The Euler Circular-Reasoning Gap: The Exponential Revisited

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"One of the most remarkable, almost astounding, formulas in all of mathematics." – Richard Feynman

The fundamental constant e has intrigued mathematicians such as Euler for centuries. Several definitions for e have been posited, for example the two main definitions, $e = \lim\limits_{x\to\infty} \left(1+\frac{1}{x}\right)^x$, or e is such that the area$(1/x, 1, e) = 1$, and ineed, if you take one for granted, you can show the other, such that the two definitions are equivalent. And, invariably, every textbook I have examined has either assumed one and shown the other, or else skipped the interplay alltogether. However, a definitional, and hence axiomatic assumption should be reserved for a statement that is "fundamentally true," and the Euler material is anything but trivial.

Realizing that the derivatives of the exponential and logarithmic functions are invariably circular in their reasoning, what I set out to do was to attempt to "close the loop," by expanding the intuition to a level of abstraction not normally achieved at the Calculus I level. In fact, Euler's Formula is usually reserved for Calculus II at the earliest, and can be attained as an example of Taylor's Expansion. Instead, in the following discourse, Euler's Formula is derived by way of exponential-growth. Passing to the complex-realm and applying DeMoivre's Theorem, it inevitably becomes circular.

Several important concepts in Calculus are reinforced by this lecture; as the number of edges of the polygon approach infinity, the polygon approaches that of a circle, which is readily grasped by the students visually. Every student, at every skill level, that attends this lecture will take something of value away from it. Many of the students have seen i before in Precalculus, and those that have not will gain a valuable exposure to this ubiquitous imaginary number.

Important Trigonometry Limits

We will take two limits for granted, that will be used in forthcoming discussion. These appear as Larson's[**1**] Theorem 1.9, the first is proven there:

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

This approximation will also used as: When $\theta \approx 0$, then $\sin\theta \approx \theta$.

A good explanation for the second limit can be found [1]:

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$$

This approximation will also be used as: When $\theta \approx 0$, then $1-\cos\theta \approx 0$, and converges faster than θ, in other words $\cos\theta \approx 1$, and it converges to 1 quickly.

[1] `http://math.hws.edu/~mitchell/Math130F12/tufte-latex/TrigLimits.pdf`

These two approximations are easily illustrated by the following table, where one sees the stark difference between cosine and sine as the angle approaches zero:

Theta(in radians)	Cosine	Sine
0.1	0.99500416527803	0.0998334166468282000
0.01	0.99995000041667	0.0099998333341666600
0.001	0.99999950000004	0.0009999998333333420
0.0001	0.99999999500000	0.0000999999998333333
0.00001	0.99999999995000	0.0000099999999998333
0.000001	0.99999999999950	0.0000009999999999998
0.0000001	1.00000000000000	0.0000001000000000000

Exponential Growth

We also provide validation for continuous exponential growth, which is worth reviewing briefly even though it was covered in Precalculus.

The constant e becomes approached as the number of compoundings n become large. Here, the growth rate is 100%, and the time interval $t = 1$. Then, we let n grow larger and see that the compounding equation begins to approach the ideal of "continuous compounding" by way of computation:

n	(1+1/n)^n		
1	2.000000000000		
10	2.593742460100		
100	2.704813829422		
1000	2.716923932236		
10000	2.718145926824		
100000	2.718268237198		
1000000	2.718280469156		
10000000	2.718281693980		
100000000	2.718281786396		
1000000000	2.718282030815		
10000000000	2.718282053235	2.71828182845905	e

Next, once we see that $(1 + 1/n)^n \approx e$, we can adapt this approximation to see that the compounding equation approximates continuous compounding for n large enough. This is the first definition that many textbooks adopt, namely that:

$e = \lim_{n\to\infty} (1 + 1/n)^n$. I will be using this as validation for Euler's Formula.

Let r be the desired growth or decay rate. Replacing n with n/r is okay, because if our rate is less than 100%, then n/r becomes larger since $r < 1$, and our calculation converges even faster. On the other hand, if $r > 1$, then it is true that n/r becomes smaller, but I claim that we only need to take n out further in that case to get our calculation to converge to the desired level of accuracy. Thus the claim is that:

$$\left(1 + \frac{1}{n/r}\right)^{n/r} \approx e$$

The next step is to divide $1/(n/r) = r/n$. Also, taking the r'th power of both sides yields: $\left(1+\frac{r}{n}\right)^n = \left(\left(1+\frac{1}{n/r}\right)^{n/r}\right)^r \approx e^r$

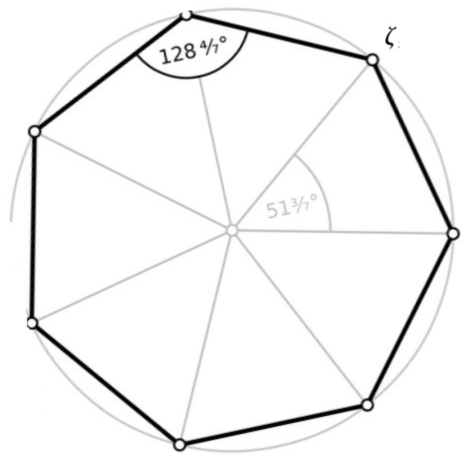
Next, take the t'th power of both sides to introduce time-increments into the equation: $\left(1+\frac{r}{n}\right)^{nt} \approx e^{rt}$. Finally, multiply both sides of the equation by the initial value P, and we have successfully derived the continuous-compounding equation[2]:

$$P\left(1+\frac{r}{n}\right)^{nt} \approx Pe^{rt}$$

DeMoivre - Polygons in the Complex Plane

One of the many many reasons that DeMoivre's theorem is so useful is that it gives us a really excellent way to map polygons.

Consider the example of an heptagon. Dividing the unit-circle into seven equal arcs reveals each to be $\theta = 2\pi/7$



One of the vertices should be $(1, 0)$. The next vertex, we label ζ, and the special math word for it is "primitive." Notice that $\zeta = \cos\theta + i\sin\theta$, since the polygon is inscribed in the unit-circle.

Notice that the coordinates of the next vertex is $\zeta_2 = \cos(2\theta) + i\sin(2\theta)$ Next, squaring the first vertex reveals: $\zeta^2 = (\cos\theta + i\sin\theta)\cdot(\cos\theta + i\sin\theta) = \cos^2\theta + i(2\cos\theta\sin\theta) - \sin^2\theta = (\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta)$. By applying double-angle trig identities, we see that $\zeta^2 = \cos(2\theta) + i\sin(2\theta)$, which is the coordinates of the next vertex! $\zeta^2 = \zeta_2$.

Cubing the primitive vertex, and applying sum-angle identities reveals

$$\begin{aligned}\zeta^3 = \zeta^2\cdot\zeta = (\cos(2\theta) + i\sin(2\theta))\cdot(\cos\theta + i\sin\theta)\\ = \cos(2\theta)\cos\theta + i\cos(2\theta)\sin\theta + i\sin(2\theta)\cos\theta - \sin(2\theta)\sin\theta\\ = (\cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta) + i(\cos(2\theta)\sin\theta + \sin(2\theta)\cos\theta)\\ = \cos(3\theta) + i\sin(3\theta)\end{aligned}$$

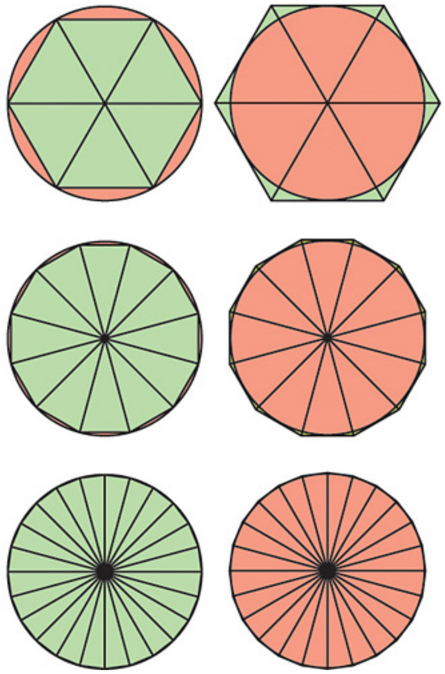
So that the m'th vertex is $\zeta^m = \cos m\theta + i\sin m\theta$, and $\zeta^7 = (1, 0) = \zeta^0$.

One could just as easily use this process to build the regular n-gon, since $\theta = 2\pi/n$, and then $\zeta = \cos\theta + i\sin\theta$, the primitive vertex, and the subsequent vertices: $\zeta^m =$

[2]I will regard this argument as common-property since it appears in numerous sources

$\cos m\theta + i \sin m\theta$.

*Notice that as the number of vertices of our regular n-gon becomes larger and larger, the polygon will begin to approximate a circle, the edges becoming finer and finer.



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Euler's Formula

For any angle θ, notice that θ/n goes to zero as n becomes larger and larger, so that for n large enough, $\theta/n \approx 0$, and: $\cos\left(\frac{\theta}{n}\right) + i \sin\left(\frac{\theta}{n}\right) \approx 1 + i\frac{\theta}{n}(*)$.

Next, recall that negative and rational exponents made perfect sense when they were explored, and irrational exponents "almost" made sense from an approximation standpoint. However, at this point we are going to jump into the deep-end of abstraction, and attempt to discuss a complex-exponent! Even the existence of i is highly questionable, to attempt to take a number to the power of i is even more mysterious. However, the result is of such fundamental importance and beauty that one wonders.

DeMoivre allows us to take the n'th power of both sides of $(*)$ to get $\cos\theta + i\sin\theta \approx \left(1 + i\frac{\theta}{n}\right)^n$. Now, whatever $e^{i\theta}$ is, it should be approximated by the same as e^r. Therefore, for n large enough, we have:

$$\cos\theta + i\sin\theta \approx \left(1 + \frac{i\theta}{n}\right)^n \approx e^{i\theta}$$

Notice that the left-most and right-most sides have no reference to n, only that n needs to be "large enough" to make the approximation "accurate enough." Now, we will do a Calculus-thing, letting n actually go to infinity will cause the approximations

[3]Image:`http://graphics8.nytimes.com/images/2010/04/04/opinion/04strogatz7/04strogatz7-custom1.jpg`

to become exact! And: $\cos\theta + i\sin\theta = e^{i\theta}$, and the complex exponential actually maps out the unit-circle, in much the same way as DeMoivre maps out a polygon!

A Foundational Limit

It is now possible to show that $\lim_{z\to 0}\frac{e^z-1}{z} = 1$, by approaching zero from a complex direction:

$$\lim_{z\to 0}\frac{e^z-1}{z} = \lim_{h\to 0}\frac{e^{ih}-1}{ih}(*)$$

Next, by Euler's Formula:

$$(*) = \lim_{h\to 0}\frac{\cos h + i\sin h - 1}{ih} = \lim_{h\to 0}\frac{1}{i}\underbrace{\frac{\cos h - 1}{h}}_{=0} + \underbrace{\frac{\sin h}{h}}_{=1} = 1$$

Derivative of the Exponential

With the previous limit, it is now possible to diferentiate the exponential function using exponent rules:

$$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x\cdot e^h - e^x}{h} = \lim_{h\to 0}\frac{e^x(e^h-1)}{h} = e^x\lim_{h\to 0}\frac{e^h-1}{h} = e^x$$

Derivative of the Natural Logarithm

Next, we differentiate the Natural Logarithm function, which is defined as the inverse function to the exponential: $\ln x = y \Leftrightarrow x = e^y$. We can now differentiate using implicit-differentiation and the derivative of the exponential: $1 = \frac{d}{dx}e^y = e^y\frac{dy}{dx}$, solving for y' yields: $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$.

Definition 1 is implied

The two definitions will fall-out from what we have shown thus far. The first is:

Let $E = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$, we want to show that $E = e$.

Since the natural logarithm is continuous, taking the logarithm of both sides yields:

$$\ln E = \lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}(*)$$

Next, since $1/x \to 0$ as $x\to\infty$, and does so in a positive/decreasing way, (*) is equal to $\lim_{t\to 0^+}\frac{\ln(1+t)}{t}$. Since $\ln 1 = 0$, can subtract it from the numerator, and since $\ln$ is differentiable, we have:

$$\ln E = \lim_{t\to 0^+} \frac{\ln(1+t)-\ln 1}{t} = \frac{d}{dx}\left[\ln x\right]_{x=1} = \left.\frac{1}{x}\right|_{x=1} = 1$$

Since $\ln E = 1$, this implies that $E = e$, as desired. Source: Appendix of Larson[**1**].

Definition 2 is implied

Namely, that by utilizing the Fundamental Theorem of Calculus:

$$\text{area}(1/x, 1, e) = \int_1^e \frac{1}{x}dx = \ln x\Big|_1^e = \ln e - \ln 1 = 1$$

Conclusion

These expoundings have arrived finally at “the most natural” definition for e, one that presupposes the platonic existence of not only the irrational $\pi = \dfrac{\text{Circumference of any Perfect-Circle}}{\text{Diameter of that same Circle}}$, but also that “unknowable” $i = \sqrt{-1}$, which at once philisophically dubious questions arise, and require further consideration. One must tread carefully or else risk hand-waiving. Indeed, one derives the third fundamental constant from the existance of the other two, that $e^{i\pi} := -1$, and all follows thusly. And whilest the reasoning remains circular at the end of all of this, it is, in fact a circle after-all, perhaps it is unavoidable.

References

1. Larson and Edwards, Calculus 9th E.*Brooks Cole Cengage Learning* (2010).

