_3$ is orthogonal to $\underset{\sim}{v}_2$ at the reference point since it is perpendicular to $\vec{a}$ which is parallel to $\underset{\sim}{v}_2$. In sum, we have, at the reference point, a set of three velocities, $\underset{\sim}{v}_1$, $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$, which are orthogonal to each other (which may not be true at other points). $\underset{\sim}{v} = \underset{\sim}{v}_1$ satisfies the generic continuity equation (4). With the prescriptions for $\underset{\sim}{v}_1$, $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$ as given above, they satisfy the following equation at all points:

$$\frac{\partial \rho}{\partial t} + Div\big(\rho(\underset{\sim}{v}_1 + \underset{\sim}{v}_2 + \underset{\sim}{v}_3)\big) = 0\ . \tag{7}$$

And if we prescribe $\underset{\sim}{v} = \underset{\sim}{v}_1 + \underset{\sim}{v}_2 + \underset{\sim}{v}_3$, this $\underset{\sim}{v}$ satisfies the generic continuity equation (4). A pause for reflecting on the logic of identifying a possible total three-dimensional velocity for equation (2) and equation (3) (hence the Schrödinger equation), and the generic continuity equation (4) in quantum mechanics, is in order.

Firstly, since the generic continuity equation (4) plays such a vital role in the derivation above, despite its ubiquitous appearance in various branches of physics, its proper derivation

[6] Note that $\rho \underset{\sim}{v}_2$ can also be expressed as the $Curl$ of a vector (see above) so that $\rho(\underset{\sim}{v}_2 + \underset{\sim}{v}_3)$ can be expressed as the $Curl$ of the sum of two vectors.

and the premise for its derivation in quantum mechanics are given in the appendix. Secondly, $\underset{\sim}{v}$ defined as above with the three components does not guarantee that it is the real velocity, or the only possible real velocity, of the particle. For example, since $Div(\rho\underset{\sim}{v}_2) = 0$, $Div(\lambda_2\rho\underset{\sim}{v}_2) = 0$ for any real λ_2 which is constant over space but can vary with time, and $\lambda_2(t)\underset{\sim}{v}_2$ is a suitable candidate to replace $\underset{\sim}{v}_2$ in equation (7) and thus in equation (4). Similarly, $Div(\lambda_3\rho\underset{\sim}{v}_3) = 0$ for any real λ_3 which is constant over space but can vary with time, so that $\lambda_3(t)\underset{\sim}{v}_3$ can replace $\underset{\sim}{v}_3$ in equation (7) and thus in equation (4). It is not inconceivable that other rotors (*curls* of vector) could be concocted to express $\underset{\sim}{v}_2$ or $\underset{\sim}{v}_3$ to satisfy equation (7) and thus equation (4). However, whatever $\underset{\sim}{v}_2$ or $\underset{\sim}{v}_3$ one comes up with, they together with $\underset{\sim}{v}_1$ should be verified through experimentation to see if they correspond to the reality in this universe before they can be accepted as credible for this universe. For this paper, since at the reference point where the particle is, we have a set of three orthogonal velocities in expressing the total velocity, $\underset{\sim}{v} = \underset{\sim}{v}_1 + \underset{\sim}{v}_2 + \underset{\sim}{v}_3$, this set has potential for corresponding to reality in this universe. As we have shown, $\underset{\sim}{v} = \underset{\sim}{v}_1 + \lambda_2\underset{\sim}{v}_2 + \lambda_3\underset{\sim}{v}_3$ can also be a suitable candidate for corresponding to reality in this universe. These ambiguities in the magnitude of $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$ precisely raise the possibility of indeterminacy in the particle's velocity. Let us look at $\lambda_2\underset{\sim}{v}_2 + \lambda_3\underset{\sim}{v}_3$ more closely.

Since at the reference point both $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$ are orthogonal to $\underset{\sim}{v}_1 \equiv \frac{\hbar}{m}\nabla S$, $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$ lie on a plane, called the $\underset{\sim}{v}_2 : \underset{\sim}{v}_3$ plane, which is perpendicular to $\underset{\sim}{v}_1$ and thus ∇S; and since the constants λ_2 and λ_3 are arbitrary (except that they are continuous in time) and $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$ are orthogonal to each other, $\lambda_2\underset{\sim}{v}_2 + \lambda_3\underset{\sim}{v}_3$ is a general expression for any vector on the $\underset{\sim}{v}_2 : \underset{\sim}{v}_3$ plane. The result is that $\underset{\sim}{v}_1$ plus any vector on the $\underset{\sim}{v}_2 : \underset{\sim}{v}_3$ plane, i.e., $\underset{\sim}{v}_1 + \lambda_2\underset{\sim}{v}_2 + \lambda_3\underset{\sim}{v}_3$, is a suitable velocity for the particle at the reference point (while $\underset{\sim}{v} = \underset{\sim}{v}_1 + \lambda_2\underset{\sim}{v}_2 + \lambda_3\underset{\sim}{v}_3$ satisfies the generic continuity equation (4) at all points). Furthermore, since $\lambda_2\underset{\sim}{v}_2 + \lambda_3\underset{\sim}{v}_3$ at the reference point on the $\underset{\sim}{v}_2 : \underset{\sim}{v}_3$ plane is perpendicular to ∇S and is therefore tangential to the S surface, one can visualise some kind of surfing motion of the particle on the S surface as it is moved forward by $\underset{\sim}{v}_1$ in the direction of ∇S. The important point to note here is that as far as the Schrödinger equation and the generic continuity equation (4) are concerned, *the surfing motion of the particle tangential to the S surface is non-determinate and therefore free.*

3. Number of Constraints and the Number of Variables for Describing the Motion of the Particle

Equation (2) and Equation (3) in S and R, i.e., the Schrödinger equation, form a deterministic system of equations. However, they give us no explicit idea about the velocity of the particle. When the generic continuity equation (4) is included, three variables corresponding to three velocity components in three dimensions over all space are introduced. These three variables together with S and R are the five variables of the system. However, we only have three equations, (2), (3) and (4) for the five variables, which means the system is under-specified and is non-deterministic – there are two degrees of freedom. If we replace the generic continuity equation (4) with the following three constraints which render this equation satisfied (as shown above): (i) the usual guiding equation, $\underset{\sim}{v}_1 = \frac{\hbar}{m}\nabla S$, (ii) $\underset{\sim}{v}_2 = \lambda_2(t)\frac{\hbar}{2m}\frac{\nabla\rho}{\rho} \wedge \vec{s}$, and (iii) $\underset{\sim}{v}_3 = \lambda_3(t)\frac{\hbar}{m}\frac{\nabla S}{\rho} \wedge \vec{a}$, then, together with equations (2) and

(3), we have a total of five constraints for the five variables of $S, R, \underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3$.[7] It may seem that the system is now closed and deterministic. However, we have in this process of replacing the generic continuity equation (4) introduced two new variables, $\lambda_2(t)$ and $\lambda_3(t)$. The result is that the system is still under-constrained and there are still two degrees of freedom. It is these two degrees of freedom which makes the Schrödinger equation, as we know it, non-determining for the velocity of a particle while it only determines one velocity component, $\underset{\sim}{v}_1$. Thus, we have the deterministic velocity component, $\underset{\sim}{v}_1$, and the non-deterministic velocity component formed by $\underset{\sim}{v}_2 + \underset{\sim}{v}_3$.

With this hindsight, one may look at the pseudo continuity equation (3) again in the following way. Given its resemblance to the generic continuity equation, one could see that a probable velocity is $\underset{\sim}{v}_1$ (called the first velocity component). However, one could have postulated that another velocity component, $\underset{\sim}{v}_s$ (called the second component), perpendicular to the first component can be added to it to form the total velocity. Equation (2) and equation (3), i.e., the Schrödinger equation, have something to say about the first velocity component (which is deterministic) but do not seem to have anything to say about the second velocity component which could then be non-deterministic. However, if one cares for the generic continuity equation to be satisfied, then one has to work out the general form of the second velocity component and how it, together with $\underset{\sim}{v}_1$, can satisfy the generic continuity equation, i.e., how it can satisfy $div(\rho \underset{\sim}{v}_s) = 0$. But one would have run into the conundrum of how one might come up with an expression for such a non-deterministic velocity. It would then require further guesses to arrive at the position as laid out in this paper. The author did not exactly take this route but another route which has some resemblance to it. He was ultimately motivated by his belief that the motion of a particle is inherently non-determinate, and hence looked for the non-deterministic component in addition to the deterministic one.

4. Conclusion

Einstein [5] was uneasy about the non-deterministic nature of the Copenhagen interpretation of quantum mechanics and suggested more constraint(s) to close the system to make it a deterministic one. He had sympathisers and there arose the deterministic pilot wave theory by de Broglie [6] and later by Bohm [1] independently. This theory was taken further by Salesi, Recami [3,4] and Esposito [2] who included a second deterministic velocity to the one derived by de Broglie and Bohm. However, this paper has shown that even though the velocity component perpendicular to the S surface is deterministic, in general the velocity of the particle on the S surface is non-deterministic and nature is not quite as tied down into a deterministic system as Einstein and others suggested. On the other hand, the Copenhagen interpretation of quantum mechanics is very non-deterministic regarding the position and momentum of a particle as it embraces the picture of the imagined particle to be a wave spread everywhere over the universe and the particle's wave function collapses only at the moment of measurement or observation. The Copenhagen interpretation raises the question of the ontology of the particle and the paradox of the Schrödinger's cat. However, this paper recovers or maintains the existence of the objective position and momentum of a particle

[7] Note that $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$ now have the factors of λ_2 and λ_3 included in their respective expressions.

before measurement (which is held by those in the pilot wave theory camp), but it also echoes the non-deterministic or contingent nature of our universe in consonance with the Copenhagen interpretation. Concerning Einstein's question: 'Does God play dice?', one can suggest that only he can answer this question.

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Appendix

A. The Generic Continuity Equation for Quantum Mechanics

It has been assumed that the generic continuity equation (4) is also valid for quantum mechanics as in other branches of physics. This is the assumed premise regarding equation (4); let this be called premise B. Is this premise B true? Let us consider an arbitrary volume V which satisfies the following condition,

$$\iint_S \rho \underset{\sim}{v} \cdot d\mathbf{s} = -\iiint_V \frac{\partial \rho}{\partial t} dv$$

where S is the surface of the volume V; i.e., the net or integrated probability flux over the surface of the volume is compensated by the change of the probability density per unit time integrated over the whole volume. Let us call this condition premise A. Is this premise true? Premise A is true if one insists that the overall probability for the arbitrary volume is conserved in time in the sense that there is no source or sink of probability in the volume, i.e., its net probability flux is the measure of its change of total probability for the volume per unit time. Since the total probability integrated over the whole physical space is equal to 1 at all times, any sink or source of probability must be simultaneously compensated by an equal source or sink respectively. In order to achieve this, it seems most reasonable to assume (as in usual treatments) that there is no sink or source of probability at any volume in space. Let us call this assumption the proto-premise. If this proto-premise is true, then the above equation (premise A) for an arbitrary volume is true. By using the divergence theorem to express the left hand side of the equation in terms of a volume integral, we have

$$\iiint_V Div(\rho \underset{\sim}{v}) dv = -\iiint_V \frac{\partial \rho}{\partial t} dv$$

which holds for an arbitrary volume and is thus virtually equivalent to equation (4). Here, we see that if the proto-premise is true, premise A is true and thus premise B – the continuity equation (4) – is also true. This equation can also be called the equation for the conservation of probability in quantum mechanics.[8] The proto-premise leading to this continuity or conservation equation, i.e., there is no sink or source of probability in any volume in space, is a highly reasonable one and can ultimately be confirmed by experiments.

[8] Equation (4) is called the equation for mass conservation or mass continuity in fluid dynamics.