Localization in interacting fermionic chains with quasi-random disorder

Vieri Mastropietro

Universitá di Milano, Via C. Saldini 50, 20133, Milano, Italy

We consider a system of fermions with a quasi-random almost-Mathieu disorder interacting through a many-body short range potential. We establish exponential decay of the zero temperature correlations, indicating localization of the interacting ground state, for weak hopping and interaction and almost everywhere in the frequency and phase; this extends the analysis in [17] to chemical potentials outside spectral gaps. The proof is based on Renormalization Group and is inspired by techniques developed to deal with KAM Lindstedt series.

1. INTRODUCTION AND MAIN RESULTS

A. Introduction

It is due to Anderson [1] the discovery that disorder can produce *localization* of independent quantum particles, consisting in the exponential decay from some point of the eigenfunctions of the one-body Schroedinger operator. The mathematical understanding of Anderson localization required the development of powerful techniques and it was finally rigorously established in the case of random [2], [3] and quasi-random (or quasi-periodic) disorder [4],[5], [6],[7].

A natural question is what happens to localization in presence of a many-body interaction, which is always present in real systems. The interplay of disorder and interaction is believed to have deep consequences on the ground state low temperature properties [8], [9], [10] and in the non equilibrium dynamics, like lack of thermalization and memory of initial state [11], [12],[13],[14],[15]. Mathematical results on localization for interacting systems are still very few [16],[17] as the breaking of the single particle description makes the problem genuinely infinite dimensional.

In this paper we consider a system of fermions on a one dimensional lattice with a quasi-random disorder described by a quasi-periodic almost-Mathieu potential $\phi_x = u \cos 2\pi(\omega x +$

θ), ω irrational, and interacting via a short range potential with coupling U. Such model is known as the *interacting Aubry-André* model [14],[18] or the *Heisenberg quasi-periodic spin chain*, and it has been recently experimentally realized in cold atoms experiments [18].

In the absence of interaction the N-particle eigenstates can be constructed from the single particle eigenstates of the Schroedinger energy operator with *almost-Mathieu* potential, for which a rather detailed mathematical knowledge exists; in particular such system shows a metal-insulator transition, with an Anderson localized insulating phase with strong disorder and a metallic extended phase at weak disorder, similar to what happens in a random three dimensional situation. The *exponential* decay of the single particle eigenstates of the *almost-Mathieu* operator, almost everywhere in ω, θ, was proved in [5] and [6] , for ε small enough, ε being the hopping, and later up to ε/u equal to $\frac{1}{2}$ in [7]. In the opposite regime $\varepsilon/u > \frac{1}{2}$ the almost Mathieu has *extended* states [20],[21],[22],[23],[24]; in particular in [20] a Diophantine condition is assumed on the phase excluding values close to $2\theta = \omega k$, k integer, corresponding to gaps [24]. In both regimes and for all irrationals the spectrum is a Cantor set [19]. The non interacting Aubry-André model has ground state correlations with a power law decay for large $\frac{\varepsilon}{u}$ [25], even in presence of interaction [26], and an exponential decay for small $\frac{\varepsilon}{u}$ [27].

The construction of the eigenvectors of the N-body Schroedinger equation with almost-Mathieu potential and interaction seems at the moment out of reach, especially for infinite N; the eigenfunctions cannot be written as product of eigenfunctions of the single particle operator and the problem is genuinely infinite dimensional. Information on the localization of the interacting ground state can be however obtained by the properties of the zero temperature grand-canonical truncated correlations of local operators, whose exponential decay with the distance is a sign of persistence of localization. We use a technique introduced in [17] based on a combination of constructive renormalization Group methods for fermions, see for instance [28] , with KAM techniques for Lindstedt series [29],[30].

Our main results can be informally stated as follow.

Almost everywhere in ω, θ, for small $\frac{\varepsilon}{u}$, $\frac{U}{u}$ the zero temperature grand canonical infinite volume truncated correlations of local operators decays exponentially for large distances

The almost everywhere condition in ω, θ is necessary even in the single particle case [7]; in particular we assume, as usual, a Diophantine property for the frequency ω and for the phase

θ. In [17] localization was proved assuming that $2\theta/\omega$ integer, corresponding to a choice of the chemical potential in one of the infinitely many gaps; here we extend such result to a full measure set of phases, where no gap is present. The result is in agreement with the qualitative phase diagram obtained by numerical simulations in [14] , in which many-body localization is found in the $(U/u, \varepsilon/u)$ plane from the origin up to an almost linear curve intersecting the points $((U/u)^*, 0)$ and $(0, 1/2)$, with $(U/u)^*$ of order 1. Our result establishes localization only for the ground state, but it possible that the method we use can be applied to prove localization of every eigenfunctions of the interacting Schroedinger almost-Mathieu equation.

B. The model

If Λ is a one dimensional lattice $\Lambda = \{x \in \mathbb{Z}, -L/2 \leq x \leq L/2\}$, L even, we introduce fermionic creation and annihilation operators a^+_x, a^-_x, $x \in \Lambda$ on the Fock space verifying $\{a^+_x, a^-_y\} = \delta_{x,y}$, $\{a^+_x, a^+_y\} = \{a^-_x, a^-_y\} = 0$. The Fock space Hamiltonian is

$$H = -\varepsilon(\sum_{x=-L/2}^{L/2-1} a^+_{x+1}a_x + \sum_{-L/2+1}^{L/2} a^+_{x-1}a^-_x) + \sum_{x=-L/2}^{L/2} \phi_x a^+_x a^-_x + U \sum_{x,y=-L/2}^{L/2} v(x-y) a^+_x a^-_x a^+_y a^-_y \quad (1)$$

with $v(x-y) = \delta_{y-x,1} + \delta_{x-y,1}$, and $\phi_x = u\cos(2\pi(\omega x + \theta))$, ω irrational. *We will choose* $u = 1$ *for definiteness.* If $a^{\pm}_{\mathbf{x}} = e^{(H-\mu N)x_0} a^{\pm}_x e^{-(H-\mu N)x_0}$, $\mathbf{x} = (x, x_0)$, $N = \sum_x a^+_x a^-_x$ and μ the chemical potential, the Grand-Canonical imaginary time 2-point correlation is

$$< \mathbf{T} a^-_{\mathbf{x}} a^+_{\mathbf{y}} > |_T = \frac{Tr e^{-\beta(H-\mu N)} \mathbf{T}\{a^-_{\mathbf{x}} a^+_{\mathbf{y}}\}}{Tr e^{-\beta(H-\mu N)}} \quad (2)$$

where $\mathbf{T}$ is the time-order product, T denotes truncation and μ is the chemical potential. In the $\varepsilon = U = 0$ the spectrum is given by $\sum_x \phi_x n_x$ with $n_x = 0, 1$ and the correlations are given by the Wick rule in terms of the fermionic 2-point function $< \mathbf{T} a^-_{\mathbf{x}} a^+_{\mathbf{y}} > |_{U=\varepsilon=0} = g(\mathbf{x}, \mathbf{y})$ with

$$g(\mathbf{x}, \mathbf{y}) = \delta_{x,y} \frac{1}{\beta} \sum_{k_0 = \frac{2\pi}{\beta}(n_0 + \frac{1}{2})} \frac{e^{-ik_0(x_0-y_0)}}{-ik_0 + \cos 2\pi(\omega x + \theta) - \mu} = \delta_{x,y} \bar{g}(x, x_0 - y_0) \quad (3)$$

If $\mu = \cos 2\pi(\omega\widehat{x} + \theta)$, $\widehat{x} \in \Lambda$ the occupation number, defined as $\bar{g}(x, 0^-)$, is at zero temperature $\chi(\cos 2\pi(\omega x + \theta) \leq \mu)$, that is the ground state is obtained by filling all the one particle states with energy $\cos 2\pi(\omega x + \theta)$ up to the level $\cos 2\pi(\omega\widehat{x} + \theta)$.

In the grand canonical ensamble the value of the chemical potential corresponding to a fixed density is a function of the interaction; therefore, if we want to fix the density, what is the more physically natural procedure, one has to properly choose the chemical potential as a function of the interaction. As the 2-point function is singular in correspondence of the chemical potential, this means that the location of the singularity of the 2-point correlation moves varying the interaction; this of course causes problems in a perturbative analysis, resulting in a lack of convergence of a naive power series expansion. It is therefore convenient, both for physical and technical reason, to write the chemical potential as a function of the interaction, and to tune it so that the singularity in the free or interacting case are the same; this corresponds to fix the density to the same value in the free or interacting case. We therefore write $\mu = \cos 2\pi(\omega\widehat{x} + \theta) + \nu$ and we choose properly the counterterm ν as a function of ε, U.

The starting point of the Renormalization Group analysis is the representation of the correlations (2) in terms of *Grassmann integrals.* Let $M \in \mathbb{N}$ and $\bar\chi(t)$ a smooth compact support function that is 1 for $t \leq 1$ and 0 for $t \geq \gamma$, with $\gamma > 1$. Let $\mathcal{D}_\beta = D_\beta \cap \{k_0 : \bar\chi(\gamma^{-M}|k_0|) > 0\}$, where $D_\beta = \{k_0 = \frac{2\pi}{\beta}(n_0 + \frac{1}{2}), n_0 \in \mathbb{Z}\}$. If $x_0 - y_0 \neq n\beta$, we can write

$$g(\mathbf{x}, \mathbf{y}) = \lim_{M\to\infty} \delta_{x,y} \frac{1}{\beta} \sum_{k_0\in\mathcal{D}_\beta} \bar\chi(\gamma^{-M}|k_0|) \frac{e^{-ik_0(x_0-y_0)}}{-ik_0 + \cos 2\pi(\omega x + \theta) - \mu} \equiv \lim_{M\to\infty} g^{(\leq M)}(\mathbf{x}, \mathbf{y}) \quad (4)$$

Because of the jump discontinuities, $g^{(\leq M)}(\mathbf{x}, \mathbf{y})$ is not absolutely convergent but is pointwise convergent and the limit is given by $g(\mathbf{x}, \mathbf{y})$ at the continuity points, while at the discontinuities it is given by the mean of the right and left limits. If $\mathcal{B}_{\beta,L} = \{\Lambda \otimes \mathcal{D}_\beta\}$, we consider the Grassmann algebra generated by the Grassmannian variables $\{\psi^\pm_{x,k_0}\}_{x,k_0\in\mathcal{B}_{\beta,L}}$ and a Grassmann integration $\int \big[\prod_{x,k_0\in\mathcal{B}_{\beta,L}} d\psi^-_{x,k_0} d\psi^+_{x,k_0}\big]$ defined as the linear operator on the Grassmann algebra such that, given a monomial $Q(\psi^-, \psi^+)$ in the variables $\psi^\pm_{x,k_0}$, its action on $Q(\psi^-, \psi^+)$ is 0 except in the case $Q(\psi^-, \psi^+) = \prod_{x,k_0\in\mathcal{B}_{\beta,L}} \psi^-_{x,k_0}\psi^+_{x,k_0}$, up to a permutation of the variables. In this case the value of the integral is determined, by using the anticommuting properties of the variables, by the condition

$$\int \Big[\prod_{x,k_0\in\mathcal{B}_{\beta,L}} d\psi^+_{x,k_0} d\psi^-_{x,k_0}\Big] \prod_{x,k_0\in\mathcal{B}_{\beta,L}} \psi^-_{x,k_0}\psi^+_{x,k_0} = 1 \quad (5)$$

We define also Grassmanian field as $\psi^\pm_{\mathbf{x}} = \frac{1}{\beta}\sum_{k_0\in\mathcal{B}_{\beta,L}} e^{\pm ik_0x_0}\psi^\pm_{x,k_0}$ with $x_0 = m_0\frac{\beta}{\gamma^M}$ and $m_0 \in (0, 1, ..., \gamma^M - 1)$. The "Gaussian Grassmann measure" (also called integration) is defined as

$$P(d\psi)=[\prod_{x,k_0\in\mathcal{B}_{\beta,L}}\beta d\psi^-_{x,k_0}d\psi^+_{x,k_0}\widehat g^{(\le M)}(x,k_0)]\exp\{-\frac{1}{\beta}\sum_{x,k_0}(\widehat g^{(\le M)}(x,k_0))^{-1}\psi^+_{x,k_0}\psi^-_{x,k_0}\} \quad (6)$$

with

$$\widehat g^{(\le M)}(x,k_0)=\frac{\bar\chi(\gamma^{-M}|k_0|)}{-ik_0+\cos 2\pi(\omega x+\theta)-\cos 2\pi(\omega\widehat x+\theta)} \quad (7)$$

We introduce the generating functional $W(\phi)$ defined in terms of the following Grassmann integral (Dirichelet boundary conditions are imposed)

$$e^{W(\phi)}=\int P(d\psi)e^{-\mathcal{V}(\psi)-\mathcal{B}(\psi,\phi)} \quad (8)$$

with

$$\mathcal{V}(\psi)=U\int d\mathbf{x}\sum_{\alpha=\pm}\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}\psi^+_{\mathbf{x}+\alpha\mathbf{e_1}}\psi^-_{\mathbf{x}+\alpha\mathbf{e_1}}+\varepsilon\int d\mathbf{x}(t^1_x\psi^+_{\mathbf{x}+\mathbf{e_1}}\psi^-_{\mathbf{x}}+t^2_x\psi^+_{\mathbf{x}-\mathbf{e_1}}\psi^-_{\mathbf{x}})$$
$$+\nu\int d\mathbf{x}\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}+\int d\mathbf{x}U\nu_C(x)\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}} \quad (9)$$

where $\int d\mathbf{x}=\sum_{x\in\Lambda}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}}dx_0$, $t^1_{L/2}=t^2_{-L/2}=0$ and $t^1_x=t^2_x=1$ otherwise and $\nu_c(x)=U(\widetilde\nu_C(x+1)+\widetilde\nu_C(x-1))$ with $\widetilde\nu_C(x)=\frac{1}{2}[\bar g(x,0^+)-\bar g(x,0^-)]$. Finally

$$\mathcal{B}(\psi,\phi)=\int d\mathbf{x}(\phi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}+\psi^+_{\mathbf{x}}\phi^-_{\mathbf{x}}) \quad (10)$$

The 2-point function is given by

$$S^{L,\beta}_2(\mathbf{x},\mathbf{y})=\frac{\partial^2}{\partial\phi^+_{\mathbf{x}}\partial\phi^-_{\mathbf{y}}}W|_0 \quad (11)$$

It is easy to check, see §1.C of [17], that the expansions in ε,U,ν of (2) and of (11) coincide in the limit $M\to\infty$; note in particular the role of the last term of (9) taking into account the fact $g(\mathbf{x},\mathbf{y})$ and $\lim_{M\to\infty}g^{(\le M)}(\mathbf{x},\mathbf{y})$ coincide everywhere except at coinciding points.

C. Main results

Our main result is the following.

Theorem 1.1 *Let us consider the 2-point function* $S^{L,\beta}_2(\mathbf{x},\mathbf{y})$ *(11) with* $\mu=\cos 2\pi(\omega\widehat x+\theta)$, $\widehat x\in\Lambda$, $\widehat x,\theta$ *non vanishing and assume that, for some* $C_0,\tau>1$

$$||\omega x||\ge C_0|x|^{-\tau},\quad ||\omega x\pm 2\theta||\ge C_0|x|^{-\tau}\quad\forall x\in\mathbb{Z}/\{0\} \quad (12)$$

with $||.||$ *is the norm on the one dimensional torus of period* 1*. There exists an* ε_0 *such that, for* $|\varepsilon|, |U| \leq \varepsilon_0$ *(*$u = 1$*),it is possible to choose a continuous function* $\nu = \nu(\varepsilon, U)$ *so that the limit* $\lim_{\beta\to\infty}\lim_{L\to\infty}\lim_{M\to\infty} S_2^{L,\beta}(\mathbf{x},\mathbf{y}) = S_2(\mathbf{x},\mathbf{y})$ *exists and for any* $N \in \mathbb{N}$

$$|S_2(\mathbf{x},\mathbf{y})| \leq Ce^{-\xi|x-y|}\log(1+\min(|x||y|))^{\tau}\frac{1}{1+(\Delta|x_0-y_0|)^N} \tag{13}$$

with $\Delta = (1+\min(|x|,|y|))^{-\tau}$, $\xi = |\log(\max(|\varepsilon|,|U|))|$.

The theorem says that the ground state correlation decays exponentially for large distances provided that the hopping ε/u and the interaction U/u are small and for a full measure set of frequencies ω and phases θ. The result confirms the phase diagram suggested by numerical experiments [13] and says that Anderson localization persists in presence of interaction, at least in the ground state. The chemical potential μ is chosen of the form $\mu = \cos 2\pi(\omega\widehat{x} + \theta)+\nu$, $\widehat{x} \in \mathbb{N}$, and the counterterm ν is chosen to fix the density to an U,ε-independent value. The Diophantine condition on the frequency (the first of (12)) is the one usually assumed for proving the localization in the almost Mathieu equation, see for instance [6]; the second condition in (12) (similar to the one considered for instance in [20]) excludes values around integer values of $\frac{2\theta}{\omega}$ integer, corresponding to one of the infinitely many gaps in the spectrum. The values $\frac{2\theta}{\omega}$ integer were previously considered in [17] and it was proved that exponential decay holds and (13) is true with Δ replaced by the gap size. The above theorem can be equivalently stated fixing the phase θ and varying the chemical potential; if we choose $\theta = 0$ and $\mu = \cos 2\pi\omega\bar{x}$, $\bar{x} \in \mathbb{R}$, than the theorem says that the two point function decays exponentially for large distances if $\bar{x}$ verify a Diophantine condition $||\omega x \pm 2\omega\bar{x}|| \geq C_1|x|^{-\tau}$, $x \neq 0$, or if $\bar{x}$ is half-integer; the first case corresponds to the chemical potential outside gaps while in the second the chemical potential is in the middle of a gap. The theorem was announced in [31].

D. Feynman Graphs expansion and small divisors

Before starting the proof of Theorem 1.1 it is useful to figure out the main difficulties of the problem, related to the presence of small divisors. Let us consider the *effective potential* defined by

$$e^{-V(\phi)} = \int P(d\psi)e^{-\mathcal{V}(\psi+\phi)} \tag{14}$$

with $\mathcal{V}(\psi)$ given by (9). We can write

$$V(\phi) = -\log \int P(d\psi)\, e^{-\mathcal{V}(\psi+\phi)} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{E}^T(-\mathcal{V}; n) \tag{15}$$

where $\mathcal{E}^T$ are the *fermionic truncated expectations*, that is, if $X(\psi+\phi)$ is a monomial

$$\mathcal{E}^T(X:n) = \frac{\partial^n}{\partial \alpha^n} \log \int P(d\psi) e^{\alpha X(\phi+\Psi)})|_{\alpha=0} \tag{16}$$

By evaluating the truncated expectations by the Wick rule, $V(\phi)$ can be written as sum over *Feynman graphs*.

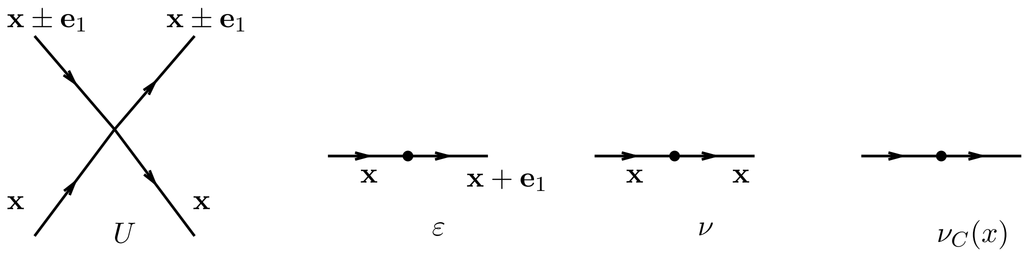

FIG. 1: Graphical representation of the four terms in $\mathcal{V}(\psi)$ eq.(9)

Each graph is obtained taking n elements represented as in Fig.1 and joining (contracting) the lines with consistent orientation so that all the n vertices are connected. Calling ℓ the contracted lines of the graph, if $\mathcal{G}_n$ is the set of all possible Feynman graphs of order n, for any graph $G \in \mathcal{G}_n$ we can associate a *value* Val(G); for instance the graphs not involving the last term in (9) have the value, if $n = n_U + n_\varepsilon + n_\nu$ and $\int d\mathbf{x} = \int dx_0 \sum_x$

$$\mathrm{Val}(G) = (-1)^\pi U^{n_U} \varepsilon^{n_\varepsilon} \nu^{n_\nu} \int d\mathbf{x}_1 ... \int d\mathbf{x}_n \prod_\ell g(\mathbf{x}_\ell, \mathbf{y}_\ell) \prod_{i \in A(G)} \phi^{\sigma_i}_{\mathbf{x}_i} \tag{17}$$

where $A(G)$ is the set of indices of the non contracted lines, ℓ are the contracted lines of the graph and $\mathbf{x}_\ell, \mathbf{y}_\ell$ the coordinates at the edge of the line, and $(-1)^\pi$ is the sign associated to the graph. With the above definitions

$$V(\phi) = \sum_{n=0}^{\infty} \sum_{G \in \mathcal{G}_n} \mathrm{Val}(G) \tag{18}$$

In the non interacting case $U = 0$ the only possible graphs are chain graphs; an example is in Fig. 2.

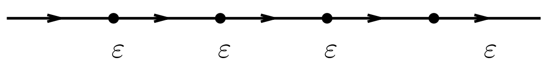

FIG. 2: A graph with $n_\varepsilon = 4$, $n_U = n_\nu = 0$

The value is

$$\varepsilon^{\nu_\rho}\nu^{n_\nu}\int\prod_{1=1}^{n} d\mathbf{x}_i\phi_{\mathbf{x}_1}[\prod_{i=1}^{n}\delta_{x_i+\alpha_i,x_{i+1}}\bar g(x_i+\alpha_i,x_{0,i}-x_{0,i+1})]\phi_{\mathbf{x}_{n+1}}=$$
$$e^{\nu_\rho}\nu^{n_\nu}\sum_{x_1}\int dx_{0,1}...dx_{0,n}\phi_{\mathbf{x}_1}\phi_{x_1+\sum_{i\le n}\alpha_i,x_{0,n}}\prod_{i=1}^{n}\bar g(x_1+\sum_{k\le i}\alpha_k,x_{0,i+1}-x_{0,i})$$

which can be rewritten as

$$\varepsilon^{\nu_\varepsilon}\nu^{n_\nu}\sum_{x_1}\int dk_0\widehat\phi_{x_1,k_0}[\prod_{k=1}^{n}\widehat g(x_1+\sum_{i\le k}\alpha_i,k_0)]\widehat\phi_{x_1+\sum_{i\le k}\alpha_i,k_0}=\varepsilon^{\nu_\varepsilon}\nu^{n_\nu}\sum_{x_1}\int dk_0 H(k_0,x_1) \quad (19)$$

In order to bound $H(k_0,x_1)$ we note that, as the frequency ω is irrational, $(\omega x)_{mod.1}$ fills densely the interval $(-1/2,1/2]$ so that the denominator $\phi_x-\mu$ can be *arbitrarily small*. Let us introduce $\bar x_+ = \widehat x \quad \bar x_- = -\widehat x - 2\theta/\omega$. If we set $x = x'+\bar x_\rho$, $\rho = \pm$, for small $(\omega x')_{\rm mod.1}$ then $\cos 2\pi(\omega(x'+\bar x_\rho)+\theta)-\cos(2\pi(\omega\widehat x+\theta)) = \rho v_0(\omega x')_{\rm mod.1}+r_{\rho,x'}$ with $r_{\rho,x'} = O(((\omega x')_{\rm mod.1})^2)$, $v_0 = \sin 2\pi(\omega\widehat x+\theta)$, so that, for small $(\omega x')_{\rm mod.1}$

$$\widehat g(x'+\bar x_\rho,k_0)\sim\frac{1}{-ik_0\pm v_0(\omega x')_{\rm mod.1}} \quad (20)$$

Note that, for $x\neq\rho\widehat x$

$$||\omega x'|| = ||\omega(x-\rho\widehat x)+2\delta_{\rho,-1}\theta||\ge C|x-\rho\widehat x|^{-\tau} \quad (21)$$

by (12). Therefore the sum of all the chain graphs of order n is bounded by $\varepsilon^n C^n||\widehat x|+|n||^{\tau n}$, a bound which does not imply convergence.

In the case of the interacting theory the graphs are much more complex and loops are present; an example is Fig. 3 whose value is the following

$$\varepsilon^4 U^2\sum_x\int dx_{0,1}...dx_{0,6}\phi_x\bar g(x;x_{0,1}-x_{0,2})g(x+1,x_{0,2}-x_{0,3})\bar g(x;x_{0,3}-x_{0,4}) \quad (22)$$
$$\bar g(x+1;x_{0,4}-x_{0,5})\bar g(x+1;x_{0,1}-x_{0,5})\bar g(x+1;x_{0,1}-x_{0,6})\bar g(x+2;x_{0,6}-x_{0,5})\phi_{x+2,x_{5,0}}$$

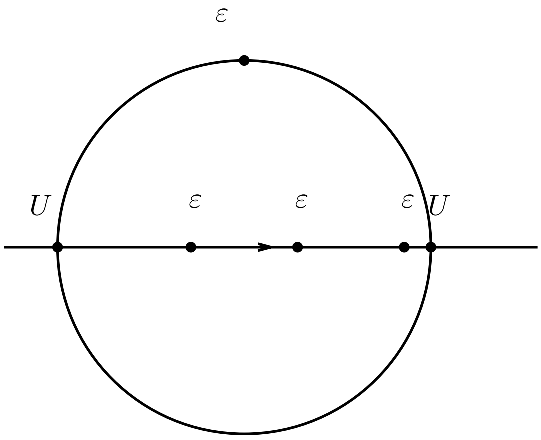

FIG. 3: A graph with $n_U = 2, n_\varepsilon = 4$

Note that, after summing over the coordinates and exploiting the kronecker deltas of the propagators connecting the vertices, only a single sum over x remains. Again each graph does not admit a bound which allow us to sum over n; in addition there is the problem that the number of graphs with loops is $O(n!^2)$ (the number of chain graphs is $O(n!)$ instead). Note that the small divisor problem of the non interacting theory is very similar to the one appearing in KAM theory; for instance the Lindstedt series can be represented in terms of graphs with no loops very similar to (19), see [29]. On the contrary the appearance of graphs with loops plagued by small divisors like (23) is the peculiar feature of localization in a many body theory.

2. PROOF OF THEOREM 1.1

A. Multiscale integration and Renormalization Group analysis

We start describing the integration of the generating function in the case $\phi = 0$ (the partition function). We introduce a function $\chi_h(t, k_0) \in C^\infty(\mathbb{T} \times \mathbb{R})$, such that $\chi_h(t, k_0) = \chi_h(-t, -k_0)$ and $\chi_h(t, k_0) = 1$, if $\sqrt{k_0^2 + v_0^2||t||_1^2} \leq a\gamma^{h-1}$ and $\chi_h(t, k_0) = 0$ if $\sqrt{k_0^2 + v_0^2||t||_1^2} \geq a\gamma^h$ with a and $\gamma > 1$ suitable constants. We define $\bar{x}_+ = \widehat{x} \quad \bar{x}_- = -\widehat{x} - 2\theta/\omega$ and we choose a so that the supports of $\chi_0(\omega(x - \widehat{x}_+), k_0)$ and $\chi_0(\omega(x - \widehat{x}_-), k_0)$ are disjoint; we also define

$\chi^{(1)}(\omega x, k_0) = 1 - \chi_0(\omega(x-\bar{x}_+), k_0) - \chi_0(\omega(x-\bar{x}_-), k_0)$. For reasons which will appear clear below, see Lemma 2.4, we choose $\gamma > 2^{\frac{1}{\tau}}$. We can write then

$$g(\mathbf{x},\mathbf{y}) = g^{(1)}(\mathbf{x},\mathbf{y}) + g^{(\leq 0)}(\mathbf{x},\mathbf{y}) \tag{23}$$

and

$$g^{(\leq 0)}(\mathbf{x},\mathbf{y}) = \sum_{\rho=\pm} g^{(\leq 0)}_\rho(\mathbf{x},\mathbf{y}) \tag{24}$$

where, for M large enough

$$\begin{aligned} g^{(1)}(\mathbf{x},\mathbf{y}) &= \frac{\delta_{x,y}}{\beta}\sum_{k_0\in D_\beta}\chi^{(1)}(\omega x,k_0)\bar{\chi}(\gamma^{-M}|k_0|)\frac{e^{-ik_0(x_0-y_0)}}{-ik_0+\cos 2\pi(\omega x+\theta)-\cos 2\pi(\omega\widehat{x}+\theta)} \\ g^{(\leq 0)}_\rho(\mathbf{x},\mathbf{y}) &= \frac{\delta_{x,y}}{\beta}\sum_{k_0\in D_\beta}\chi_0(\omega(x-\bar{x}_\rho),k_0)\frac{e^{-ik_0(x_0-y_0)}}{-ik_0+\cos 2\pi(\omega x+\theta)-\cos 2\pi(\omega\widehat{x}+\theta)} \end{aligned} \tag{25}$$

We use the following property; if $P_g(d\psi)$ is a Gaussian Grassmann integration with propagator g and $g = g_1 + g_2$, then $P_g(d\psi) = P_{g_1}(d\psi_1)P_{g_2}(d\psi_2)$, in the sense that for every polynomial f

$$\int P_g(d\psi)f(\psi) = \int P_{g_1}(d\psi_1)\int P_{g_2}(d\psi_2)f(\psi_1+\psi_2)\ . \tag{26}$$

By using such property

$$e^{W(0)} = \int P(d\psi)e^{-\mathcal{V}(\psi)} = \int P(d\psi^{(\leq 0)})\int P(d\psi^{(1)})e^{-\mathcal{V}(\psi^{(\leq 0)}+\psi^{(1)})} \tag{27}$$

where $P(d\psi^{(1)})$ and $P(d\psi^{(\leq 0})$ are gaussian Grassmann integrations with propagators respectively $g^{(1)}(\mathbf{x},\mathbf{y})$ and $g^{(\leq 0)}(\mathbf{x},\mathbf{y})$ and $\psi^{(1)}$ and $\psi^{(\leq 0)}$ are independent Grassmann variables. We can write

$$\int P(d\psi^{(1)})e^{-\mathcal{V}(\psi^{(\leq 0)}+\psi^{(1)})} = e^{\sum_{n=0}^\infty \frac{(-1)^n}{n!}\mathcal{E}_1^T(\mathcal{V};n)} \equiv e^{-\beta L E_0 - \mathcal{V}^{(0)}(\psi^{(\leq 0)})} \tag{28}$$

where $\mathcal{E}_1^T$ is the fermionic truncated expectation with respect to $P(d\psi^{(1)})$. By the above definition

$$\mathcal{V}^{(0)} = \sum_{n=1}^\infty\sum_{x_1}\int dx_{0,1}....\sum_{x_n}\int dx_{0,n}W_n^{(h)}(\mathbf{x}_1,...,\mathbf{x}_n)[\prod_{i=1}^n \psi^{(\varepsilon_i)(\leq 0)}_{\mathbf{x}'_i,\rho_i}] \tag{29}$$

with $\mathbf{x} = \mathbf{x}' + \bar{\mathbf{x}}_\rho$, $\bar{\mathbf{x}}_\rho = (\bar{x}_\rho, 0)$ and E_0 is a constant; moreover

$$e^{W(0)} = e^{-\beta L E_0}\int P(d\psi^{(\leq 0)})e^{-\mathcal{V}^{(0)}(\psi^{(\leq 0})} \tag{30}$$

Note that the kernel $W_n^{(h)}(\mathbf{x}_1,...,\mathbf{x}_n)$ contains in general Kronecker or Dirac deltas, and we define the L_1 norm as they would be positive functions, *e.g.* if $W(\mathbf{x}_1,\mathbf{x}_2,..\mathbf{x}_n) = \delta(\sum_j \eta_j \mathbf{x}_j)\bar W(\mathbf{x}_1,..,\mathbf{x}_n)$ then $|W|_{L_1} = \int d\mathbf{x}_1..d\mathbf{x}_n \delta(\sum_j \eta_j \mathbf{x}_j)|\bar W(\mathbf{x}_1,..,\mathbf{x}_n)|$. It was proved in Lemma 2.1 [17] that the constant E_0 and the kernels $W_n^{(0)}$ are given by power series in U,ε,ν convergent for $|U|,|\varepsilon|,|\nu|\le\varepsilon_0$, for ε_0 small enough and independent of β, L. They satisfy the following bounds:

$$|W_n^{(0)}|_{L_1}\le L\beta C^n\varepsilon_0^{k_n}\ , \tag{31}$$

for some constant $C>0$ and $k_n=\max\{1,n-1\}$. Moreover the limit $M\to\infty$ exists and is reached uniformly.

We describe the integration of $\psi^{(\le 0)}$ inductively. Assume that we have integrated the fields $\psi^{(0)}...\psi^{(h+1)}$ obtaining

$$e^{-\beta L E_0}\int P(d\psi^{(\le 0)})e^{-\mathcal{V}^{(0)}(\psi^{(\le 0)})} = e^{-\beta L E_h}\int P(d\psi^{(\le h)})e^{-\mathcal{V}^{(h)}(\psi^{(\le h)})} \tag{32}$$

where $P(d\psi^{(\le h)})$ is the gaussian grassman integration with propagator, $\rho=\pm$

$$g_\rho^{(\le h)}(\mathbf{x}',\mathbf{y}') = \delta_{x',y'}\bar g_\rho^{(\le h)}(x',x_0-y_0) \tag{33}$$

with, if $x=x'+\bar x_\rho$

$$g_\rho^{(\le h)}(x',x_0-y_0) = \int dk_0 e^{-ik_0(x_0-y_0)}\chi_h(\omega x',k_0)\frac{1}{-ik_0+v_0\rho(\omega x')_{\text{mod}.1}+r_{\rho,x'}} \tag{34}$$

and the corresponding fields are denoted by $\psi^{(\varepsilon,\le h)}_{\mathbf{x}',\rho}$. The effective potential $\mathcal{V}^{(h)}$ can be written as sum of terms (see §2 B below) of the form

$$\sum_{x_1'}\int dx_{0,1}....\int dx_{0,n}H^{(h)}_{n;\rho_1,..,\rho_n}(x_1';x_{0,1},.,x_{0,n})[\prod_{i=1}^n\psi^{\varepsilon_i(\le h)}_{\mathbf{x}_i',\rho_i}] \tag{35}$$

and x_i' are functions of x_1. There is an important constraint on the ρ indices; if $x_i'=x_j'$ then $\rho_i=\rho_j$. This follows from the second of (12), implying that $\frac{2\theta}{\omega}\notin\mathbb{Z}/\{0\}$; indeed as $x_i-x_j=M\in\mathbb{Z}$ and $x_i'=x_j'$ then $(\bar x_{\rho_i}-\bar x_{\rho_j})+M=0$, so that $\rho_i=\rho_j$ as $\bar x_+=\widehat x$ and $\bar x_-=-\widehat x-2\theta/\omega$ and $\widehat x\in\mathbb{Z}$.

We call *resonances* the contribution to $\mathcal{V}^{(h)}$ of the form (35) such that $x_i'=x_1'\equiv x'$ for any $i=1,..,n$; in the resonances $\rho_i'=\rho_1'$ for any $i=1,..,n$.

In order to perform the integration of the field $\psi^{(h)}$ we have to split the effective potential as $\mathcal{V}^{(h)}=\mathcal{L}\mathcal{V}^{(h)}+\mathcal{R}\mathcal{V}^{(h)}$ where $\mathcal{R}=1-\mathcal{L}$ and $\mathcal{R}$ is defined in the following way.

1. If $n=2$ then $\mathcal{R}=1$ if (35) is non resonant, while if (35) is resonant

$$\mathcal{R}\sum_{x'}\int dx_{0,1}dx_{0,2}H^{(h)}_{2;\rho,\rho}(x';x_{0,1},x_{0,2})\psi^{+(\le h)}_{x',x_{0,1},\rho}\psi^{-(\le h)}_{x',x_{0,2},\rho} \tag{36}$$
$$=\sum_{x'}\int dx_{0,1}dx_{0,2}\{H^{(h)}_{2;\rho,\rho}(x';x_{0,1},x_{0,2})\psi^{+(\le h)}_{x',x_{0,1},\rho}\psi^{-(\le h)}_{x',x_{0,2},\rho}-H^{(h)}_{2;\rho,\rho}(0;x_{0,1},x_{0,2})\psi^{+(\le h)}_{x',x_{0,1},\rho}\psi^{-(\le h)}_{x',x_{0,1},\rho}\}$$

2. If $n=4$ $\mathcal{R}=1$ if (35) is non resonant, while if (35) is resonant

$$\mathcal{R}\sum_{x'}\int\prod_{i=1}^4 dx_{0,i}H^{(h)}_{4;\rho,\rho,\rho,\rho}(x';x_{0,1},x_{0,2},x_{0,3},x_{0,4})\psi^{+(\le h)}_{x',x_{0,1},\rho}\psi^{+(\le h)}_{x',x_{0,2},\rho}\psi^{-(\le h)}_{x',x_{0,3},\rho}\psi^{-(\le h)}_{x',x_{0,4},\rho}= \tag{37}$$
$$\sum_{x'}\int\prod_{i=1}^4 dx_{0,i}H^{(h)}_{4;\rho,\rho,\rho,\rho}(x';x_{0,1},x_{0,2},x_{0,3},x_{0,4})D^{+(\le h)}_{x',x_{0,1},x_{0,2}\rho}\psi^{+(\le h)}_{x',x_{0,2},\rho}D^{-(\le h)}_{x',x_{0,3},x_{0,4}\rho}\psi^{-(\le h)}_{x',x_{0,4},\rho}$$

where

$$D^{\pm(\le h)}_{x',x_{0,1},x_{0,2},\rho}=\psi^{\pm(\le h)}_{x',x_{0,1},\rho}-\psi^{\pm(\le h)}_{x',x_{0,2},\rho} \tag{38}$$

That is, the $\mathcal{R}$ operation simply consists in replacing the fields $\psi^{\pm(\le h)}_{x',x_{0,i},\rho}\psi^{\pm(\le h)}_{x',x_{0,j},\rho}$ with $D^{\pm(\le h)}_{x',x_{0,i},x_{0,j},\rho}\psi^{\pm(\le h)}_{x',x_{0,j},\rho}$

3. If $n\ge 6$, the $\mathcal{R}$ operation consists in replacing any monomial of fields with the same x,ε in (35), that is $\psi^{\varepsilon(\le h)}_{x',x_{0,1},\rho}\prod_i\psi^{\varepsilon(\le h)}_{x',x_{0,i},\rho}$, with

$$\psi^{\varepsilon(\le h)}_{x',x_{0,1},\rho}\prod_i D^{\varepsilon(\le h)}_{x',x_{0,1},x_{0,i},\rho} \tag{39}$$

.

By the above definitions

$$\mathcal{L}\mathcal{V}^{(h)}=\gamma^h\nu_h\sum_\rho\sum_{x'}\int dx_0\psi^{+(\le h)}_{\mathbf{x}',\rho}\psi^{-(\le h)}_{\mathbf{x}',\rho} \tag{40}$$

The ν_h coefficients are *independent* from ρ and real, as (8) is invariant under parity $x\to -x$, $\alpha\to-\alpha$ (in the limit $L\to\infty$), and this implies invariance under the transformation $\psi^{\pm(h)}_{x_0,x',\rho}\to\psi^{\pm(h)}_{x_0,-x',-\rho}$; therefore, if $\varepsilon=\pm$

$$\sum_{\rho,x'}\int dx_0dy_0H^{(h)}_{2,\rho}(x',x_0,y_0)\psi^{+(\le h)}_{x',x_0,\rho}\psi^{+(\le h)}_{x',x_0,\rho}=\sum_{\rho,x'}\int dx_0dy_0H^{(h)}_{2,-\rho,}(-x',x_0,y_0)\psi^{+(\le h)}_{x',x_0,\rho}\psi^{+(\le h)}_{x',x_0,\rho} \tag{41}$$

so that the independence from ρ of ν_h follows. Moreover $(g^{(k)})^*(x,k_0)=g^{(k)}(x,-k_0)$ so that $(\widehat{H}^{(h)}_{2,\rho}(x',k_0))^*=\widehat{H}^{(h)}_{2,\rho}(x',-k_0)$, and this implies reality. In writing (40) we have also used that $\psi^{\pm}_{x',x_{0,1},\rho}\psi^{\pm}_{x',x_{0,1},\rho}=0$, so that there is no contribution from non bilinear terms . With the above definitions we finally write (32) as

$$\int P(d\psi^{(\le h-1)})\int P(d\psi^{(h)})e^{-\mathcal{L}\mathcal{V}^{(h)}-\mathcal{R}\mathcal{V}^{(h)}}=e^{-\beta L\widetilde{E}_h}\int P(d\psi^{(\le h-1)})e^{-\mathcal{V}^{(h-1)}(\psi^{(\le h-1)})} \tag{42}$$

where $P(d\psi^{(\le h-1)})$ have propagator $g^{(\le h-1)}$ coinciding with (34) with $h-1$ replacing h, and $P(d\psi^{(h)})$ has propagator $g^{(h)}$ coinciding with $g^{(\le h-1)}$ with χ_{h-1} replaced by $f_h=\chi_h-\chi_{h-1}$, with f_h a smooth compact support function vanishing for $c_1\gamma^{h-1}\le\sqrt{k_0^2+v_0^2||\omega x'||_1^2}\le c_2\gamma^{h+1}$, for a suitable constants c_1,c_2. From the r.h.s. of (42), the procedure can be iterated. The single scale propagator $g^{(h)}$ verifies the following bound, for any integer N and a suitable constant C_N

$$|\bar{g}^{(h)}_\rho(x',x_0-y_0)|\le\frac{C_N}{1+(\gamma^h|x_0-y_0|)^N} \tag{43}$$

which can be easily obtained integrating by parts.

The above procedure allows to write the $W(0)$ (27) in terms of an expansion in the *running coupling constants* ν_k, with $k\le 0$; as it is clear from the above construction, they verify a recursive equation of the form

$$\nu_{h-1}=\gamma\nu_h+\beta^{(h)}(\nu_h,..\nu_0;\varepsilon;U) \tag{44}$$

We will describe more explicitly such expansion in the following section.

B. Tree expansion

The effective potential $\mathcal{V}^{(h)}$ can be written as sum over Gallavotti *trees*, defined in the following way. Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $n\ge 1$ points, the *endpoints* of the *unlabeled tree*, so that r is not a branching point. n will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order, and their number is bounded by 4^n. We shall also consider the set $\mathcal{T}_{h,n}$ of the *labeled trees* with n endpoints (to be called simply trees in the following); they are defined by associating some labels with the unlabeled trees. In particular, we associate a label $h\le 0$ with the root. Moreover, we introduce a family of vertical lines, labeled by an integer taking values in $[h,2]$, and we represent any tree $\tau\in\mathcal{T}_{h,n}$ so that, if v is an endpoint

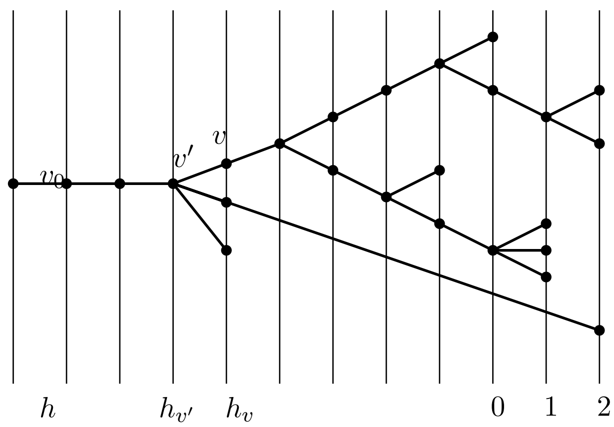

FIG. 4: A tree $\tau \in \mathcal{T}_{h,n}$ with its scale labels.

or a non trivial vertex, it is contained in a vertical line with index $h_v > h$, to be called the *scale* of v, while the root r is on the line with index h. In general, the tree will intersect the vertical lines in set of points different from the root, the endpoints and the branching points; these points will be called *trivial vertices*. Every vertex v of a tree will be associated to its scale label h_v, defined, as above, as the label of the vertical line whom v belongs to. Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$.

There is only one vertex immediately following the root, which will be denoted v_0; its scale is $h+1$. Given a vertex v of $\tau \in \mathcal{T}_{h,n}$ that is not an endpoint, we can consider the subtrees of τ with root v, which correspond to the connected components of the restriction of τ to the vertices $w \geq v$; the number of endpoint of these subtrees will be called N_v. If a subtree with root v contains only v and one endpoint on scale $h_v + 1$, it will be called a *trivial subtree*. With each endpoint v of scale $h_v \leq 1$ we associate $\mathcal{L}\mathcal{V}^{(h_v - 1)}$, and there is the constrain that $h_v = h_{v'} + 1$, if v' is the non trivial vertex immediately preceding it or v_0; to the end-points of scale $h_v = 2$ are associated one of the terms contributing to $\mathcal{V}$ and there is not such a constrain. The set of field labels associated with the endpoint v will be called I_v; if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v. Finally with each trivial or non trivial vertex $v > v_0$, $h_v \leq 0$, which is not an endpoint, we associate the $\mathcal{R} = 1 - \mathcal{L}$ operator, acting on the corresponding kernel.

If $h \le -1$ the effective potential can be written in the following way:

$$\mathcal{V}^{(h)}(\psi^{(\le h)}) + L\beta E_{h+1} = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} V^{(h)}(\tau, \psi^{(\le h)}) \tag{45}$$

where, if v_0 is the first vertex of τ and $\tau_1, .., \tau_s$ $(s = s_{v_0})$ are the subtrees of τ with root v_0, $V^{(h)}(\tau, \psi^{(\le h)})$ is defined inductively by the relation, if $s > 1$

$$V^{(h)}(\tau, \psi^{(\le h)}) = \frac{(-1)^{s+1}}{s!} \mathcal{E}^T_{h+1}[\bar{V}^{(h+1)}(\tau_1, \psi^{(\le h+1)}); ..; \bar{V}^{(h+1)}(\tau_s, \psi^{(\le h+1)})] \tag{46}$$

where $\bar{V}^{(h+1)}(\tau_i, \psi^{(\le h+1)})$:

1. it is equal to $\mathcal{R}\mathcal{V}^{(h+1)}(\tau_i, \psi^{(\le h+1)})$, with $\mathcal{R}$ given by (36),(37),(39) if the subtree τ_i is non trivial;
2. if τ_i is trivial, it is equal to $\mathcal{L}\mathcal{V}^{(h+1)}$.

Starting from the above inductive definition, the effective potential can be written in a more explicit way. We associate with any vertex v of the tree a subset P_v of I_v, the *external fields* of v, and the set $\mathbf{x}_v$ of all space-time points associated with one of the end-points following v. The subsets P_v must satisfy various constraints. First of all, $|P_v| \ge 2$, if $v > v_0$; moreover, if v is not an endpoint and $v_1, \ldots, v_{S_v}$ are the $S_v \ge 1$ vertices immediately following it, then $P_v \subseteq \cup_i P_{v_i}$; if v is an endpoint, $P_v = I_v$. If v is not an endpoint, we shall denote by Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The union $\mathcal{I}_v$ of the subsets $P_{v_i} \setminus Q_{v_i}$ is, by definition, the set of the *internal fields* of v, and is non empty if $S_v > 1$. Given $\tau \in \mathcal{T}_{h,n}$, there are many possible choices of the subsets P_v, $v \in \tau$, compatible with all the constraints. We shall denote $\mathcal{P}_\tau$ the family of all these choices and $\mathbf{P}$ the elements of $\mathcal{P}_\tau$. With these definitions, we can rewrite $\mathcal{V}^{(h)}(\tau, \psi^{(\le h)})$ as

$$\mathcal{V}^{(h)}(\tau, \psi^{(\le h)}) = \sum_{\mathbf{P} \in \mathcal{P}_\tau} \mathcal{V}^{(h)}(\tau, \mathbf{P}) \quad \bar{\mathcal{V}}^{(h)}(\tau, \mathbf{P}) = \int d\mathbf{x}_{v_0} \widetilde{\psi}^{(\le h)}(P_{v_0}) K^{(h+1)}_{\tau,\mathbf{P}}(\mathbf{x}_{v_0}) \, , \tag{47}$$

where $K^{(h+1)}_{\tau,\mathbf{P}}(\mathbf{x}_{v_0})$ is defined inductively and $\widetilde{\psi}^{(h_v)}(P_v) = \prod_{f \in P_v} \psi^{\varepsilon(f)(h_v)}_{\mathbf{x}'(f),\rho(f)}$.

Given a tree τ and $\mathbf{P} \in \mathcal{P}_\tau$, we shall define the χ-vertices are the vertices v of τ , such that $\mathcal{I}_v$ (the union of the subsets $P_{v_i} \setminus Q_{v_i}$ defined before (47), that is the set of lines contracted in v) is non empty; note that $|V_\chi|$ is smaller than $4n$. We call $\bar{v}'$ is the first vertex $\in V_\chi$ following v. The tree structure provides an arrangement of endpoints into a hierarchy of *clusters*, see Fig.5. Given a cluster with scale h_v, one can imagine that the

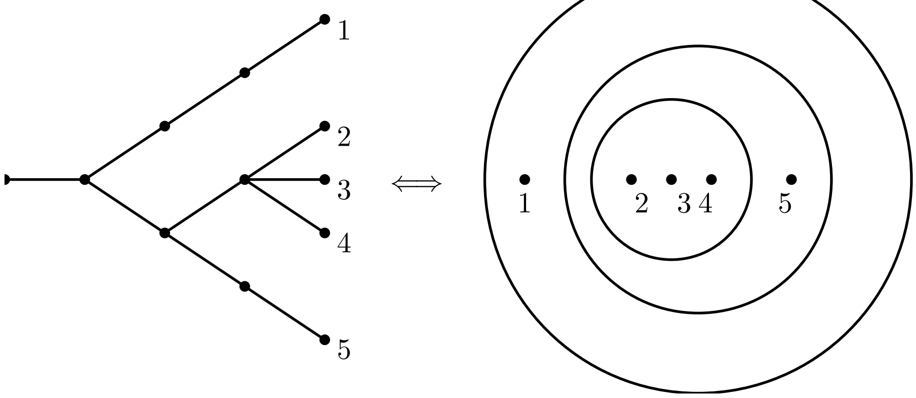

FIG. 5: A tree of order 5 and the corresponding clusters. Only the vertices $v \in V_\chi$ are represented.

fields $\widetilde{\psi}^{(h_v)}(P_{v_1} \setminus Q_{v_1})$,..,$\widetilde{\psi}^{(h_v)}(P_{v_{S_v}} \setminus Q_{v_{S_v}})$ are external to the S_v inner clusters, and the $\mathcal{E}^T_{h_v}$ operation contracts them in pairs.

In order to get the final form of our expansion, we need a convenient representation for the truncated expectation. Let us put $P_i := P_{v_i} \setminus Q_{v_i}$; moreover we order in an arbitrary way the sets $P^{\pm}_{v_i} := \{f \in P_{v_i}, \varepsilon(f) = \pm\}$, we call $f^{\pm}_{ij}$ their elements and we define $\mathbf{x}^{(i)} = \cup_{f\in P_i^-}\mathbf{x}(f)$, $\mathbf{y}^{(i)} = \cup_{f\in P_i^+}\mathbf{y}(f)$, $\mathbf{x}_{ij} = \mathbf{x}(f^-_{ij})$, $\mathbf{y}_{ij} = \mathbf{x}(f^+_{ij})$. A couple $l := (f^-_{ij}, f^+_{i'j'}) := (f^-_l, f^+_l)$ will be called a line joining the fields with labels $f^-_{ij}, f^+_{i'j'}$. Then, we use the *Brydges-Battle-Federbush* formula saying that , if $S_v > 1$,

$$\mathcal{E}^T_{h_v}(\widetilde{\psi}^{(h_v)}(P_i), \cdots, \widetilde{\psi}^{(h_v)}(P_{S_v}))) = \sum_{T_v} \prod_{l\in T_v} \left[\delta_{x_l,y_l} \bar{g}^{(h_v)}_{\rho l}(x'_l, x_{0,l} - y_{0,l})\right] \int dP_T(\mathbf{t}) \det G^{h_v,T}(\mathbf{t}) \;, \tag{48}$$

where T_v is a set of lines forming an *anchored tree graph* between the clusters of points $\mathbf{x}^{(i)} \cup \mathbf{y}^{(i)}$, see Fig.6, that is T_v is a set of lines, which becomes a tree graph if one identifies all the points in the same cluster. Moreover $\mathbf{t} = \{t_{ii'} \in [0,1], 1 \le i,i' \le S_v\}$, $dP_{T_v}(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{ii'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i \in \mathbb{R}^{S_v}$ of unit norm.

$$G^{h_v,T}_{ij,i'j'} = t_{ii'}\delta_{x_{ij},y_{i'j'}}\bar{g}^{(h_v)}_{\rho_{ij}}(x_{ij}, x_{0,ij} - y_{0,i'j'}) \;, \tag{49}$$

with $(f^-_{ij}, f^+_{i'j'})$ not belonging to T_v.

We define $\bar{T}_v = \bigcup_{w\ge v} T_w$ starting from T_v and attaching to it the trees $T_{v_1},..,T_{v_{S_v}}$ associated to the vertices $v_1,..,v_{S_v}$ following v, and repeating this operation until the end-points are reached. The tree $\bar{T}_v$ is composed by a set of lines, representing propagators with scale

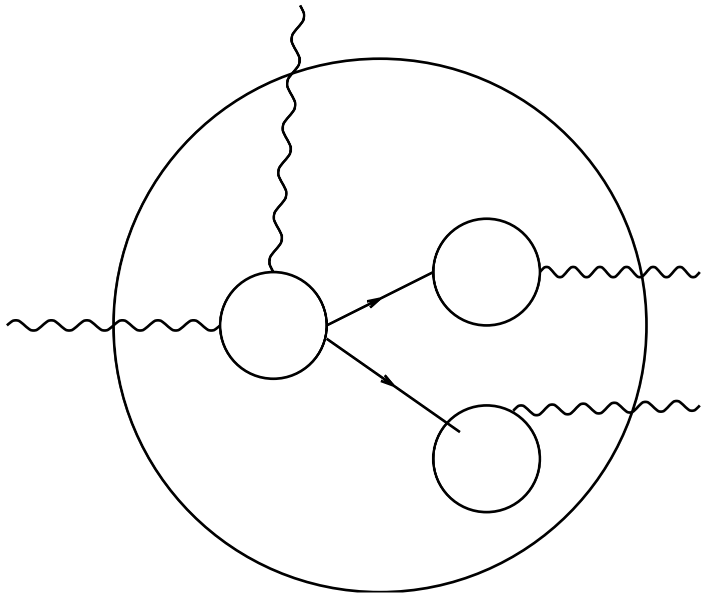

FIG. 6: A symbolic representation of a contribution to (48); the solid lines represent the propagators $g^{(h_v)}$ in T_v connecting the $S_v = 3$ clusters, represented as circles, the wiggly lines are the external fields $\widetilde{\psi}(P_v)$; the fields in the determinant are not represented

$\geq h_v$, connecting end-points w of the tree τ. Note that, contrary to T_v, the vertices of $\bar{T}_v$ are connected with at most four lines. By writing the truncated expectations as in (48) we write $\mathcal{V}^{(h)}$ as sum over T_v, for any v; by summing the Kronecker deltas in the propagators belonging to T_v the coordinate x' of the external fields $\widetilde{\psi}(P_v)$ are determined according to the following rule. To each line coming in or out w is associated a factor $\delta_w^{i_w}$, where i_w is a label identifying the lines connected to w. The vertices w (which correspond to the end-points of τ) can be of type U, ν or ν_h, and a) $\delta_w^i = 0$ if w corresponds to a ν or ν_h end-point; b) $\delta_w^i = \pm 1$ if it corresponds to an ε end-point; c) $\delta_w^i = (0, \pm 1)$ if it corresponds to a U end-point.

According to the above definitions, consider two vertices w_1, w_2 in $\bar{T}_v$ such that x'_{w_1} and x'_{w_2} are coordinates of the external fields, and let be c_{w_1,w_2} the path (vertices and lines) in $\bar{T}_v$ connecting w_1 with w_2 (in the example in Fig. 7 the path is composed by w_1, w_a, w_b, w_c, w_2 and the corresponding lines) ; as the path is a linear tree there is a natural orientation in the vertices, and we we call i_w the label of the line exiting from w in c_{w_1,w_2}. We call $|c_{w_1,w_2}|$

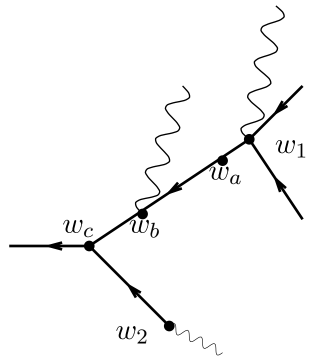

FIG. 7: A tree $\bar{T}_v$ with attached wiggly lines representing the external lines P_v; the lines represent propagators with scale $\geq h_v$ connecting w_1, w_a, w_b, w_c, w_2, representing the end-points following v in τ.

the number of vertices in c_{w_1,w_2}. The following relation holds

$$x'_{w_1} - x'_{w_2} = (\bar{x}_{\rho_{\ell_{w_2}}} - \bar{x}_{\rho_{\ell_{w_1}}}) + \sum_{w \in c_{w_1,w_2}} \delta_w^{i_w} \tag{50}$$

This implies, in particular, that the coordinates of the external fields $\widetilde{\psi}(P_{v_0})$ are determined once that the choice of a single one of them and of $\tau, \bar{T}_{v_0}$ and $\mathbf{P}$ is done. If, using (50), the coordinates x' of the fields $\widetilde{\psi}(P_v)$ are the *same* we say that v is a *resonant vertex*, while if the coordinates are different is called *non resonant vertex*; the set of resonant vertices in V_χ is denoted by H_χ and the set of non-resonant vertices is denoted by L_χ. If $v_1, \ldots, v_{S_v}$ are the $S_v \geq 1$ vertices following the vertex v, we define

$$S_v = S_v^L + S_v^H + S_v^2 \tag{51}$$

where S_v^L is the number of *non resonant* vertices following v, S_v^H is the number of *resonant* vertices following v, while S_v^2 is the number of trivial trees with root v associated to end-points.

C. Renormalization

In order to get the final form of our expansion we need to write more explicitly the action of the renormalization operation $\mathcal{R}$; we can write the r.h.s. of (36) as

$$\int dx_{0,1}dx_{0,2}\{H^{(h)}_{2;\rho,\rho}(x';x_{0,1},x_{0,2})-H^{(h)}_{2;\rho,\rho}(0;x_{0,1},x_{0,2})\}\psi^{+(\leq h)}_{x',x_{0,1},\rho}\psi^{-(\leq h)}_{x',x_{0,2},\rho}+$$
$$H^{(h)}_{2;\rho,\rho}(0;x_{0,1},x_{0,2})D^{+(\leq h)}_{x',x_{0,1},x_{0,2}\rho}\psi^{-(\leq h)}_{x',x_{0,1},\rho}\} \tag{52}$$

The second term in (52) consists in replacing the ψ fields with D-fields; the same effect is produced by the $\mathcal{R}$ operation in (37), (39). The propagators associated to the D fields are

$$\bar g^{(h)}(x',x_{0,1}-z_0)-\bar g^{(h)}(x',x_{0,i}-z_0) \tag{53}$$

which can be conveniently rewritten as

$$(x_{0,1}-x_{0,i})\int_0^1 dt\partial\bar g^{(h)}(x',\widehat{x}_{0,1i}(t)-z_0) \tag{54}$$

where $\widehat{x}_{0,1i}(t)=x_{0,1}+t(x_{0,i}-x_{0,1})$ is an interpolated point between $x_{0,1}$ and $x_{0,2}$. Note that the "zero" factor $(x_{0,1}-x_{0,i})$, produces an extra γ^{-k} in the bounds and the extra derivative produces an extra γ^h; the final factor is γ^{h-k}. The difference $H^{(h)}_{2;\rho,\rho}(x';x_{0,1},x_{0,2})-H^{(h)}_{2;\rho,\rho}(0;x_{0,1},x_{0,2})$ in (52) can be written as a sum of terms in which a propagator $\bar g^{(k)}(x'+y;z_0)$ is replaced by

$$\bar g^{(k)}(x'+y;z_0)-\bar g^{(k)}(y;z_0) \tag{55}$$

which can be rewritten as

$$\bar g^{(k)}(x'+y,z_0)-g^{(k)}(y,z_0)= \tag{56}$$
$$(\omega x')\int dk_0e^{-ik_0z_0}\int_0^1\frac{\partial}{\partial t\omega x'}\frac{f_h(\omega y+t\omega x',k_0)}{-ik_0+\cos 2\pi(\omega y+\omega\bar x_\rho+t\omega x'+\theta)-\cos 2\pi(\omega\bar x_\rho+\theta)}$$

Note that $(\omega x')\sim\gamma^h$ for the compact support properties of the propagators associated to $\psi^{\leq h}$, while the derivative produces an extra γ^{-k}; therefore the final effect is again to produce an extra γ^{h-k} factor in the bounds.

D. Renormalized Graphs expansion

We can write the truncated expectations in terms of the Wick rule, and this leads to a representation of the effective potential in terms of renormalized Feynman graphs

$$\mathcal{V}^{(h)}(\psi^{(\leq h)})=\sum_{n=1}^\infty\sum_{\tau\in\mathcal{T}_{h,n}}\sum_{G\in\mathcal{G}(\tau)}\mathrm{Val}(G) \tag{57}$$

where $\mathcal{G}(\tau)$ is the set of renormalized Feynman graphs; with respect to the Feynman graph described in §1.D, each propagator carries an index h_v, if $v \in V_\chi$ is the minimal cluster containing the propagator, see Fig.8. If we do not take into account the $\mathcal{R}$ operation, an

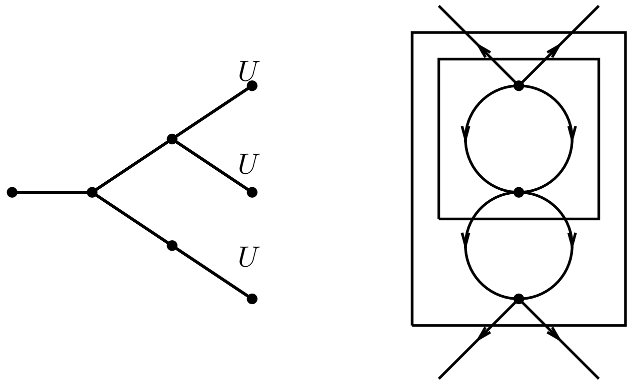

FIG. 8: A tree τ (only the vertices $v \in V_\chi$ are represented), the corresponding clusters, represented as boxes, and a Feynman graph; the propagators have scale h_{v_1} and h_{v_2} respectively.

immediate bound for each Feynman graph is, if $|\nu_h|, |U|, |\varepsilon| \leq \varepsilon_0$, and remembering that S_v is the number of clusters contained in the cluster v

$$\varepsilon_0^n C^n \prod_{v \in V_\chi} \gamma^{-(S_v-1)h_v} \tag{58}$$

The above estimate is obtained considering a tree of propagators connecting all vertices, bounding by a constant the propagators not belonging to such tree and by γ^{-h_v} the integrals of each one of the $S_v - 1$ propagators in the tree connecting the vertices in the cluster v. Note that, as $h_v < 0$, the above estimate is unbounded when summed over h_v. By definition we can rewrite (59) as

$$\varepsilon_0^n C^n \prod_{v \in V_\chi} \gamma^{-(S_v^H + S_v^L + S_v^2 - 1)h_v} \tag{59}$$

If we take into account the $\mathcal{R}$ operation than, by §2.C, we get the following bound

$$\varepsilon_0^n C^n \prod_{v \in V_\chi} \gamma^{-(S_v^H + S_v^L + S_v^2 - 1)h_v} \prod_{v \in H_\chi} \gamma^{(h_{\bar{v}'} - h_v)} \tag{60}$$

where $\bar{v}'$ is the first vertex $\in V_\chi$ following v and the last factor can compensate the term $\gamma^{-S_v^H h_v}$, proportional to the number of resonant terms, from the first factor in (60); this is indeed the reason why we introduce the $\mathcal{R}$ operation. It remains however the term $\gamma^{-S_v^L h_v}$, proportional to the number of non resonant terms; one cannot define the $\mathcal{R}$ operation for the non resonant terms, as in that way an infinite number of relevant terms with any number of fields is produced (in absence of the resonance condition the local part is non vanishing), and we could not control their flow. As we will see in the following section, the contribution of the non resonant terms is controlled using the Diophantine conditions.

E. The non resonant terms

Consider a non resonant vertex v and x'_{w_1} and x'_{w_2} are coordinates of two external fields, with $x'_{w_1} - x'_{w_2}$ given by (50). The Diophantine conditions imply a relation between the scale h_v and the number of vertices between w_2 and w_1 in $\bar{T}_v$.

Lemma 2.1 *Given $\tau, \mathbf{P}, \mathbf{T}$, let us consider $v \in L_\chi$ and w_1, w_2 two vertices in $\bar{T}_v$, see (50), with $x'_{w_1} \neq x'_{w_2}$; then*

$$|c_{w_1,w_2}| \geq A\gamma^{\frac{-h_{\bar{v}'}}{\tau}} \tag{61}$$

with a suitable constant A.

Proof. Note that $||\omega x'_{w_i}||_1 \leq c v_0^{-1} \gamma^{h_{\bar{v}'}-1}$, $i = 1, 2$ by the compact support properties of the propagator; therefore by using (50) and the Diophantine condition, if

$$2cv_0^{-1}\gamma^{h_{\bar{v}'}} \geq ||(\omega x'_{w_1})|| + ||(\omega x'_{w_2})|| \geq ||\omega(x'_{w_1} - x'_{w_2})|| = \tag{62}$$

$$||(\bar{x}_{\rho_{\ell_{w_2}}} - \bar{x}_{\rho_{\ell_{w_1}}})\omega + \omega \sum_{w \in c_{w_1,w_2}} \delta_w^{i_w}|| \tag{63}$$

If $\rho_{\ell_{w_2}} = \rho_{\ell_{w_1}}$ by the first of (12) we get

$$2cv_0^{-1}\gamma^{h_{\bar{v}'}} \geq \frac{C_0}{|\sum_{w \in c_{w_1,w_2}} \delta_w^{i_w}|^{-\tau}} \tag{64}$$

.

If $\rho_{\ell_{w_2}} = \varepsilon$, $\rho_{\ell_{w_1}} = -\varepsilon$, $\varepsilon = \pm$ then

$$||(\bar{x}_{\rho_{\ell_{w_2}}} - \bar{x}_{\rho_{\ell_{w_1}}})\omega + \omega \sum_{w \in c_{w_1,w_2}} \delta_w^{i_w}|| = ||2\varepsilon\omega\widehat{x} + 2\varepsilon\theta + \omega \sum_{w \in c_{w_1,w_2}} \delta_w^{i_w}|| \tag{65}$$

and if $\sum_{w\in c_{w_1,w_2}}\delta_w^{i_w}+2\varepsilon\widehat{x}\neq 0$ by the second of (12)

$$2cv_0^{-1}\gamma^{h_{\bar v'}}\geq\frac{C_0}{|2\varepsilon\widehat{x}+\sum_{w\in c_{w_1,w_2}}\delta_w^{i_w}|^{-\tau}}\geq\frac{C_0}{(2|\widehat{x}|+|\sum_{w\in c_{w_1,w_2}}\delta_w^{i_w}|)^{-\tau}}\geq\frac{C_0}{|\sum_{w\in c_{w_1,w_2}}\delta_w^{i_w}|^{-\tau}} \tag{66}$$

Finally if $\sum_{w\in c_{w_1,w_2}}\delta_w^{i_w}+2\varepsilon\widehat{x}=0$ then $cv_0^{-1}\gamma^{h_{\bar v'}}\geq||2\theta||\geq||2\theta||\frac{|2\widehat{x}|^\tau}{|\sum_{w\in c_{w_1,w_2}}\delta_w^{i_w}|^\tau}$. The fact that $|\sum_{w\in c_{w_1,w_2}}\delta_w^{i_w}|\leq|c_{w_1,w_2}|$ ends the proof. ∎

Lemma 2.1 says that there is a relation between the number of end-points following $v\in L_\chi$ and the scales of the external lines coming out from v. In particular the U,ε-endpoints with scale $h_v=2$ have $|c_{w_1,w_2}|=1$, hence the scale of the first vertex $v\in V_\chi$ preceding the end-point is bounded by a constant.

Lemma 2.2 *Given* $\tau,\mathbf{P},\mathbf{T}$ *the following inequality holds, for any* $0<c<1$

$$c^n\leq\prod_{v\in L_\chi}c^{A\gamma^{\frac{-h_{\bar v'}}{\tau}}2^{h_{\bar v'}-1}} \tag{67}$$

Proof. If $v\in V_\chi$ and $N_v=\sum_{i,v_i^*>v}1$ is the number of end-points following v in τ then

$$c^n\leq\prod_{v\in V_\chi}c^{N_v2^{h_{\bar v'}-1}} \tag{68}$$

Indeed we can write

$$c=\prod_{h=-\infty}^{0}c^{2^{h-1}} \tag{69}$$

Given a tree $\tau\in\mathcal{T}_{h,n}$, we consider an end-point v^* and the path in τ from v^* to the root v_0; to each vertex $v\in V_\chi$ in such path with scale h_v we associate a factor $c^{2^{h_v-2}}$; repeating such operation for any end-point, the vertices v followed by N_v end-points are in N_v paths, therefore we can associate to them a factor $c^{N_v2^{h_v-2}}$; finally we use that $c^{2^{h_v-2}}<c^{2^{h_{\bar v'}-2}}$.

Note that if v is non resonant, there exists surely two external fields with coordinates x'_1,x'_2 such that $x'_1\neq x'_2$; note that

$$N_v\geq|c_{w_1,w_2}|\geq A\gamma^{\frac{-h_{\bar v'}}{\tau}} \tag{70}$$

therefore, by (68), (67) follows, ∎.

By combing the above results we get the following final lemma which will play a crucial role in the following. We choose $\gamma^{\frac{1}{\tau}}/2\equiv\gamma^\eta>1$; for instance $\gamma=2^{2\tau}$, $\eta=\frac{1}{2\tau}$.

Lemma 2.3 *Given* $\tau, \mathbf{P}, \mathbf{T}$ *the following inequality holds*

$$[\prod_{v\in V_\chi} \gamma^{-4h_v S^L_v}][\prod_{v\in L_\chi} c^{A\gamma^{\frac{-h_{v'}}{\tau}} 2^{h_{v'}}}] \leq \bar{C}^n \tag{71}$$

with $\bar{C} = [\frac{3}{|\log|c||A}]^3 e^{-3}]$.

Proof As we assumed $\gamma^{\frac{1}{\tau}}/2 \equiv \gamma^\eta > 1$ than, for any N

$$c^{A\gamma^{\frac{-h}{\tau}} 2^h} = e^{-|\log c|A\gamma^{-\eta h}} \leq \gamma^{N\eta h} \frac{N}{|\log|c||A]^N e^N} \tag{72}$$

as $e^{-\alpha x} x^N \leq [\frac{N}{\alpha}]^N e^{-N}$. Therefore, by choosing $N = 4/\eta$ we get

$$\prod_{v\in L_\chi} c^{A\gamma^{\frac{-h_{\bar{v}'}}{\tau}} 2^{h_{\bar{v}'}}} \leq \bar{C}^n \prod_{v\in V_\chi} \gamma^{4S^L_v h_v} \tag{73}$$

■

F. Renormalized expansion

We write the expansion for the kernels of the effective potential in a way more suitable for the final bounds. Given a contribution with fixed $\tau, \mathbf{P}, T$ to $\mathcal{V}^{(h)}$, we consider the vertex v in τ with smallest h_v on which the $\mathcal{R}$ operation acts non trivially. Let us consider first the case of a resonance with two external lines. We consider a D-field associated to one of the external lines, see the second term in (52), and we write it as (54). We decompose the zero as $(x_{0,1} - x_{0,j}) = \sum_k (x_{0,k} - x_{0,k+1})$, where $(x_{0,k} - x_{0,k+1})$ is a zero corresponding to one of the lines l of the tree graph $\bar{T}_v$. If the corresponding propagator has scale h_w, and if $w > w' > w''... > v$, we add an index to one of the external D-fields (if present) of each vertex between w and v indicating that, in the next iteration, one has not to write the corresponding difference of propagators as (54) (the contribution of the two terms is written separately). The reason is that one gets in the bounds an extra factor $\gamma^{h_v - h_w} = \gamma^{h_{w'} - h_w} \gamma^{h_{w''} - h_{w'}}...$, so that the gain of the $\mathcal{R}$ operation on the intermediate vertices is already obtained (for more details see for instance §3 of [32] in a similar case). Similarly we proceed for the first term in (52). If v is resonant and has 4 external fields, we proceed in the same way; if there is a zero of order 2 in a propagator in w, we add an index to two of the D-fields (if present) of each vertex between w and v saying that one has not to write the corresponding difference of propagators as (54). Finally assume that v has more than 6 external fields (resonant or non resonant); we call $\bar{\rho}, \bar{\varepsilon}$ the labels of the external fields whose number is maximal; we define this set m_v and $|m_v| \geq |P_v|/4$. We consider a tree $\bar{T}_v$ and we define a pruning operation associating to it another tree $\widehat{T}_v$ eliminating from $\bar{T}_v$ all the trivial vertices w in $\bar{T}_v$ not associated to any external line with label $\bar{\rho}, \bar{\varepsilon}$, and all the subtrees not containing any external line with label $\bar{\rho}, \bar{\varepsilon}$ (see Fig. 9 for an example), so that there is an external line associated to all end-points. The vertices w of $\widehat{T}_v$ are then only non trivial vertices or trivial

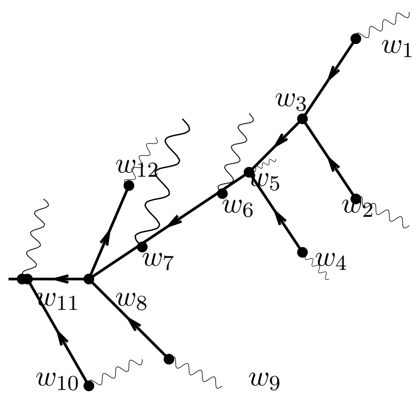

FIG. 9: In the picture the lines represent the propagators with scale $\leq h_v$ in $\widehat{T}_v$ and the wiggly lines represent the external lines P_v with label $\bar{\rho}$; note that, by definition of the pruning operation, all the end-points have associated wiggly lines, contrary to what happens in $\bar{T}_v$, see Fig. 7.

vertices with external lines $\bar{\rho}, \bar{\varepsilon}$; all the end-points have associated an external line. We define a procedure to group in two sets the fields in m_v. We start considering the end-points w_a immediately followed by vertices w_b with external lines (in the figure w_4, w_{10}), and we say that the couple of fields in w_a, w_b is of type 1 if $x'_{w_a} = x'_{w_b}$, while it is of type 2 if $x'_{w_a} \neq x'_{w_b}$. We now prune tree $\widehat{T}_v$ canceling the end-points w_a already considered and the resulting subtrees with no external lines; in the resulting tree we select an end-point w_a immediately followed by vertices w_b, and again such a couple can be of type 1 or 2. We again prune the tree and we continue unless there are no end-points w followed by vertices with wiggly line. Then in the resulting tree we consider (if they are present, otherwise the tree is trivial and the procedure ends) a couple of endpoints followed by the same non trivial vertex (in the picture w_1, w_2); we call them w_a, w_b and we proceed exactly as above distinguishing the two kind of couples. We then cancel such end-points w_a, w_b and the subtrees not containing external lines, so that the end-points are associated to external lines; we consider end-points followed by non trivial vertices with no external lines, and we proceed in the same way. If the resulting tree has again end-points with external lines followed by vertices with external lines (in the picture w_5), we prune such vertices as described above and we continue in this way so that at the end all except at most one vertex with external lines are considered. Note that by construction the paths c_{w_a,w_b} in $\bar{T}_v$ do not overlap; for instance in Fig.9 the paths are $c_{w_{10},w_{11}}$, c_{w_4,w_5}, c_{w_1,w_2}, c_{w_5,w_6}, c_{w_6,w_7}, $c_{w_7,w_{12}}, c_{w_9,w_{11}}$. Therefore, given a vertex v in the tree τ, we have paired all the external fields with index $\bar{\rho}, \bar{\varepsilon}$, whose number is $|m_v| \geq |P_v|/4$, in couples both with the same x' or with different x'. We say that a field is of type 1 if it belongs to a couple with the same x' and of type 2 if belongs to a couple with different x' (if it belongs to 2 couples of different kind, we follow the order of the construction). The number of fields of type 1 is $|m_v^1|$ and type 2 is $|m_v^2|$ and $|m_v| = |m_v^1| + |m_v^2|$. In a couple of fields with the same x' one is surely a D-fields; we then write it as (54) which will produce in the bounds a factor $\gamma^{(h_{\bar{v}'}-h_v)}$. Note that the zero is decomposed along the path connecting the two fields; as the paths $c_{w,w'}$ are non overlapping by construction, the order of such zero is at most 1; by this fact we get in the bounds a factor $\gamma^{(h_{\bar{v}'}-h_v)\frac{|m_v^1|}{2}}$. Again an index is added to the D fields associated to vertices between the vertex of the zero and v, as done above. On the other hand given w, w' with $x'_w \neq x'_{w'}$, we have $|c_{w,w''}| \geq B\gamma^{-h_{\bar{v}'}/\tau}$ by lemma 2.1; moreover by Lemma 2.2 we can associate to each $v \in V_\chi$ a factor $c^{N_v 2^{h_{\bar{v}}-1}}$ with N_v the vertices in $\bar{T}_v$; as the paths $c_{w,w'}$ are non overlapping, we get one factor $c^{|c_{w,w'}|2^{h_{v'}}} \leq c^{B\gamma^{-h_{\bar{v}'}/\tau}2^{h_{\bar{v}'}}}$ for each of the couples . The above procedure is the iterated in the vertices $\widehat{v}$ following v in τ, taking into account the indices saying that some of the D fields is not written as (54).

We add an index α to distinguish the terms generated by this procedure, so that we can write

$$V^{(h)} = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} \sum_{T \in \mathbf{T}} \sum_{\mathbf{P} \in \mathcal{P}_\tau} \sum_{\alpha \in A_T} \sum_{x} \int dx_{0,v_0} H_{\tau,\mathbf{P},T,\alpha}(x, x_{0,v_0}) \prod_{f \in P_{v_0}} \psi^{(\leq h)\varepsilon(f)}_{\widehat{\mathbf{x}}'(f),\rho(f)} \tag{74}$$

and

$$H_{\tau,\mathbf{P},T,\alpha}(x, x_{0,v_0}) = K_{\tau,\mathbf{P},T,\alpha} \prod_{v \text{ not e.p.}} \frac{1}{S_v!} \int dP_T(\mathbf{t}) \det \widetilde{G}_\alpha^{h_v,T_v}(\mathbf{t}_v) \tag{75}$$

$$\prod_{l \in T_v} \partial^{q_\alpha(f_l^+)}_{\gamma^{h_v} x_{0,l}} \partial^{q_\alpha(f_l^-)}_{\gamma^{h_v} x_{0,l}} \partial^{\widetilde{q}_\alpha(f_l^+)}_{\gamma^{-h_v}\omega x'_l} (\gamma^{h_l}(x_{0,l} - y_{0,l}))^{b_\alpha(l)} (\gamma^{-h_v}(\omega x'_l))^{\widetilde{b}_\alpha(l)} \bar{g}^{(h_v)}_{\rho_l}(x'_l; x_{0,l} - y_{0,l}))\Big|\Big]$$

where $\mathbf{T}$ is the set of the tree graphs on $\mathbf{x}_{v_0}$, obtained by putting together an anchored tree graph T_v for each non trivial vertex v, A_T is a set of indices which allows to distinguish the different terms produced by the non trivial $\mathcal{R}$ operations and the iterative decomposition of the zeros $G_\alpha^{h_v,T_v}(\mathbf{t}_v)$ has elements

$$G^{h_v,T_v}_{\alpha,ij,i'j'} = t_{v,i,i'}\delta_{x_{ij},y_{i'j'}}(\omega x_{ij})^{\widetilde{q}_\alpha(f^+_{ij})}\partial^{q_\alpha(f^+_{ij})}_{\gamma^h x_{0ij}}\partial^{q_\alpha(f^-_{ij})}_{\gamma^h x_{0ij}} g^{(h)}(x_{ij}, x_{0,ij} - y_{0,i'j'}) \tag{76}$$

The indices $q_\alpha, \widetilde{q}_\alpha, b_\alpha, \widetilde{b}_\alpha \in (0,3)$ are such that, by construction and for $c < 1$

$$|K_{\tau,\mathbf{P},T,\alpha}| \leq c^{-n} \prod_{v\in V_\chi} \gamma^{(\alpha_v+\beta_v)(h_{\bar{v}'}-h_v)}\gamma^{-\alpha|P_v|} \tag{77}$$

with $\bar{v}'$ the first vertex belonging to V_χ following v in τ and, by construction

1. if v is resonant then $\alpha_v = 1$;

2. If v is resonant and $|P_v| \geq 4$ then $\beta_v = 1$

The factor $\prod_{v\in V_\chi} \gamma^{(\alpha_v+\beta_v)(h_{\bar{v}'}-h_v)}$ is obtained by the action of $\mathcal{R}$ on the resonant term; the factor $\gamma^{-\alpha|P_v|}$ is obtained, as discussed above (74), by the action of $\mathcal{R}$ on the terms with more than 6 lines and by Lemma 2.2

$$\prod_{v\in V_\chi} c^{\gamma^{-h_{\bar{v}'}/\tau} 2^{h_{\bar{v}'}}|m^2_v|} \prod_{v\in V_\chi} \gamma^{\frac{1}{82}|m^1_v|(h_{\bar{v}'}-h_v)} \leq \prod_{v\in V_\chi} \gamma^{-\alpha|P_v|} \tag{78}$$

Note that

$$\prod_{v\in V_\chi} \gamma^{(\alpha_v+\beta_v)(h_{\bar{v}'}-h_v)} = \prod_{v\in H_\chi} \gamma^{(1+\beta_v)(h_{\bar{v}'}-h_v)} \tag{79}$$

Regarding the flow equation for ν_h we get by construction

$$\nu_{h-1} = \gamma\nu_h + \gamma^{-h} \sum_{n\geq 2}\sum_{\tau\in\mathcal{T}_{h,n}}\sum_{T\in\mathbf{T}}\sum_{\mathbf{P}\in\mathcal{P}_\tau}\sum_{\alpha\in A_T} \int dx_{0,v_0} H_{\tau,\mathbf{P},T}(0, x_{0,v_0}) \tag{80}$$

Note that on the first vertex of the trees v_0 the $\mathcal{L}$ operation acts; therefore, as $\mathcal{LR} = 0$, necessarily $v_0 \in V_\chi$.

G. Bounds for the effective potential

In this section we get a bound for the kernels of the effective potential defined in (74).

Lemma 2.4 *If $n = n_\nu + n_U + \nu_\varepsilon$ the following bound holds*

$$\frac{1}{\beta L} \sum_{\tau\in\mathcal{T}_{h,n}}\sum_{T\in\mathbf{T}}\sum_{\mathbf{P}\in\mathbf{P}_\tau}\sum_{x} \int dx_{0,v_0} |H_{\tau,\mathbf{P},T,\alpha}(x, x_{0,v_0})| \leq C^n \gamma^{h_{v_0}} (\sup_{k\geq h} |\nu_k||)^{n_\nu} |U|^{n_U} |\varepsilon|^{n_\varepsilon} \tag{81}$$

where C is a suitable constant.

Proof We start from (75) and, in order to bound the matrix $\widetilde{G}^{h,T}_{ij,i'j'}$, we introduce an Hilbert space $\mathcal{H} = \ell^2 \otimes \mathbb{R}^s \otimes L^2(\mathbb{R}^1)$ so that

$$\widetilde{G}^{h,T}_{ij,i'j'} = \Big(\mathbf{v}_{x_{ij}} \otimes \mathbf{u}_i \otimes A(x_{0,ij}-, x_{ij}) \ , \ \mathbf{v}_{y_{i',j'}} \otimes \mathbf{u}_{i'} \otimes B(y_{0,i'j'}-, x_{ij})\Big) \ , \tag{82}$$

where $\mathbf{v} \in \mathbb{R}^L$ are unit vectors such that $(\mathbf{v}_i, \mathbf{v}_j) = \delta_{ij}$, $\mathbf{u} \in \mathbb{R}^s$ are unit vectors $(u_i, u_i) = t_{ii'}$, and A, B are vectors in the Hilbert space with scalar product

$$(A, B) = \int dz_0 A(x', x_0 - z_0) B^*(x', z_0 - y_0) \tag{83}$$

given by

$$A(x', x_0 - z_0) = \frac{1}{\beta} \sum_{k_0} e^{-ik_0(x_0 - z_0)} \sqrt{f_h(\omega x', k_0)}$$

$$B(x', y_0 - z_0) = \frac{1}{\beta} \sum_{k_0} \frac{e^{-ik_0(y_0 - z_0)} \sqrt{f_h(\omega x', k_0)}}{-ik_0 + \cos 2\pi(\omega x' + \bar{x}_\rho + \theta) - \cos 2\pi(\bar{x}_\rho + \theta)}$$

Moreover

$$||A_h||^2 = \int dz_0 |A_h(x', z_0)|^2 \leq C\gamma^{-3h} \ , \qquad ||B_h||^2 \leq C\gamma^{3h} \ , \tag{84}$$

for a suitable constant C. Therefore by Gram-Hadamard indequality we get:

$$|\det \widetilde{G}^{h_v, T_v}(\mathbf{t}_v)| \leq C^{\sum_{i=1}^{S_v} |P_{v_i}| - |P_v| - 2(S_v - 1)} \ . \tag{85}$$

By using (79),(78),(67),(71) we get

$$\frac{1}{L\beta} \sum_x \int dx_{0,v_0} |H_{\tau,\mathbf{P},T,\alpha}(x, x_{0,v_0})| \leq c^{-n} [\prod_v \frac{1}{S_v!}][\prod_{v \in V_\chi} \gamma^{4h_v S^L_v}][\prod_{v \in H_\chi} \gamma^{(1+\beta_v)(h_{\bar{v}'} - h_v)}][\prod_{v \in V_\chi} \gamma^{-\alpha|P_v|}]$$
$$[\prod_{v \in V_\chi} \gamma^{-h_v(S^H_v + S^L_v - 1)}](\sup_{k \geq h} |\nu_k||)^{n_\nu} |U|^{n_U} |\varepsilon|^{n_\varepsilon} \tag{86}$$

where $\beta_v = 1$ if v is a resonant cluster with more than 2 external lines. Note that

$$[\prod_{v \in V_\chi} \gamma^{-h_v(S^H_v + S^L_v - 1)}][\prod_{v \in H_\chi} \gamma^{h_{\bar{v}'} - h_v}] \leq \gamma^{h_{v_0}} [\prod_{v \in V_\chi} \gamma^{-h_v(S^H_v + S^L_v)}][\prod_{v \in H_\chi} \gamma^{h_{\bar{v}'}}] \tag{87}$$

as $v_0 \notin H_\chi$ so that $\prod_{v \in V_\chi} \gamma^{h_v} \leq \gamma^{h_{v_0}} \prod_{v \neq v_0, v \in H_\chi} \gamma^{h_v}$. Moreover

$$[\prod_{v \in V_\chi} \gamma^{-h_v S^H_v}][\prod_{v \in H_\chi} \gamma^{h_{\bar{v}'}}] = 1 \tag{88}$$

so that

$$[\prod_{v \in V_\chi} \gamma^{-h_v(S^H_v + S^L_v - 1)}][\prod_{v \in H_\chi} \gamma^{h_{\bar{v}'} - h_v}] \leq \gamma^{h_{v_0}} [\prod_{v \in V_\chi} \gamma^{-h_v S^L_v}] \tag{89}$$

We get

$$\frac{1}{L\beta}\sum_x\int dx_{v_0}|H_{\tau,\mathbf{P},T,\alpha}(x,\mathbf{x}_{v_0})|\le$$
$$\gamma^{h_{v_0}}[\prod_v\frac{1}{S_v!}][\prod_{v\in V_\chi}\gamma^{3h_vS_v^L}][\prod_{v\in H_\chi}\gamma^{\beta_v(h_{\bar v'}-h_v)}][\prod_{v\in V_\chi}\gamma^{-\alpha|P_v|}](\sup_{k\ge h}|\nu_k||)^{n_\nu}|U|^{n_U}|\varepsilon|^{n_\varepsilon}\tag{90}$$

Note that $\sum_{\mathbf{P}}[\prod_{v\in V_\chi}\gamma^{-\alpha|P_v|}]\le C^n$, see for instance §3.7 of [28] for a proof; moreover $\sum_{\mathbf{T}}[\prod_v\frac{1}{S_v!}]\le C^n$, see Lemma 2.4 of [28]. The sum over the trees τ is done performing the sum of unlabeled trees and the sum over scales. The unlabeled trees can be bounded by 4^n by Caley formula, and the sum over the scales reduces to the sum over h_v, with $v\in V_\chi$, as given a tree with such scales assigned, the others are of course determined. It remains to prove that

$$\sum_{\{h_v\}}][\prod_{v\in V_\chi}\gamma^{3h_vS_v^L}][\prod_{v\in H_\chi}\gamma^{\beta_v(h_{\bar v'}-h_v)})]\le C^n\tag{91}$$

We can write $\sum_{\{h_v\}}=\sum_{h_v,v\in V_\chi \atop S_v^L\ge 1}+\sum_{h_v,v\in V_\chi \atop S_v^L=0}$; for the first sum we can simply use $[\prod_{v\in H_\chi}\gamma^{\beta_v(h_{\bar v'}-h_v)})]<1$ so that

$$\sum_{h_v,v\in V_\chi \atop S_v^L\ge 1}\prod_{v\in V_\chi}\gamma^{3h_vS_v^L}\le C^n\tag{92}$$

Regarding the second sum, we have to sum scales of vertices followed by vertices $v_1,..,v_{S_v}$ which are all resonant. We can still distinguish two cases; or $|P_{v_i}|=2$, $i=1,..,S_v$, that is the inner clusters of the cluster v have two external lines, or not. In the last case, there is surely a j such that $|P_{v_j}|\ge 4$; we call the scale of such inner cluster $h_{\widehat v}$ and $\widehat v'=v$, we can extract from the product $[\prod_{v\in H_\chi}\gamma^{\beta_v(h_{\bar v'}-h_v}]$ a factor $\gamma^{(h_v-h_{\widehat v})}$ and we can use such factor to sum over $h_v\le h_{\widehat v}$, see Fig.10. Otherwise, see Fig. 11, the inner resonant clusters have all 2 external lines; therefore such clusters are connected by propagators with the same coordinate and momentum of the external lines, so that there is no sum over h_v as $h_v=h_{v'}+1$ by the support properties of the propagators. ∎

Lemma 2.3 implies convergence of the expansion for the kernels of the effective potential, provided that ε,U and ν_k are small enough; this last condition is ensured by choosing properly the counterterm ν as a function of ε,U, as we will show below.

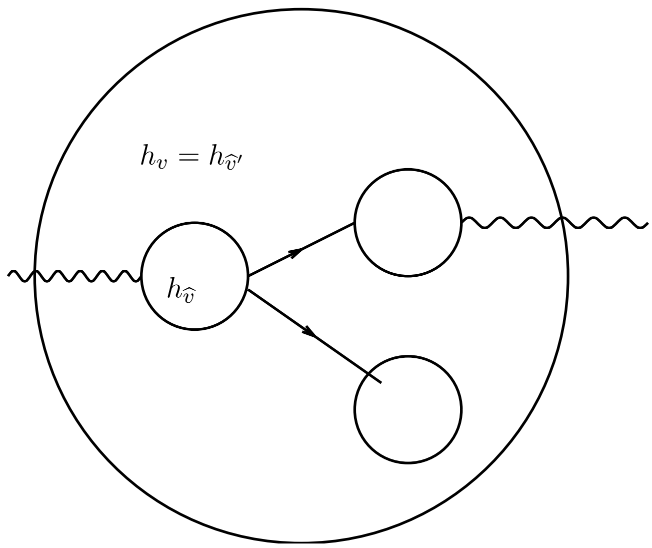

FIG. 10: A cluster v with $S_v = 3$, $S_v^L = 0$; one inner cluster has more that 2 external lines.

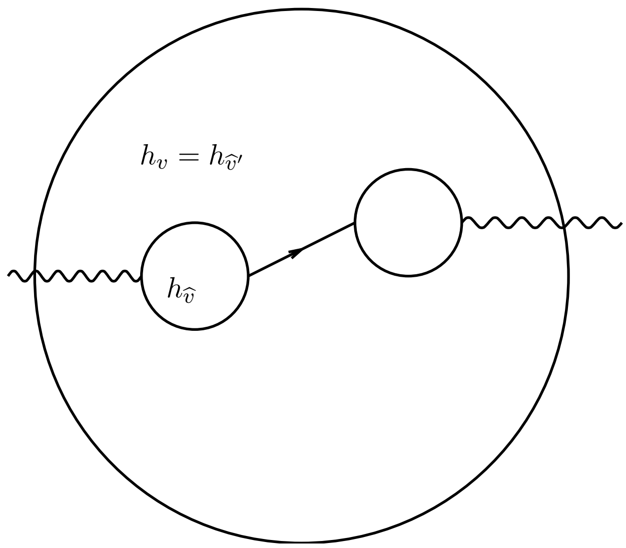

FIG. 11: A cluster v with $S_v = 2$, $S_v^L = 0$; all inner clusters have two external lines

H. Choice of the counterterm ν

The above lemma ensures convergence provided that ν_k are small for any k. We can write (80) as

$$\nu_{h-1} = \gamma\nu_h + \sum_{n=2}^{\infty} \beta_n^{(h)} \tag{93}$$

Lemma 2.5 *If $n = n_\nu + n_U + n_\varepsilon$ then $\beta_n^{(h)} = 0$ if $n_\varepsilon = n_U = 0$; moreover*

$$|\beta_n^{(h)}| \leq C^n (\sup_{k\geq h} |\nu_k||)^{n_\nu} |U|^{n_U} |\varepsilon|^{n_\varepsilon} \tag{94}$$

Proof By (80)

$$\gamma^h \beta_n^{(h)} = \sum_{\tau\in\mathcal{T}_{h,n}} \sum_{T\in\mathbf{T}} \sum_{\mathbf{P}\in\mathcal{P}_\tau} \sum_{\alpha\in A_T} \int dx_{0,v_0} H_{\tau,\mathbf{P},T}(0, x_{0,v_0}) \tag{95}$$

and $v_0 \in V_\chi$. If $n_\varepsilon = n_U = 0$ the only contributions is from chain graphs, whose value is given by a product of $\widehat{g}^k(x', k_0)$, $k \geq h$ computed at $k_0 = x' = 0$, hence they are vanishing by the compact support properties of the propagators $\widehat{g}^k(0;0) = 0$. Moreover the r.h.s. of (95) verifies the same bound as the r.h.s. of (81) with γ^h replacing $\gamma^{h_{v_0}}$ as $h_{v_0} = h$ as $v_0 \in V_\chi$ because $\mathcal{LR} = 0$. ■

It remains to prove that we can choose ν so that ν_h is small. First we write

$$\nu_h = \gamma^{-h+1}(\nu + \sum_{k=h+1}^{1} \gamma^k \beta_k) \tag{96}$$

We introduce a sequence of $\nu_k^{(n)}$ such that $\nu_h^{(0)} = 0$ and

$$\nu_h^{(n)} = \gamma^{-h+1}(\nu + \sum_{k=h+1}^{1} \gamma^k \beta_k^{(n-1)}) \tag{97}$$

where $\beta_k^{(n-1)}$ is obtained from β_k replacing ν_k with $\nu_k^{(n-1)}$.

Lemma 2.6 *Setting*

$$\nu^{(n)} = -\sum_{k=h+1}^{1} \gamma^k \beta_k^{(n-1)} \tag{98}$$

then the sequence $\nu_h^{(n)}$, $h \leq 1$, $\nu_1 \equiv \nu$, converges uniformly to ν_h with $|\nu_h| \leq C \max(|\varepsilon|, |U|)$

Proof We show by induction that

$$|\nu_h^{(n)} - \nu_h^{(n-1)}| \leq C(\max(|\varepsilon|, |U|))^n \qquad |\beta_h^{(n)} - \beta_h^{(n-1)}| \leq C(\max(|\varepsilon|, |U|))^n \tag{99}$$

For $n = 1$ then $\nu_h^{(1)} = 0$ for $h \leq 1$ and $\nu_1^{(1)}$, by (94), is such that $|\nu_1^{(1)}| \leq C \max(|\varepsilon|, |U|)$, for ε, U small enough. If $n > 1$ then $\beta_h^{(n)} - \beta_h^{(n-1)}$ can be written as sum of terms in which there is at least a $\nu_h^{(n)} - \nu_h^{(n-1)}$, hence (99) follows by (94), as $|\beta_h^{(n)} - \beta_h^{(n-1)}| \leq \bar{C} \max(|\varepsilon|, |U|)|\nu_h^{(n)} - \nu_h^{(n-1)}|$ as $\beta_n^{(h)}$ is vanishing if $n_\varepsilon = n_U = 0$. Therefore uniform convergence follows. ■

I. The 2-point function

We have finally to get a bound for the two-point function, which can be written as

$$S(\mathbf{x},\mathbf{y}) = \sum_{n=2}^{\infty} H_n(\mathbf{x},\mathbf{y}) \tag{100}$$

where $H_n(\mathbf{x},\mathbf{y})$ is sum over trees with n end-points and any value of h_{v_0}, among which there are 2 special end-points associated to the external lines and $n-2$ are associated normal end-points of type ε, U, ν. Note that there is necessarily a path c_{w_1,w_2} in $\widehat{T}_v$ connecting the points w_1, with $\mathbf{x}_{w_1} = \mathbf{x}$ and w_2 with $\mathbf{x}_{w_2} = \mathbf{y}$ such that by (50) $|x-y| \le |c_{w_1,w_2}|$; moreover $|c_{w_1,w_2}| \le n$ so that $H_n = 0$ for $n < |x-y|$. Therefore with respect to the bound to the effective potential (81) there is an extra $\gamma^{h_{v_0}}$ for a missing integral due to the fact that $\mathbf{x},\mathbf{y}$ are fixed and and extra $\gamma^{-2h_{v_0}}$ for the presence of the external lines. The sum over the scales is bounded by $|\bar{h}|$ with

$$\begin{aligned}&\gamma^{-\bar{h}} \le \max_{k\in 0,n}\max_{\rho=\pm 1} \frac{1}{||\omega(x+k) - \omega\rho\widehat{x} - 2\delta_{\rho,-1}\theta||} \le \\ &C(1+\min\{|x|,|y|\}+n)^\tau \le C(1+\min\{|x|,|y|\})^\tau(1+\frac{n}{1+\min\{|x|,|y|\}})^\tau\end{aligned} \tag{101}$$

so that in conclusion

$$\begin{aligned}&|S(\mathbf{x},\mathbf{y})| \le \sum_{n\ge|x-y|} (\max(|\varepsilon|,|U|))^n C^n \log[(1+\min\{|x|,|y|\})^\tau(1+\frac{n}{1+\min\{|x|,|y|\}})^\tau] \\ &\le e^{-\frac{\alpha}{2}|\log max(|\varepsilon|,|U|)||x-y|} \log[(1+\min\{|x|,|y|\})^\tau]\end{aligned} \tag{102}$$

We can get another bound, which is better for large $|x_0 - y_0|$; by integrating by parts and using that each derivative carry an extra $\gamma^{-h_{v_0}}$ one gets

$$|S(\mathbf{x},\mathbf{y})| \le e^{-\frac{\alpha}{2}|\log max(|\varepsilon|,|U|)||x-y|} \frac{C_N}{1+(\min\{|x|,|y|\}^{-\tau}|x_0-y_0|)^N} \tag{103}$$

and combining the above two bounds, Theorem 1.1 follows.

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