Nonlinear dissipation for some systems of critical NLS equations in two dimensions

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Abstract. We prove the global well-posedness in $H^1(\mathbb{R}^2, \mathbb{C}^N)$ for certain systems of the critical Nonlinear Schrödinger equations coupled linearly or nonlinearly with nonlinear supercritical dissipation terms, generalizing the previous result of [8] obtained for a single equation of this kind.

Keywords: Nonlinear Schrödinger equations, arrest of collapse, nonlinear dissipation.

AMS subject classification: 35Q55, 35Q60

1 Introduction

There has been a substantial amount of work on nonlinear partial differential equations involving dissipative terms accomplished in recent years. Article [8] is devoted to the studies of the focusing, critical in two dimensions (with the cubic nonlinearity) Nonlinear Schrödinger (NLS) equation. It was established that when linear or nonlinear dissipative terms are incorporated in such an equation, it becomes globally well -posed in $H^1(\mathbb{R}^2)$. Work [5] is a numerical approach to the studies of singular solutions of the critical and supercritical NLS equation with the nonlinear dissipation. In paper [1] the authors show the global well-posedness for the cubic NLS with nonlinear damping when the external quadratic confining potential is present. The present article is the generalization of the ideas of [8] to the case of certain systems of NLS equations with the cubic nonlinearities, which are critical in two dimensions. We show the arrest of collapse occuring when the supercritical nonlinear dissipation is involved in such systems of equations. Apparently, such an effect has similarities with the enhanced binding in nonrelativistic Quantum Electrodynamics (QED) (see e.g. [2], [4], [6]). It is well known that in $\mathbb{R}^3$ the Schrödinger operator with a negative, shallow, short-range potential does not possess square integrable bound states. It was established that when the quantized radiation field in the Pauli-Fierz model is turned on, the particle absorbs the energy from it, its mass is getting increased and the negative eigenvalues with corresponding eigenfunctions belonging to $L^2(\mathbb{R}^3)$ appear. Certain systems of coupled NLS equations without dissipation were studied in [9] and [10] from the point of view of understanding the spectral stability of solitary waves. In the present work we will be using the $H^1(\mathbb{R}^2, \mathbb{C}^N)$ Sobolev space equipped with the norm

$$\|\psi\|^2_{H^1(\mathbb{R}^2,\mathbb{C}^N)} := \|\psi\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} + \|\nabla\psi\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} = \sum_{k=1}^{N}\{\|\psi_k\|^2_{L^2(\mathbb{R}^2)} + \|\nabla\psi_k\|^2_{L^2(\mathbb{R}^2)}\}$$

for a vector-function $\psi = (\psi_1, \psi_2, ..., \psi_N)^T$. The inner product of two square integrable functions $f, g \in L^2(\mathbb{R}^2)$ will be designated as

$$(f, g)_{L^2(\mathbb{R}^2)} := \int_{\mathbb{R}^2} f(x)\bar{g}(x)dx.$$

The first part of the article deals with the system of N focusing critical NLS equations in two dimensions with the supercritical nonlinear dissipation, coupled linearly, such that

$$i\frac{\partial\psi_k}{\partial t} = -\Delta\psi_k - i\delta_k|\psi_k|^p\psi_k - |\psi_k|^2\psi_k + \sum_{s=1}^{N} a_{ks}\psi_s, \quad 1 \le k \le N. \tag{1.1}$$

The coupling matrix here and in system (1.2) is assumed to be arbitrary, constant in space and time and Hermitian, such that $a_{ks} = \bar{a_{sk}}$ for all $1 \le s, k \le N$. Here and in (1.2) $\delta_k > 0, \ 1 \le k \le N$ are meant to be arbitrary constants as well. Let us denote $\delta_{min} := min\{\delta_k\}_{k=1}^N > 0$. For the supercritical power involved in the nonlinear dissipative term both in (1.1) and (1.2) we have $p := 2(1 + \alpha)$ with some constant $\alpha > 0$. The initial condition for systems (1.1) and for (1.2) analogously would be $\psi(x, 0) = \psi_0(x) \in H^1(\mathbb{R}^2, \mathbb{C}^N)$. In the absence of nonlinear dissipation in two dimensions, for system (1.1) we have the blow up, which can be studied by virtue of the fairly standard scaling argument for NLS type equations (see e.g. Section 6 of [3]). The situation in (1.2) will depend on the choice of coefficients a_{ks}. Our first main result is as follows.

Theorem 1. *For every initial condition $\psi(x, 0) \in H^1(\mathbb{R}^2, \mathbb{C}^N)$, there is a unique mild solution $\psi(x, t), \ t \in [0, \infty)$ of (1.1) with $\psi(x, t) \in H^1(\mathbb{R}^2, \mathbb{C}^N)$.*

The second part of the article is devoted to the studies of the system of critical NLS equations in two dimensions with the nonlinear supercritical dissipation coupled nonlinearly, namely

$$i\frac{\partial\psi_k}{\partial t} = -\Delta\psi_k - i\delta_k|\psi_k|^p\psi_k - \frac{1}{N}\sum_{s=1}^{N} a_{ks}|\psi_s|^2\psi_k, \quad 1 \le k \le N. \tag{1.2}$$

We also assume for the system above that for all $k, s = 1, ..., N$ we have a_{ks} real valued and symmetric, namely $a_{ks} = a_{sk}$ and $|a_{ks}| \leq a$ with some constant $a > 0$. Our second main statement is as follows.

Theorem 2. *For an arbitrary initial condition $\psi(x, 0) \in H^1(\mathbb{R}^2, \mathbb{C}^N)$, there exists a unique mild solution $\psi(x, t),\ t \in [0, \infty)$ of system (1.2) with $\psi(x, t) \in H^1(\mathbb{R}^2, \mathbb{C}^N)$.*

First we turn our attention to the case of the supercritical nonlinear dissipation in a system of critical NLS equations coupled linearly.

2 The system of focusing, critical NLS with the nonlinear dissipation coupled linearly

Proof of Theorem 1. We rewrite the system of equations (1.1) as

$$\frac{\partial \psi_k}{\partial t} = i\Delta\psi_k + (F[\psi])_k, \quad 1 \leq k \leq N \tag{2.1}$$

with $(F[\psi])_k := -\delta_k|\psi_k|^p\psi_k + i|\psi_k|^2\psi_k - i\sum_{s=1}^N a_{ks}\psi_s$. The mild solution of our system satisfies the Duhamel's principle

$$\psi_k(x, t) = e^{it\Delta}\psi_{0,k}(x) + e^{it\Delta}\int_0^t e^{-is\Delta}(F[\psi(s)])_k ds, \quad 1 \leq k \leq N$$

in $H^1(\mathbb{R}^2, \mathbb{C}^N)$ for $t \in [0, T)$. The local well-posedness for our problem can be established using the Strichartz estimates (see e.g. Section 4 of [3] for the standard argument for the NLS type equations). Moreover,

$$\lim_{t\to T^-}\|\psi(t)\|_{H^1(\mathbb{R}^2,\mathbb{C}^N)} = \infty$$

if T is finite. With a slight abuse of notations we will be using the same letter T in such context in the proof of the consecutive theorem as well. Our goal is to establish that this solution is in fact global in time. Using the system of equations (2.1), we easily compute for $t \in [0, T)$:

$$\frac{d}{dt}\|\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} = -2\sum_{k=1}^N \delta_k \int_{\mathbb{R}^2} |\psi_k(t)|^{p+2}dx - 2Im\sum_{k=1}^N(\psi_k(t), \sum_{s=1}^N a_{ks}\psi_s(t))_{L^2(\mathbb{R}^2)}.$$

Note that the last expression in the right side of the identity above vanishes since the coupling matrix in our system is Hermitian as assumed. Thus, we arrive at

$$\frac{d}{dt}\|\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} = -2\sum_{k=1}^N \delta_k \int_{\mathbb{R}^2} |\psi_k(t)|^{p+2}dx \leq 0, \quad t \in [0, T), \tag{2.2}$$

which is analogous to formula (3.13) of [8] proven for the single NLS equation. Hence

$$\|\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} \le \|\psi_0\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)}, \quad t \in [0,T) \tag{2.3}$$

and for $t \in [0,T)$ we have

$$\|\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} + 2\sum_{k=1}^{N} \delta_k \int_0^t \int_{\mathbb{R}^2} |\psi_k(x,s)|^{p+2} dx ds = \|\psi_0\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)}. \tag{2.4}$$

Therefore, the $L^2(\mathbb{R}^2,\mathbb{C}^N)$ norm of our mild solution is well under control. Using the system of equations (2.1) and taking the sufficiently regular solutions, we evaluate

$$\frac{d}{dt}\|\nabla\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} =$$

$$= 2\sum_{k=1}^{N} \delta_k Re(\bar{\psi}_k(t)\Delta\psi_k(t), |\psi_k(t)|^p)_{L^2(\mathbb{R}^2)} - 2\sum_{k=1}^{N} Im(\bar{\psi}_k(t)\Delta\psi_k(t), |\psi_k(t)|^2)_{L^2(\mathbb{R}^2)}. \tag{2.5}$$

In the computation above we used that

$$Im\sum_{k=1}^{N}(-\Delta\psi_k(t), \sum_{s=1}^{N} a_{ks}\psi_s(t))_{L^2(\mathbb{R}^2)} = 0.$$

Indeed, since the coupling matrix for our system is constant and Hermitian as assumed, its product with the Laplacian operator is self-adjoint and therefore, the term above vanishes. We will make use of the trivial identity

$$\bar{\psi}_k\Delta\psi_k - \psi_k\Delta\bar{\psi}_k = div(\bar{\psi}_k\nabla\psi_k - \psi_k\nabla\bar{\psi}_k), \quad 1 \le k \le N \tag{2.6}$$

to obtain

$$Im(\bar{\psi}_k\Delta\psi_k, |\psi_k|^2)_{L^2(\mathbb{R}^2)} = Im\int_{\mathbb{R}^2} \psi_k^2 (\nabla\bar{\psi}_k)^2 dx. \tag{2.7}$$

A straightforward computation yields that the first term in the right side of (2.5) equals to

$$-2\sum_{k=1}^{N} \delta_k Re\int_{\mathbb{R}^2} \bar{\psi}_k(t)\nabla\psi_k(t).\nabla|\psi_k(t)|^p dx - 2\sum_{k=1}^{N} \delta_k \int_{\mathbb{R}^2} |\nabla\psi_k(t)|^2|\psi_k(t)|^p dx. \tag{2.8}$$

Thus we arrive at

$$\frac{d}{dt}\|\nabla\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} =$$

$$= -2Im\sum_{k=1}^{N}\int_{\mathbb{R}^2} \psi_k^2(t)(\nabla\bar{\psi}_k(t))^2 dx - 2\sum_{k=1}^{N}\delta_k Re\int_{\mathbb{R}^2} \psi_k(t)\nabla\bar{\psi}_k(t).\nabla|\psi_k(t)|^p dx -$$

$$-2\sum_{k=1}^{N}\delta_k\int_{\mathbb{R}^2}|\nabla\psi_k(t)|^2|\psi_k(t)|^p dx,\quad t\in[0,T).\tag{2.9}$$

A trivial calculation gives us

$$Re\int_{\mathbb{R}^2}\psi_k\nabla|\psi_k|^p.\nabla\bar{\psi}_k dx=\frac{p}{4}\int_{\mathbb{R}^2}[\nabla|\psi_k|^2]^2|\psi_k|^{p-2}dx\geq 0,\quad 1\leq k\leq N.\tag{2.10}$$

Let us introduce the following auxiliary quantity

$$P_k(t):=\int_{\mathbb{R}^2}|\psi_k|^p|\nabla\psi_k|^2dx,\quad 1\leq k\leq N.\tag{2.11}$$

We use the Hölder's inequality to obtain the following estimate from above

$$\int_{\mathbb{R}^2}|\psi_k|^2|\nabla\psi_k|^{\frac{4}{p}}|\nabla\psi_k|^{2-\frac{4}{p}}dx\leq\Big(P_k(t)\Big)^{\frac{2}{p}}\|\nabla\psi_k\|_{L^2(\mathbb{R}^2)}^{\frac{2(p-2)}{p}}.\tag{2.12}$$

Note that $p>2$ as assumed. We recall the Young's inequality

$$ab\leq\frac{a^p}{p}+\frac{b^q}{q}\tag{2.13}$$

for $a,b\geq 0$ and $\frac{1}{p}+\frac{1}{q}=1$. In the context of our work the conjugate exponents are $\frac{p}{2}$ and $\frac{p}{p-2}$. This gives us the following upper bound for the right side of inequality (2.12)

$$\delta_kP_k(t)+\frac{p-2}{p}\Big(\frac{2}{\delta_kp}\Big)^{\frac{2}{p-2}}\|\nabla\psi_k\|_{L^2(\mathbb{R}^2)}^2.\tag{2.14}$$

Let us introduce the constant

$$C(\delta_{min},p):=2\frac{p-2}{p}\Big(\frac{2}{\delta_{min}p}\Big)^{\frac{2}{p-2}}>0.\tag{2.15}$$

Hence, due to the estimates above, we arrive at the differential inequality

$$\frac{d}{dt}\|\nabla\psi(t)\|_{L^2(\mathbb{R}^2,\mathbb{C}^N)}^2\leq C(\delta_{min},p)\|\nabla\psi(t)\|_{L^2(\mathbb{R}^2,\mathbb{C}^N)}^2,\quad t\in[0,T).\tag{2.16}$$

Apparently, differential inequality (2.16) yields the bound

$$\|\nabla\psi(x,t)\|_{L^2(\mathbb{R}^2,\mathbb{C}^N)}^2\leq\|\nabla\psi_0\|_{L^2(\mathbb{R}^2,\mathbb{C}^N)}^2e^{C(\delta_{min},p)t},\quad t\in[0,T).\tag{2.17}$$

Note that similarly to single NLS equations studied in [8], the supercritical nonlinear dissipation gives us the factor exponentially growing in time. Via the blow-up alternative, (2.17) implies that the system of NLS equations (1.1) is globally well-posed in $H^1(\mathbb{R}^2,\mathbb{C}^N)$. ∎

3 The system of critical NLS with nonlinear dissipation coupled nonlinearly

Proof of Theorem 2. Clearly, the system of equations (1.2) can be easily written as

$$\frac{\partial \psi_k}{\partial t} = i\Delta\psi_k + (G[\psi])_k, \quad 1 \le k \le N \tag{3.1}$$

with $(G[\psi])_k := -\delta_k|\psi_k|^p\psi_k + \frac{i}{N}\sum_{s=1}^{N} a_{ks}|\psi_s|^2\psi_k$. The mild solution of system (3.1) satisfies the Duhamel's principle

$$\psi_k(x,t) = e^{it\Delta}\psi_{0,k}(x) + e^{it\Delta}\int_0^t e^{-is\Delta}(G[\psi(s)])_k ds, \quad 1 \le k \le N$$

in $H^1(\mathbb{R}^2, \mathbb{C}^N)$ for $t \in [0,T)$ and the local well- posedness can be established via the standard argument for NLS type equations by applying the Strichartz estimates (see e.g. Section 4 of [3]). Furthermore,

$$\lim_{t\to T^-} \|\psi(t)\|_{H^1(\mathbb{R}^2,\mathbb{C}^N)} = \infty$$

if T is finite. We are going to prove that such solution is global in time. A straightforward computation yields that estimates (2.2), (2.3) and (2.4) hold here as well. Using system (3.1) and considering sufficiently regular solutions, we obtain for $t \in [0,T)$:

$$\frac{d}{dt}\|\nabla\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} = 2\sum_{k=1}^{N}\delta_k Re(\bar{\psi}_k(t)\Delta\psi_k(t), |\psi_k(t)|^p)_{L^2(\mathbb{R}^2)} -$$

$$-2\sum_{k=1}^{N} Im\frac{1}{N}\sum_{s=1}^{N} a_{ks}(\bar{\psi}_k(t)\Delta\psi_k(t), |\psi_s(t)|^2)_{L^2(\mathbb{R}^2)}.$$

For the first term in the right side of the identity above we will use formula (2.8). A straightforward computation yields

$$Im(\bar{\psi}_k\Delta\psi_k, |\psi_s|^2)_{L^2(\mathbb{R}^2)} = -Im(\bar{\psi}_k\nabla\psi_k, \nabla|\psi_s|^2)_{L^2(\mathbb{R}^2,\mathbb{C}^2)}.$$

Clearly, via the Schwarz inequality we obtain

$$\left|Im(\bar{\psi}_k\nabla\psi_k, \bar{\psi}_s\nabla\psi_s)_{L^2(\mathbb{R}^2,\mathbb{C}^2)}\right| \le \|\bar{\psi}_k\nabla\psi_k\|_{L^2(\mathbb{R}^2)}\|\bar{\psi}_s\nabla\psi_s\|_{L^2(\mathbb{R}^2)},$$

$$\left|Im(\bar{\psi}_k\nabla\psi_k, \psi_s\nabla\bar{\psi}_s)_{L^2(\mathbb{R}^2,\mathbb{C}^2)}\right| \le \|\bar{\psi}_k\nabla\psi_k\|_{L^2(\mathbb{R}^2)}\|\psi_s\nabla\bar{\psi}_s\|_{L^2(\mathbb{R}^2)}.$$

Therefore, we arrive at

$$-2\sum_{k=1}^{N} Im\frac{1}{N}\sum_{s=1}^{N} a_{ks}(\bar{\psi}_k\Delta\psi_k, |\psi_s|^2)_{L^2(\mathbb{R}^2)} \le 4a\sum_{k=1}^{N}\|\bar{\psi}_k\nabla\psi_k\|^2_{L^2(\mathbb{R}^2)}, \tag{3.2}$$

such that

$$\frac{d}{dt}\|\nabla\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} + 2\sum_{k=1}^N \delta_k Re\int_{\mathbb{R}^2} \bar{\psi_k}(t)\nabla\psi_k(t).\nabla|\psi_k(t)|^p dx +$$

$$+2\sum_{k=1}^N \delta_k \int_{\mathbb{R}^2} |\nabla\psi_k(t)|^2|\psi_k(t)|^p dx \leq 4a\sum_{k=1}^N \|\bar{\psi_k}(t)\nabla\psi_k(t)\|^2_{L^2(\mathbb{R}^2)}.$$

We use identity (2.10) for the second term in the left side of the inequality above. The argument analogous to (2.12) and (2.14) yields

$$4a\sum_{k=1}^N \int_{\mathbb{R}^2} |\psi_k|^2|\nabla\psi_k|^2 dx \leq \sum_{k=1}^N 2\delta_k P_k(t) + \tilde{C}(\delta_{min}, p)\|\nabla\psi\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)}$$

with $P_k(t)$ given by (2.11) and

$$\tilde{C}(\delta_{min}, p) := 4a\frac{p-2}{p}\Big(\frac{4a}{\delta_{min}p}\Big)^{\frac{2}{p-2}} > 0.$$

Hence we obtain the differential inequality

$$\frac{d}{dt}\|\nabla\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} \leq \tilde{C}(\delta_{min}, p)\|\nabla\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)}, \quad t\in[0,T)$$

Finally, we arrive at the bound

$$\|\nabla\psi(x,t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} \leq \|\nabla\psi_0\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} e^{\tilde{C}(\delta_{min},p)t}, \quad t\in[0,T). \tag{3.3}$$

Analogously to (2.17), the supercritical nonlinear dissipation gives us the factor exponentially growing in time. By means of the blow-up alternative, this implies that the system of NLS equations (1.2) is globally well-posed in $H^1(\mathbb{R}^2, \mathbb{C}^N)$. ■

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