On the well-posedness of the magnetic Schrödinger-Poisson system in $\mathbb{R}^3$

Jean-Marie Barbaroux
Aix Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France
e-mail: barbarou@univ-tln.fr

Vitali Vougalter
University of Toronto, Department of Mathematics,
Toronto, Ontario, M5S 2E4, Canada
e-mail: vitali@math.toronto.edu

Abstract. We prove global existence and uniqueness of strong solutions for the Schrödinger-Poisson system in the repulsive Coulomb case in $\mathbb{R}^3$ in the presence of a smooth magnetic field.

Keywords: Schrödinger-Poisson system, functional spaces, density matrices, global existence and uniqueness, magnetic fields

AMS subject classification: 82D10, 82C10

1 Introduction

The present article is devoted to the studies of the global well-posedness of the magnetic Schrödinger-Poisson system in the space of three dimensions. Such system is relevant to the description of many-body non relativistic quantum particles in the mean-field limit (for instance, in plasma), when the system interacts with an external magnetic field. Consider non relativistic quantum particles in $\mathbb{R}^3$. The particles interact via the electrostatic field they collectively generate. In the mean-field limit, the density matrix $\rho(t)$ describing the *mixed* state at time t of the system satisfies the Hartree-von Neumann equation

$$\begin{cases} i\partial_t \rho(t) = [H_{A,V}, \rho(t)], \quad x \in \mathbb{R}^3, \quad t \geq 0 \\ -\Delta V = n(t,x), \quad n(t,x) = \rho(t,x,x), \ \rho(0) = \rho_0. \end{cases} \tag{1.1}$$

The magnetic Hamiltonian is given by

$$H_{A,V} := (-i\nabla + A)^2 + V(t,x), \tag{1.2}$$

where the magnetic vector potential $A \in \mathbb{C}^1(\mathbb{R}^3,\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3,\mathbb{R}^3)$ with $divA = 0$. Here, $(-i\nabla + A)^2$ stands for the magnetic Laplacian on $L^2(\mathbb{R}^3)$; see [4, 5] for a derivation of such system of equations in the *non-magnetic* case. Since $\rho(t)$ is a nonnegative, self-adjoint trace-class operator acting on $L^2(\mathbb{R}^3)$, its kernel can, for every $t \in \mathbb{R}_+$, be decomposed with respect to an orthonormal basis of $L^2(\mathbb{R}^3)$. Let us represent the kernel of the initial data ρ_0 in the form

$$\rho_0(x,y) = \sum_{k\in\mathbb{N}} \lambda_k \psi_{0,k}(x)\overline{\psi_{0,k}(y)}, \tag{1.3}$$

where $\{\psi_{0,k}\}_{k\in\mathbb{N}}$ stands for an orthonormal basis of $L^2(\mathbb{R}^3)$ and coefficients

$$\underline{\lambda} := \{\lambda_k\}_{k\in\mathbb{N}} \in \ell^1\ ,\quad \lambda_k \geq 0\ ,\quad \sum_{k\in\mathbb{N}} \lambda_k = 1.$$

We will prove below, that there exists a one-parameter family of complete orthonormal bases of $L^2(\mathbb{R}^3)$, $\{\psi_k(t)\}_{k\in\mathbb{N}}$ for $t \in \mathbb{R}_+$, such that the kernel of the solution $\rho(t)$ to (1.1) can be represented as

$$\rho(t,x,y) = \sum_{k\in\mathbb{N}} \lambda_k \psi_k(t,x)\overline{\psi_k(t,y)}. \tag{1.4}$$

Let us note that, the coefficients $\underline{\lambda}$ are *independent* of t, and thus the same as those in ρ_0, which is due to the fact that the operators $-iH_{A,V}$ and $\rho(t)$ form a Lax pair in the original equation (1.1). When substituting (1.4) in (1.1), the one-parameter family of orthonormal vectors $\{\psi_k(t)\}_{k\in\mathbb{N}}$ is seen to satisfy the magnetic Schrödinger-Poisson system

$$\begin{cases} i\dfrac{\partial\psi_k}{\partial t} = (-i\nabla + A)^2\psi_k + V[\Psi]\psi_k, \quad k\in\mathbb{N}, & (1.5)\\ -\Delta V[\Psi] = n[\Psi], \quad \Psi := \{\psi_k\}_{k=1}^\infty, & (1.6)\\ n[\Psi](t,x) = \displaystyle\sum_{k=1}^\infty \lambda_k|\psi_k(t,x)|^2, & (1.7)\\ \psi_k(t=0,x) = \psi_{0,k}(x), \quad k\in\mathbb{N}. & (1.8) \end{cases}$$

The potential function $V[\Psi]$ satisfies the Poisson equation (1.6).

We remind that $V[\Psi]$ has the explicit Newtonian potential integral representation (see e.g. [7, 10]).

$$V[\Psi](x,t) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{n[\Psi](t,y)}{|x-y|}\mathrm{d}y. \tag{1.9}$$

We will establish in Lemma 6 below that solutions of (1.5)-(1.7) preserve the orthonormality of $\{\psi_k(t)\}_{k\in\mathbb{N}}$.

We introduce the magnetic Sobolev norms for functions:

$$\|f\|^2_{H^1_A(\mathbb{R}^3)} := \|f\|^2_{L^2(\mathbb{R}^3)} + \|(-i\nabla + A)f\|^2_{L^2(\mathbb{R}^3)}, \tag{1.10}$$

$$\|f\|^2_{H^2_A(\mathbb{R}^3)} := \|f\|^2_{L^2(\mathbb{R}^3)} + \|(-i\nabla + A)^2 f\|^2_{L^2(\mathbb{R}^3)}. \tag{1.11}$$

The usual Sobolev norms $\|f\|^2_{H^1(\mathbb{R}^3)}$ and $\|f\|^2_{H^2(\mathbb{R}^3)}$ will be used when the magnetic vector potential $A(x)$ vanishes. The state space for our magnetic Schrödinger-Poisson system is given by

$$\mathcal{L} := \{(\Psi, \underline{\lambda}) \mid \Psi = \{\psi_k\}_{k=1}^\infty \subset H^2_A(\mathbb{R}^3) \text{ is a complete orthonormal system in } L^2(\mathbb{R}^3),$$

$$\underline{\lambda} = \{\lambda_k\}_{k=1}^\infty \in \ell^1, \quad \lambda_k \geq 0, \ k \in \mathbb{N}, \quad \sum_{k=1}^\infty \lambda_k \int_{\mathbb{R}^3} |(-i\nabla + A)^2 \psi_k|^2 \mathrm{d}x < \infty\}.$$

Let us define the inner product for fixed $\underline{\lambda} \in \ell^1, \quad \lambda_k \geq 0$, and for sequences of square integrable functions $\Phi := \{\phi_k\}_{k=1}^\infty$ and $\Psi := \{\psi_k\}_{k=1}^\infty$ as

$$(\Phi, \Psi)_X := \sum_{k=1}^\infty \lambda_k (\phi_k, \psi_k)_{L^2(\mathbb{R}^3)}.$$

Clearly, it induces the norm

$$\|\Phi\|_X := (\sum_{k=1}^\infty \lambda_k \|\phi_k\|^2_{L^2(\mathbb{R}^3)})^{\frac{1}{2}}.$$

Let us introduce the corresponding Hilbert space

$$X := \{\Phi = \{\phi_k\}_{k=1}^\infty \mid \phi_k \in L^2(\mathbb{R}^3), \ \forall \ k \in \mathbb{N}, \ \|\Phi\|_X < \infty\}.$$

Let us also give the Hilbert space defining strong solutions

$$Z := \{\Phi = \{\phi_k\}_{k=1}^\infty \mid \phi_k \in H^2(\mathbb{R}^3), \ \forall \ k \in \mathbb{N}, \ \|\Phi\|_Z := \left(\sum_{k=1}^\infty \lambda_k \|\phi_k\|^2_{H^2(\mathbb{R}^3)}\right)^{\frac{1}{2}} < \infty\}$$

Our main result is as follows.

Theorem 1. *For every initial state $(\Psi(x,0), \underline{\lambda}) \in \mathcal{L}$, there exists a unique mild solution $\Psi(x,t)$, $t \in [0,\infty)$, of (1.5)-(1.8) with $(\Psi(x,t), \underline{\lambda}) \in \mathcal{L}$. This is also a unique strong global solution in X, i.e., $\Psi \in C([0,\infty); Z) \cap C^1([0,\infty); X)$.*

Proving the global well-posedness of the Schrödinger-Poisson system plays a critical role in establishing the existence and nonlinear stability of stationary states, i.e. the nonlinear bound states of the Schrödinger-Poisson system, which was done in the non magnetic case in [7, 12]. These issues in the semi-relativistic regime were addressed recently in [1], [2], [3]. The case of the magnetic Schrödinger-Poisson system in a bounded domain of $\mathbb{R}^3$ was treated in [6]. The corresponding one dimensional problem was studied in [15]. The existence of solutions for a single Nonlinear Schrödinger (NLS) equation with a magnetic field was established in [9], see also [8].

2 Proof of global well-posedness

Let us make a fixed choice of $\underline{\lambda} = \{\lambda_k\}_{k=1}^{\infty} \in \ell^1$, with $\lambda_k \geq 0$ and $\sum_{k\in\mathbb{N}} \lambda_k = 1$ for the sequence of coefficients determined by the initial data ρ_0 of the Hartree-von Neumann equation (1.1) via (1.4), for $t = 0$.

Let us introduce inner products $(\cdot,\cdot)_{Y_A}$ and $(\cdot,\cdot)_{Z_A}$ inducing the magnetic Sobolev norms

$$\|\Phi\|_{Y_A} := \left(\sum_{k=1}^{\infty} \lambda_k \|\phi_k\|^2_{H^1_A(\mathbb{R}^3)}\right)^{\frac{1}{2}} \quad and \quad \|\Phi\|_{Z_A} := \left(\sum_{k=1}^{\infty} \lambda_k \|\phi_k\|^2_{H^2_A(\mathbb{R}^3)}\right)^{\frac{1}{2}}. \tag{2.1}$$

We define the corresponding Hilbert spaces

$$Y_A := \{\Phi = \{\phi_k\}_{k=1}^{\infty} \mid \phi_k \in H^1_A(\mathbb{R}^3),\ \forall\ k \in \mathbb{N},\ \|\Phi\|_{Y_A} < \infty\}$$

and

$$Z_A := \{\Phi = \{\phi_k\}_{k=1}^{\infty} \mid \phi_k \in H^2_A(\mathbb{R}^3),\ \forall\ k \in \mathbb{N},\ \|\Phi\|_{Z_A} < \infty\}$$

respectively. The notations $\|\Phi\|_Y,\ \|\Phi\|_Z$ will be used when the magnetic vector potential $A(x)$ vanishes in $\mathbb{R}^3$, similarly to Section 3 of [7]. We have the following equivalence of magnetic and non magnetic norms.

Lemma 2. *Assume that the vector potential $A(x) \in C^1(\mathbb{R}^3,\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3,\mathbb{R}^3)$ and the Coulomb gauge is chosen, namely*

$$divA = 0. \tag{2.2}$$

a) Let $f(x) \in H^1_A(\mathbb{R}^3)$. Then the norms $\|f\|_{H^1(\mathbb{R}^3)}$ and $\|f\|_{H^1_A(\mathbb{R}^3)}$ are equivalent.

b) Let $f(x) \in H^2_A(\mathbb{R}^3)$. Then the norms $\|f\|_{H^2(\mathbb{R}^3)}$ and $\|f\|_{H^2_A(\mathbb{R}^3)}$ are equivalent.

c) Let $\Phi(x) \in Y_A$. Then the norms $\|\Phi\|_{Y_A}$ and $\|\Phi\|_Y$ are equivalent.

d) Let $\Phi(x) \in Z_A$. Then the norms $\|\Phi\|_{Z_A}$ and $\|\Phi\|_Z$ are equivalent.

Proof. In the argument below, with a slight abuse of notations C will denote a finite, positive constant. Since the vector potential $A(x)$ is bounded in $\mathbb{R}^3$, as assumed, we easily obtain

$$\|(-i\nabla + A)f\|_{L^2(\mathbb{R}^3)} \leq \|\nabla f\|_{L^2(\mathbb{R}^3)} + C\|f\|_{L^2(\mathbb{R}^3)}.$$

Therefore,

$$\|f\|_{H^1_A(\mathbb{R}^3)} \leq C\|f\|_{H^1(\mathbb{R}^3)}.$$

Clearly,

$$\|\nabla f\|_{L^2(\mathbb{R}^3)} \leq \|(-i\nabla + A)f\|_{L^2(\mathbb{R}^3)} + \|Af\|_{L^2(\mathbb{R}^3)} \leq \|(-i\nabla + A)f\|_{L^2(\mathbb{R}^3)} + C\|f\|_{L^2(\mathbb{R}^3)},$$

which yields

$$\|f\|_{H^1(\mathbb{R}^3)} \leq C\|f\|_{H^1_A(\mathbb{R}^3)}$$

and completes the proof of the part a) of the lemma. By virtue of (2.2), we have

$$(-i\nabla + A)^2 = -\Delta - 2iA\nabla + A^2.$$

Evidently,

$$\|\Delta f\|_{L^2(\mathbb{R}^3)} \le \|(-i\nabla + A)^2 f\|_{L^2(\mathbb{R}^3)} + 2\|A\nabla f\|_{L^2(\mathbb{R}^3)} + \|A^2 f\|_{L^2(\mathbb{R}^3)}.$$

Obviously, we have

$$\|A^2 f\|_{L^2(\mathbb{R}^3)} \le C\|f\|_{L^2(\mathbb{R}^3)}, \quad \|A\nabla f\|_{L^2(\mathbb{R}^3)} \le C\|\nabla f\|_{L^2(\mathbb{R}^3)}.$$

By means of the result of the part a), the right side of the second inequality above can be estimated from above by $C\|f\|_{H^1_A(\mathbb{R}^3)}$. Edivently, this expression has an upper bound $C\|f\|_{H^2_A(\mathbb{R}^3)}$. Therefore,

$$\|\Delta f\|_{L^2(\mathbb{R}^3)} \le C\|f\|_{H^2_A(\mathbb{R}^3)}$$

and

$$\|f\|_{H^2(\mathbb{R}^3)} \le C\|f\|_{H^2_A(\mathbb{R}^3)}.$$

Clearly,

$$\|(-i\nabla + A)^2 f\|_{L^2(\mathbb{R}^3)} \le \|\Delta f\|_{L^2(\mathbb{R}^3)} + 2\|A\nabla f\|_{L^2(\mathbb{R}^3)} + \|A^2 f\|_{L^2(\mathbb{R}^3)}.$$

Therefore, we have

$$\|(-i\nabla + A)^2 f\|^2_{L^2(\mathbb{R}^3)} \le C\|f\|^2_{H^2(\mathbb{R}^3)},$$

such that

$$\|f\|_{H^2_A(\mathbb{R}^3)} \le C\|f\|_{H^2(\mathbb{R}^3)},$$

which completes the proof of the part b) of the lemma. The statements of parts c) and d) follow easily from the ones of a) and b) using norm definitions (2.1). ■

Let $\Psi = \{\psi_m\}_{m=1}^{\infty}$ be a wave function and the magnetic kinetic energy operator acts on it $(-i\nabla + A)^2\Psi$ componentwise. We have the following two auxiliary lemmas.

Lemma 3. *The domain of the magnetic kinetic energy operator $(-i\nabla + A)^2$ on X is given by $D((-i\nabla+A)^2) = Z_A = Z$, and the operator $(-i\nabla+A)^2$ generates the strongly continuous one parameter group $e^{-it(-i\nabla+A)^2}$, $t \in \mathbb{R}$ on X.*

Proof. Since the magnetic potential A is a bounded function, we get the result from the generalization of the properties of $(-i\nabla + A)^2$ on $L^2(\mathbb{R}^3)$. ■

Let us rewrite the magnetic Schrödinger-Poisson system for $x \in \mathbb{R}^3$ into the form

$$\frac{\partial \Psi}{\partial t} = -i(-i\nabla + A)^2\Psi + F[\Psi(x,t)], \ \textit{where } F[\Psi] := i^{-1}V[\Psi]\Psi, \tag{2.3}$$

$$-\Delta V[\Psi] = n[\Psi],$$

$$n[\Psi](x,t) = \sum_{k=1}^{\infty} \lambda_k |\psi_k(x,t)|^2$$

and derive the following auxiliary result.

Lemma 4. *Given an initial state* $(\Psi(x,0),\underline{\lambda}) \in \mathcal{L}$*, there exists* $T \in [0,\infty]$ *such that the magnetic Schrödinger-Poisson system* (1.5)-(1.8) *admits a unique mild solution* Ψ *in* Z_A *on a time interval* $[0,T)$*, which solves the integral equation.*

$$\Psi(t) = e^{-i(-i\nabla+A)^2 t}\Psi(0) + \int_0^t e^{-i(-i\nabla+A)^2(t-s)}F[\Psi(s)]\mathrm{d}s \tag{2.4}$$

in Z_A*. Moreover,* Ψ *is a unique strong solution in* X*,* $\Psi \in C([0,T);Z_A) \cap C^1(]0,T[;X)$*.*

Proof. By means of the result [10, Proposition 3.2], we have for Ψ and Φ in $Z_A = Z$

$$\|F[\Psi] - F[\Phi]\|_Z \le C(\|\Phi\|_Z, \|\Psi\|_Z)\|\Psi - \Phi\|_Z,$$

where the constant $C(\|\Phi\|_Z, \|\Psi\|_Z)$ depends in a monotone increasing way on $\|\Phi\|_Z$ and $\|\Psi\|_Z$. Therefore, using the equivalence of magnetic and non magnetic norms proved in Lemma 2, we obrain that the map $F : Z_A \to Z_A$ is locally Lipschitz continuous.

By virtue of [13, Theorem 1.7 §6], along with the Lipschitz property of F, we obtain that the magnetic Schrödinger-Poisson system admits a unique mild solution Ψ in Z_A on a time interval $[0,T)$. This solution solves the integral equation

$$\Psi(t) = e^{-i(-i\nabla+A)^2 t}\Psi(0) + \int_0^t e^{-i(-i\nabla+A)^2(t-s)}F[\Psi(s)]\mathrm{d}s$$

in Z_A. Moreover,

$$\lim_{t\nearrow T}\|\Psi(t)\|_{Z_A} = \infty$$

if T is finite. We also obtain from [13, Theorem 1.7 §6] that Ψ is a unique strong solution in X on the same time interval.

Let us establish the conservation of energy for the solutions to our magnetic Schrödinger-Poisson system in the following sense.

Lemma 5. *For the unique mild solution (2.4) of the magnetic Schrödinger-Poisson system (1.5)-(1.7) and for any value of time* $t \in [0,T)$ *the identity*

$$\|\Psi(x,t)\|^2_{Y_A} + \frac{1}{2}\|\nabla V[\Psi(x,t)]\|^2_{L^2(\mathbb{R}^3)} = \|\Psi(x,0)\|^2_{Y_A} + \frac{1}{2}\|\nabla V[\Psi(x,0)]\|^2_{L^2(\mathbb{R}^3)} \tag{2.5}$$

holds.

Proof. Complex conjugation of our magnetic Schrödinger-Poisson system (1.5) yields

$$-i\frac{\partial\bar{\psi}_k}{\partial t} = (i\nabla + A)^2\bar{\psi}_k + V[\Psi(x,t)]\bar{\psi}_k, \quad k \in \mathbb{N}. \tag{2.6}$$

Let us add the k-th equation of (1.5) multiplied by $\frac{\partial\bar{\psi}_k}{\partial t}$, and the k-th equation in (2.6) multiplied by $\frac{\partial\psi_k}{\partial t}$. We arrive at

$$\frac{\partial}{\partial t}\|(-i\nabla + A)\psi_k\|^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} V[\Psi(x,t)]\frac{\partial}{\partial t}|\psi_k|^2\mathrm{d}x = 0, \quad k \in \mathbb{N}.$$

Multiplying by λ_k, and summing over k, we easily derive

$$\frac{\partial}{\partial t}\|\Psi(x,t)\|^2_{Y_A} + \int_{\mathbb{R}^3} V[\Psi(x,t)]\frac{\partial}{\partial t}n[\Psi(x,t)]\mathrm{d}x = 0. \tag{2.7}$$

It can be trivially shown that

$$\frac{\partial}{\partial t}\|\nabla V[\Psi(x,t)]\|^2_{L^2(\mathbb{R}^3)} = 2\int_{\mathbb{R}^3} V[\Psi(x,t)]\frac{\partial}{\partial t}n[\Psi(x,t)]\mathrm{d}x.$$

Substituting this equality in (2.7) yields the result of the lemma. ∎

Below we establish a conservation law for the density.

Lemma 6. *Suppose that the initial condition $\{\psi_{0,k}(x)\}_{k\in\mathbb{N}}$ of the Schrödinger-Poisson system (1.5)-(1.7) is a complete orthonormal system in $L^2(\mathbb{R}^3)$. Let T be given by Lemma 4. Then for any $t\in[0,T)$, the set $\{\psi_k(\cdot,t)\}_{k\in\mathbb{N}}$ remains a complete orthonormal system in $L^2(\mathbb{R}^3)$.*

Moreover, the X-norm of the solutions is preserved,

$$\|\Psi(x,t)\|_X = \|\Psi(x,0)\|_X, \ t\in[0,T).$$

Proof. We have, using (1.5)

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi_k,\psi_l) = -i((-i\nabla + A)^2 + V_\Psi)\psi_k,\psi_l) + i(\psi_k,(-i\nabla + A)^2 + V_\Psi)\psi_l) = 0.$$

This yields

$$(\psi_k(x,t),\psi_l(x,t))_{L^2(\mathbb{R}^3)} = (\psi_k(x,0),\psi_l(x,0))_{L^2(\mathbb{R}^3)} = \delta_{k,l}, \quad k,l\in\mathbb{N},$$

where $\delta_{k,l}$ stands for the Kronecker symbol. Hence, for $k\in\mathbb{N}$,

$$\|\psi_k(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} = \|\psi_k(\cdot,0)\|^2_{L^2(\mathbb{R}^3)}.$$

Thus for $t\in[0,T)$, the X-norm is preserved,

$$\|\Psi(\cdot,t)\|_X = (\sum_{k=1}^{\infty}\lambda_k\|\psi_k(x,t)\|^2_{L^2(\mathbb{R}^3)})^{\frac{1}{2}} = (\sum_{k=1}^{\infty}\lambda_k\|\psi_k(x,0)\|^2_{L^2(\mathbb{R}^3)})^{\frac{1}{2}} = \|\Psi(\cdot,0)\|_X.$$

For the unique given solution $\Psi(t)$ of our magnetic Schrödinger-Poisson system on $[0,T)$ given by Lemma 4, we obtain the time-dependent magnetic single particle Hamiltonian

$$H_{A,V_\Psi}(t) = (-i\nabla + A)^2 + V_\Psi(t,x)$$

with the scalar potential V_Ψ satisfying $-\Delta V_\Psi(t,x) = n[\Psi(t)]$, and given by the integral representation (1.9). The components of $\Psi(t)$ thus satisfy the *non-autonomous* magnetic

Schrödinger equation $i\partial_t \psi_k(t,x) = H_{A,V_\Psi}(t)\psi_k(t,x)$, for $k \in \mathbb{N}$, on the time interval $[0,T)$. Note that $V_\Psi \in L^\infty(([0,T)]; L^\infty(\mathbb{R}^3))$. Indeed, by means of [10, Lemma 3.3], we have

$$\|V_\Psi\|_{L^\infty(\mathbb{R}^3)} \leq C\|\Psi\|_X\|\Psi\|_Y.$$

Then by virtue of the equivalence of magnetic and non magnetic norms established in Lemma 2, we arrive at

$$\|V_\Psi\|_{L^\infty(\mathbb{R}^3)} \leq C\|\Psi\|_{Y_A}^2.$$

and thus, according to Lemma 5, V_Ψ is uniformly bounded on $[0,T)$.

Moreover, from [10, Lemma 3.4-Lemma 3.5] and the regularity of Ψ stated in Lemma 4, we derive that $t \mapsto V_\Psi(t)$ is a continuously differentiable L^∞ valued function on $[0,T)$.

Therefore, using [14, Theorem X.71], there exists a propagator, denoted by abuse of notation $e^{-i\int_0^t H_{A,V_\Psi}(\tau)d\tau}$ such that for $t \in [0,T)$,

$$\psi_k(x,t) = e^{-i\int_0^t H_{A,V_\Psi}(\tau)d\tau}\psi_k(x,0), \ k \in \mathbb{N}. \tag{2.8}$$

Let us consider an arbitrary function $f \in L^2(\mathbb{R}^3)$. Evidently, we have the expansion

$$f(x) = \sum_{k=1}^{\infty}(f(y), \psi_k(y,0))_{L^2(\mathbb{R}^3)}\psi_k(x,0)$$

and analogously

$$e^{i\int_0^t H_{A,V_\Psi}(\tau)d\tau}f(x) = \sum_{k=1}^{\infty}(e^{i\int_0^t H_{A,V_\Psi}(\tau)d\tau}f(y), \psi_k(y,0))_{L^2(\mathbb{R}^3)}\psi_k(x,0).$$

Therefore, by means of (2.8) we obtain the expansion

$$f(x) = \sum_{k=1}^{\infty}(f(y), \psi_k(y,t))_{L^2(\mathbb{R}^3)}\psi_k(x,t)$$

for $t \in [0,T)$, thus proving that $\{\psi_k(t)\}$ is complete. ■

Armed with Lemma 2-Lemma 5 proved above, we now proceed to the establishment of our main result.

Proof of Theorem 1. Due to the above Lemmas, it only remains to show that the solution is global in time.

Let us apply the norm $\|.\|_{Z_A}$ to both sides of (2.4), to obtain

$$\|\Psi(t)\|_{Z_A} \leq \|\Psi(0)\|_{Z_A} + \int_0^t \|F[\Psi(s)]\|_{Z_A}\mathrm{d}s.$$

We have

$$\|F[\Psi(s)]\|_{Z_A} \leq C\|\Psi\|_{Z_A},$$

which can be proven analogously to the argument of Lemma 3.9 of [10] using the energy conservation statement of Lemma 5. Hence

$$\|\Psi(t)\|_{Z_A} \leq \|\Psi(0)\|_{Z_A} + \int_0^t C\|\Psi(s)\|_{Z_A}\mathrm{d}s.$$

Gronwall's lemma gives us

$$\|\Psi(t)\|_{Z_A} \leq \|\Psi(0)\|_{Z_A} e^{Ct}, \quad t \in [0, T).$$

By means of the blow-up alternative, this implies that our magnetic Schrödinger-Poisson system is globally well-posed in Z_A. ■

References

[1] W. Abou Salem, T. Chen, V. Vougalter. On the well-posedness of the semi-relativistic Schrödinger-Poisson system. Dyn. Partial Differ. Edu. **9** (2012), no. 2, 121–132.

[2] W. Abou Salem, T. Chen, V. Vougalter. On the generalized semi-relativistic Schrödinger-Poisson system in $\mathbb{R}^n$. Doc. Math. **18** (2013), 343–357.

[3] W. Abou Salem, T. Chen, V. Vougalter. Existence and nonlinear stability of stationary states for the semi-relativistic Schrödinger-Poisson system. Ann. Henri Poincare **15** (2014), no. 6, 1171–1196.

[4] I. Anapolitanos. Rate of convergence towards the Hartree-von Neumann limit in the mean-field regime. Lett. Math. Phys. **98** (2011), no. 1, 1–31.

[5] I. Anapolitanos, I. M. Sigal. The Hartree-von Neumann limit of many body dynamics. Preprint [http://arxiv.org/abs/0904.4514].

[6] J.M. Barbaroux, V. Vougalter Existence and nonlinear stability of stationary states for the magnetic Schrödinger-Poisson system. Preprint 2016.

[7] F. Brezzi, P. Markowich. The three-dimensional Wigner-Poisson problem: existence, uniqueness and approximation. Math. Methods Appl. Sci. **14** (1991), no. 1, 35–61.

[8] T. Cazenave, M.J. Esteban. On the stability of stationary states for nonlinear Schrödinger equations with an external magnetic field. Mat. Apl. Comput. **7** (1988), no. 3, 155–168.

[9] A. De Bouard. Nonlinear Schroedinger equations with magnetic fields. Differential Integral Equations **4** (1991), no. 1, 73–88.

[10] R. Illner, P.F. Zweifel, H. Lange. Global existence, uniqueness and asymptotic behaviour of solutions of the Wigner-Poisson and Schrödinger-Poisson systems. Math. Methods Appl. Sci. **17** (1994), no. 5, 349–376.

[11] E. Lieb, M. Loss. Analysis, Graduate Studies in Mathematics, Volume 14. American Mathematical Society, Providence, RI, 1997.

[12] P.A. Markowich, G. Rein, G. Wolansky. Existence and nonlinear stability of stationary states of the Schrödinger-Poisson system. J.Statist. Phys. **106** (2002), no. 5-6, 1221–1239.

[13] A. Pazy. Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, Berlin (1983).

[14] M. Reed, B. Simon. Methods of Modern Mathematical Physics II: Fourier Analysis and Self-Adjointness, Academic Press, New York, 1975.

[15] H. Steinrück. The one-dimensional Wigner-Poisson problem and its relation to the Schrödinger-Poisson problem. SIAM J. Math. Anal. **22** (1991), no. 4, 957–972.