ON THE NUMBER OF EIGENVALUES OF THE DISCRETE ONE-DIMENSIONAL DIRAC OPERATOR WITH A COMPLEX POTENTIAL

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ABSTRACT. In this paper we define a one-dimensional discrete Dirac operator on $\mathbb{Z}$. We study the eigenvalues of the Dirac operator with a complex potential. We obtain bounds on the total number of eigenvalues in the case where V decays exponentially at infinity. We also estimate the number of eigenvalues for the discrete Schrödinger operator with complex potential on $\mathbb{Z}$. That is we extend the result obtained by Hulko [16] to the whole $\mathbb{Z}$.

1. Introduction and Main Results

Eigenvalues of Schrödinger operators with complex potentials have been studied intensively during the last 10 –15 years. Specialists have been interested in the bounds on the number of eigenvalues which hold not only in an asymptotic regime, but always. The most interesting of such estimates are bounds involving some simple and computationally accessible quantities, such as L^p norms of potentials. A good description of the related results could be found in [3]. One should point out that the number of eigenvalues of a continuous Schrödinger operators with complex potentials was recently estimated by Frank, Laptev and Safronov in [12]. Besides the above noted paper, there are two papers worth mensioning, [28] and [29] (by Stepin) which apply to one and three dimensional cases.

The difference between the self-adjoint and non-selfadjoint cases is illustrated by the remarkable result of Pavlov [24]. This result shows that for any $0 < q < 1/2$ and any $\lambda > 0$ there is a (real) potential V satisfying $|V(x)| \leq Ce^{-cx^q}$, for some $C, c > 0$ and a complex number σ, such that the operator $\dfrac{-d^2}{dx^2} + V$ in $L^2(0,\infty)$, with the boundary condition $\psi'(0) = \sigma\psi(0)$, has an infinite number of eigenvalues accumulating to λ. On the other hand, Pavlov [23] has also showed that if $|V(x)| \leq Ce^{-cx^{1/2}}$ for some $C, c > 0$ then the number of eigenvalues of the operator $\dfrac{-d^2}{dx^2} + V$ in $L^2(0,\infty)$, with any boundary condition of the form $\psi'(0) = \sigma\psi(0)$, is finite. A similar theorem holds in three dimensions.

In spite of the fact that the results mentioned above were proven in 1966 and 1967, the "uniform" bounds on the number of eigenvalues were obtained only in 2016 in the article [12]. In our paper, we establish similar bounds for the discrete Dirac operator.

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Such an operator is well known in the continuous case, however, in the discrete case it has not been defined yet (or at least we do not know if it has). In this paper, we define a discrete Dirac operator on the Hilber space $\mathfrak{H} = \ell^2(\mathbb{Z}, \mathbb{C}^2)$. We then show that the spectrum of the free Dirac operator coincides with $[-\sqrt{5}, -1] \cup [1, \sqrt{5}]$ and is absolutely continuous. Moreover, we expand the results obtained by Hulko in [16] to the whole $\mathbb{Z}$ and use them to establish the upper bound for the number of eigenvalues $\mathcal{N}_D$ of the aforementioned Dirac operator.

1.1. **Definition of the Operator.** Let $\mathfrak{H} = \ell^2(\mathbb{Z}, \mathbb{C}^2)$ be the Hilbert space of square summable sequences of vectors on $\mathbb{Z}$. Let $V : \mathfrak{H} \mapsto \mathfrak{H}$ be the operator of multiplication by a bounded complex-valued function on $\mathbb{Z}$. We define the free Dirac operator on $\mathfrak{H}$ by

$$D_0 = \begin{bmatrix} 1 & S-1 \\ S^*-1 & -1 \end{bmatrix}.$$

Here the operator S is the shift operator in $\ell^2(\mathbb{Z}, \mathbb{C})$ and is defined as follows:

$$(Su)_n = u_{n-1}.$$

And finally we define the operator D_V to be $D_V = D_0 + V$.

In this paper we prove the following two theorems regarding the Dirac operator and two more theorems in Section 3 which deal with the extension of the Schrödinger operator to the whole $\mathbb{Z}$:

Theorem 1.1. *The number $\mathcal{N}_D$ of eigenvalues of D_V in $\ell^2(\mathbb{Z}, \mathbb{C}^2)$, counting algebraic multiplicities, satisfies*

$$\mathcal{N}_D \le \frac{\left(\|V\|_{\infty,q}\left(4\Lambda^{1+\frac{\varepsilon}{2}} + 2\sqrt{2}\right) + \|V\|_{\infty,q}^2\right)^2}{\ln \Lambda} \left(\frac{\Lambda^2}{\Lambda^2 - 1}\right)^2 \left(\frac{\Lambda^\varepsilon + 1}{\Lambda^\varepsilon - 1}\right)^2 + 4,$$

where Λ is any constant greater than 1, $q = \frac{1}{\Lambda^{2+\varepsilon}}$, and $\|V\|_{\infty,q} = \sup_{-\infty<n<\infty} |V_n q^{-|n|}|$.

Theorem 1.2. *The number $\mathcal{N}_D$ of eigenvalues of D_V in $\ell^2(\mathbb{Z}, \mathbb{C}^2)$, counting algebraic multiplicities, satisfies*

$$\mathcal{N}_D \le \frac{\|V\|_{\infty,q}\left(4\Lambda^{1+\frac{\varepsilon}{2}} + 2\sqrt{2} + \|V\|_{\infty,q}\right)}{\ln \Lambda} \frac{\sqrt{2}\Lambda^2}{\Lambda^2 - 1} \left(\frac{\Lambda^{\varepsilon/2} + 1}{\Lambda^{\varepsilon/2} - 1}\right)^2 + 4,$$

where Λ is any constant greater than 1, $q = \frac{1}{\Lambda^{2+\varepsilon}}$, and $\|V\|_{\infty,q} = \sup_{-\infty<n<\infty} |V_n q^{-|n|}|$.

The additional "+4" term is related to the number of edges of the spectrum of the free Dirac operator. Under small perturbations of D_0, the eigenvalues of D_V appear near the edges of the a.c. spectrum.

One may claim that Theorem 1.2 implies Theorem 1.1, since the results are very similar. At first glance we see that the numerator over the $\ln(\Lambda)$ in the Theorem 1.1

is squared, as opposed to it being linear in the Theorem 1.2. This may provide a false impression to draw such a conclusion. However, after some analysis, it can be seen that it is not the case. In other words, Theorem 1.1 does not imply Theorem 1.2, nor does Theorem 1.2 imply Theorem 1.1. And so the two results are independent of each other.

1.2. **Reduction of the Dirac Operator to Schrödinger Operator.**

After we square the free Dirac Operator D_0 we get the following:

$$D_0^2 = \begin{bmatrix} 1 & S-1 \\ S^*-1 & -1 \end{bmatrix} \begin{bmatrix} 1 & S-1 \\ S^*-1 & -1 \end{bmatrix} = \begin{bmatrix} 3-(S+S^*) & 0 \\ 0 & 3-(S+S^*) \end{bmatrix} = 3 - H_0,$$

where H_0 is the free discrete Schrödinger operator on $\mathbb{Z}$. We know that the absolutely continuous spectrum of the H_0 is $\sigma_c(H_0) = [-2, 2]$. As a result, the continuous spectrum of D_0 then must correspond to $\sigma_c(D_0) = [-\sqrt{5}, -1] \cup [1, \sqrt{5}]$. Next, if we look at the square of the operator D_V, we will get

$$D_V^2 = (D_0 + V)^2 = D_0^2 + D_0 V + V D_0 + V^2 = (3 - H_0) + Q\,,$$

where $Q = D_0 V + V D_0 + V^2$.

The estimate for H_0 was recently obtained by Hulko on $\mathbb{N}$ (see [16]). In this paper, while finding estimates for the Dirac Operator, we will also extend the results obtained by Hulko to the whole $\mathbb{Z}$. Doing so will come in useful in our estimation for D_V. In the following section we will prove some useful results which later will be used to prove Theorems for both, the Dirac operator and the extension of the Schrödinger operator.

2. Preliminary Results

Before we can prove the theorems above, however, we will need to prove a few preliminary results. We consider space $\mathfrak{H} = \ell^2(\mathbb{Z})$, and let H_0 denote the free Jacobi operator on $\mathbb{Z}$. The results we obtain in this section will be similar to those obtained by Hulko in [16] and will be used in both, Section 3 and Section 4, in estimating Schrödinger and Dirac operators.

The first two result will be very similar to Proposition 2.1 in [16]. The following two result will resemble very closely the Corollary 2.2 in [16].

Proposition 2.1. *Let $0 < R < 1$. Let $a(\cdot)$ be a function in the circle $\Omega := \{k : |k| > R\}$ of the form $a(k) = a_0(k) e^{\frac{ck}{k^2-1}}$, where $a_0(k)$ is meromorphic and $c \in \mathbb{R}$. Assume that $a(\cdot)$ is continuous on the compliment of an open circle without +1 and -1, has poles at $k = \pm 1$ of order n, and satisfies*

$$a(k) = 1 + O(|k|^{-1}) \qquad \text{as } |k| \to \infty \text{ in } \Omega = \{k : |k| > R\}. \tag{2.1}$$

Assume also that for some $A \geq 1$,

$$|a(k)| \leq A, \qquad \text{if } |k| = R\,. \tag{2.2}$$

Then the zeroes k_j *of* $a(\cdot)$ *in* Ω*, repeated according to their multiplicities, satisfy*

$$R^{2n} \prod_j \left(\frac{|k_j|}{R} \right) \leq A. \tag{2.3}$$

Proof. The proof of the Proposition will be similar to the proof of Proposition 2.1 in [16]. We want to look at the function $log(a(k)) = \log\left[a_0(k)e^{\frac{ck}{k^2-1}}\right] = \log[a_0(k)] + \dfrac{ck}{k^2-1}$, but $\log[a_0(k)]$ in not analytic in Ω. To make it analytic we need to get rid of all zeros and poles in $a_0(k)$. To do so we introduce a Bluschke product as follows

$$B(k) = \left(\prod_j \frac{k-k_j}{R - R^{-1}\overline{k_j}k} \right) \frac{(R-R^{-1}k)^n}{(k-1)^n} \frac{(R+R^{-1}k)^n}{(k+1)^n}.$$

As a result, the function $\log\left[a_0(k)/B(k)\right]$ exists and is analytic in Ω, $|B(k)| = 1$ on $C_R := \partial\Omega = \{k : |k| = R\}$. Then

$$\log\left[\frac{a_0(k)}{B(k)}\right] = \alpha_0 + \frac{\alpha_1}{k} + \frac{\alpha_2}{k^2} + \cdots \Rightarrow \frac{1}{k}\log\left[\frac{a_0(k)}{B(k)}\right] = \frac{\alpha_0}{k} + \frac{\alpha_1}{k^2} + \frac{\alpha_2}{k^3} + \cdots.$$

By residue calculus we get the following:

$$\int_{C_R} \log\left[\frac{a(k)}{B(k)}\right]\frac{dk}{k} = \int_{C_R} \left(\log\left[\frac{a_0(k)}{B(k)}\right] + \frac{ck}{k^2+1} \right) \frac{dk}{k} = 2\pi i\alpha_0 + 0 = 2\pi i \alpha_0, \tag{2.4}$$

since $\displaystyle\int_{C_R} \frac{c}{k^2-1}dk = 0$.

So to calculate the integral we just compute α_0 as follows:

$$\begin{aligned}\alpha_0 &= \lim_{k\to\infty} \log\left[\frac{a(k)}{B(k)}\right] = \lim_{k\to\infty}\log\left[\frac{a_0(k)}{B(k)}\right] = \log\left[\lim_{k\to\infty}\frac{a_0(k)}{B(k)}\right] \\ &= \log\left[\left(\prod_j \frac{-\overline{k_j}}{R}\right)(-1)^n R^{2n}\right]\end{aligned} \tag{2.5}$$

As a result, from 2.4 and 2.5, we get:

$$\int_{C_R} \log\left[\frac{a(k)}{B(k)}\right]\frac{dk}{k} = 2\pi i \log\left[\left(\prod_j \frac{-\overline{k_j}}{R}\right)(-1)^n R^{2n}\right]$$

After the change of variables $k = R\,e^{i\varphi}$ the equation above becomes

$$\begin{aligned}\int_0^{2\pi} \ln\left|\frac{a\left(R\,e^{i\varphi}\right)}{B\left(R\,e^{i\varphi}\right)}\right| d\varphi &= 2\pi \ln\left|(-1)^n R^{2n}\left(\prod_j \frac{-\overline{k_j}}{R}\right)\right| \\ &= 2\pi\ln\left[R^{2n}\left(\prod_j \frac{|\overline{k_j}|}{R}\right)\right]\end{aligned} \tag{2.6}$$

On the other hand we have the following estimate

$$\int_0^{2\pi} \ln \left| \frac{a\,(R\,e^{i\varphi})}{B\,(R\,e^{i\varphi})} \right| d\varphi \leq 2\pi \ln A\,, \tag{2.7}$$

since $|B(k)| = 1$ and $|a(k)| \leq A$ on C_R.

Inequality 2.3 follows from 2.6 and 2.7. □

Corollary 2.2. *Let $0 < R < 1$. Let $a(\cdot)$ be a meromorphic function in $\{k : \ |k| > R\}$ with poles of order n at $k = \pm 1$ and satisfying* (2.1)*. Assume that, for any $R' > R$ sufficiently close to R, condition* (2.2) *holds with R replaced by R'. Then the number*

$$\mathcal{N} := \#\{j : \ |k_j| \geq 1\}$$

of zeroes k_j of $a(\cdot)$ in $\{k : |k| \geq 1\}$, repeated according to their multiplicities, satisfies

$$\mathcal{N} \leq \frac{\ln A}{\ln 1/R} + 2n\,.$$

Proof of this Corollary is analogous to the proof of Corollary 2.2 in [16].

2.1. **Some results about classes of compact operators and determinants.** Let $1 \leq p < \infty$. We say that a compact operator T belongs to the Schatten class $\mathfrak{S}_p$ if its p^{th} Schatten norm is finite. In other words, $T \in \mathfrak{S}_p$ if $s_j(T)$ satisfy

$$\|T\|^p_{\mathfrak{S}_p} := \sum_j s^p_j(T) < \infty,$$

where the functional $\|\cdot\|_{\mathfrak{S}_p}$ is the norm on $\mathfrak{S}_p$.

For $K \in \mathfrak{S}_n$, $n \in \mathbb{N}$, we can define the n-th order regularized determinant $\det{}_n(1+K)$ as follows:

$$\det{}_n(1+K) := \prod_j \left((1+\lambda_j(K)) \exp\left(\sum_{m=1}^{n-1} \frac{(-1)^m}{m} \lambda_j(K)^m \right) \right)\,,$$

where $\lambda_j(K)$ denote the eigenvalues of K, repeated according to algebraic multiplicities.

Lemma 2.3. *Let $n \in \mathbb{N}$ and let $K \in \mathfrak{S}_n$. Then*

$$\ln|\det{}_n(1+K)| \leq \Gamma_n \|K\|^n_{\mathfrak{S}_n}\,,$$

where Γ_n is a positive constant independent of K. In particular,

$$\Gamma_1 = 1 \quad \textit{and} \quad \Gamma_2 = 1/2\,. \tag{2.8}$$

Proof. To prove the lemma, let $f(z) := (1+z)\exp\left(\sum_{m=1}^{n-1} \frac{(-1)^m}{m} z^m \right)$. Then $\ln|f(z)|$ can be bounded by a constant times $|z|^n$ for small $|z|$ and by a constant times $|z|^{n-1}$

for large $|z|$. Thus, $\ln|f(z)| \le \Gamma_n |z|^n$, and so

$$\ln|\det{}_n(1+K)| \le \Gamma_n \sum_j |\lambda_j(K)|^n$$

By Weyl's inequality [26, Thm. 1.15], the sum on the right side does not exceed $\|K\|^n_{\mathfrak{S}_n}$. A simple computation shows that for $n=1$ and $n=2$ one can take $\Gamma_1 = 1$ and $\Gamma_2 = 1/2$, respectively (see [27]). □

Now let us recall the Birman-Schwinger principle for the special case, where H_0 is a bounded self-adjoint operator and $V = G^* G_0$, and G_0 and G are compact operators. Now, set

$$H = H_0 + V.$$

In this case the Birman–Schwinger principle states that $z \in \rho(H_0)$ is an eigenvalue of H if and only if -1 is an eigenvalue of the Birman–Schwinger operator $G_0(H_0 - z)^{-1}G^*$. Moreover, the corresponding geometric multiplicities coincide.

The next lemma states that the algebraic multiplicities of eigenvalues of H can also be characterized by the zeros of the of the determinant of the Birman-Schwinger operator mentioned above.

Lemma 2.4. *Let $n \in \mathbb{N}$. Assume that $G_0(H_0 - \zeta)^{-1}G^* \in \mathfrak{S}_n$ for all $\zeta \in \rho(H_0)$. Then the function $\zeta \mapsto \det{}_n(1 + G_0(H_0 - \zeta)^{-1}G^*)$ is analytic in $\rho(H_0)$. A point $z \in \rho(H_0)$ is an eigenvalue of H if and only if $\det{}_n(1 + G_0(H_0 - z)^{-1}G^*) = 0$. Moreover, the order of the zero coincides with the algebraic multiplicity of the corresponding eigenvalue.*

It is well known that the function $\zeta \mapsto \det{}_n(1 + G_0(H_0 - \zeta)^{-1}G^*)$ is analytic (see, e.g., [27]). The result about the algebraic multiplicity in the case $n = 1$ is well knowns also. The result for the general n is essentially due to [18], and the extension of the proof to the present setting is due to [10].

2.2. **Resolvent bounds.**

In this section we collect trace ideal bounds for the Birman–Schwinger operator

$$K(k) = \sqrt{V}(H_0 - z)^{-1}\sqrt{|V|}, \qquad z = k + k^{-1}, \quad |k| \ge 1\,. \tag{2.9}$$

Where $\sqrt{V}$ is defined as $\sqrt{V(x)} = \dfrac{V(x)}{\sqrt{|V(x)|}}$ if $V(x) \neq 0$ and $\sqrt{V(x)} = 0$ if $V(x) = 0$.

Please recall that the space is $\mathfrak{H} = \ell^2(\mathbb{Z})$, and H_0 in (2.9) denotes the free Jacobi operator on $\mathbb{Z}$. From the matrix representation of V it is easy to see that, if V is compactly supported, then $K(k)$ admits an analytic continuation to $\mathbb{C} \setminus \{0\}$. The following propositions give bounds on the Hilbert–Schmidt norm of $K(k)$.

Proposition 2.5. *For any $k \in \mathbb{C} \setminus \{0\}$ with $|k| < 1$,*

$$\|K(k)\|_{\mathfrak{S}_2} \le \frac{1}{1 - |k|^2} \sum_{n=-\infty}^{\infty} |k|^{-2|n|} |V_n|\,,$$

Proof. The kernel of $(H_0 - z)^{-1}$ is given by

$$g_k(n,m) = \frac{-k}{k^2-1}\left(k^{-|n-m|}\right) .$$

The equation above can be estimated as follows:

$$|g_k(n,m)| \leq \frac{1}{1-|k|^2}|k|^{-(|n|+|m|)} .$$

After plugging the estimation above into the identity

$$\|K(k)\|^2_{\mathfrak{S}_2} = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}|V_n||g_k(n,m)|^2|V_m|$$

we obtain the desired bound. □

Proposition 2.6. *For any $k \in \mathbb{C}\setminus\{0\}$ with $|k|<1$,*

$$\|K(k)\|_{\mathfrak{S}_1} \leq \frac{1}{1-|k|^2}\Big(\sum_{n=-\infty}^{\infty}|k|^{-|n|}|V_n|^{1/2}\Big)^2 ,$$

Proof. The kernel of $(H_0 - z)^{-1}$ is defined by

$$g_k(n,m) = \frac{-k}{k^2-1}\left(k^{-|n-m|}\right) .$$

The equation above can be estimated as follows:

$$|g_k(n,m)| \leq \frac{1}{1-|k|^2}|k|^{-(|n|+|m|)} .$$

After plugging the estimation above into the identity

$$\|K(k)\|_{\mathfrak{S}_1} \leq \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}|V_n|^{1/2}|g_k(n,m)||V_m|^{1/2}$$

we obtain the desired bound. □

3. Extension of One-Dimentional Discrete Schrödinger Operator to $\mathbb{Z}$

In this section we will extend the result obtained by Hulko in [16] to the whole $\mathbb{Z}$. Many of the results we acquire in this section will be very similar to the results in [16].

We consider the Hilbert space $\mathfrak{H} = \ell^2(\mathbb{Z},\mathbb{C})$. Let the Schrödinger operator H be defined on $\mathfrak{H}$ as follows:

$$(H\,u)_j = \sum_{|l-j|=1} u_l + V_j u_j, \tag{3.1}$$

where $H = H_0 + V$. The spectrum of the self-adjoint operator $H_0 = H - V$ coincides with the interval $[-2,2]$ and is absolutely continuous. Moreover, the sequence V_j decays exponentially fast. The following two theorems extend the result by Hulko in [16] to the whole $\mathbb{Z}$.

Theorem 3.1. *The number $\mathcal{N}_S$ of eigenvalues of H in $\ell^2(\mathbb{Z})$, counting algebraic multiplicities, satisfies*

$$\mathcal{N}_S \leq \frac{1}{2\ln\Lambda}\left(\frac{\Lambda^2}{\Lambda^2-1}\sum_{n=-\infty}^{\infty}\Lambda^{2|n|}|V_n|\right)^2 + 2,$$

for any $\Lambda > 1$.

Before proving the theorem we would like to note that the class of potentials for which $k = \pm 1$ are poles of $\det_1(1+X)$ forms a dense subset of the space of potentials for which $\|V\|_{\infty,q} < \infty$. This fact allows us to apply Proposition 2.1 to the functions $a(k) := \det{}_1(1+K(k))$ and $a(k) := \det{}_2(1+K(k))$ in proofs of all the theorems in this paper.

Proof. Suppose V is compactly supported. The Birman–Schwinger operators $K(k)$ from (2.9) can be extended analytically to $\mathbb{C}\setminus\{0\}$, as discussed in Section 2.2. The same proof shows that the operators are not only meromorphic with respect to the infinity norm, but even with respect to the norm in $\mathfrak{S}_2$.

We will apply Corollary 2.2 to the function $a(k) := \det{}_2(1+K(k))$, then we get the following estimates:

$$\begin{aligned}
\ln|a(K)| &\leq \frac{1}{2}\|K\|^2_{\mathfrak{S}_2} && \text{by Lemma 2.3}\\
&\leq \frac{1}{2}\left(\frac{1}{1-|k|^2}\sum_{n=-\infty}^{\infty}|k|^{-2|n|}|V_n|\right) && \text{by Proposition 2.5}\\
&\leq \frac{1}{2}\left(\frac{1}{1-R^2}\sum_{n=-\infty}^{\infty}R^{-2|n|}|V_n|\right) && \text{since } |k| > R
\end{aligned}$$

So plugging the estimate above into result in Corollary 2.2 for $ln(A)$ and setting $\Lambda = 1/R$ we get the following result

$$\#\{j:\ \operatorname{Im} k_j \geq 0\} \leq \frac{1}{2\ln\Lambda}\left(\frac{\Lambda^2}{\Lambda^2-1}\sum_{n=-\infty}^{\infty}\Lambda^{2|n|}|V_n|\right)^2 + 2\,.$$

Now, by Lemma 2.4 for $|k_j| > 1$, the $k_j + k_j^{-1}$ coincide with the eigenvalues of H, counting algebraic multiplicities.

The established bound for compact V can be easily extended to the general case with the use of the continuity argument. Hence the result in Theorem 3.1 holds. □

Theorem 3.2. *The number $\mathcal{N}_S$ of eigenvalues of H in $\ell^2(\mathbb{Z})$, counting algebraic multiplicities, satisfies*

$$\mathcal{N}_S \le \frac{1}{\ln \Lambda} \frac{\Lambda^2}{(\Lambda^2-1)} \left(\sum_{n=-\infty}^{\infty} \Lambda^{|n|} |V_n|^{1/2} \right)^2 + 2,$$

for any $\Lambda > 1$.

The proof of Theorem 3.2 will be almost identical to the proof of Theorem 3.1 with the few slight differences. If we consider $a(k) := \det{}_1(1+K(k)) = \det(1+K(k))$ instead of $a(k) := \det{}_2(1+K(k))$, substitute $\|K\|_{\mathfrak{S}_1}^1$ for $\|K\|_{\mathfrak{S}_2}^2$, and apply Proposition 2.6 in place of Proposition 2.5, the result will follow.

4. Estimation for the Dirac Operator

In this section we will prove our estimations for the Dirac operator D_V.

Let us first recall that $D_V^2 = (3 - H_0) + Q$, where $Q = D_0 V + V D_0 + V^2$. We then apply the Birman-Schwinger principle in the regular case to conclude that the number of eigenvalues, λ, of D_V s.t. $\lambda \in \rho(D_0)$ corresponds to the number of zeros of the function $d(k) = det_p(1 + Q(H_0 - \lambda)^{-1})$, for any $p \in \mathbb{N}$.

For our further estimations it will be usefull to define a new operator W as follows: $W_n = q^{\frac{|n|}{2}}$, for some $q \in (0,1)$. Another useful result is that for any operators A and B the equality $\sigma_p(AB) = \sigma_p(BA)$ holds. As a result, we can rewrite the equation as follows:

$$\begin{aligned} Q(H_0 - z)^{-1} &= \frac{1}{W} Q(H_0 - z)^{-1} W \\ &= \frac{1}{W}(D_0 V + V D_0 + V^2)(H_0 - z)^{-1} W \\ &= \frac{1}{W} D_0 V \frac{1}{W} W (H_0 - z)^{-1} W + \frac{1}{W} V D_0 \frac{1}{W} W (H_0 - z)^{-1} W + \frac{1}{W} V^2 (H_0 - z)^{-1} W \end{aligned}$$

It is also known that for any two Schatten class operators S_1 and S_2, their Schatten norm can be estimated by $\|S_1 S_2\|_{\mathfrak{S}_p} \le \|S_1\|_\infty \|S_2\|_{\mathfrak{S}_p}$. Applying this property we get the following estimate:

$$\begin{aligned} \|Q(H_0 - z)^{-1}\|_{\mathfrak{S}_p} &= \left\| \frac{1}{W} D_0 V \frac{1}{W} W (H_0 - z)^{-1} W + \frac{1}{W} V D_0 \frac{1}{W} W (H_0 - z)^{-1} W + \frac{1}{W} V^2 (H_0 - z)^{-1} W \right\|_{\mathfrak{S}_p} \\ &\le \left\| \frac{1}{W} D_0 V \frac{1}{W} \right\|_\infty \|W(H_0 - z)^{-1} W\|_{\mathfrak{S}_p} + \left\| \frac{1}{W} V D_0 \frac{1}{W} \right\|_\infty \|W(H_0 - z)^{-1} W\|_{\mathfrak{S}_p} \\ &\quad + \left\| \frac{1}{W} V^2 \frac{1}{W} \right\|_\infty \|W(H_0 - z)^{-1} W\|_{\mathfrak{S}_p} \\ &= \|W(H_0 - z)^{-1} W\|_{\mathfrak{S}_p} \left(\left\| \frac{1}{W} D_0 V \frac{1}{W} \right\|_\infty + \left\| \frac{1}{W} V D_0 \frac{1}{W} \right\|_\infty + \left\| \frac{1}{W} V^2 \frac{1}{W} \right\|_\infty \right) \qquad (\star) \end{aligned}$$

Now, applying Proposition 2.5 while substituting V in the proposition with W^2, we get the following:

$$\left\|W(H_0-z)^{-1}W\right\|_{\mathfrak{S}_2} \leq \frac{1}{1-|k|^2}\sum_{n=-\infty}^{\infty}|k|^{-2|n|}q^{|n|} \qquad (\dagger)$$

Similarly, applying Proposition 2.6 while substituting V with W^2, we get

$$\left\|W(H_0-z)^{-1}W\right\|_{\mathfrak{S}_1} \leq \frac{1}{1-|k|^2}\left(\sum_{n=-\infty}^{\infty}|k|^{-|n|}q^{\frac{|n|}{2}}\right)^2 \qquad (\dagger\dagger)$$

To estimate $\left\|\frac{1}{W}D_0V\frac{1}{W}\right\|_\infty$ we decompose the operator D_0 into the sum of two operators T_S and T_0:

$$D_0 = \begin{bmatrix} 1 & S-1 \\ S^*-1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & S \\ S^* & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = T_S + T_0$$

Then $\left\|\frac{1}{W}D_0V\frac{1}{W}\right\|_\infty \leq \left\|\frac{1}{W}T_SV\frac{1}{W}\right\|_\infty + \left\|\frac{1}{W}T_0V\frac{1}{W}\right\|_\infty$. Notice that the eigenvalues of T_0 are $\pm\sqrt{2}$ and T_0 is self-adjoint, so $\|T_0\|_\infty = \sqrt{2}$. As a result, since T_0 does not depend on W, we get the inequality,

$$\left\|\frac{1}{W}T_0V\frac{1}{W}\right\|_\infty \leq \|T_0\|_\infty\left\|\frac{V}{W^2}\right\|_\infty = \sqrt{2}\left\|\frac{V}{W^2}\right\|_\infty = \sqrt{2}\sup|V_nq^{-|n|}| := \|V\|_{\infty,q} \ .$$

The estimation of $\left\|\frac{1}{W}T_SV\frac{1}{W}\right\|_\infty$, however, will be a little more complicated:

We remind the reader that $W=\{W_n\}$ is acting on the Hilbert space $\mathfrak{H} = \ell^2(\mathbb{Z},\mathbb{C}^2)$. Then for every $\xi = \begin{Bmatrix}\alpha_n\\ \beta_n\end{Bmatrix}_{n\in\mathbb{Z}} \in \mathfrak{H}$, $W\xi = \begin{Bmatrix}W_n\alpha_n\\ W_n\beta_n\end{Bmatrix}_n = \begin{Bmatrix}q^{\frac{|n|}{2}}\alpha_n\\ q^{\frac{|n|}{2}}\beta_n\end{Bmatrix}_n$.

Furthermore

$$T_SW\xi = \begin{Bmatrix}q^{\frac{|n-1|}{2}}\alpha_{n-1}\\ q^{\frac{|n+1|}{2}}\beta_{n+1}\end{Bmatrix}_n \quad \text{and} \quad \frac{1}{W}T_SW\xi = \begin{Bmatrix}\frac{q^{\frac{|n-1|}{2}}}{q^{\frac{|n|}{2}}}\alpha_{n-1}\\ \frac{q^{\frac{|n+1|}{2}}}{q^{\frac{|n|}{2}}}\beta_{n+1}\end{Bmatrix}_n .$$

As a result, we get the following estimate for $\left\|\frac{1}{W}T_SW\right\|_\infty$:

$$\left\|\frac{1}{W}T_SW\right\|_\infty \leq \sup_n\left|\frac{q^{\frac{|n-1|}{2}}}{q^{\frac{|n|}{2}}}\right| + \sup_n\left|\frac{q^{\frac{|n+1|}{2}}}{q^{\frac{|n|}{2}}}\right| = q^{\frac{-1}{2}} + q^{\frac{-1}{2}} = 2q^{\frac{-1}{2}} .$$

Similarly, we get the estimate for $\left\|WT_S\frac{1}{W}\right\|_\infty$ to be

$$\left\|WT_S\frac{1}{W}\right\|_\infty \leq \sup_n \left|\frac{q^{\frac{|n|}{2}}}{q^{\frac{|n-1|}{2}}}\right| + \sup_n \left|\frac{q^{\frac{|n|}{2}}}{q^{\frac{|n+1|}{2}}}\right| = q^{\frac{-1}{2}} + q^{\frac{-1}{2}} = 2q^{\frac{-1}{2}}\,.$$

Hence

$$\left\|\frac{1}{W}D_0V\frac{1}{W}\right\|_\infty = \left\|\frac{1}{W}VD_0\frac{1}{W}\right\|_\infty = \left\|\frac{V}{W^2}\right\|_\infty\left(\left\|\frac{1}{W}T_SW\right\|_\infty + 1\right) = \|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+1\right)$$

The only thing left for us to do is to note that

$$\left\|\frac{V^2}{W^2}\right\|_\infty \leq \left\|\frac{V^2}{W^4}\right\|_\infty \leq \left\|\frac{V}{W^2}\right\|_\infty^2 = \|V\|_{\infty,q}^2\ .$$

As a result, from (†) and all the estimates above, we get the following estimate for the Schatten p-norm of the operator $Q(H_0-z)^{-1}$:

$$\begin{aligned}\|Q(H_0-z)^{-1}\|_{\mathfrak{S}_p} &\leq \|W(H_0-z)^{-1}W\|_{\mathfrak{S}_p}\left(\left\|\frac{1}{W}DV\frac{1}{W}\right\|_\infty + \left\|\frac{1}{W}VD\frac{1}{W}\right\|_\infty + \left\|\frac{1}{W}V^2\frac{1}{W}\right\|_\infty\right)\\ &\leq \left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right) + \|V\|_{\infty,q}^2\right)\|W(H_0-z)^{-1}W\|_{\mathfrak{S}_p} \qquad (\star\star)\end{aligned}$$

4.1. **Proof of Theorems 1.1 and 1.2.** We will apply similar technique to the one we applied in the proof of Theorem 3.1.

Proof of Theorem 1.1. Suppose V is compactly supported. The Birman–Schwinger operators $K(k)$ from (2.9) can be extended analytically to $\mathbb{C}\setminus\{0\}$, as discussed in Section 2.2. The same proof shows that the operators are not only meromorphic with respect to the infinity norm, but even with respect to the norm in $\mathfrak{S}_2$.

We will apply Corollary 2.2 with $n=2$ to the function $a(k) := \det{}_2(1+Q(H_0-z)^{-1})$. to get the following estimates:

$$\begin{aligned}\ln|a(K)| &\leq \frac{1}{2}\|Q(H_0-z)^{-1}\|_{\mathfrak{S}_2}^2 && \text{by Lemma 2.3}\\ &\leq \frac{1}{2}\left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|_{\infty,q}^2\right)^2\|W(H_0-z)^{-1}W\|_{\mathfrak{S}_2}^2 && \text{by } (\star\star)\\ &\leq \frac{1}{2}\left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|_{\infty,q}^2\right)^2\left(\frac{1}{1-|k|^2}\sum_{n=-\infty}^{\infty}|k|^{-2|n|}q^{|n|}\right)^2 && \text{by (†)}\\ &\leq \frac{1}{2}\left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|_{\infty,q}^2\right)^2\left(\frac{1}{1-R^2}\sum_{n=-\infty}^{\infty}R^{-2|n|}q^{|n|}\right)^2 && \text{since } |k|>R\\ &\leq \frac{1}{2}\left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|_{\infty,q}^2\right)^2\left(\frac{1}{1-R^2}\left(\frac{2}{1-qR^{-2}}-1\right)\right)^2\end{aligned}$$

Note that in order for the last inequality to make sense we must enforce the following restriction on q : $q = R^{2+\varepsilon}$, $\varepsilon > 0$. So plugging the estimate above into result in Corollary 2.2 for $ln(A)$ and setting $\Lambda = 1/R$ yields the following inequality:

$$\#\{j : \operatorname{Im} k_j \geq 0\} \leq \frac{\left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|^2_{\infty,q}\right)^2}{2\ln\Lambda}\left(\frac{\sqrt{2}\Lambda^2}{\Lambda^2-1}\right)^2\left(\frac{\Lambda^\varepsilon+1}{\Lambda^\varepsilon-1}\right)^2+4\,.$$

The extra $\sqrt{2}$ in front of the $\dfrac{\Lambda^2}{\Lambda^2-1}$ term is due to the fact that H_0, is an orthogonal sum of two operators, as can be seen in subsection 1.2. Now, by Lemma 2.4 for $|k_j| > 1$, the $k_j + k_j^{-1}$, coincide with the eigenvalues of H, counting algebraic multiplicities.

The established bound for compact V can be easily extended to the general case with the use of the continuity argument. Hence the result in Theorem 1.1 holds. □

The proof for Theorem 1.2 will be very similar to the proof of Theorem 1.1.

Proof of Theorem 1.2. Suppose V is compactly supported. The Birman–Schwinger operators $K(k)$ from (2.9) can be extended analytically to $\mathbb{C}\setminus\{0\}$, as discussed in Section 2.2. The same proof shows that the operators are not only meromorphic with respect to the infinity norm, but even with respect to the norm in $\mathfrak{S}_1$.

We will apply Corollary 2.2 with $n = 2$ to the function $a(k) := \det{}_1(1+Q(H_0-z)^{-1})$, then we get the following estimates:

$$\begin{aligned}
\ln|a(K)| &\leq \|Q(H_0-z)^{-1}\|_{\mathfrak{S}_1} && \text{by Lemma 2.3}\\
&\leq \left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|^2_{\infty,q}\right)\left\|W(H_0-z)^{-1}W\right\|_{\mathfrak{S}_1} && \text{by } (\star\star)\\
&\leq \left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|^2_{\infty,q}\right)\frac{1}{1-|k|^2}\left(\sum_{n=-\infty}^{\infty}|k|^{-|n|}q^{\frac{|n|}{2}}\right)^2 && \text{by } (\dagger\dagger)\\
&\leq \left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|^2_{\infty,q}\right)\frac{1}{1-R^2}\left(\sum_{n=-\infty}^{\infty}R^{-|n|}q^{\frac{|n|}{2}}\right)^2 && \text{since } |k|>R\\
&\leq \frac{1}{2}\left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|^2_{\infty,q}\right)^2\frac{1}{1-R^2}\left(\frac{2}{1-q^{\frac{1}{2}}R^{-1}}-1\right)^2
\end{aligned}$$

So after plugging the estimate above into result in Corollary 2.2 for $ln(A)$ and setting $\Lambda = 1/R$ we get the following result

$$\#\{j : \operatorname{Im} k_j \geq 0\} \leq \frac{\left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|^2_{\infty,q}\right)}{\ln\Lambda}\frac{\sqrt{2}\Lambda^2}{\Lambda^2-1}\left(\frac{\Lambda^{\varepsilon/2}+1}{\Lambda^{\varepsilon/2}-1}\right)^2+4\,.$$

Now, by Lemma 2.4 for $|k_j| > 1$, the $k_j + k_j^{-1}$, coincide with the eigenvalues of H, counting algebraic multiplicities.

The established bound for compact V can be easily extended to the general case with the use of the continuity argument. Hence the result in Theorem 1.2 holds. □

Most of the papers listed below contain results on the eigenvalues of non-selfadjoint operators. More specifically,those are the articles [1]-[15], [17], [19]-[25] and [28]-[29]. The remaining references were needed for technical reasons.

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