Fractional convexity maximum principle*

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Abstract

We construct an anisotropic, degenerate, fractional operator that nevertheless satisfies a strong form of the maximum principle. By applying such an operator to the concavity function associated to the solution of an equation involving the usual fractional Laplacian, we obtain a fractional form of the celebrated convexity maximum principle devised by Korevaar in the 80's. Some applications are discussed.

1 Introduction

The celebrated *convexity maximum principle* was proved by Nick Korevaar [13, 14] to answer a question posed by his advisor, prof. Robert Finn, concerning convexity of capillary surfaces in convex pipes. Korevaar's idea gave birth to a number of subsequent contributions, especially due to Kawohl [8, 9] and Kennington [10, 11, 12]. To be more specific, in order to prove the convexity of a continuous function $u(x)$ in a convex domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, the *concavity function*

$$C(x, y) = 2\, u(\tfrac{x+y}{2}) - u(x) - u(y), \quad x, y \in \Omega \tag{1.1}$$

was introduced (see [9, p. 113, (3.30)]). One may also deal with the function $c(x, y, \lambda) = (1 - \lambda)\, u(x) + \lambda\, u(y) - u((1 - \lambda)\, x + \lambda y)$ for $x, y \in \Omega$ and $\lambda \in [0, 1]$ as in [12, p. 687], but we prefer to keep $\lambda = \frac{1}{2}$ for simplicity. This is enough because $u(x)$ is continuous.

The convexity of $u(x)$ in Ω is equivalent to the inequality $C(x, y) \leq 0$ in the Cartesian product $\Omega^2 = \Omega \times \Omega$. In order to prove this inequality, the first step amounts to exclude that $C(x, y)$ attains an interior, positive maximum, i.e., to prove a maximum principle. This is the reason why such kind of result became known as *convexity maximum principle*. Concerning the method of proof, in the mentioned

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papers the conclusion is obtained by contradiction, arguing at an interior point (x_0, y_0) where the concavity function supposedly becomes extremal. In [6], instead, an elliptic degenerate inequality satisfied by $C(x, y)$ is constructed, starting from the equation satisfied by $u(x)$. For instance, if a function $u \in C^2(\Omega)$ is a classical solution of the torsion equation $-\Delta u = 1$ in Ω, then the following equality holds:

$$\sum_{i=1}^{N} \left(\frac{\partial^2 C}{\partial x_i\, \partial x_i} + 2\frac{\partial^2 C}{\partial x_i\, \partial y_i} + \frac{\partial^2 C}{\partial y_i\, \partial y_i} \right) = 0 \quad \text{in } \Omega^2. \tag{1.2}$$

Equation (1.2), although degenerate, implies a maximum principle (see [6]). Independently of the method used for excluding that $C(x, y)$ has interior positive maxima, in order to conclude that $C(x, y) \le 0$ in the whole domain Ω^2 it is necessary to ensure that $C(x, y) \le 0$ at the boundary of Ω^2, i.e., when at least one of x, y lies on $\partial\Omega$ (here we are assuming $u \in C^0(\overline{\Omega})$). Unfortunately this turns out to be a difficult task, not only from a technical point of view, but also because the claim is false, even in very simple cases. For instance, if Ω is a smooth, convex, bounded domain, then the solution v of

$$\begin{cases} -\Delta v = 1 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

is known to be *power-concave* in the sense that the function

$$u(x) = v^{\frac{1}{2}}(x) \tag{1.4}$$

is concave (see [9, p. 120, Example 3.4] and [12, p. 697, Remark 4.2.1]). Thus, the point of view is slightly changed. In fact, thanks to the exponent $\frac{1}{2}$ in (1.4), the gradient Du becomes infinite along $\partial\Omega$, and this implies that the concavity function $C(x, y)$ cannot attain a negative minimum on $\partial(\Omega^2)$. Hence, if u were not concave, the function $C(x, y)$ would attain a negative minimum in the interior of Ω^2. Furthermore, a minimum principle holds (see [9, p. 116, Theorem 3.13] and [12, p. 691, Theorem 3.1]). It follows that the minimum of $C(x, y)$ over $\overline{\Omega}^2$ equals zero, and therefore $\sqrt{v(x)}$ is a concave function.

Different approaches have also been used for proving that solutions of elliptic PDE's are convex or concave: for instance, comparison of $u(x)$ with its convex envelope [2] and constant-rank Hessian theorems [15].

Problem (1.3) also provides an example where all strategies for proving concavity must necessarily fail. Indeed, let $\Omega \subset \mathbb{R}^2$ be an equilateral triangle. In this case the solution v is known explicitly, and we have $Dv = 0$ at each vertex of $\partial\Omega$: therefore the (positive) function v is *not concave.* Nevertheless, since v is power-concave, then *its level sets are convex:* this was initially proved by Makar-Limanov [17] in dimension 2 (see also [1, 16]).

In the present paper we extend the convexity maximum principle to continuous functions $u \in C^0(\mathbb{R}^N)$ satisfying the equation

$$(-\Delta)^s\, u(x) = f(u) \qquad \text{in } \Omega, \tag{1.5}$$

where $(-\Delta)^s$, $s \in (0,1)$, is the *fractional Laplacian*

$$(-\Delta)^s u(x_0) = c_{N,s} \, \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x_0) - u(x)}{|x_0 - x|^{N+2s}} \, dx \tag{1.6}$$
$$= c_{N,s} \lim_{\varepsilon \to 0^+} \int_{|x_0 - x| > \varepsilon} \frac{u(x_0) - u(x)}{|x_0 - x|^{N+2s}} \, dx.$$

Here P.V. stands for *principal value*, and the constant $c_{N,s}$ (which is found, for instance, in [3, Remark 3.11]) is given by

$$c_{N,s} = \frac{4^s \, s \, \Gamma(\frac{N}{2} + s)}{\pi^{\frac{N}{2}} \, \Gamma(1 - s)}.$$

A continuous function $u \colon \mathbb{R}^N \to \mathbb{R}$ is a solution of (1.5) if the integral in (1.6) converges for all $x_0 \in \Omega$, and if the equation in (1.5) is satisfied pointwise. To give an idea of the applications of the tools developed afterwards, let us quote a statement that holds under rather simple assumptions on $f(t)$.

Theorem 1.1 (Convexity maximum principle, sample statement 1)**.** *Let $u \in C^0(\mathbb{R}^N)$ be a solution of (1.5) in a convex, bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$. Suppose that the function $f(t)$ in (1.5) is monotone non-increasing and convex. If $C(x, y) \leq 0$ for all $(x, y) \notin \Omega^2$, then $C(x, y) \leq 0$ in all of $\mathbb{R}^{2N}$.*

We also put into evidence the following surprising property of convex functions in two variables satisfying equation (1.5) in a (possibly unbounded, or even very small) convex domain Ω.

Theorem 1.2 (Convexity maximum principle, sample statement 2)**.** *Let $s \in [\frac{1}{2}, 1)$, and let $u \colon \mathbb{R}^2 \to \mathbb{R}$ be a convex function in the plane satisfying equation (1.5) in a convex domain Ω. Suppose that the function f in (1.5) is negative, and that $-1/f(t)$ is a convex function. If there exist two distinct points $x_0, y_0 \in \Omega$ such that $u(\frac{x_0+y_0}{2}) = \frac{u(x_0)+u(y_0)}{2}$, then the graph of u over $\mathbb{R}^2$ is a ruled surface.*

The convexity maximum principle is obtained by showing that $C(x, y)$ satisfies a degenerate inequality (see Section 4) extending (1.2) to the fractional case. The inequality is constructed by introducing in Section 2 a convenient degenerate operator, denoted by $(-\Delta_A)^s$, which is proved to satisfy the maximum principle. The computation of $(-\Delta_A)^s C$ in terms of $(-\Delta)^s u$ is done in Section 3. The two sample statements given above are proved in the last section.

Equations posed in *the exterior* of a convex body K have also been considered in the literature: confining ourselves to fractional operators, we mention that the solution of

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u = 0 & \text{in } \mathbb{R}^N \setminus K; \\ u = 1 & \text{in } K; \\ \lim\limits_{|x| \to +\infty} u(x) = 0 \end{cases}$$

is shown to have *convex level sets* in [18].

2 Degenerate anisotropic fractional Laplacian

The following definition introduces a linear, non-local operator, denoted by $(-\Delta_A)^s$, which includes the fractional Laplacian as a special case. Apart from being non-local, such an operator may also be degenerate due to the fact that the domain of integration $A(\boldsymbol{x}_0)$ indicated in (2.1) is allowed to have lower dimension than the whole space. Accordingly, a *degenerate* strong maximum principle holds (see Theorem 2.2).

Definition 2.1. (Degenerate anisotropic fractional Laplacian) Let G be an open subset of $\mathbb{R}^m$, $m \geq 1$. For $\boldsymbol{x}_0 \in G$, choose an affine subspace $A(\boldsymbol{x}_0) \subset \mathbb{R}^m$ of positive dimension $k = k(\boldsymbol{x}_0) \leq m$ passing through $\boldsymbol{x}_0$. The operator $(-\Delta_A)^s$ is defined as follows:

$$(-\Delta_A)^s\, w(\boldsymbol{x}_0) = c_{k(\boldsymbol{x}_0),s}\ \text{P.V.}\int_{A(\boldsymbol{x}_0)} \frac{w(\boldsymbol{x}_0) - w(\boldsymbol{x})}{|\boldsymbol{x}_0 - \boldsymbol{x}|^{N+2s}}\, d\mathcal{H}^{k(\boldsymbol{x}_0)}(\boldsymbol{x}) \tag{2.1}$$

provided that the integral in the right-hand side is well defined. The notation $d\mathcal{H}^{k(\boldsymbol{x}_0)}$ represents the $k(\boldsymbol{x}_0)$-dimensional Hausdorff measure.

When $k(\boldsymbol{x}) \equiv m$, i.e., when $A(\boldsymbol{x}) = \mathbb{R}^m$ for all $\boldsymbol{x} \in G$, the operator $(-\Delta_A)^s$ is non-degenerate and coincides with the usual fractional Laplacian $(-\Delta)^s$. In such a case, a strong minimum principle is found in [7]. Apart from considering the more general case $0 < k(\boldsymbol{x}) \leq m$, here we also put into evidence that the conclusion *propagates* to the whole space (see Remark (4)).

Theorem 2.2 (Anisotropic strong maximum principle)**.** *Let k be any function from G to the set $\{1, \ldots, m\}$, and let A be a function that associates to every $\boldsymbol{x} \in G$ a $k(\boldsymbol{x})$-dimensional affine subspace $A(\boldsymbol{x}) \subset \mathbb{R}^m$. Let $w\colon \mathbb{R}^m \to \mathbb{R}$ be an upper semicontinuous function satisfying*

$$(-\Delta_A)^s\, w(\boldsymbol{x}) \leq b(\boldsymbol{x})\, w(\boldsymbol{x}) \quad \text{in } G \tag{2.2}$$

where $b\colon G \to \mathbb{R}$ is any real-valued function.

(*i*) *Assume that G is bounded, $w \leq 0$ in $\mathbb{R}^m \setminus G$, and $b \leq 0$ in G. Then $w \leq 0$ in all of $\mathbb{R}^m$.*

(*ii*) *If $w \leq 0$ in all of $\mathbb{R}^m$, and if $w(\boldsymbol{x}_0) = 0$ at some $\boldsymbol{x}_0 \in G$, then $w(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in A(\boldsymbol{x}_0)$.*

Remarks. (1) Claim (*ii*) holds even though G is unbounded, and irrespectively for the sign of b.

(2) The two claims may be used together: indeed, under the assumptions of Claim (*i*), it follows that w is non-positive and therefore Claim (*ii*) applies.

(3) The degeneracy of the operator $(-\Delta_A)^s$ for $k < m$ reflects on Claim (*ii*): indeed, from the equality $w(\boldsymbol{x}_0) = 0$ it is not possible to deduce $w = 0$ in all of $\mathbb{R}^m$ as in the non-degenerate case $k(\boldsymbol{x}_0) = m$.

(4) The non-local character of the operator $(-\Delta_A)^s$, for $k \le m$, also appears in Claim (ii): the claim asserts that $w(\boldsymbol{x}) = 0$ for all $\boldsymbol{x}$ in the affine subspace $A(\boldsymbol{x}_0)$, i.e. even though $\boldsymbol{x} \notin G$ and independently from the geometry (connectedness) of G. By contrast, a similar result does not hold for the Laplacian. For example, if $-\Delta u \le 0$ in an open set $\Omega \subset \mathbb{R}^N$, and if $u = 0$ on $\partial\Omega$ and $u(\boldsymbol{x}_0) = 0$ at some $\boldsymbol{x}_0 \in \Omega$, then the function u may well be negative in some connected component of Ω distinct from the one containing $\boldsymbol{x}_0$.

Proof of Theorem 2.2. Claim (i). If w were positive somewhere in G, then, by the compactness of $\overline{G}$ and using the fact that $w \le 0$ in $\mathbb{R}^m \setminus G$, w would reach its (positive) maximum at some $\boldsymbol{x}_0 \in G$. By (2.1) we may write

$$\begin{aligned}(-\Delta_A)^s\, w(\boldsymbol{x}_0) \;&\ge\; c_{k(\boldsymbol{x}_0),s}\ \text{P.V.}\!\int_{G\cap A(\boldsymbol{x}_0)} \frac{w(\boldsymbol{x}_0) - w(\boldsymbol{x})}{|\boldsymbol{x}_0 - \boldsymbol{x}|^{N+2s}}\, d\mathcal{H}^{k(\boldsymbol{x}_0)}(\boldsymbol{x})\\ &+\; c_{k(\boldsymbol{x}_0),s}\int_{A(\boldsymbol{x}_0)\setminus G} \frac{w(\boldsymbol{x}_0)}{|\boldsymbol{x}_0 - \boldsymbol{x}|^{N+2s}}\, d\mathcal{H}^{k(\boldsymbol{x}_0)}(\boldsymbol{x}).\end{aligned}$$

The first integral is non-negative because $w(\boldsymbol{x}_0) = \max w$. Concerning the second integral, we have omitted P.V. because $\boldsymbol{x}_0$ is interior to G. Furthermore, since $k(\boldsymbol{x}_0) > 0$ and G is bounded, the difference $A(\boldsymbol{x}_0) \setminus G$ has an infinite $k(\boldsymbol{x}_0)$-dimensional measure. This and $w(\boldsymbol{x}_0) > 0$ imply that the second integral is strictly positive. Consequently, we get that $(-\Delta_A)^s\, w(\boldsymbol{x}_0) > 0$. However, $b(\boldsymbol{x}_0) \le 0$ by assumption, hence $b(\boldsymbol{x}_0)\, w(\boldsymbol{x}_0) \le 0$, thus contradicting (2.2). Thus, we must have $w \le 0$ in all of $\mathbb{R}^m$, as claimed.

Claim (ii). Suppose, by contradiction, that $w(\boldsymbol{x}_1) < 0$ at some $\boldsymbol{x}_1 \in A(\boldsymbol{x}_0)$. Then, by upper semicontinuity, there exists $\varepsilon_1 > 0$ such that $-w(\boldsymbol{x}) \ge \varepsilon_1$ for all $\boldsymbol{x}$ in the ball $B_1 = B(\boldsymbol{x}_1, \varepsilon_1)$. By reducing ε_1 if necessary, we may assume that $\boldsymbol{x}_0 \notin \overline{B}_1$, thus avoiding singularities in the second integral below. Recalling that $w(\boldsymbol{x}_0) = 0$, we may write

$$\begin{aligned}(-\Delta_A)^s\, w(\boldsymbol{x}_0) \;&\ge\; c_{k(\boldsymbol{x}_0),s}\ \text{P.V.}\!\int_{A(\boldsymbol{x}_0)\setminus B_1} \frac{-w(\boldsymbol{x})}{|\boldsymbol{x}_0 - \boldsymbol{x}|^{N+2s}}\, d\mathcal{H}^{k(\boldsymbol{x}_0)}(\boldsymbol{x})\\ &+\; c_{k(\boldsymbol{x}_0),s}\int_{B_1\cap A(\boldsymbol{x}_0)} \frac{\varepsilon_1}{|\boldsymbol{x}_0 - \boldsymbol{x}|^{N+2s}}\, d\mathcal{H}^{k(\boldsymbol{x}_0)}(\boldsymbol{x}).\end{aligned}$$

As before, the first integral non-negative because now $w \le 0$ in $\mathbb{R}^m$. Furthermore, the second integral is strictly positive because the intersection $B_1 \cap A(\boldsymbol{x}_0)$ has a positive $k(\boldsymbol{x}_0)$-dimensional measure. Hence we get $(-\Delta_A)^s\, w(\boldsymbol{x}_0) > 0$. However, since $b(\boldsymbol{x}_0)\, w(\boldsymbol{x}_0) = 0$, a contradiction with (2.2) is reached. Thus, we must have $w(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in A(\boldsymbol{x}_0)$, and the proof is complete. □

3 Fundamental expansion

In order to investigate the convexity of a solution u of (1.5), we will apply the operator $(-\Delta_A)^s$ introduced in (2.1) to the concavity function $C(x,y)$. To this aim we let $m = 2N$ and $k(\boldsymbol{x}) = k(x,y) \equiv N$. Furthermore, in the present section we suitably choose the subspace $A(\boldsymbol{x}) = A(x,y)$ and give an expression of $(-\Delta_A)^s\, C$ in terms of $(-\Delta)^s\, u$. To this purpose, we start from a spectral analysis of the matrix M in (3.2). Such a matrix was used in [6] as the characteristic matrix of a local operator to be applied to $C(x,y)$. It is worth recalling that the idea of a rotation of the coordinate frame in order to give a PDE a more convenient form goes back to d'Alembert, who investigated the wave equation (see [5, p. 216]).

Proposition 3.1 (Spectral analysis)**.** *Let I be the $N \times N$ unit matrix, $N \geq 1$, and let σ, τ be two real numbers such that $\sigma^2 + \tau^2 > 0$. Furthermore, let $\omega \in [0, 2\pi)$ be the angle determined uniquely by*

$$\cos\omega = \frac{\sigma}{\sqrt{\sigma^2+\tau^2}}, \qquad \sin\omega = \frac{\tau}{\sqrt{\sigma^2+\tau^2}}. \tag{3.1}$$

Then, the $2N \times 2N$ symmetric matrix $M = M(\sigma,\tau)$ given by

$$M = \begin{pmatrix} \sigma^2\, I & \sigma\tau\, I \\ \sigma\tau\, I & \tau^2\, I \end{pmatrix} \tag{3.2}$$

is transformed into a diagonal matrix by means of the orthogonal, symmetric matrix $P = P(\omega)$ defined as follows:

$$P = \begin{pmatrix} (\cos\omega)\, I & (\sin\omega)\, I \\ (\sin\omega)\, I & (-\cos\omega)\, I \end{pmatrix}. \tag{3.3}$$

More precisely, we have

$$P^T M\, P = (\sigma^2+\tau^2) \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

where the exponent ^{T} denotes transposition, and 0 is the $N \times N$ null matrix. The matrix $M(\sigma,\tau)$ has two distinct eigenvalues: the eigenvalue $\lambda_0 = 0$ and the eigenvalue $\lambda_1 = \sigma^2 + \tau^2$, each one of multiplicity N. The corresponding eigenspaces $V_0(\omega)$ and $V_1(\omega)$ are given by

$$V_0(\omega) = \left\{ (x,y) \in \mathbb{R}^{2N} \;\middle|\; \tbinom{x}{y} = P\,\tbinom{0}{\eta},\ \eta \in \mathbb{R}^N \setminus \{0\} \right\};$$

$$V_1(\omega) = \left\{ (x,y) \in \mathbb{R}^{2N} \;\middle|\; \tbinom{x}{y} = P\,\tbinom{\xi}{0},\ \xi \in \mathbb{R}^N \setminus \{0\} \right\}.$$

Proof. All claims are easily verified by computation. ∎

We can now prove the following fundamental lemma, which gives an expansion of $(-\Delta_A)^s\, C$ in terms of $(-\Delta)^s\, u$ provided that A is defined as in (3.4).

Lemma 3.2 (Fundamental expansion)**.** *Let $u\colon \mathbb{R}^N \to \mathbb{R}$ be a continuous function such that the fractional Laplacian $(-\Delta)^s\, u(x)$ is well defined for all x in a convex domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$. Fix $x_0, y_0 \in \Omega$, and let $z_0 = (x_0 + y_0)/2$. Choose an angle $\omega \in [0, 2\pi)$ and define the N-dimensional affine subspace $A \subset \mathbb{R}^{2N}$ as follows:*

$$\begin{aligned} A &= (x_0, y_0) + V_1(\omega) \\ &= \left\{ (x,y) \in \mathbb{R}^{2N} \;\middle|\; \binom{x}{y} = \binom{x_0}{y_0} + P\binom{\xi}{0},\ \xi \in \mathbb{R}^N \setminus \{0\} \right\}. \end{aligned} \tag{3.4}$$

Then

$$\begin{aligned} (-\Delta_A)^s\, C(x_0, y_0) &= 2\left(\tfrac{|\cos\omega + \sin\omega|}{2}\right)^{2s} (-\Delta)^s\, u(z_0) \\ &\quad - |\cos\omega|^{2s}\, (-\Delta)^s\, u(x_0) - |\sin\omega|^{2s}\, (-\Delta)^s\, u(y_0). \end{aligned} \tag{3.5}$$

Proof. Since the operator $(-\Delta_A)^s$ is linear, and by (1.1), we start by computing $(-\Delta_A)^s\, w(x_0, y_0)$, where $w(x,y) = u(z)$ and $z = \frac{x+y}{2}$. By means of the matrix $P = P(\omega)$ defined in (3.3), we perform the change of variables $\binom{x}{y} = \binom{x_0}{y_0} + P\binom{\xi}{0}$ and find

$$\begin{aligned} (-\Delta_A)^s\, u(\tfrac{x+y}{2})_{|(x_0,y_0)} &= c_{N,s}\ \mathrm{P.V.}\int_A \frac{u(z_0) - u(z)}{|(x_0,y_0) - (x,y)|^{N+2s}}\, d\mathcal{H}^N(x,y) \\ &= c_{N,s}\ \mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(z_0) - u(z_0 + \frac{\cos\omega + \sin\omega}{2}\,\xi)}{|\xi|^{N+2s}}\, d\xi. \end{aligned}$$

In the case when $\cos\omega + \sin\omega = 0$, we immediately obtain $(-\Delta_A)^s\, u(\frac{x+y}{2})_{|(x_0,y_0)} = 0$. Otherwise we take $z = z_0 + \frac{\cos\omega+\sin\omega}{2}\,\xi$ as the new variable of integration. Since $dz = (\frac{|\cos\omega+\sin\omega|}{2})^N\, d\xi$, we arrive at

$$\begin{aligned} (-\Delta_A)^s\, u(\tfrac{x+y}{2})_{|(x_0,y_0)} &= \left(\tfrac{|\cos\omega+\sin\omega|}{2}\right)^{2s} c_{N,s}\ \mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(z_0) - u(z)}{|z_0 - z|^{N+2s}}\, dz \\ &= \left(\tfrac{|\cos\omega+\sin\omega|}{2}\right)^{2s} (-\Delta)^s\, u(z_0). \end{aligned}$$

Note that the last equality collects the case $\cos\omega + \sin\omega = 0$ as well. To proceed further, let us compute $(-\Delta_A)^s\, w(x_0, y_0)$ where the function w, different from before, is given by $w(x,y) = u(x)$. Denote by $\pi_1(x,y) = x$ the first canonical projection over $\mathbb{R}^N$. Using again the change of variables $\binom{x}{y} = \binom{x_0}{y_0} + P\binom{\xi}{0}$ we find

$$\begin{aligned} (-\Delta_A)^s\, u(\pi_1(x,y))_{|(x_0,y_0)} &= c_{N,s}\ \mathrm{P.V.}\int_A \frac{u(x_0) - u(x)}{|(x_0,y_0) - (x,y)|^{N+2s}}\, d\mathcal{H}^N(x,y) \\ &= c_{N,s}\ \mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(x_0) - u(x_0 + (\cos\omega)\,\xi)}{|\xi|^{N+2s}}\, d\xi. \end{aligned}$$

If $\cos\omega = 0$ we immediately obtain $(-\Delta_A)^s\, u(\pi_1(x,y))_{|(x_0,y_0)} = 0$. Otherwise we use $x = x_0 + (\cos\omega)\,\xi$ as the new variable of integration. Since $dx = |\cos\omega|^N\, d\xi$, we arrive at

$$\begin{aligned}(-\Delta_A)^s\, u(\pi_1(x,y))_{|(x_0,y_0)} &= |\cos\omega|^{2s}\, c_{N,s}\ \mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(x_0)-u(x)}{|x_0-x|^{N+2s}}\, dx \\ &= |\cos\omega|^{2s}\, (-\Delta)^s\, u(x_0).\end{aligned}$$

The equality above continues to hold when $\cos\omega = 0$. Finally, a similar computation shows that $(-\Delta_A)^s\, u(\pi_2(x,y))_{|(x_0,y_0)} = |\sin\omega|^{2s}\,(-\Delta)^s\, u(y_0)$, where $\pi_2(x,y) = y$ is the second canonical projection over $\mathbb{R}^N$. The lemma follows. □

4 A non-local inequality

We establish a non-local inequality of the form (2.2) satisfied by the function $C(x,y)$ in the set $G = \Omega^2$. More precisely, we give conditions on the function f in (1.5) sufficient to obtain such an inequality through the expansion (3.5). We state for first the general assumption (4.1), then we discuss some special cases where such an assumption holds.

Lemma 4.1 (Non-local inequality)**.** *Let Ω be a convex domain in $\mathbb{R}^N$, and let u be a (continuous) solution of (1.5). Denote by $U = \{\, t \in \mathbb{R} \mid t = u(x) \text{ for some } x \in \Omega\,\}$ the interval described by $u(x)$ as x ranges in Ω. Suppose that for every couple of real numbers $t_1, t_2 \in U$ there exists an angle $\omega = \omega(t_1,t_2) \in [0, 2\pi)$ such that*

$$2\left(\tfrac{|\cos\omega+\sin\omega|}{2}\right)^{2s} f(\tfrac{t_1+t_2}{2}) - |\cos\omega|^{2s} f(t_1) - |\sin\omega|^{2s} f(t_2) \le 0. \tag{4.1}$$

Then for every $(x_0,y_0) \in G$ we may define the N-dimensional affine subspace $A = A(x_0,y_0) \subset \mathbb{R}^{2N}$ by letting $\omega = \omega(u(x_0), u(y_0))$ in (3.4), and we have

$$(-\Delta_A)^s\, C(x_0,y_0) \le b(x_0,y_0)\, C(x_0,y_0) \quad \textit{for all } (x_0,y_0) \in G \tag{4.2}$$

where the coefficient $b(x_0,y_0)$ is given by

$$b(x_0,y_0) = \begin{cases} 2\left(\frac{|\cos\omega+\sin\omega|}{2}\right)^{2s} \frac{f(u(\frac{x_0+y_0}{2})) - f(\frac{u(x_0)+u(y_0)}{2})}{C(x_0,y_0)}, & C(x_0,y_0) \ne 0; \\ 0 & C(x_0,y_0) = 0. \end{cases}$$

Proof. The conclusion follows from Lemma 3.2 by using assumption (4.1) and the identity

$$2\left(\tfrac{|\cos\omega+\sin\omega|}{2}\right)^{2s} f(u(z_0)) = 2\left(\tfrac{|\cos\omega+\sin\omega|}{2}\right)^{2s} f(\tfrac{u(x_0)+u(y_0)}{2}) + b(x_0,y_0)\, C(x_0,y_0).$$

□

Remarks. (1) If the function f is monotone non-increasing then $b(x,y) \le 0$ in G. The last inequality is an assumption of Claim (i) of the maximum principle (Theorem 2.2).

(2) Assumption (4.1) is satisfied if $f(t) \ge 0$ for all $t \in U$. This is readily seen by letting $\omega(t_1,t_2) = \frac{3}{4}\pi$ for all $t_1,t_2 \in U$, so that $\cos\omega + \sin\omega = 0$.

(3) If f is a convex function (hence, in particular, if f is constant) then assumption (4.1) holds with $\omega(t_1,t_2) \equiv \frac{\pi}{4}$.

A further condition implying (4.1) involves the *harmonic concavity* of the function $g = -f$. For the present purposes, it is convenient to adopt the following definition:

Definition 4.2. (Harmonic concavity) A non-negative function g defined in an interval $U \subset \mathbb{R}$ is *harmonic concave* if

$$g(\tfrac{t_1+t_2}{2}) \ge \frac{2\, g(t_1)\, g(t_2)}{g(t_1) + g(t_2)}$$

for every $t_1,t_2 \in U$ such that $g(t_1) + g(t_2) > 0$.

If g is concave, then it is harmonic concave (see [12, p. 688]). In comparison to the definition in [6, 11, 12], the present one is restricted to the case $g \ge 0$: this because we will consider the power function g^{2s-1} in the proof of the following proposition. In the realm of continuous, non-negative functions, all the mentioned definitions coincide. Continuity enters in this equivalence because the definition here (as well as in [6]) involves just the middle point $\frac{t_1+t_2}{2}$ instead of the whole interval $\lambda\, t_1 + (1-\lambda)\, t_2$, $\lambda \in (0,1)$, as in [11, 12]. Finally, it is worth recalling that a *positive* continuous function g is harmonic concave if and only if $1/g$ is convex.

Proposition 4.3. *Suppose* $s \in [\frac{1}{2},1)$. *If* $f(t) \le 0$ *for all* $t \in U$, *and if the function* $g = -f$ *is harmonic concave, then* (4.1) *holds.*

Proof. Fix $t_1,t_2 \in U$. If $f(t_1) = f(t_2) = 0$, we may take ω arbitrarily and (4.1) holds because $f(\frac{t_1+t_2}{2}) \le 0$. For later purposes we choose $\omega = \frac{5}{4}\pi$. If, instead, $f(t_1) + f(t_2) > 0$, then we let $\omega \in (\frac{3}{4}\pi, \frac{7}{4}\pi)$ be the angle determined by (3.1) with $\sigma = f(t_2)$ and $\tau = f(t_1)$. Since $|\sigma+\tau| = -\sigma-\tau$, the target condition (4.1) may be rewritten as

$$\left(\tfrac{-\sigma-\tau}{2}\right)^{2s} g(\tfrac{t_1+t_2}{2}) - \tfrac{|\sigma|^{2s}\, g(t_1) + |\tau|^{2s}\, g(t_2)}{2} \ge 0. \tag{4.3}$$

If either $|\sigma| = g(t_2) = 0$ or $|\tau| = g(t_1) = 0$, then (4.3) trivially holds. Otherwise, since g is harmonic concave, in order to prove (4.3) it is enough to check that

$$\left(\tfrac{g(t_2)+g(t_1)}{2}\right)^{2s-1} g(t_1)\, g(t_2) - \tfrac{(g(t_2))^{2s}\, g(t_1) + (g(t_1))^{2s}\, g(t_2)}{2} \ge 0.$$

Dividing by $g(t_1)\, g(t_2)$ we get the equivalent inequality

$$\left(\tfrac{g(t_2)+g(t_1)}{2}\right)^{2s-1} - \tfrac{(g(t_2))^{2s-1} + (g(t_1))^{2s-1}}{2} \ge 0,$$

which holds true because the power function a^{2s-1} with $s \in [\frac{1}{2},1)$ is concave in the variable $a > 0$. □

5 Applications

Let us prove the two sample statements given in the Introduction. We start proving the following, generalized form of Theorem 1.1, which also applies to non-convex functions $f(t)$.

Theorem 5.1. *Let $u \in C^0(\mathbb{R}^N)$ be a solution of* (1.5) *in a convex, bounded domain Ω. Suppose that the function $f(t)$ in* (1.5) *satisfies* (4.1) *and is monotone non-increasing when t ranges in the interval U, image of the domain Ω trough the function u. If*

$$u(\tfrac{x+y}{2}) \leq \tfrac{u(x)+u(y)}{2}$$

whenever $x, y \notin \Omega$, as well as when $x \in \Omega$ and $y \notin \Omega$, then u is convex in $\mathbb{R}^N$.

Proof. The assumptions on u imply $C(x,y) \leq 0$ in $\mathbb{R}^{2N} \setminus \Omega^2$, those on f imply that C satisfies inequality (4.2) in the bounded domain $G = \Omega^2$, with $b \leq 0$. Since Ω is bounded, the theorem follows from Claim (i) of Theorem 2.2. □

The statement in Theorem 1.2 is a special case of the following, which makes use of the notion of harmonic concavity.

Theorem 5.2. *Let $s \in [\frac{1}{2}, 1)$, and let $u\colon \mathbb{R}^2 \to \mathbb{R}$ be a convex function in the plane satisfying equation* (1.5) *in a convex domain Ω. Suppose that the function f in* (1.5) *is non-positive and harmonic concave. If there exist two distinct points $x_0, y_0 \in \Omega$ such that $u(\frac{x_0+y_0}{2}) = \frac{u(x_0)+u(y_0)}{2}$ then the graph of u over $\mathbb{R}^2$ is a ruled surface.*

Proof. Observe, firstly, that since u is convex by assumption then whenever $C(x,y) = 0$ the graph of u contains the line segment whose endpoints are $(x, u(x)), (y, u(y)) \in \mathbb{R}^{N+1}$. This will be repeatedly used in the sequel.

In order to prove the theorem, let us apply the fractional convexity maximum principle. The assumptions on f imply that $C(x,y)$ satisfies inequality (4.2) in $G = \Omega^2$. The assumptions on u imply $C(x,y) \leq 0$ in $\mathbb{R}^4$, and there exists $(x_0, y_0) \in G$ such that $C(x_0, y_0) = 0$. Hence by Claim (ii) of Theorem 2.2 we have

$$C(x,y) = 0 \text{ for all } (x,y) \in A = A(x_0, y_0). \tag{5.1}$$

The two-dimensional affine subspace $A \subset \mathbb{R}^4$ is given by (3.4), where the angle ω is chosen as in the proof of Proposition 4.3: if $f(u(x_0)) = f(u(y_0)) = 0$ then $\omega = \frac{5}{4}\,\pi$, otherwise ω is determined by (3.1) with $\sigma = f(u(y_0))$ and $\tau = f(u(x_0))$. Since $f \leq 0$ by assumption, we get $\omega \in [\pi, \frac{3}{2}\,\pi]$. In conclusion, by (5.1) we may write $C(x,y) = 0$ for all $x, y \in \mathbb{R}^2$ given by

$$\begin{cases} x = x_0 + (\cos\omega)\,\xi \\ y = y_0 + (\sin\omega)\,\xi \end{cases}$$

as ξ ranges in $\mathbb{R}^2$. Letting $\xi = \lambda\,(x_0 - y_0)$ for $\lambda \in \mathbb{R}$, and recalling the initial observation, we see that the graph of u contains the whole straight line passing through $(x_0, u(x_0))$ and $(y_0, u(y_0))$.

Now let us turn our attention to the points $x_1 \in \mathbb{R}^2$ such that $x_1 \neq x_0+\lambda\,(x_0-y_0)$ for every $\lambda \in \mathbb{R}$. Since $\cos\omega$ and $\sin\omega$ cannot vanish simultaneously, without loss of generality suppose $\cos\omega \neq 0$. Then every x_1 as above is given by $x_1 = x_0 + (\cos\omega)\,\xi_1$ for a convenient $\xi_1 \in \mathbb{R}^2$, which in its turn defines a particular point $y_1 = y_0 + (\sin\omega)\,\xi_1$. Since $\cos\omega, \sin\omega \leq 0$, and since $x_0 \neq y_0$ by assumption, we have $y_1 \neq x_1$. Arguing as before we get that the graph of u contains the whole straight line passing through $(x_1, u(x_1))$ and $(y_1, u(y_1))$. Since x_1 is arbitrary, the graph of u is a ruled surface, as claimed. □

Let us conclude the paper by explaining why the present method cannot be used to prove that the solution to

$$\begin{cases} (-\Delta)^s\, u = -1 & \text{in } \Omega; \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.2}$$

where Ω is a smooth, convex, bounded domain, is concave in Ω. Essentially, the present method fails because the solution u of (5.2), which is positive in Ω and vanishes outside, is *not* concave in the whole space. Nevertheless, in view of the results in [19] concerning the boundary behavior of u, we may expect that *the restriction* of u to the domain Ω is concave.

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