Geometrically induced spectral effects in tubes with a mixed Dirichlet-Neumann boundary

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Abstract: We investigate spectral properties of the Laplacian in $L^2(Q)$, where Q is a tubular region in $\mathbb{R}^3$ of a fixed cross section, and the boundary conditions combined a Dirichlet and a Neumann part. We analyze two complementary situations, when the tube is bent but not twisted, and secondly, it is twisted but not bent. In the first case we derive sufficient conditions for the presence and absence of the discrete spectrum showing, roughly speaking, that they depend on the direction in which the tube is bent. In the second case we show that a constant twist raises the threshold of the essential spectrum and a local slowndown of it gives rise to isolated eigenvalues. Furthermore, we prove that the spectral threshold moves up also under a sufficiently gentle periodic twist.

Keywords: Laplacian, Dirichlet-Neuman boundary, tube, discrete spectrum

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1 Introduction

Relations between spectral properties and geometry belong to trademark topics in mathematical physics. A particularly interesting class of problems concerns spectra of the Laplacians and related operators in tubular regions which has various applications, among others they are used to model waveguide effects in quantum systems. The turning point here was the seminal observation that 'bending means binding', that is that the Dirichlet Laplacian in a tube of a fixed cross section which is bent but asymptotically straight has a nonempty discrete spectrum. It inspired a long series of investigations, for a survey we refer to the monograph [11] and the bibliography there.

A nontrivial geometry can be manifested not only in the shape of the tube but also in the boundary conditions entering the definition of the Laplacian. A simple but striking example can be found in [6]: an infinite planar strip of constant width whose one boundary is Dirichlet and the other Neumann exhibits a discrete spectrum provided the Dirichlet boundary is bent 'inward' while in the opposite case the spectral threshold remains preserved. One is naturally interested whether this effect has three-dimensional analogue. The geometry is substantially richer in this case, of course, nevertheless our first main result provides an affirmative answer of a sort to this question, namely that some bending directions are favorable from the viewpoint of the discrete spectrum existence and some are not.

Another class of geometric deformations are tube twistings. In general, they act in the way opposite to bendings: to produce bound states of the Dirichlet Laplacian supported by a locally twisted tube of a non-circular cross section, an additional attractive interaction must exceed some critical strength [8]. On the other hand, a discrete spectrum may arise in a tube which is constantly twisted and the twist is locally slowed down [10]. Note that these results have a two-dimensional analogue, namely a Hardy inequality in planar strips where the Dirichlet and Neumann condition suddenly 'switch sides' [12] and the appearance of a nontrivial discrete spectrum when a sufficiently long purely Neumann segment is inserted in between [7].

In the second part of the paper we examine twisted tubes with a mixed Dirichlet-Neumann boundary. We show that the effect of twisting and its local slowdowns is present again, now it may occur also if the tube cross section itself exhibits a rotational symmetry but the boundary conditions violate it. Furthermore, we consider a wider class of tubes where the twist is not constant along the tube but only periodic and ask whether in this case too the threshold of the essential spectrum moves up; we prove this property for twists that are sufficiently gentle. The bending and twisting, constant and periodic, results are presented and proved in Sections 4, 5, and 6, respectively. Before coming to it, we collect in two preliminary sections the needed properties of the tubes and the operators involved.

2 Preliminaries: geometry of the waveguide

Let us begin with a curve $\ell : \mathbb{R} \to \mathbb{R}^3$ that will play the role of waveguide axis supposed to be a C^3-diffeomorfism of the real axis $\mathbb{R}$ onto $\ell(\mathbb{R})$. Without loss of generality we may parametrize it by its arc length, that is, to assume that $\dot{\ell}_1(z)^2 + \dot{\ell}_2(z)^2 + \dot{\ell}_3(z)^2 = 1$, where by $\dot{\ell}_j$ we denote the derivative of function ℓ_j with respect to the variable z. Dealing with the curve ℓ, we want to associate with it the Frenet frame, i.e. the orthonormal triad of smooth vector fields $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ called respectively the tangent, normal, and binormal vectors, defined as follows

$$\mathbf{t} = \dot{\ell}\,, \quad \mathbf{n} = \kappa^{-1}\ddot{\ell}\,, \quad \mathbf{b} = \mathbf{t} \times \mathbf{n}\,.$$

where the cross denotes the vector product in $\mathbb{R}^3$ and $\kappa := |\ddot{\ell}|$ is the curvature of ℓ. Put like that, the Frenet frame may not exist, in particular, because it is necessary to assume that $\kappa > 0$ holds to make sense of the definition of the normal and binormal. If a part of ℓ is a straight line segment, i.e. $\kappa = 0$ holds on it identically, one can employ any fixed triad one element of which coincides with the tangent vector. With a slight abuse of terminology we will say that ℓ possesses a *global Frenet frame* if triads corresponding to its straight and non-straight parts can glued together smoothly, modulo a rotation of the Frenet parts on a fixed angle around the appropriate tangent vector vector, see [8, 9] or [11, Sec. 1.3].

In such a case the Serret-Frenet formulæ give

$$\dot{\mathbf{t}} = \kappa\mathbf{n}\,, \quad \dot{\mathbf{n}} = -\kappa\mathbf{t} + \tau\mathbf{b}\,, \quad \dot{\mathbf{b}} = -\tau\mathbf{n}\,, \tag{2.1}$$

where τ stands for the torsion of the curve ℓ. Given a function $\beta \in C^1(\mathbb{R})$ we introduce further a general moving frame $\{\mathbf{T}^\beta, \mathbf{N}^\beta, \mathbf{B}^\beta\}$ by

$$\mathbf{T}_\beta = \mathbf{t}\,, \quad \mathbf{N}_\beta = \mathbf{n}\cos\beta - \mathbf{b}\sin\beta\,, \quad \mathbf{B}_\beta = \mathbf{n}\sin\beta + \mathbf{b}\cos\beta\,; \tag{2.2}$$

the equations (2.1) show that this triad elements satisfy the relations

$$\begin{aligned} \dot{\mathbf{T}}_\beta &= \kappa(\mathbf{N}_\beta\cos\beta + \mathbf{B}_\beta\sin\beta)\,, \\ \dot{\mathbf{N}}_\beta &= -\kappa\mathbf{T}_\beta\cos\beta + (\tau - \dot{\beta})\mathbf{B}_\beta\,, \\ \dot{\mathbf{B}}_\beta &= -\kappa\mathbf{T}_\beta\sin\beta - (\tau - \dot{\beta})\mathbf{N}_\beta\,. \end{aligned} \tag{2.3}$$

A particular choice $\beta(z) = \int_{-\infty}^{z} \tau(s)ds$ yields the so-called *parallel transport frame*, in the physics literature often referred to as the *Tang frame*.

Let next ω be a two-dimensional bounded domain with the boundary $\partial\omega$ supposed to be C^2. The waveguide we are going to consider is defined as the tube $Q_{\ell,\beta}$ obtained by moving the cross-section ω along the reference curve ℓ keeping its position fixed with respect to the frame (2.2). More precisely, we set

$$Q_{\ell,\beta} = \{X \in \mathbb{R}^3 \colon X = \ell(x_3) + x_1 \mathbf{N}_\beta + x_2 \mathbf{B}_\beta,\ x' = (x_1, x_2) \in \omega,\ z = x_3 \in \mathbb{R}\}. \tag{2.4}$$

Denoting $a := \sup_{x' \in \omega} |x'|$ and assuming that

$$a\|\kappa\|_\infty < 1 \tag{2.5}$$

one can check easily that the formula (2.4) induces a local C^1-diffeomorfism between $Q_{\ell,\beta}$ and the straight tube $Q = \omega \times \mathbb{R}$. We will assume, without repeating it every time, that this diffeomorphism is *global*, in other words, that the tube $Q_{\ell,\beta}$ has no self-intersections.

With the eye on the definition of the operators in the next section we divide the boundary $\partial\omega$ into two parts. One denoted as γ_D is assumed to be a union of a finite number of arcs, each of a positive measure, while its complement $\partial\omega \setminus \gamma_D$ is denoted as γ_N. The pair (ω, γ_D) is called *rotationally invariant* if ω and γ_D are both rotationally invariant with respect to the origin. From the viewpoint of this paper, this trivial case that can occur only if ω is a disc, an annulus, or a family of concentric annuli, and each connected component of $\partial\omega$ is circle belonging to only one of the sets γ_D, γ_N. In the following we will consider only *rotationally non-invariant* pairs (ω, γ_D).

Let us now specify two types of geometric deformations which we will consider in this paper. We say that the tube $Q_{\ell,\beta}$ is *bent* if the reference curve ℓ is not a straight line, that is, the curvature κ does not vanish identically. Furthermore, the tube $Q_{\ell,\beta}$ is said to be *twisted* if the pair (ω, γ_D) is not rotationally invariant and $\tau - \dot\beta \neq 0$. Looking at the equations (2.3) it is obvious that these perturbations are mutually independent. For the sake of simplicity, we will study separately the following two cases:

(i) *bending without twisting:* $\kappa \neq 0$ and $\tau - \dot\beta = 0$, and

(ii) *twisting without bending:* $\kappa = 0$ and $\tau - \dot\beta \neq 0$.

3 Preliminaries: definition of the operator

Before introducing the operator of our interest, we need a couple of auxiliary notions. We denote by λ_1 the lowest eigenvalue of the problem

$$-\Delta'\psi = \lambda\psi \ \text{ in } \omega\,, \quad \psi = 0 \ \text{ on } \gamma_D\,, \quad \partial_n\psi = 0 \ \text{ on } \gamma_N\,, \tag{3.1}$$

where $\Delta' = \nabla' \cdot \nabla'$ stands for the Laplace operator with respect to the variables x' and ∂_n is the outward normal derivative. The corresponding eigenfunction normalized in $L^2(\omega)$ will be denote by ψ_1; we note that ψ_1 can be chosen positive in ω and it satisfies the integral identity

$$(\nabla'\psi_1, \nabla'\phi)_\omega = \lambda_1(\psi_1, \phi)_\omega \qquad \forall\phi \in H_0^1(\omega, \gamma_D)\,, \tag{3.2}$$

where $(\cdot,\cdot)_\omega$ is the natural scalar product in the Lebesgue space $L^2(\omega)$ and $H_0^1(\omega, \gamma_D)$ consists of functions from the first Sobolev space that vanish on γ_D. If (ω, γ_D) is not rotationally invariant we have

$$\partial_\varphi\psi_1 \neq 0 \quad \Leftrightarrow \quad \int_\omega |\partial_\varphi\psi_1|^2\,\mathrm{d}x' \neq 0\,. \tag{3.3}$$

By ∂_φ we denote here the first-order differential operator $x_1\partial_2 - x_2\partial_1$ which corresponds to differentiation with respect to the polar angle φ in the plane (x_1, x_2), or up to the imaginary unit, to the angular momentum operator generating rotations around the tangent vector to the tube axis.

The main object of our interest is the Laplace operator $\widetilde{T}_{\ell,\beta}$ on $L^2(Q_{\ell,\beta})$ with mixed Dirichlet-Neumann boundary conditions that can be associated with the closed quadratic form

$$\widetilde{a}_{\ell,\beta}[u] = \int_{Q_{\ell,\beta}} |\nabla_X u|^2\,\mathrm{d}X, \quad u \in H_0^1(Q_{\ell,\beta}, \Gamma_{\ell,\beta})\,,$$

where $\Gamma_{\ell,\beta} := \{X \in \mathbb{R}^3 \colon X = \ell(x_3) + x_1\mathbf{N}_\beta + x_2\mathbf{B}_\beta,\ x' \in \gamma_D,\ x_3 \in \mathbb{R}\}$.

The diffeomorphism (2.4) can be used to map $\widetilde{T}_{\ell,\beta}$ in the usual way [11, Secs. 1.3 and 1.7] to an operator on the straight tube Q in which the geometry is encoded in the coefficients. Specifically, in case (i) the operator $\widetilde{T}_{\ell,\beta}$ is unitarily equivalent to operator $T_{\ell,\beta}$ associated with the quadratic form

$$a_{\ell,\beta}[u] := \int_Q \left[g(|\partial_1 u|^2 + |\partial_2 u|^2) + g^{-1}|\partial_3 u|^2\right]\mathrm{d}x\,, \quad u \in H_0^1(Q, \Gamma, g)\,, \tag{3.4}$$

in the weighted Lebesgue space $L^2(Q, g)$ with the scalar product

$$(u, v)_{Q,g} = (ug, v)_Q \,,$$

where $g(x) := 1 - (x_1 \cos\beta(x_3) + x_2 \sin\beta(x_3))\kappa(x_3)$. In particular, when the bending is absent, $\kappa = 0$, and the cross-section remain fixed in the parallel transport frame, $\dot\beta = \tau$, the spectrum of $T_{\ell,\beta}$ is found by separation of variables: it is purely continuous and equal to $[\lambda_1, +\infty)$, where λ_1 is the eigenvalue appearing in (3.2). The question we address in the next section is under which circumstances can a bending of such a tube give rise to a nonempty discrete spectrum below the threshold λ_1.

Similarly, in case (ii) the operator $\widetilde{T}_{\ell,\beta}$ is unitarily equivalent to the operator T_β associated with the quadratic form

$$a_\beta[u] := \int_Q \left[|\partial_1 u|^2 + |\partial_2 u|^2 + |(\partial_3 + \dot\beta \partial_\varphi)u|^2 \right] \mathrm{d}x \,, \quad u \in H_0^1(Q, \Gamma_D) \,, \quad (3.5)$$

in the space $L^2(Q)$. Here the role of an unperturbed system will be played by tubes with a constant twisting; in Sec. 5 below we will discuss what happens if the twisting rate is modified locally.

4 The effect of bending

Let us first focus on spectral properties of $T_{\ell,\beta}$ if the tube exhibits a bending without twisting, i.e. the case (i) indicated in Sec. 2. To state the results, we need to introduce two quantities,

$$A_j = \frac{1}{2} \int_{\partial\omega} n_j |\psi_1|^2 \, \mathrm{d}x' = \int_\omega \psi_1 \partial_j \psi_1 \, \mathrm{d}x' \,, \quad j = 1, 2 \,,$$

where (n_1, n_2) are the components of outward normal to the boundary $\partial\omega$. Note that while we use modulus in the first expressions as it is common in quantum mechanics, we suppose that the function ψ_1 is real-valued.

Theorem 4.1. *If there exists a compact interval $I \subset \mathbb{R}$ such that*

$$\int_I \kappa(x_3)(A_1 \cos\beta(x_3) + A_2 \sin\beta(x_3)) \, \mathrm{d}x_3 < 0$$

and $\kappa(x_3)(A_1 \cos\beta(x_3) + A_2 \sin\beta(x_3)) \le 0$ holds for all $x_3 \in \mathbb{R} \setminus I$, then

$$\inf \sigma(T_{\ell,\beta}) < \lambda_1 \,.$$

In particular, if the curvature κ is compactly supported, the discrete spectrum of $T_{\ell,\beta}$ is nonempty.

Proof. It is sufficient to find a trial function $u \in H_0^1(Q_{\ell,\beta}, \Gamma_{\ell,\beta})$ such that

$$a_{\ell,\beta}[u] - \lambda_1 \|u\|_{Q,g}^2 < 0\,, \tag{4.1}$$

where $a_{\ell,\beta}$ is the quadratic form (3.4). We will seek it in the form $u(x) = v(x_3)\psi_1(x')$, where v is a smooth function with compact support such that $v(x_3) = 1$ for $x_3 \in I$ and

$$\|\partial_3 v\|_{L^2(\mathbb{R})} = \delta\,, \tag{4.2}$$

where δ is a parameter to be chosen. The assumption (2.5) in combination with relation (4.2) imply that there is a $C > 0$ such that

$$\int_Q g^{-1}|\partial_3 u|^2 \,\mathrm{d}x \le C\delta^2\,,$$

At the same time, the remaining part of the quadratic form in question is

$$\begin{aligned}
&\int_Q g(x)(|\nabla' u(x)|^2 - \lambda_1 |u(x)|^2)\,\mathrm{d}x = \\
&= -\int_{\mathbb{R}} \kappa(x_3)\cos\beta(x_3)|v(x_3)|^2\,\mathrm{d}x_3 \int_\omega x_1(|\nabla'\psi_1(x')|^2 - \lambda_1|\psi_1(x')|^2)\,\mathrm{d}x' - \\
&\quad - \int_{\mathbb{R}} \kappa(x_3)\sin\beta(x_3)|v(x_3)|^2\,\mathrm{d}x_3 \int_\omega x_2(|\nabla'\psi_1(x')|^2 - \lambda_1|\psi_1(x')|^2)\,\mathrm{d}x'\,,
\end{aligned}$$

where we have employed the explicit formula of $g(x)$ and relation (3.2). Next we note that

$$A_j = -\int_\omega x_j(|\nabla'\psi_1(x')|^2 - \lambda_1|\psi_1(x')|^2)\,\mathrm{d}x'\,, \quad j = 1, 2\,.$$

Indeed, let us insert $\phi(x') = x_j\psi_1(x')$ into the integral identity (3.2) and rewrite it in the following way

$$\begin{aligned}
-\int_\omega x_j(|\nabla'\psi_1(x')|^2 - \lambda_1|\psi_1(x')|^2)\,\mathrm{d}x' &= \int_\omega \psi_1(x')\nabla' x_j \cdot \nabla'\psi_1(x')\,\mathrm{d}x' \\
&= \int_\omega \psi_1(x')\partial_j\psi_1(x')\,\mathrm{d}x'\,.
\end{aligned}$$

Under the stated assumptions the whole expression is negative and choosing δ small enough we can make relation (4.1) satisfied. We know that for the tube without bending, $\kappa = 0$, the spectrum is purely essential and equal to $[\lambda_1, +\infty)$ and it easy to check that it remains preserved under a compactly supported perturbation, hence the spectrum below λ_1 must be in such a case discrete and nonempty. ■

On the other hand, we can state a condition under which the bending does not move $\inf \sigma(T_{\ell,\beta})$ down, which means, in particular, the absence of eigenvalues for (non-twisted) tubes with a compactly supported curvature.

Theorem 4.2. *If for all $x_3 \in \mathbb{R}$ and $x' \in \omega$ the inequality*

$$\kappa(x_3)(\partial_1\psi_1(x')\cos\beta(x_3) + \partial_2\psi_1(x')\sin\beta(x_3)) \geq 0 \tag{4.3}$$

is valid, we have $\inf \sigma(T_{\ell,\beta}) \geq \lambda_1$.

Proof. Fix an arbitrary function $u \in C_0^\infty(Q)$. Since ψ_1 is positive in ω we can write it as $u(x) = \psi_1(x')v(x)$ with some $v \in H^1(Q)$. The 'shifted' quadratic form entering (4.1) can be estimated from below by neglecting the non-negative term containing the derivative with respect to the longitudinal variable x_3,

$$a_{\ell,\beta}[u] - \lambda_1\|u\|_{Q,g}^2 \geq \int_Q g(x)\left(|\nabla' u(x)|^2 - \lambda_1|u(x)|^2\right) \mathrm{d}x. \tag{4.4}$$

The above described factorization yields the formula

$$|\nabla'(\psi_1 v)|^2 = |\psi_1|^2|\nabla' v|^2 + \nabla'\psi_1 \cdot \nabla'(\psi_1 v^2)\,,$$

which allows us to split the last integral in (4.4) into two parts,

$$J_1 = \int_Q g(x)|\psi_1(x')|^2|\nabla' v(x)|^2 \mathrm{d}x \geq 0$$

and

$$J_2 = \int_Q g(x)\left(\nabla'\psi_1(x') \cdot \nabla'(\psi_1(x')v^2(x)) - \lambda_1|\psi_1(x')|^2|v(x)|^2\right) \mathrm{d}x\,.$$

Using the explicit form of $g(x)$ in combination with (3.2) we get

$$J_2 = -\int_Q x_1 \cos\beta(x_3)\kappa(x_3)\big(\nabla'\psi_1(x')\cdot\nabla'(\psi_1(x')v^2(x)) - \lambda_1|\psi_1(x')|^2|v(x)|^2\big)\,\mathrm{d}x$$
$$-\int_Q x_2 \sin\beta(x_3)\kappa(x_3)\big(\nabla'\psi_1(x')\cdot\nabla'(\psi_1(x')v^2(x)) - \lambda_1|\psi_1(x')|^2|v(x)|^2\big)\,\mathrm{d}x.$$

Inserting next the function $x_j\psi_1(x')|v(x',x_3)|^2$ into the integral identity (3.2) as a test function with x_3 as parameter, we obtain

$$\int_\omega x_j \nabla'\psi_1(x')\cdot\nabla'(\psi_1(x')v^2(x))\mathrm{d}x' - \lambda_1\int_\omega x_j|\psi_1(x')|^2|v(x)|^2\mathrm{d}x' =$$
$$= -\int_\omega \psi_1(x')v^2(x)\nabla' x_j\cdot\nabla'\psi_1(x')\,\mathrm{d}x' = -\int_\omega \psi_1(x')v^2(x)\partial_j\psi_1(x')\,\mathrm{d}x'\,,$$

and consequently,

$$J_2 = \int_Q \kappa(x_3)(\partial_1\psi_1(x')\cos\beta(x_3) + \partial_2\psi_1(x')\sin\beta(x_3))\psi_1(x')v^2(x)\,\mathrm{d}x\,,$$

which together the assumption (4.3) shows that $a_{\ell,\beta}[u] - \lambda_1\|u\|^2_{Q,g} \geq 0$ holds for any $u \in C_0^\infty(Q)$. To finish the proof it is enough to observe that this set is dense in $H_0^1(Q)$ and also, *mutatis mutandis*, in $H_0^1(Q,\Gamma)$. ■

In the particular case where the waveguide axis is a planar curve and the Jacobian depends only on the coordinates x_1 and x_3, in other words, $\tau(x_3) = \beta(x_3) = 0$ holds for all $x_3 \in \mathbb{R}$, we have $\mathbf{N} = \mathbf{n}$, $\mathbf{B} = \mathbf{b}$ and the quadratic form expression simplifies to

$$a_{\ell,0}[u] = \int_Q (1 - x_1\kappa(x_3))^{-1}|\partial_3 u(x)|^2 + (1 - x_1\kappa(x_3))(|\nabla' u(x)|^2)\,\mathrm{d}x\,;$$

then the above theorems lead to the following conclusions:

Corollary 4.3. *If there is a compact interval $I \subset \mathbb{R}$ such that*

$$A_1\int_I \kappa(x_3)\,\mathrm{d}x_3 < 0$$

and $A_1\kappa(x_3) \leq 0$ for all $x_3 \in \mathbb{R}\setminus I$, then $\inf\sigma(T_{\ell,0}) < \lambda_1$.

Corollary 4.4. *If for all $x_3 \in \mathbb{R}$ and for all $x' \in \omega$ the inequality*

$$\kappa(x_3)\partial_1\psi_1(x') \geq 0$$

is valid, then $\inf \sigma(T_{\ell,0}) \geq \lambda_1$.

It is instructive to compare these results with the spectral properties of a bent and asymptotically straight two-dimensional guide which has one boundary Dirichlet and one Neumann. As mentioned in the introduction, it is known [6] that the discrete spectrum of such a system is nonvoid if the total bending of the strip is positive (in fact, nonnegative) and the Dirichlet boundary faces 'inward', and on the other hand, the spectral threshold remains preserved if the bend is simple, i.e. the curvature does not change sign, and the Dirichlet condition is on the 'outward' side. This nicely fits with the above corollaries if we realize that the normal, in other words, the x_1 direction points ''inwards' in a bend and the outward normal derivative of ψ_1 is zero at the Neumann segment(s) of the boundary and negative at the Dirichlet one(s). Note also that while a waveguide with a rectangular cross-section and two flat sides does not satisfy our boundary smoothness requirement, the above reasoning can nevertheless be carried through. If the bent sides of such a rectangular tube are Dirichlet and Neumann and a fixed condition is chosen on each of the flat sides, by separation of variables we get a direct correspondence between the said two-dimensional properties and the results obtained here.

5 Twisting without bending

Let us now turn to the second class of geometric perturbations indicated in Sec. 2 and discuss the situation when the waveguide with a mixed Dirichlet-Neumann boundary is twisted waveguide. Recall first how the situation looks like for tubes with purely Dirichlet boundary. If the twisting is only local there it does not affect the essential spectrum of the Laplacian, and moreover, it does stabilize it against negative perturbations – see, e.g., [4, 8]. This has consequences such as the absence of weakly coupled bound states of Schrödinger operators in twisted waveguides [8, 13]. If the twist is not local but constant, then it even increases the threshold of the essential spectrum of the Laplacian, and moreover, any local slowdown of the constant twisting rate induces at least one bound state of the corresponding operator [10].

Our aim in this section is to show that the behavior of twisted tubes with a mixed Dirichlet-Neumann boundary is similar. We thus suppose that $\dot\beta_0(x_3) = \alpha$ and introduce a model eigenvalue problem on the cross section,

$$-\Delta\psi^\alpha - \alpha^2\partial_\varphi^2\psi^\alpha = \lambda\psi^\alpha \text{ in } \omega\,, \quad \psi^\alpha = 0 \text{ on } \gamma_D\,, \quad \partial_n\psi^\alpha = 0 \text{ on } \gamma_N\,.$$

We denote the smallest eigenvalue of this problem by λ_1^α, the corresponding normalized eigenfunction in $L^2(\omega)$ will be ψ_1^α; we note that ψ_1^α can be supposed to be positive without loss of generality and that it satisfies the integral identity

$$(\nabla'\psi_1^\alpha, \nabla'\phi)_\omega + \alpha^2(\partial_\varphi\psi_1^\alpha, \partial_\varphi\phi)_\omega = \lambda_1^\alpha(\psi_1^\alpha, \phi)_\omega \qquad \forall\phi \in H_0^1(\omega, \gamma_D)\,.$$

Proposition 5.1. *If $\dot\beta_0 = \alpha$ is constant the spectrum of the positive self-adjoint operator T_{β_0} coincides with the interval $[\lambda_1^\alpha, +\infty)$.*

Proof. Similarly to the proof of Theorem 4.2, we consider function from a core of T_{β_0} writing them as $u(x) = \psi_1^\alpha(x')v(x)$ with $v \in H^1(Q)$, then a direct calculation shows that

$$a_{\beta_0}[u] - \lambda_1^\alpha\|u\|_Q^2 = \int_Q |\psi_1^\alpha(x')|^2\big(|\nabla'v(x)|^2 + |(\partial_3 + \alpha\partial_\varphi)v(x)|^2\big)\,\mathrm{d}x \geq 0$$

and by the density argument we obtain the estimate $\inf\sigma(T_{\beta_0}) \geq \lambda_1^\alpha$. To complete the proof it is sufficient to construct in the standard way Weyl sequences for any $\lambda \in [\lambda_1^\alpha, +\infty)$. ■

In this way a constant twisting, $\dot\beta_0 = \alpha$, changes the essential spectrum in a way depending on α. Consider next a local slowdown of the twist. Let $\theta = \theta(x_3)$ be a C^1-function supported in a compact interval I and assume that the rotation angle β is of the form

$$\dot\beta(x_3) = \alpha - \theta(x_3)\,. \tag{5.1}$$

From the compactness of $\operatorname{supp}\theta$ it follows by a standard perturbation argument that

$$\inf\sigma_{\mathrm{ess}}(T_\beta) = \inf\sigma(T_{\beta_0}) = \lambda_1^\alpha\,.$$

Proposition 5.2. *Let $\dot\beta$ be given by the formula* (5.1) *and*

$$\int_{\mathbb{R}} (|\dot\beta(x_3)|^2 - \alpha^2)\,\mathrm{d}x_3 < 0\,,$$

then $\inf\sigma(T_\beta) < \lambda_1^\alpha$, *and consequently,* $\sigma_{\mathrm{disc}}(T_\beta) \neq \varnothing$.

Proof. We employ trial functions $u \in H_0^1(Q, \Gamma_D)$ of the factorized form $u(x) = \psi_1^\alpha(x')v(x_3)$, where v is a smooth function such that $v(x_3) = 1$ if $x_3 \in I$; it is easy to check that for any $\delta > 0$ one can choose v so that $\|\partial_3 v\|_{L^2(\mathbb{R})} = \delta$. A straightforward calculation then yields

$$a_\beta[u] - \lambda_1^\alpha\|u\|_Q^2 = \delta^2 + \int_{\mathbb{R}} (|\dot\beta(x_3)|^2 - \alpha^2)\,\mathrm{d}x_3 \int_\omega |\partial_\varphi\psi_1^\alpha(x')|^2\,\mathrm{d}x'.$$

Taking into account relation (3.3) we find that for δ small enough we have $a_\beta[u] - \lambda_1^\alpha\|u\|_Q^2 < 0$ which completes the proof. ∎

6 Periodic twisting

Now we are going to consider a more general situation, bending still absent and the twisting is non-constant but periodic leading to a band-gap structure of the spectrum. It is natural to expect that a higher twisting rate could increase the spectral threshold. Our aim here is to demonstrate that it is indeed the case provided the twisting is gentle. To state the result let us denote by $\beta \in C^2(\mathbb{R})$ the twisting function with a positive and 1-periodic derivative $\dot\beta$. Let $\lambda_\dagger(\beta)$ be the spectral threshold of the Laplacian with the mixed Dirichlet-Neumann boundary conditions in the twisted tube Q_β. The main result of the present section is the following theorem:

Theorem 6.1. *Let $\vartheta_1, \vartheta_2 \in C^2(\mathbb{R})$ with $\dot\vartheta_1$, $\dot\vartheta_2$ being positive 1-periodic functions and*

$$\int_0^1 |\dot\vartheta_1|^2\,\mathrm{d}x_3 < \int_0^1 |\dot\vartheta_2|^2\,\mathrm{d}x_3\,,$$

then there exists an $\varepsilon_0 > 0$ such that the inequality $\lambda_\dagger(\varepsilon\vartheta_1) < \lambda_\dagger(\varepsilon\vartheta_2)$ holds for all $\varepsilon \in (0, \varepsilon_0)$.

We will prove the theorem in several steps.

6.1 Formulation of the problem

If the function $\dot\beta$ is 1-periodic the spectrum of positive self-adjoint operator T_β is known to be purely essential having a band-gap structure,

$$\sigma(T_\beta) = \bigcup_n \mathcal{B}_n(\beta)\,. \tag{6.1}$$

One naturally expects it to be absolutely continuous, however, this property has been so far established in some cases only [11, Chap. 9]. Among spectral properties of periodic waveguides, the gap opening for Laplace operator with various boundary conditions has been discussed in many papers – cf., e.g., [1, 5, 14, 16] – in particular, for the case of Dirichlet Laplacian in periodically twisted waveguide see [3].

To study the spectrum (6.1) we use Floquet-Bloch theory and decompose the operator T_β into a direct integral of the operator family $\{T_\beta(\eta)\}$ parametrized by the quasi-momentum $\eta \in [-\pi, \pi]$. The fiber operators $T_\beta(\eta)$ can be defined through their quadratic forms

$$a_{\beta,\eta}[U] = \int_\Omega |\nabla' U|^2 + |(\partial_3 + \dot\beta\partial_\varphi)U|^2 \,\mathrm{d}x\,, \quad U \in \mathcal{H}_\eta\,,$$

related as usual to the corresponding sesquilinear forms $A_{\beta,\eta}$ by $a_{\beta,\eta}[U] = A_{\beta,\eta}(U,U)$, where $\Omega = \omega \times [0,1]$ is the periodicity cell and the form domain $\mathcal{H}_\eta$ consists of functions $U \in H_0^1(\Omega, \gamma_D \times [0,1])$ satisfying a quasi-periodicity condition

$$U(x',1) = e^{i\eta}U(x',0)\,, \quad x' \in \omega\,.$$

The inner product in $\mathcal{H}_\eta$ is given by $\langle U, V\rangle = (\nabla U, \nabla V)_\Omega$. It is not difficult to check that the quadratic form $a_{\beta,\eta}$ is positive and closed, and therefore associated with a unique self-adjoint operator $T_\beta(\eta)$. Due to the compactness of the periodicity cell the latter has a compact resolvent, and as a consequence, the spectrum of operator $T_\beta(\eta)$ is purely discrete, in other words, a sequence of eigenvalues

$$0 < \Lambda_{1,\beta}(\eta) \le \Lambda_{2,\beta}(\eta) \le \Lambda_{3,\beta}(\eta) \le \dots \tag{6.2}$$

accumulating only at infinity; as the inequalities (6.2) suggest, except the first some on them may not be simple. We denote the corresponding eigenfunctions by $U_{k,\beta} \in \mathcal{H}_\eta$, where for the sake of simplicity the dependence on

η will usually not be shown. We can choose them so that they satisfy the orthogonality property

$$(U_{j,\beta}, U_{k,\beta})_\Omega = \delta_{j,k}\,, \quad j,k = 1,2,3,\dots\,,$$

where $\delta_{j,k}$ is the Kronecker symbol. The band functions $\eta \mapsto \Lambda_{j,\beta}$ are known to be continuous and 2π-periodic so the sets

$$\mathcal{B}_j(\beta) = \{\Lambda_{j,\beta}(\eta) |\, \eta \in [-\pi,\pi]\} \subset [0,+\infty)\,,$$

the spectra bands, are closed intervals. In this notation the overall spectral threshold can be written as

$$\lambda_\dagger(\beta) = \inf \sigma(T_\beta) = \inf_{\eta\in[-\pi,\pi]} \Lambda_{1,\beta}(\eta)\,.$$

In the absence of twisting, $\beta = 0$, the eigenvalues and eigenfunctions of the fiber operator are easily found explicitly,

$$V_{j,k}(x,\eta) = e^{i\eta x_3} e^{2\pi i k x_3} \psi_j(x')\,,\ M_{j,k}(\eta) = (\eta + 2\pi k)^2 + \lambda_j\,,\ k \in \mathbb{Z},\ j \in \mathbb{N}, \tag{6.3}$$

where (ψ_j, λ_j) is the jth eigenpair of the problem (3.1). The family $\{\psi_j\}_{j=1}^\infty$ can be chosen to be othonormal in $L^2(\omega)$ and the eigenvalue sequence $\{\lambda_j\}_{j=1}^\infty$ is conventionally ordered in the non-decreasing way counting multiplicities. Rearranging the sequence $\{M_{j,k}(\eta)\}$ in the ascending order we obtain (6.2) for $\beta = 0$, in particular, $\Lambda_{1,0}(\eta) = \lambda_1 + \eta^2$; we note that these eigenvalues are simple unless $\eta = \pm\pi$.

6.2 Small twisting, simple estimates

Let us introduce a positive parameter $\varepsilon \in (0,1)$ and discuss the properties of the spectrum $\sigma(T_{\varepsilon\beta})$ as $\varepsilon \to 0$ for a given function β. We start with the following simple lemma.

Lemma 6.2. *There is a constant C_β such that for all $\varepsilon \in (0,1)$, $\eta \in [-\pi,\pi]$, and $U \in \mathcal{H}_\eta$ the estimate*

$$|a_{\varepsilon\beta,\eta}[U] - a_{0,\eta}[U]| \le C_\beta\, \varepsilon \|\nabla U\|_\Omega^2 \tag{6.4}$$

holds.

Proof. In view of (3.5) we have

$$a_{\varepsilon\beta,\eta}[U] - a_{0,\eta}[U] = 2\varepsilon \operatorname{Re}(\partial_3 U, \dot\beta\partial_\varphi U)_\Omega + \varepsilon^2\|\dot\beta\partial_\varphi U\|_\Omega^2 ,$$

and since $|\partial_\varphi U| \le \sup_{x\in\Omega}|x|\,|\nabla U|$, we get the desired estimate. ■

Corollary 6.3. *There is an $\varepsilon(\beta) \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon(\beta))$ and $\eta \in [-\pi, \pi]$ the estimate*

$$a_{\varepsilon\beta,\eta}[U] \ge \frac{1}{2}\|\nabla U\|_\Omega^2$$

holds.

Proof. Observing that $\|\nabla U\|_\Omega^2 = a_{0,\eta}[U]$ and using (6.4) we obtain

$$\left|a_{\varepsilon\beta,\eta}[U] - \|\nabla U\|_\Omega^2\right| \le C_\beta\,\varepsilon\|\nabla U\|_\Omega^2 ,$$

thus for all $\varepsilon \in (0, C_\beta^{-1})$ the inequality

$$a_{\varepsilon\beta,\eta}[U] \ge (1 - C_\beta\varepsilon)^{-1}\|\nabla U\|_\Omega^2$$

is valid and it is enough to take $\varepsilon(\beta) = (2C_\beta)^{-1}$. ■

This allows us to estimate the twist effect on the fiber operator eigenvalues.

Lemma 6.4. *To any k there exists a constant $C_{k,\beta}$ such that*

$$|\Lambda_{k,\varepsilon\beta}(\eta) - \Lambda_{k,0}(\eta)| \le C_{k,\beta}\varepsilon .$$

holds for all $\varepsilon \in (0, \varepsilon(\beta))$ and $\eta \in [-\pi, \pi]$.

Proof. Due to the min-max principle, cf. [2] or [15], we have

$$\Lambda_{k,\varepsilon\beta}(\eta) = \sup_E \inf_{V\in E\setminus\{0\}} \frac{a_{\varepsilon\beta,\eta}[V]}{\|V\|_\Omega^2} ,$$

where E stands for any subspace in $\mathcal{H}_\eta$ of codimension $k-1$. Since the sequence $\{U_{j,0}\}_{j=1}^k$ is chosen orthonormal in $L^2(\Omega)$ and each E is infinite-dimensional we can within it an element U of the form

$$U = \sum_{j=1}^k \alpha_j U_{j,0} , \quad \sum_{j=1}^k |\alpha_j|^2 = 1 .$$

Consequently,

$$\inf_{V\in E\setminus\{0\}} \frac{a_{\varepsilon\beta,\eta}[V]}{\|V\|^2_\Omega} \le a_{\varepsilon\beta,\eta}[U] \le a_{0,\eta}[U] + (a_{\varepsilon\beta,\eta}[U] - a_{0,\eta}[U])\,.$$

By definition of the form $a_{0,\eta}$ we have

$$\|\nabla U\|^2_\Omega = a_{0,\eta}[U] = \sum_{j=1}^{k} |\alpha_j|^2 \Lambda_{j,0}(\eta) \le \Lambda_{k,0}(\eta)\,,$$

and in combination with inequality (6.4) this yields the estimate

$$\Lambda_{k,\varepsilon\beta}(\eta) \le \Lambda_{k,0}(\eta)(1 + C_\beta \varepsilon)\,. \tag{6.5}$$

In a similar way one can write

$$\Lambda_{k,0}(\eta) = \sup_E \inf_{V\in E\setminus\{0\}} \frac{a_{0,\eta}[V]}{\|V\|^2_\Omega}\,,$$

where E runs though the same family of subspaces as above. Repeating the argument we find $U \in E$ of the form $U = \sum_{j=1}^k \alpha_j U_{j,\varepsilon\beta}$ with $\sum_{j=1}^k |\alpha_j|^2 = 1$ and infer that

$$\inf_{V\in E\setminus\{0\}} \frac{a_{0,\eta}[V]}{\|V\|^2_\Omega} \le \Lambda_{k,\varepsilon\beta}(\eta)(1 + 2C_\beta\, \varepsilon)$$

and

$$\Lambda_{k,0}(\eta) \le \Lambda_{k,\varepsilon\beta}(\eta)(1 + 2C_\beta\, \varepsilon)\,. \tag{6.6}$$

The inequalities (6.5) and (6.6) imply the claim of the Lemma. ■

6.3 Asymptotic procedure

Now we are going to present, in (6.7) and (6.8) below, asymptotic expansions for the eigenvalues $\Lambda_{k,\varepsilon\beta}(\eta)$ and for the eigenfunctions $U_{k,\varepsilon\beta}$ as $\varepsilon \to 0$. We consider only simple eigenvalues $\Lambda_{k,0}(\eta)$ which is sufficient for our purpose. Regarding the eigenpair $(U_{k,\varepsilon\beta}, \Lambda_{k,\varepsilon\beta}(\eta))$ as a perturbation of $(U_{k,0}, \Lambda_{k,0}(\eta))$ it is natural to seek the asymptotic formulæ in the form

$$\Lambda_{k,\varepsilon\beta}(\eta) = \Lambda_{k,0}(\eta) + \varepsilon\Lambda'_k(\eta) + \varepsilon^2\Lambda''_k(\eta) + \widetilde{\Lambda}^\varepsilon_k(\eta), \tag{6.7}$$

$$U_{k,\varepsilon\beta} = U_{k,0} + \varepsilon U'_k + \varepsilon^2 U''_k + \widetilde{U}^\varepsilon_k, \tag{6.8}$$

where $\Lambda'_k(\eta)$, $\Lambda''_k(\eta)$, U'_k, U''_k are the coefficient to be determined and $\widetilde{\Lambda}^\varepsilon_k(\eta)$, $\widetilde{U}^\varepsilon_k$ the remainders to be estimated. To begin with, we write the operator $T_{\varepsilon\beta}(\eta)$ as

$$T_{\varepsilon\beta}(\eta) = T_0(\eta) + \varepsilon T_1(\eta) + \varepsilon^2 T_2(\eta), \tag{6.9}$$

where $T_0(\eta)$ is the operator associated with the quadratic form $a_{0,\eta}$, while $T_1(\eta)$ and $T_2(\eta)$ correspond to the forms

$$b_{1,\eta}[U] := 2\operatorname{Re}(\partial_3 U, \dot\beta\partial_\varphi U)_\Omega \quad \text{and} \quad b_{2,\eta}[U] := \|\dot\beta\partial_\varphi U\|^2_\Omega$$

in $\mathcal{H}(\eta)$, respectively. Substituting from (6.7)–(6.9) into the eigenvalue equation $T_{\varepsilon\beta(\eta)}U_{k,\varepsilon\beta} = \Lambda_{k,\varepsilon\beta}U_{k,\varepsilon\beta}$ and collecting terms of order ε and ε^2 we get

$$T_0(\eta)U'_k - \Lambda_{k,0}(\eta)U'_k = \Lambda'_k(\eta)U_{k,0} - T_1(\eta)U_{k,0}\,, \tag{6.10}$$

$$T_0(\eta)U''_k - \Lambda_{k,0}(\eta)U''_k = \Lambda''_k(\eta)U_{k,0} - T_2(\eta)U_{k,0} + \Lambda'_k(\eta)U'_k - T_1(\eta)U'_k\,. \tag{6.11}$$

Since the eigenvalue $\Lambda_{k,0}$ is supposed to be simple and the resolvent of $T_0(\eta)$ is compact, equation (6.10) has by Fredholm alternative one solvability condition, namely its right-hand side must be orthogonal to $U_{k,0}$ in $L^2(\Omega)$. Due to the normalization condition we get

$$\Lambda'_k(\eta) = (T_1(\eta)U_{k,0}, U_{k,0})_\Omega = 2\operatorname{Re}(\partial_3 U_{k,0}, \dot\beta\partial_\varphi U_{k,0})_\Omega = 0\,, \tag{6.12}$$

where we have used the representation (6.3) of the function $U_{k,0}$ and the following simple observation,

$$(\psi_n, \partial_\varphi\psi_n)_\omega = 0 \quad \forall n \in \mathbb{N}\,.$$

In view of (6.12) the equations (6.10) and (6.11) simplify to

$$T_0(\eta)U'_k - \Lambda_{k,0}(\eta)U'_k = -T_1(\eta)U_{k,0}\,, \tag{6.13}$$

$$T_0(\eta)U''_k - \Lambda_{k,0}(\eta)U''_k = \Lambda''_k(\eta)U_{k,0} - T_2(\eta)U_{k,0} - T_1(\eta)U'_k\,. \tag{6.14}$$

Note that the solution U'_k to (6.13) is determined up to a multiple of the eigenfunction $U_{k,0}$, hence without loss of generality we may assume that

$$(U'_k, U_{k,0})_\Omega = 0\,. \tag{6.15}$$

In the same way, using the solvability condition of (6.14), we obtain

$$\Lambda''_k(\eta) = (T_2(\eta)U_{k,0}, U_{k,0})_\Omega + 2\operatorname{Re}(\partial_3 U'_k, \dot\beta U_{k,0})_\Omega$$

$$= \|\dot\beta\partial_\varphi\psi_n\|_\Omega^2 + \Lambda_{k,0}(\eta)\|U_k'\|_\Omega^2 - \|\nabla U_k'\|_\Omega^2 \,.$$

Here we assume that the function $U_{k,0}$ is represented as

$$U_{k,0}(x,\eta) = e^{i\eta x_3}e^{2\pi i j x_3}\psi_n(x')$$

for some j and n and as before we choose U_k'' in such a way that

$$(U_k'', U_{k,0})_\Omega = 0\,.$$

Summarizing the above reasoning, we have reached the following conclusions:

Lemma 6.5. *Let $I \subset (-\pi,\pi)$ be a compact set such that $\Lambda_{k,0}(\eta)$ is simple for all $\eta \in I$. Then for all $\eta \in I$ the equations* (6.13) *and* (6.14) *have unique solutions U_k' and U_k'' orthogonal to $U_{k,0}$. Moreover, functions U_k' and U_k'' satisfy the estimate*

$$\max\{\|U_k'\|_{\mathcal{H}_\eta}, \|U_k''\|_{\mathcal{H}_\eta}\} \le c_{k,I} \tag{6.16}$$

with a constant $c_{k,I}$ independent of η.

Now we are in position to formulate the main result of this section:

Theorem 6.6. *Let the eigenvalue $\Lambda_{k,0}(\eta)$ of operator $T_0(\eta)$ be simple for η from a compact set $I \subset [-\pi,\pi]$, then there exists a $c_{k,\beta,I}$ such that for all $\varepsilon \in (0,\varepsilon(\beta))$ and $\eta \in I$ one has*

$$\left|\Lambda_{k,\varepsilon\beta}(\eta) - \Lambda_{k,0}(\eta) - \varepsilon^2\Lambda_k''(\eta)\right| \le c_{k,\beta,I}\,\varepsilon^3\,.$$

Before proceeding with the proof of the Theorem 6.6 a bit of preliminary work is needed. Let us first introduce a new scalar product

$$\langle U,V\rangle_{\varepsilon,\eta} := A_{\varepsilon\beta,\eta}[U,V] \tag{6.17}$$

in the space $\mathcal{H}_\eta$. According to Lemma 6.2 and Corollary 6.3 the corresponding norm is uniformly equivalent to the gradient norm for all $\varepsilon \in (0,\varepsilon(\beta))$. The Hilbert space with the inner product (6.17) will be denoted by $\mathcal{H}_{\varepsilon,\eta}$. Next we define the operator $L_{\varepsilon,\eta}$ by the formula

$$\langle L_{\varepsilon,\eta}U,V\rangle_{\varepsilon,\eta} = (U,V)_\Omega\,;$$

it is easy to check that it is compact and self-adjoint and its discrete spectrum consists of the eigenvalue sequence

$$\nu_{k,\varepsilon}(\eta) = (\Lambda_{k,\varepsilon\beta}(\eta))^{-1} . \tag{6.18}$$

Proof of Theorem 6.6: According to Lemma 6.4 it is sufficient to prove that there is a constant $c_{k,\beta,I} > 0$ such that the interval

$$(\Lambda_{k,0}(\eta) + \varepsilon^2\Lambda_k''(\eta) - c_{k,\beta,I}\varepsilon^3, \Lambda_{k,0}(\eta) + \varepsilon^2\Lambda_k''(\eta) + c_{k,\beta,I}\varepsilon^3)$$

contains at least one member of the sequence $\{\Lambda_{j,\varepsilon\beta}(\eta)\}_{j\geq 1}$. Equivalently, it is enough to establish the existence of at least one eigenvalue from the sequence (6.18) in the interval

$$((\Lambda_{k,0}(\eta) + \varepsilon^2\Lambda_k''(\eta))^{-1} - c'_{k,\beta,I}\varepsilon^3, (\Lambda_{k,0}(\eta) + \varepsilon^2\Lambda_k''(\eta))^{-1} + c'_{k,\beta,I}\varepsilon^3)$$

for some $c'_{k,\beta,I} > 0$. To do this we construct a function $W \in \mathcal{H}_{\varepsilon,\eta} \setminus \{0\}$ such that

$$\|L_{\varepsilon,\eta}W - (\Lambda_{k,0}(\eta) + \varepsilon^2\Lambda_k''(\eta))^{-1}W\|_{\mathcal{H}_{\varepsilon,\eta}} \leq c'_{k,\beta,I}\varepsilon^3\|W\|_{\mathcal{H}_{\varepsilon,\eta}} ,$$

namely, we put $W := U_{k,0} + \varepsilon U_k' + \varepsilon^2 U_k''$. Then

$$\|(L_{\varepsilon,\eta}-(\Lambda_{k,0}(\eta)+\varepsilon^2\Lambda_k''(\eta))^{-1})W\|_{\mathcal{H}_{\varepsilon,\eta}} = \sup_V\langle(L_{\varepsilon,\eta}-(\Lambda_{k,0}(\eta)+\varepsilon^2\Lambda_k''(\eta))^{-1})W, V\rangle_{\varepsilon,\eta} ,$$

where $V \in \mathcal{H}_{\varepsilon,\eta}$ runs over over unit-length vectors,

$$\|V\|_{\mathcal{H}_{\varepsilon,\eta}} = 1 . \tag{6.19}$$

Let us observe the following chain of equalities

$$\begin{aligned}\tau :=&\langle(L_{\varepsilon,\eta} - (\Lambda_{k,0}(\eta) + \varepsilon^2\Lambda_k''(\eta))^{-1})W, V\rangle_{\varepsilon,\eta} \\ =&(W, V)_\Omega - (\Lambda_{k,0}(\eta) + \varepsilon^2\Lambda_k''(\eta))^{-1}A_{\varepsilon\beta,\eta}[W, V] = \\ =&(W, V)_\Omega - (\Lambda_{k,0}(\eta) + \varepsilon^2\Lambda_k''(\eta))^{-1}((T_0(\eta) + \varepsilon T_1(\eta) + \varepsilon^2 T_2(\eta))W, V)_\Omega \\ =&(\Lambda_{k,0}(\eta) + \varepsilon^2\Lambda_k''(\eta))^{-1} \\ &\times (\varepsilon^3\Lambda''(\eta)U_k' + \varepsilon^4\Lambda''(\eta)U_k'' - \varepsilon^3 T_2 U' - \varepsilon^3 T_1 U'' - \varepsilon^4 T_2 U'', V)_\Omega .\end{aligned}$$

Since the expression $(\Lambda_{k,0}(\eta)+\varepsilon^2\Lambda_k''(\eta))^{-1}$ is bounded from above on I, in view of the estimates (6.16) and normalization condition (6.19) we can conclude that there is a $C_{k,\beta,I}$ such that

$$|\tau| \leq C_{k,\beta,I}\,\varepsilon^3$$

holds for all ε small enough. To complete the proof it is sufficient to observe that by virtue of the estimates (6.16) combined with Lemma 6.2 and Corollary 6.3 the expression $\|W\|_{\mathcal{H}_{\varepsilon,\eta}}$ is uniformly bounded away from zero for $\eta \in I$ and $\varepsilon \in (0, \varepsilon(\beta))$. □

Corollary 6.7. *There is a* $\mathbf{c}_\beta > 0$ *such that for all* $\eta \in [-\pi/2, \pi/2]$ *we have*

$$|\Lambda_{1,\varepsilon\beta}(\eta) - \lambda_1 - \eta^2| \leq \mathbf{c}_\beta \varepsilon^2 \,.$$

6.4 Concluding the proof

We now able to prove the main result of this section, Theorem 6.1. Let us divide the interval $[-\pi, \pi]$ into three parts. On the first part, $I_1 := \{\eta \colon |\eta| > \pi/2\}$, we write using Lemma 6.4

$$\Lambda_{1,\varepsilon\beta}(\eta) \geq \Lambda_{1,0}(\eta) - C_{1,\varepsilon}\varepsilon = \lambda_1 + \eta^2 - C_{1,\varepsilon}\varepsilon \geq \lambda_1 + (\pi/2)^2 - C_{1,\varepsilon}\varepsilon \,.$$

On the second part, $I_2 := \{\eta \colon |\eta| \in [\alpha\varepsilon, \pi/2]\}$, we use Corollary 6.7 to obtain

$$\Lambda_{1,\varepsilon\beta}(\eta) \geq \lambda_1 + \eta^2 - \mathbf{c}_\beta \varepsilon^2 \geq \lambda_1 + (\alpha^2 - \mathbf{c}_\beta)\varepsilon^2 \,;$$

the parameter α will be specified below. As for the third part, $I_3 := \{\eta \colon |\eta| < \alpha\varepsilon\}$, applying the estimate from Theorem 6.6 we get

$$\Lambda_{1,\varepsilon\beta}(\eta) \geq \lambda_1 + \eta^2 + \varepsilon^2 \Lambda_1''(\eta) - c_{1,\beta,I_3}\varepsilon^3 \geq \lambda_1 + \varepsilon^2 \Lambda_1''(\eta) - c_{1,\beta,I_3}\varepsilon^3 \,,$$

where $\Lambda_1''(\eta)$ can be calculated by the formula

$$\begin{aligned} \Lambda_1''(\eta) &= \|\dot\beta \partial_\varphi \psi_1\|_\Omega^2 + \Lambda_{1,0}(\eta)\|U_1'\|_\Omega^2 - \|\nabla U_1'\|_\Omega^2 \\ &\geq \int_0^1 |\dot\beta(x_3)|^2 \,\mathrm{d}x_3 \int_\omega |\partial_\varphi \psi_1(x')|^2 \,\mathrm{d}x' - C\alpha^2\varepsilon^2 \,. \end{aligned}$$

The last inequality holds true, because the solution of the equation (6.13) for $k = 1$ with orthogonality condition (6.15) satisfies the estimate

$$\begin{aligned} \|U_1'\|_\Omega + \|\nabla U_1'\|_\Omega &\leq C\|T_1(\eta)U_{1,0}\|_\Omega \\ &= C|\eta|\|\dot\beta\|_{L^2(0,1)}\|\partial_\varphi \psi_1\|_{L^2(\omega)} \,. \end{aligned}$$

Now we put $\alpha^2 := \mathbf{c}_\beta + \int_0^1 |\dot\beta(x_3)|^2 \, \mathrm{d}x_3 \int_\omega |\partial_\varphi \psi_1(x')|^2 \, \mathrm{d}x \, \mathrm{d}y$ and obtain for all sufficiently small ε the estimate

$$\lambda_\dagger(\varepsilon\beta) \geq \lambda_1 + \varepsilon^2 \int_0^1 |\dot\beta(x_3)|^2 \, \mathrm{d}x_3 \int_\omega |\partial_\varphi \psi_1(x')|^2 \, \mathrm{d}x' - C_\beta \varepsilon^3 \, .$$

Together with the result of Theorem 6.6 for $\eta = 0$ and $k = 1$ we get in this way the asymptotic expansion

$$\lambda_\dagger(\varepsilon\beta) = \lambda_1 + \varepsilon^2 \int_0^1 |\dot\beta(x_3)|^2 \, \mathrm{d}x_3 \int_\omega |\partial_\varphi \psi_1(x')|^2 \, \mathrm{d}x' + O(\varepsilon^3) \, .$$

The claim of Theorem 6.1 is then a simple consequence of this formula and relation (3.3). □

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