ON THE NUMBER OF EIGENVALUES OF THE POLYHARMONIC OPERATOR OF FOURTH ORDER ON $\mathbb{R}^3$ WITH A COMPLEX POTENTIAL

ARTEM HULKO

Abstract. In this paper we study the polyharmonic operator of the fourth order on $\mathbb{R}^3$ with a complex potential, which decays exponentially at infinity. We obtain the bounds on the number of eigenvalues of the said operator.

1. Introduction and Main Results

We consider the operator

$$H = (-\Delta)^2 + V(x), \qquad x \in \mathbb{R}^3$$

with a complex valued exponentially decaying potential V. We obtain an estimate for the total number $\mathcal{N}$ of eigenvalues of the operator H in the complex plane $\mathbb{C}$ less the positve real line. Our work is motivated by a recent result of Frank, Laptev and Safronov [12] which gives a similar estimate for the number of eigenvalues of a Schrödinger operator. While Shrödinger operators are the most intensively studied operators in mathematical physics, polyharmonic operators of higher order have also been considered, as they too have some interesting applications.

Theorem 1.1. *The number $\mathcal{N}$ of eigenvalues of H in $L^2(\mathbb{R}^3)$, counting algebraic multiplicities, satisfies*

$$\mathcal{N} \le \frac{1}{\varepsilon^3}\left(2\varepsilon \left\|V\right\|_{1,\varepsilon} + \frac{\varepsilon+2}{64\pi^2}\left\|V\right\|_{1,\varepsilon}^2 + \frac{3\gamma}{64\pi^4}\left\|V\right\|_{1,\varepsilon}^3\right) + 1,$$

for any $\varepsilon > 0$, where $\left\|V\right\|_{1,\varepsilon} = \int_{\mathbb{R}^3} |V(x)|e^{\varepsilon|x|}dx$ and $\gamma > 0$ is the best constant satisfying the inequality $|det_3(1+X)| \le e^{\gamma\|X\|_{\mathfrak{S}_2}^3}$.

Note that the potential V in our estimate must decay exponentially fast so that $\left\|V\right\|_{1,\varepsilon} = \int_{\mathbb{R}^3} |V(x)|e^{\varepsilon|x|}dx$ is finite. It turns out that such exponential decay at infinity is not needed in order to guarantee that the number of eigenvalues of the corresponding Schrödinger operator is finite. It was B. Pavlov who used the notion of quasi-analyticity to obtain certain criteria for the number of eigenvalues to be finite (see [24], [25]). However, his methods do not lead to estimates for their total number N. One should also mention that Pavlov established that if the potential decays slower than $\gamma \exp(-\alpha|x|^{1/2})$ then the number of eigenvalues of the corresponding one-dimensional Schrödinger operator might be infinite.

Author expresses his gratitude to Dr. Oleg Safronov for his suggestions and remarks.

2. Relation between operator of fourth order and the Schrödinger operator

In this section we show that it is possible to express the resolvent of the polyharmonic operator of the fourth order through the difference of resolvents of two Schrödinger operators.

Let us first consider the case when $A = (-\Delta)^2$. Then, using the difference of two sqaures formula, we can express the operator $(A-z)^{-1}$ as follows:

$$(A-z)^{-1} = ((-\Delta)^2 - z)^{-1} = (-\Delta - k)^{-1}(-\Delta + k)^{-1}, \text{ where } k^2 = z.$$

Next, recall the Hilbert's identity, which states that for two operators T and S the following holds: $T^{-1} - S^{-1} = T^{-1}(S-T)S^{-1}$. Applying this result we get the following identity:

$$(-\Delta - k)^{-1} - (-\Delta + k)^{-1} = (-\Delta - k)^{-1}(2k)(-\Delta + k)^{-1} = 2k(A-z)^{-1}$$

Now solving for $(A-z)^{-1}$ yields the following difference of the two resolvent Schrödinger operators of the first order :

$$(A-z)^{-1} = \frac{1}{2k}\left[(-\Delta - k)^{-1} - (-\Delta + k)^{-1}\right], \tag{2.1}$$

where $k^2 = z$. Consequently, we can apply results we already know for first order Schrödinger operator to get the desired bound.

3. Resolvent Bounds

In this section we obtain some results which will be very useful for the proof of the Theorem 1.1.

Proposition 3.1. *Let $a(w)$ be a function in $\Omega := \{w \in \mathbb{C} : Im\, w \geq -\varepsilon,\ Re\, w \geq -\varepsilon\}$, such that $a(w) = 1 + O\left(\frac{1}{|w|^3}\right)$ as $|w| \to \infty$, and $\ln|a(w)| \leq \frac{D}{|w|^3}$ if w lies on the boundary of Ω. Moreover, assume that $a(w) = a_0(w)e^{f(w)}$, where $a_0(w)$ is meromorphic, having only one pole of order n at $w=0$ and $f(w)$ is analytic everywhere except $w = 0$. Then the number of zeros N of $a(w)$ in the first quadrant satisfies*

$$N \leq \frac{\left|\int_{\partial\Omega_R} f(w)(w + (1+i)\varepsilon)dw\right|}{2\pi\varepsilon^2} + \frac{3D}{\pi\varepsilon^3} + n\,,$$

where $\Omega_R = \{w \in \Omega : |w| \leq R\}$, for any $R > 0$.

Proof. To get the desired estimate we would like to look at the function $\log(a(w)) = \log[a_0(w)e^{f(w)}] = \log[a_0(w)] + f(w)$. However, $\log[a_0(w)]$ is not analytic in Ω, due to a_0 having a pole at $w = 0$, as well as possibly having zeros in Ω. To make it analytic we need to get rid of the pole and all the zeros. To do so we introduce the following Blaschke product:

$$B(k) = \left(\frac{(w+(1+i)\varepsilon)^2 - ((1-i)\varepsilon)^2}{(w+(1+i)\varepsilon)^2 - ((1+i)\varepsilon)^2} \right)^n \prod_j \frac{(w+(1+i)\varepsilon)^2 - (w_j+(1+i)\varepsilon)^2}{(w+(1+i)\varepsilon)^2 - (\overline{w_j}+(1-i)\varepsilon)^2},$$

where w_j are zeros of $a_0(w)$. Let $\partial\Omega_R = I_R \cup J_R \cup C_R$, where $C_R = \{w \in \Omega : |w| = R\}$, I_R is the line $\{w \in \Omega : w = t - i\varepsilon \text{ and } |w| \le R\}$, and J_R is the line $\{w \in \Omega : w = it - \varepsilon \text{ and } |w| \le R\}$. Then the function $\log[a_0(w)/B(w)]$ exists and is analytic in Ω, and $|B(w)| = 1$ on $\partial\Omega_R$ for every R. Now applying residue calculus yields the following equality

$$\begin{aligned}\operatorname{Re}\left[2\pi i Res\left(f(w)(w+(1+i)\varepsilon), w=0\right)\right] &= \operatorname{Re}\int_{\partial\Omega_R}\left(\log\left[\frac{a_0(w)}{B(w)}\right] + f(w)\right)(w+(1+i)\varepsilon)dw \\ = \operatorname{Re}\int_{C_R}\log\frac{a(w)}{B(w)}(w+(1+i)\varepsilon)dw &+ \operatorname{Re}\int_{I_R}\log\frac{a(w)}{B(w)}(w+(1+i)\varepsilon)dw \\ &+ \operatorname{Re}\int_{J_R}\log\frac{a(w)}{B(w)}(w+(1+i)\varepsilon)dw\end{aligned} \tag{3.1}$$

As a result, from the equation above, after moving the last two integrals to the left side and using the triangle inequality, we get the following

$$\begin{aligned}\left|\int_{\partial\Omega_R} f(w)(w+(1+i)\varepsilon)dw\right| &+ \operatorname{Re}\int_{I_R}\log\frac{a(w)}{B(w)}(w+(1+i)\varepsilon)dw \\ + \operatorname{Re}\int_{J_R}\log\frac{a(w)}{B(w)}(w+(1+i)\varepsilon)dw &\ge -\operatorname{Re}\int_{C_R}\log\frac{a(w)}{B(w)}(w+(1+i)\varepsilon)dw\end{aligned} \tag{3.2}$$

Now our goal is to estimate the right hand side from below and the left hand side from above. So first we will estimate the integral on the right. We know that the integral of a function over a curve can be estimated by the maximum value of the function on that curve multiplied by the length of the curve. Namely,

$$\left|\int_{C_R}\psi(z)dz\right| \le \max_{z\in C_R}(\psi(z))|C_R|$$

Here $|C_R|$ denotes the length of the curve C_R. Now, since $\ln|a(w)| \le \dfrac{D}{|w|^3}$ by our assumption, we get the following estimate for the integral

$$\begin{aligned}\lim_{R\to\infty}\operatorname{Re}\int_{C_R}\log[a(w)](w+(1+i)\varepsilon)dw &\le \lim_{R\to\infty}\left\{\frac{D}{|R|^3}|(R+(1+i)\varepsilon)||C_R|\right\} \\ &= \lim_{R\to\infty}\left\{\frac{D\pi R}{2|R|^3}|(R+(1+i)\varepsilon)|\right\} = 0\end{aligned} \tag{3.3}$$

And so we only need to estimate $\lim_{R\to\infty} \operatorname{Re} \int_{C_R} \log B(w)(w+(1+i)\varepsilon)dw$ from below. After setting $\xi = w + (1+i)\varepsilon$ at first, then setting $\xi = Re^{i\varphi}$ and then computing the integral we arrive at the following inequality

$$\lim_{R\to\infty} \operatorname{Re} \int_{C_R} \log B(w)(w+(1+i)\varepsilon)dw = 4i\varepsilon^2 n \left(\frac{\pi i}{4}\right) + 2\left(\frac{\pi}{4}\right)\sum_j \operatorname{Im}(w_j + (1+i)\varepsilon)^2$$
$$\geq -2n\varepsilon^2\pi + \pi(2\varepsilon^2)N = 2\pi\varepsilon^2(N-n) \tag{3.4}$$

where N is the number of zeros w_j of the function $a_0(w)$ in the first quadrant. The last inequality is due to the fact that if we set $w = a+bi$ and compute $(w+(1+i)\varepsilon)^2$ we see that $\operatorname{Im}(w+(1+i)\varepsilon)^2 = 1(a+\varepsilon)(b+\varepsilon) \geq 2\varepsilon^2$. Hence, since we are taking the sum over all zeros of $a_0(w)$ which lie in the first quadrant, the desired inequality holds.

We now will estimate second and third terms on the left hand side of 3.2. Note that

$$\begin{aligned}\operatorname{Re} \int_{I_R} \log \frac{a(w)}{B(w)}(w+(1+i)\varepsilon)dw &= \int_{I_R} \log\left|\frac{a(w)}{B(w)}\right|(w+(1+i)\varepsilon)dw, \\ &\qquad \text{since } (w+(1+i)\varepsilon)dw \text{ is real} \\ &= \int_{I_R} \log|a(w)|(w+(1+i)\varepsilon)dw, \\ &\qquad \text{since } |B(w)| = 1 \text{ on the boundary} \\ &\leq \int_{I_R} \frac{D}{|w|^3}(w+(1+i)\varepsilon)dw\end{aligned}$$

Analogous arguments show that the integral over the boundary J_R will yield the same estimate as above. So for $w \in I_R$ (i.e. $w = t - i\varepsilon$), or $w \in J_R$ (i.e. $w = it - \varepsilon$), after appropriate change of variables and integration we get

$$\begin{aligned}\int_{J_R\,(\text{or } I_R)} \log\left|\frac{a(w)}{B(w)}\right|(w+(1+i)\varepsilon)dw &\leq \frac{D}{\varepsilon}\int_{-1}^{\infty} \frac{s+1}{(s^2+1)^{3/2}}ds \\ \text{setting } u = s^2+1: \qquad &= \frac{D}{\varepsilon}\int_2^{\infty}\left(\frac{1}{2u^{3/2}\sqrt{u-1}} + \frac{1}{2u^{3/2}}\right)du + \\ &\quad + \frac{2D}{\varepsilon}\int_1^2\left(\frac{1}{2u^{3/2}\sqrt{u-1}} + \frac{1}{2u^{3/2}}\right)du \\ &= \frac{D}{\varepsilon}\left(\sqrt{1-\frac{1}{u}} - \frac{1}{u^{1/2}}\right)\Bigg|_2^{\infty} + \frac{2D}{\varepsilon}\left(\sqrt{1-\frac{1}{u}} - \frac{1}{u^{1/2}}\right)\Bigg|_1^2 \\ &= \frac{3D}{\varepsilon},.\end{aligned} \tag{3.5}$$

Now combining inequalities 3.4, 3.5 and 3.2 we get a bound for the number of zeros N of $a(w)$ as follows

$$2\pi\varepsilon^2(N-n) \le \left|\int_{\partial\Omega_R} f(w)(w+(1+i)\varepsilon)dw\right| + \frac{6D}{\varepsilon}.$$

The colclusion of the proposition follows immediately. □

3.1. Classes of compact operators and determinants.

Recall the Birman-Schwinger principle for the special case, where H_0 is a self-adjoint operator and V is bounded. In this case the Birman-Schwinger principle states that for $z \in \rho(H_0)$, z is the eigenvalue of the operator $H_0 + V$ if and only if -1 is an eigenvalue of the operator $X := W_1(H_0 - z)^{-1}W_2$, where $W_1W_2 = V$. Moreover, the corresponding multiplicities coincide.

Let us also recall a well known result that for an operator H_0 and a point $z \in \rho(H_0)$. The function $\varsigma \mapsto det_n(1 + W_1(H_0 - \varsigma)^{-1}W_2)$ is analytic on $\rho(H_0)$. And a point $z \in \rho(H_0)$ is an eigenvalue of $H_0 + V$ if and only if $det_n(1 + W_1(H_0 - z)^{-1}W_2) = 0$. Moreover, the order of the zero coincides with the algebraic multiplicity of the corresponding eigenvalue. [27, 18, 10] This means that the algebraic multiplicities of eigenvalues of H can also be characterized by zeros of the of the determinant of the Birman-Schwinger operator mentioned above.

It is well known that the integral kernel for the operator $(-\Delta - z)^{-1}$ in the case when dimension $d = 3$ is given by $\rho_0(x, y) = \dfrac{e^{ik|x-y|}}{4\pi|x-y|}$, where $k^2 = z$, $\operatorname{Im} k > 0$. So from this result and equation 2.1 we can derive the integral kernel for $(A - z)^{-1}$ to be the following:

$$\rho(x,y) = \frac{1}{2k}\left[\frac{e^{i\sqrt{k}|x-y|}}{4\pi|x-y|} - \frac{e^{i\sqrt{-k}|x-y|}}{4\pi|x-y|}\right] = \frac{1}{2k}\left[\frac{e^{i\sqrt{k}|x-y|} - 1 - (e^{i\sqrt{-k}|x-y|} - 1)}{4\pi|x-y|}\right].$$

Note that for any $z \in \mathbb{C}\backslash\mathbb{R}_+$, $w = \sqrt[4]{z}$ is always located in the first quadrant. However, as we look at the proposition 3.1, we consider all values of $w \in \Omega$, keeping in mind that $w^4 = z$ if w is in the first quadrant. Also using the inequality

$$\left|e^{i(a+bi)\tau} - 1\right| \le \left|\int_0^\tau i(a+bi)e^{i(a+bi)s}ds\right| \le \int_0^\tau |i||a+bi|\left|e^{i(a+bi)s}ds\right| \le \tau|a+bi|$$

we arrive at the following estimate for $\rho(x, y)$:

$$\rho(x,y)| \le \frac{1}{2|k|}\frac{|\sqrt{k}|\left|e^{\varepsilon_1(|x|+|y|)}\right||x-y| + |\sqrt{-k}|\left|e^{\varepsilon_1(|x|+|y|)}\right||x-y|}{4\pi|x-y|} = \frac{e^{\varepsilon_1(|x|+|y|)}}{4\pi\sqrt{|k|}}. \quad (3.6)$$

Next, we use this result to find a bound for the Hilbert-Schmidt norm of the operator $X = W_1(H_0 - z)^{-1}W_2$. Simple calculations show us that the Hilbert-Schmidt norm of X can be estimated as so:

$$\begin{aligned}
\|X\|^3_{\mathfrak{S}_2} &= \left[\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |W_1(x)\rho(x,y)W_2(y)|^2 dxdy\right]^{3/2} \\
&\leq \left[\int_{\mathbb{R}^3} |W_2(y)|^2 \int_{\mathbb{R}^3} |W_1(x)|^2 |\rho(x,y)|^2 dxdy\right]^{3/2}, \\
&\qquad \text{since } \rho(x,y)=\rho(y,x) \\
&\leq \left[\left(\frac{1}{4\pi\sqrt{|k|}}\right)^2 \int_{\mathbb{R}^3} |W_2(y)|^2 \left|e^{2\varepsilon_1|y|}\right| \int_{\mathbb{R}^3} |W_1(x)|^2 \left|e^{2\varepsilon_1|x|}\right| dxdy\right]^{3/2}, \qquad (3.7) \\
&\qquad \text{by inequality 3.6} \\
&= \left[\frac{1}{16\pi^2|k|}\int_{\mathbb{R}^3} |V(y)| \left|e^{2\varepsilon_1|y|}\right| dy \int_{\mathbb{R}^3} |V(x)| \left|e^{2\varepsilon_1|x|}\right| dx\right]^{3/2} \\
&= \frac{1}{64\pi^3|k|^{3/2}}\left(\int_{\mathbb{R}^3} \left|e^{2\varepsilon_1|x|}\right| |V(x)|dx\right)^3.
\end{aligned}$$

4. Theorem Proof

As discussed in the section above we will be looking at $det_3(1+W_1(H_0-\varsigma)^{-1}W_2)$. We know that

$$\begin{aligned}
det_3(1+W_1(H_0-\varsigma)^{-1}W_2) &= det_1(1+W_1(H_0-\varsigma)^{-1}W_2)e^{-Tr(K)+\frac{Tr(K^2)}{2}} \\
&= \left[\prod_j (1+k_j)\right] e^{-Tr(X)+\frac{Tr(X^2)}{2}}.
\end{aligned}$$

Note that the product has a pole of order 1 at $k=0$. Moreover, the function in the exponent will then have a pole of order 2 at $k=0$. Let us note that $|det_3(1+X)| \leq e^{\gamma\|X\|^3_{\mathfrak{S}_3}} \leq e^{\gamma\|X\|^3_{\mathfrak{S}_2}}$, for some $\gamma>0$. The proof of this statement can be found in Lemma 2.3 in [17]; it is essentially due to Weyls inequality [27, Thm. 1.15]. Now if we set $w^2=k$ and apply Proposition 3.1 to the function $a(w)=\det_3(1+X(w))$ we will get an estimate for the number of zeros of the function $\det_3(1+X(w))$. However, to get an explicit bound we still need to compute $\int_{\partial\Omega_R}\left(-Tr(X)+\frac{Tr(X^2)}{2}\right)(w+(1+i)\varepsilon)dw$. This is exactly what we will be doing in the next two subsections.

4.1. **Estimating** $\left|\int_{\partial\Omega_R} -Tr(X)(w+(1+i)\varepsilon)dw\right|$.

After a simple Taylor expansion and integration we get that $Tr(X)=\left(\frac{i+1}{8w\pi}\right)\int_{\mathbb{R}^3} V(x)dx$.

Now using residue calculus we get

$$\begin{aligned}\int_{\partial\Omega_R} Tr(X)(w+(1+i)\varepsilon)dx &= \int_{\partial\Omega_R}\left(\frac{i+1}{8w\pi}\right)\int_{\mathbb{R}^3} V(x)dxdw\\ &= (i+1)\int_{\mathbb{R}^3} V(x)dx\int_{\partial\Omega_R}\left(1+\frac{(1+i)\varepsilon}{w}\right)dw\\ &= -4\pi\int_{\mathbb{R}^3} V(x)dx\end{aligned}$$

This yields the following estimate from above:

$$\left|\int_{\partial\Omega_R} -Tr(X)(w+(1+i)\varepsilon)dx\right| = \left|-4\pi\int_{\mathbb{R}^3} V(x)dx\right| \le 4\pi\int_{\mathbb{R}^3}|V(x)|e^{\varepsilon|x|}dx \quad (4.1)$$

4.2. **Estimating** $\left|\int_{\partial\Omega_R}\frac{Tr(X^2}{2}(w+(1+i)\varepsilon)dw\right|$**.** Note that

$$\begin{aligned}Tr(X^2) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} V(y)\rho^2(x,y)V(x)dydx\\ &= \frac{i}{32w^2\pi^2}\left(\int_{\mathbb{R}^3} V(x)dx\right)^2 - \frac{i+1}{32w\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} V(x)|x-y|V(y)dydx+\psi(w),\end{aligned} \quad (4.2)$$

where $\psi(w)$ is analytic at $w=0$. Now when we use residue calculus to integrate $Tr(X^2)$ over $\partial\Omega_R$ we get the following bound

$$\begin{aligned}\left|\int_{\partial\Omega_R}\frac{Tr(X^2)}{2}(w+(1+i)\varepsilon)dw\right| &= \left|\int_{\partial\Omega_R}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} V(y)\rho^2(x,y)V(x)dydx\right|\\ &= \left|\frac{\varepsilon}{16\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} V(x)|x-y|V(y)dydx + \frac{i}{32\pi}\left(\int_{\mathbb{R}^3} V(x)dx\right)^2\right|\\ &\le \frac{\varepsilon}{16\pi}\left(\int_{\mathbb{R}^3}|V(x)(1+|x|)dx\right)^2 + \frac{i}{32\pi}\left(\int_{\mathbb{R}^3} V(x)dx\right)^2\\ &\le \frac{1}{16\pi\varepsilon}\left(\int_{\mathbb{R}^3}|V(x)|e^{\varepsilon|x|}dx\right)^2 + \frac{|i|}{32\pi}\left(\int_{\mathbb{R}^3}|V(x)|e^{\varepsilon|x|}dx\right)^2\\ &= \frac{\varepsilon+2}{32\pi\varepsilon}\left(\int_{\mathbb{R}^3}|V(x)|e^{\varepsilon|x|}dx\right)^2\end{aligned} \quad (4.3)$$

Note that the last inequality is valid, since, due to our assumption, the potential decays exponentially at infinity, forcing the integrals to be finite.

4.3. **Proof of the Theorem.** Combining estimates 4.1 and 4.3 we get the following inequality:

$$\left|\int_{\partial\Omega_R}\left(-Tr(X)+\frac{Tr(X^2)}{2}\right)(w+(1+i)\varepsilon)dw\right| \le \left(\frac{4\pi}{\|V\|_{1,\varepsilon}}+\frac{\varepsilon+2}{32\pi\varepsilon}\right)\|V\|_{1,\varepsilon}^2, \quad (4.4)$$

where $\|V\|_{1,\varepsilon} = \int_{\mathbb{R}^3} |V(x)|e^{\varepsilon|x|}dx$. Then we apply Proposition 3.1 to the function $a(w) = det_3(1+X)$ with $a_0(w) = det_1(1+X)$ and $f(w) = -Tr(X) + \frac{Tr(X^2)}{2}$, as discussed in the beginning of this section. After this we use the estimate 4.4. Moreover, we set $D = \frac{\gamma}{64\pi^3}\left(\int_{\mathbb{R}} e^{2\varepsilon_1|x|}|V(x)|dx\right)^3$ due to 3.7. Choosing $\varepsilon_1 = \varepsilon/2$ we conclude that the total number $\mathcal{N}$ of zeros of the function $det_3(1+X)$ satisfies

$$\mathcal{N} \leq \frac{1}{\varepsilon^3}\left(2\varepsilon \|V\|_{1,\varepsilon} + \frac{\varepsilon+2}{64\pi^2}\|V\|^2_{1,\varepsilon} + \frac{3\gamma}{64\pi^4}\|V\|^3_{1,\varepsilon}\right) + 1$$

where $\|V\|_{1,\varepsilon} = \int_{\mathbb{R}^3} |V(x)|e^{\varepsilon|x|}dx$. Then by the argument in the second paragraph of the subsection 3.1, the same estimate gives us the bound for the number of eigenvalues of the operator H.

□

References

[1] A. A. Abramov, A. Aslanyan, E. B. Davies, *Bounds on complex eigenvalues and resonances.* J. Phys. A **34** (2001), 57–72.

[2] A. Borichev, L. Golinskii, S. Kupin, *A Blaschke-type condition and its application to complex Jacobi matrices.* Bull. London Math. Soc. **41** (2009), 117–123.

[3] E. B. Davies, *Non-self-adjoint differential operators.* Bull. London Math. Soc. **34** (2002), no. 5, 513–532.

[4] E. B. Davies, J. Nath, *Schrödinger operators with slowly decaying potentials.* J. Comput. Appl. Math. **148** (2002), 1–28.

[5] M. Demuth, G. Katriel, *Eigenvalue inequalities in terms of Schatten norm bounds on differences of semigroups, and application to Schrödinger operators.* Ann. Henri Poincaré **9** (2008), no. 4, 817–834.

[6] M. Demuth, M. Hansmann, G. Katriel, *On the discrete spectrum of non-selfadjoint operators.* J. Funct. Anal. **257** (2009), no. 9, 2742–2759.

[7] M. Demuth, M. Hansmann, G. Katriel, *Eigenvalues of non-selfadjoint operators: A comparison of two approaches*, in: Mathematical Physics, Spectral Theory and Stochastic Analysis, Springer, 2013, 107–163.

[8] A. Enblom, *Estimates for eigenvalues of Schrödinger operators with complex-valued potentials.* Preprint (2015), arXiv:1503.06337.

[9] R. L. Frank, *Eigenvalue bounds for Schrödinger operators with complex potentials.* Bull. Lond. Math. Soc. **43** (2011), no. 4, 745–750.

[10] R. L. Frank, *Eigenvalue bounds for Schrödinger operators with complex potentials. III.* Preprint (2015), http://arxiv.org/pdf/1510.03411v1.pdf

[11] R. L. Frank, A. Laptev, E. H. Lieb, R. Seiringer, *Lieb–Thirring inequalities for Schrödinger operators with complex-valued potentials.* Lett. Math. Phys. **77** (2006), 309–316.

[12] R. L. Frank, A. Laptev and O. Safronov, *On the number of eigenvalues of Schrödinger operators with complex potentials*, to appear

[13] R. L. Frank, A. Laptev, R. Seiringer, *A sharp bound on eigenvalues of Schrödinger operators on the half-line with complex-valued potentials.* Spectral theory and analysis, 39–44, Oper. Theory Adv. Appl. **214**, Birkhäuser/Springer Basel AG, Basel, 2011.

[14] R. L. Frank, J. Sabin, *Restriction theorems for orthonormal functions, Strichartz inequalities and uniform Sobolev estimates*. Preprint (2014), http://arxiv.org/pdf/1404.2817.pdf

[15] R. L. Frank, B. Simon, *Eigenvalue bounds for Schrödinger operators with complex potentials. III.* J. Spectr. Theory, to appear.

[16] A. Hulko, *On the number of eigenvalues of the discrete one-dimentional Schrödinger operator with a complex potential.* Bull. Math. Sci., to appear.

[17] A. Hulko, *On the number of eigenvalues of the discrete one-dimensional Dirac operator with a complex potential.* Preprint (2017), https://www.ma.utexas.edu/mp_arc/c/17/17-46.pdf

[18] A. Laptev, O. Safronov, *Eigenvalue estimates for Schrödinger operators with complex potentials*. Comm. Math. Phys. **292** (2009), 29–54.

[19] Y. Latushkin, A. Sukhtayev, *The algebraic multiplicity of eigenvalues and the Evans function revisited*. Math. Model. Nat. Phenom. **5** (2010), no. 4, 269–292.

[20] R. M. Martirosjan, *On the spectrum of the non-selfadjoint operator* $\Delta u + cu$ *in three dimensional space*. (Russian) Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **10** (1957), no. 1, 85–111.

[21] R. M. Martirosjan, *On the spectrum of various perturbations of the Laplace operator in spaces of three or more dimensions*. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **24** (1960), 897–920.

[22] Kh. Kh. Murtazin, *Spectrum of the nonself-adjoint Schrödinger operator in unbounded regions*. (Russian) Mat. Zametki **9** (1971) 19–26. English translation: Math. Notes **9** (1971), 12–16.

[23] M. A. Naĭmark, *Investigation of the spectrum and the expansion in eigenfunctions of a non-selfadjoint operator of the second order on a semi-axis*. (Russian) Trudy Moskov. Mat. Obšč. **3** (1954), 181–270.

[24] B. S. Pavlov, *On a non-selfadjoint Schrödinger operator*. (Russian) 1966 Probl. Math. Phys., No. 1, Spectral Theory and Wave Processes (Russian) pp. 102–132 Izdat. Leningrad. Univ., Leningrad.

[25] B. S. Pavlov, *On a non-selfadjoint Schrödinger operator. II*. (Russian) 1967 Problems of Mathematical Physics, No. 2, Spectral Theory, Diffraction Problems (Russian) pp. 133–157 Izdat. Leningrad. Univ., Leningrad.

[26] O. Safronov, *On a sum rule for Schrödinger operators with complex potentials*. Proc. Amer. Math. Soc. **138** (2010), no. 6, 2107–2112.

[27] B. Simon, *Trace ideals and their applications*. Second edition. Amer. Math. Soc., Providence, RI, 2005.

[28] B. Simon, *Notes on infinite determinants of Hilbert space operators*. Advances in Math. **24** (1977), no. 3, 244–273.

[29] S. A. Stepin, *Complex potentials: bound states, quantum dynamics and wave operators*. Semigroups of operators - theory and applications, 287–297, Springer Proc. Math. Stat. **113**, Springer, Cham, 2015.

[30] S. A. Stepin, *Estimate for the number of eigenvalues of the nonselfadjoint Schrödinger operator*. (Russian) Dokl. Akad. Nauk **455** (2014), no. 4, 394–397; translation in Dokl. Math. **89** (2014), no. 2, 202–205.

Artem Hulko, Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA

E-mail address: `ahulko@uncc.edu`