SUPERLINEAR NONLOCAL FRACTIONAL PROBLEMS WITH INFINITELY MANY SOLUTIONS

ZHANG BINLIN, GIOVANNI MOLICA BISCI, AND RAFFAELLA SERVADEI

Abstract. In this paper we study the existence of infinitely many weak solutions for equations driven by nonlocal integrodifferential operators with homogeneous Dirichlet boundary conditions. A model for these operators is given by the fractional Laplacian

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\, dy\,, \quad x \in \mathbb{R}^n$$

where $s \in (0,1)$ is fixed.

We consider different superlinear growth assumptions on the nonlinearity, starting from the well-known Ambrosetti–Rabinowitz condition. In this framework we obtain three different results about the existence of infinitely many weak solutions for the problem under consideration, by using the Fountain Theorem. All these theorems extend some classical results for semilinear Laplacian equations to the nonlocal fractional setting.

Contents

Key words and phrases. Fractional Laplacian, nonlocal problems, variational methods, critical point theory, Fountain Theorem, integrodifferential operators, superlinear nonlinearities.

2010 AMS Subject Classification: Primary: 49J35, 35A15, 35S15; Secondary: 47G20, 45G05.

The first author was supported by the Research Foundation of Heilongjiang Educational Committee (No. 12541667). The second and the third author were supported by the GNAMPA Project 2014 *Proprietà geometriche e analitiche per problemi non-locali*. The third author was supported by the MIUR National Research Project *Variational and Topological Methods in the Study of Nonlinear Phenomena* and by the ERC grant ϵ (*Elliptic Pde's and Symmetry of Interfaces and Layers for Odd Nonlinearities*).

1. Introduction and main results

In this paper we are concerned with the existence of infinitely many weak solutions of the nonlocal fractional equations whose prototype of order $s \in (0,1)$ is given by

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{1.1}$$

which is the counterpart of this Laplace equation

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega\,. \end{cases} \tag{1.2}$$

Here $(-\Delta)^s$ is the fractional Laplace operator, which, up to normalization factors, may be defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\, dy\,, \quad x \in \mathbb{R}^n\,.$$

In recent years, a great attention has been focused on the study of fractional and non-local operators of elliptic type, both for the pure mathematical research and for concrete real-world applications. Fractional and nonlocal operators appear in many fields such as, among the others, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves. This is one of the reason why, recently, nonlocal fractional problems are widely studied in the literature in many different contexts. Just to name a few, we recall, for instance, the following papers and the references therein: [9, 10, 32] for regularity results, [4, 5, 8, 18, 25, 26, 29, 36, 43] for the existence of solutions, [17, 27, 35] for multiplicity of solutions and [19, 28, 30] for Kirchhoff nonlocal fractional problems.

In [38] the authors considered the following general nonlocal problem

$$\begin{cases} -\mathcal{L}_K u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases} \tag{1.3}$$

Here Ω is an open bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $n > 2s$, $s \in (0,1)$, while $\mathcal{L}_K$ is the integrodifferential operator defined as follows

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} \Big(u(x+y)+u(x-y)-2u(x)\Big) K(y)\, dy\,, \quad x \in \mathbb{R}^n\,, \tag{1.4}$$

with the kernel $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ such that

$$mK \in L^1(\mathbb{R}^n), \text{ where } m(x) = \min\{|x|^2, 1\}\,; \tag{1.5}$$

$$\text{there exists } \theta > 0 \text{ such that } K(x) \geqslant \theta |x|^{-(n+2s)} \text{ for any } x \in \mathbb{R}^n \setminus \{0\}\,; \tag{1.6}$$

$$K(x) = K(-x) \text{ for any } x \in \mathbb{R}^n \setminus \{0\}\,. \tag{1.7}$$

A prototype for K is given by the singular kernel $K(x) = |x|^{-(n+2s)}$ which gives rise to the fractional Laplace operator $-(-\Delta)^s$. In [38] an existence theorem for problem (1.3) has been proved by using the Mountain Pass Theorem, when the nonlinear term f has a superlinear and subcritical growth.

Motivated by an evident and increasing interest in the current literature on fractional elliptic problems, here we are interested in the existence of infinitely many weak solutions of problem (1.3) under the same superlinear growth assumptions on f adopted in [38], that is $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a function verifying the following standard conditions

$$f \in C(\overline{\Omega} \times \mathbb{R}) \tag{1.8}$$

$$\begin{aligned}&\text{there exist } a_1, a_2 > 0 \text{ and } q \in (2, 2^*), 2^* = 2n/(n-2s)\,, \text{ such that}\\ &|f(x,t)| \leqslant a_1 + a_2|t|^{q-1} \text{ for any } x \in \Omega, t \in \mathbb{R}\,;\end{aligned} \tag{1.9}$$

$$\begin{aligned}&\text{there exist } \mu > 2 \text{ and } r > 0 \text{ such that for any } x \in \Omega, t \in \mathbb{R},\ |t| \geqslant r\\ &0 < \mu F(x,t) \leqslant t f(x,t)\,,\end{aligned} \tag{1.10}$$

where the function F is the primitive of f with respect to the second variable, that is

$$F(x,t) = \int_0^t f(x,\tau)d\tau\,. \tag{1.11}$$

When looking for infinitely many solutions, it is natural requiring some symmetry on the nonlinearity. In the sequel we will assume that the following further assumption on f is satisfied

$$f(x,-t) = -f(x,t) \text{ for any } x \in \Omega,\ t \in \mathbb{R}\,. \tag{1.12}$$

As a model for f we can take the function $f(x,t) = a(x)|t|^{q-2}t$, with $a \in C(\overline{\Omega})$ and $q \in (2, 2^*)$.

The first result of this paper is the following theorem.

Theorem 1. *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying* (1.5)–(1.7) *and let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a function verifying* (1.8)–(1.10) *and* (1.12)*.*

Then, the problem (1.3) *has infinitely many solutions $u_j \in X_0$, $j \in \mathbb{N}$, whose energy $\mathcal{J}_K(u_j) \to +\infty$ as $j \to +\infty$.*

In the literature assumption (1.10) is well-known and it is called *Ambrosetti–Rabinowitz condition*, since it was originally introduced by Ambrosetti and Rabinowitz in [3], where, as an application of the famous Mountain Pass Theorem, they obtained the existence of nontrivial solutions of problem (1.2), under superlinear and subcritical growth conditions on the right-hand side.

A lot of works concerning superlinear elliptic boundary value problem have been written by using this usual Ambrosetti–Rabinowitz condition (see, for instance, [42, 44] and the references therein), whose role consists in ensuring the boundedness of the Palais–Smale sequences of the energy functional associated with the problem under consideration.

The Ambrosetti–Rabinowitz condition is a superlinear growth assumption on the nonlinearity f. Indeed, from (1.10) it follows that for some a_3, $a_4 > 0$

$$F(x,t) \geqslant a_3|t|^\mu - a_4 \text{ for any } (x,t) \in \overline{\Omega} \times \mathbb{R}\,, \tag{1.13}$$

see, for instance, [39, Lemma 4]. However, there are many functions which are superlinear at infinity, but do not satisfy the Ambrosetti–Rabinowitz condition. At this purpose, we would note that from (1.13) and the fact that $\mu > 2$, it follows that

$$\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^2} = +\infty \text{ uniformly for any } x \in \overline{\Omega}\,. \tag{1.14}$$

Of course, also condition (1.14) characterizes the nonlinearity f to be superlinear at infinity. It is easily seen that the function

$$f(x,t) = t\log(1+|t|) \tag{1.15}$$

verifies condition (1.14) and does not satisfy (1.13) (and so, as a consequence, does not verify (1.10)).

In order to study the superlinear problem (1.2) in [20] Jeanjean introduced the following assumption on f:

$$\begin{aligned}&\text{there exists } \gamma \geqslant 1 \text{ such that for any } x \in \Omega\\ &\mathcal{F}(x,t') \leqslant \gamma \mathcal{F}(x,t) \text{ for any } t, t' \in \mathbb{R} \text{ with } 0 < t' \leqslant t\,,\end{aligned} \tag{1.16}$$

where

$$\mathcal{F}(x,t) = \frac{1}{2}\, t f(x,t) - F(x,t)\,. \tag{1.17}$$

It is easy to see that the function (1.15) satisfies also the condition (1.16).

In recent years, condition (1.16) was often applied to consider the existence of nontrivial solutions for the superlinear problem (1.2) without the Ambrosetti–Rabinowitz condition, for example, see [2, 15, 21, 22, 23]. For other papers treating superlinear problems without the Ambrosetti-Rabinowitz condition we refer to [13, 16, 20, 24, 33, 45, 46] and references therein.

In this framework our result is the following one:

Theorem 2. *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying* (1.5)–(1.7) *and let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a function verifying* (1.8), (1.9), (1.12), (1.14) *and* (1.16).

Then, the problem (1.3) *has infinitely many solutions $u_j \in X_0$, $j \in \mathbb{N}$, whose energy $\mathcal{J}_K(u_j) \to +\infty$ as $j \to +\infty$.*

According to [23, Proposition 2.3], the condition (1.16) is weaker than the following assumption:

$$\text{the function } t \mapsto \frac{f(\cdot,t)}{t} \text{ is increasing in } t \geqslant 0 \text{ and decreasing in } t \leqslant 0. \tag{1.18}$$

However, both (1.16) and (1.18) are global conditions, and hence they are not very satisfactory. For this reason, we replace the condition (1.18) with the following local condition introduced by Liu in [22]:

$$\begin{gathered} \text{there exists } \bar{t} > 0 \text{ such that for any } x \in \Omega \\ \text{the function } t \mapsto \frac{f(x,t)}{t} \text{ is increasing in } t \geqslant \bar{t} \text{ and decreasing in } t \leqslant -\bar{t}. \end{gathered} \tag{1.19}$$

Under this assumption, our main result reads as follows:

Theorem 3. *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying* (1.5)–(1.7) *and let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a function verifying* (1.8), (1.9), (1.12), (1.14) *and* (1.19).

Then, the problem (1.3) *has infinitely many solutions $u_j \in X_0$, $j \in \mathbb{N}$, whose energy $\mathcal{J}_K(u_j) \to +\infty$ as $j \to +\infty$.*

We would remark that, due to the symmetry assumption (1.12), if u is a weak solution of problem (1.3), then also $-u$ does. Hence, our results assure the existence of infinitely many pairs $\{u_j, -u_j\}_{j\in\mathbb{N}}$ of weak solutions.

The strategy in order to get the multiplicity results stated here above consists in looking for infinitely many critical points for the Euler-Lagrange functional associated with problem (1.3), namely here we will apply the Fountain Theorem proved by Bartsch in [6]. As usual for critical point theorems, we have to study the compactness properties of the functional together with its geometric features. With respect to the compactness, we will prove that the functional satisfies the classical Palais–Smale condition when the nonlinearity verifies the Ambrosetti-Rabinowitz assumption, while, for a right-hand side satisfying other superlinear conditions (see (1.14) and (1.16) or (1.19)), the Cerami condition will be considered. In both cases the main difficulty relies in the proof of the boundedness of the Palais–Smale (or Cerami) sequence.

As for the geometry of the functional, we will show that it is negative in ball of a suitable finite-dimensional subspace of X_0 and positive in ball of an infinite-dimensional subspace. For the negativity of the functional we will mainly use the equivalence of the norms in finite-dimensional spaces, while for the other geometric feature we will need a more careful analysis, strictly related to the superlinear assumptions on the nonlinear term f.

Finally, we would note that Theorem 1, Theorem 2 and Theorem 3 represent the nonlocal counterpart of [6, Theorem 3.7], [23, Theorem 1.1] and [22, Theorem 1.4], respectively. We would also point out that in [26, Theorem 3.1] the author proved the existence of infinitely many weak solutions of problem (1.3) requiring conditions (1.8)–(1.10) and (1.12), but exploiting a method different from the one used here and, precisely, a symmetric version of the Mountain Pass Theorem for even functionals.

Furthermore, in [39] the authors studied the nonlocal problem

$$\begin{cases} -\mathcal{L}_K u - \lambda u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases} \tag{1.20}$$

and proved the existence of a nontrivial solution for it, for any $\lambda \in \mathbb{R}$, as an application of the Mountain Pass Theorem and of the Linking Theorem (see [3, 31]). Motivated by this existence result, in the last part of this paper we study the existence of infinitely many solutions for (1.20), under all the different superlinear assumptions on f we considered above. Namely, we prove that the following results hold true:

Theorem 4. *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying* (1.5)*–*(1.7) *and let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a function verifying* (1.8)*–*(1.10) *and* (1.12)*.*

Then, for any $\lambda \in \mathbb{R}$ the problem (1.20) *has infinitely many solutions $u_j \in X_0$, $j \in \mathbb{N}$, whose energy $\mathcal{J}_{K,\lambda}(u_j) \to +\infty$ as $j \to +\infty$.*

Theorem 5. *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying* (1.5)*–*(1.7) *and let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a function verifying* (1.8)*,* (1.9)*,* (1.12)*,* (1.14) *and* (1.16)*.*

Then, for any $\lambda \in \mathbb{R}$ the problem (1.20) *has infinitely many solutions $u_j \in X_0$, $j \in \mathbb{N}$, whose energy $\mathcal{J}_{K,\lambda}(u_j) \to +\infty$ as $j \to +\infty$.*

Theorem 6. *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying* (1.5)*–*(1.7) *and let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a function verifying* (1.8)*,* (1.9)*,* (1.12)*,* (1.14) *and* (1.19)*.*

Then, for any $\lambda \in \mathbb{R}$ the problem (1.20) *has infinitely many solutions $u_j \in X_0$, $j \in \mathbb{N}$, whose energy $\mathcal{J}_{K,\lambda}(u_j) \to +\infty$ as $j \to +\infty$.*

The proofs of Theorem 4, Theorem 5 and Theorem 6 rely on the same arguments used for Theorem 1, Theorem 2 and Theorem 3, respectively. We just need, in some step of the proofs, some careful estimates of the term $\lambda \|u\|^2_{L^2(\Omega)}$.

Theorem 4 is the nonlocal counterpart of [44, Corollary 3.9], where the limit case as $s \to 1$ (that is the Laplace case) was considered. Finally, we would recall that in [35] the existence of infinitely many solutions for (1.20) was proved under assumptions on f different from the ones considered here and just for the case when $q \in (2, 2^* - 2s/(n-2s))$, but in presence of a perturbation $h \in L^2(\Omega)$.

The paper is organized as follows. In Section 2 we will present some necessary preliminary notions and results. In Section 3 we will discuss the compactness properties of the energy functional associated with the problem under consideration. Section 4 will be devoted to the proofs of the main results of the paper. Finally, in Section 5 we will study problem(1.20) and we will prove the related multiplicity results.

2. Preliminaries

In this section we give some preliminary results which will be used in the sequel.

2.1. **The functional space X_0.** Problems (1.1) and (1.3) have a variational nature and, in order to study them from this point of view, what we first need is to give a suitable variational formulation for them.

The natural spaces where finding solutions for these problems are the fractional Sobolev spaces. On the other hand, the variational formulation needs to encode the Dirichlet datum $u = 0$ in $\mathbb{R}^n \setminus \Omega$. In order to overcome this problem, in [37] (see also [38, 39]) the authors considered a new functional space, denoted by X_0, which is inspired to the fractional Sobolev spaces (but it is not equivalent to them) and seems to be the good space for writing the variational formulation of our problems.

The space X_0 is defined as

$$X_0 := \big\{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\big\}\,,$$

where the functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

$$\text{the map } (x,y) \mapsto (g(x)-g(y))\sqrt{K(x-y)} \text{ is in } L^2\big((\mathbb{R}^n \times \mathbb{R}^n) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), dxdy\big)$$

(here $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$). Moreover, X_0 is endowed with the norm

$$X_0 \ni g \mapsto \|g\|_{X_0} := \Big(\int_{\mathbb{R}^n \times \mathbb{R}^n} |g(x)-g(y)|^2 K(x-y)\,dx\,dy\Big)^{1/2} \tag{2.1}$$

and $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space (for this see [38, Lemma 7]), with scalar product

$$\langle u,v\rangle_{X_0} := \int_{\mathbb{R}^n \times \mathbb{R}^n} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\,K(x-y)\,dx\,dy\,. \tag{2.2}$$

The usual fractional Sobolev space $H^s(\Omega)$ is endowed with the so-called *Gagliardo norm* (see, for instance [1, 14]) given by

$$\|g\|_{H^s(\Omega)} := \|g\|_{L^2(\Omega)} + \Big(\int_{\Omega\times\Omega} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\,. \tag{2.3}$$

It is easy to see that, even in the model case in which $K(x) = |x|^{-(n+2s)}$, the norms in (2.1) and (2.3) are not the same: this makes the space X_0 not equivalent to the usual fractional Sobolev spaces and the classical fractional Sobolev space approach not sufficient for studying our problem from a variational point of view.

Just for completeness, we would recall that both the spaces X and X_0 are non-empty, since $C^2_0(\Omega) \subseteq X_0$ (see [37, Lemma 5.1]), and that for a generale kernel K satisfying conditions (1.5)–(1.7), the following inclusion holds true

$$X_0 \subseteq \big\{g \in H^s(\mathbb{R}^n) : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\big\}\,,$$

while, when $K(x) = |x|^{-(n+2s)}$, the space X_0 can be characterized as follows

$$X_0 = \big\{g \in H^s(\mathbb{R}^n) : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\big\}\,.$$

For further details on X and X_0 we refer to [37, 38, 39, 40], where various properties of these spaces were proved. While, for more details on the fractional Sobolev spaces we refer to [14] and to the references therein.

In the sequel, we also need some properties of the spectrum of the operator $-\mathcal{L}_K$ (for a complete study we refer to [34, Proposition 2.3], [39, Proposition 9 and Appendix A] and [41, Proposition 4]). We recall that the eigenvalue problem

$$\begin{cases} -\mathcal{L}_K u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases} \tag{2.4}$$

possesses a divergent sequence of positive eigenvalues

$$\lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \lambda_{k+1} \leqslant \ldots\,,$$

whose corresponding eigenfunctions will be denoted by e_k. From [39, Proposition 9], we know that $\{e_k\}_{k\in\mathbb{N}}$ can be chosen in such a way that this sequence provides an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in X_0.

2.2. **Weak solutions and energy functional of the problem.** Along this paper we are interested in the existence of infinitely many weak solutions for problem (1.3), that is on solutions of the following problem

$$
\begin{cases} \displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^n}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dx\,dy=\int_\Omega f(x,u(x))\varphi(x)dx \\ \hfill \forall\,\varphi\in X_0 \\ u\in X_0. \end{cases} \tag{2.5}
$$

The weak formulation (2.5) represents the Euler-Lagrange equation of the energy functional $\mathcal{J}_K : X_0\to\mathbb{R}$ given by

$$
\mathcal{J}_K(u):=\frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}|u(x)-u(y)|^2K(x-y)\,dx\,dy-\int_\Omega F(x,u(x))\,dx\,, \tag{2.6}
$$

where F is the function defined in (1.11). We would remark that $\mathcal{J}_K\in C^1(X_0)$ thanks to the assumptions on f and also due to the embedding properties of X_0 into the classical Lebesgue spaces (see [38, Lemma 6 and Lemma 8] and [39, Lemma 9]).

In order to prove our main results, stated in Theorem 1, Theorem 2 and Theorem 3, we will apply the Fountain Theorem due to Bartsch (see [6]), which, under suitable compactness and geometric assumptions on a functional, provides the existence of an unbounded sequence of critical value for it.

3. Verification of the compactness conditions

The compactness assumption required by the Fountain Theorem is the well-known *Palais–Smale condition* (see, for instance, [42, 44] and references therein), which in our framework reads as follows:

$\mathcal{J}_K$ satisfies the *Palais–Smale compactness condition* at level $c\in\mathbb{R}$
if any sequence $\{u_j\}_{j\in\mathbb{N}}$ in X_0 such that
$$\mathcal{J}_K(u_j)\to c \text{ and } \sup\Big\{\big|\langle\,\mathcal{J}_K'(u_j),\varphi\,\rangle\big|\,:\,\varphi\in X_0\,,\|\varphi\|_{X_0}=1\Big\}\to 0 \text{ as } j\to+\infty,$$
admits a subsequence strongly convergent in X_0.

In the case when the right-hand side in problem (1.3) satisfies conditions (1.9) and (1.10), in the sequel we will prove that the corresponding energy functional $\mathcal{J}_K$ verifies the Palais–Smale condition. While, when removing the Ambrosetti–Rabinowitz condition (1.10) and replacing it with assumptions (1.14) and (1.16) or (1.19), we will show that $\mathcal{J}_K$ verifies another compactness assumption, say the well-know *Cerami condition*, which in our setting can be written as follows:

$\mathcal{J}_K$ satisfies the *Cerami compactness condition* at level $c\in\mathbb{R}$
if any sequence $\{u_j\}_{j\in\mathbb{N}}$ in X_0 such that
$$\mathcal{J}_K(u_j)\to c \text{ and } (1+\|u_j\|)\sup\Big\{\big|\langle\,\mathcal{J}_K'(u_j),\varphi\,\rangle\big|\,:\,\varphi\in X_0\,,\|\varphi\|_{X_0}=1\Big\}\to 0$$
as $j\to+\infty$, admits a subsequence strongly convergent in X_0.

The Cerami condition was introduced by Cerami in [11, 12] as a weak version of the Palais–Smale condition. Hence, the Fountain Theorem holds true also under this compactness assumption. We would remark that if a functional satisfies the Palais–Smale condition or the Cerami condition, then it verifies the deformation condition, that is it fits with the requirements of the Deformation Theorem.

3.1. **Nonlinearities satisfying the Ambrosetti–Rabinowitz condition.** In this framework we prove the following result about the compactness of the functional $\mathcal{J}_K$:

Proposition 7. *Let $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ be a function verifying conditions* (1.8)–(1.10). *Then, $\mathcal{J}_K$ satisfies the Palais–Smale condition at any level $c\in\mathbb{R}$.*

Proof. Let $c \in \mathbb{R}$ and let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in X_0 such that

$$\mathcal{J}_K(u_j) \to c \tag{3.1}$$

and

$$\sup\Big\{\big|\langle\, \mathcal{J}_K'(u_j), \varphi\,\rangle\big| \,:\, \varphi \in X_0\,, \|\varphi\|_{X_0} = 1\Big\} \to 0 \tag{3.2}$$

as $j \to +\infty$.

We proceed by steps: first of all we show that the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0 and then that it admits a subsequence strongly convergent in X_0.

Step 1. *The sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0.* For any $j \in \mathbb{N}$ by (3.1) and (3.2) it easily follows that there exists $\kappa > 0$ such that

$$\Big|\langle \mathcal{J}_K'(u_j), \frac{u_j}{\|u_j\|_{X_0}}\rangle\Big| \leqslant \kappa$$

and

$$|\mathcal{J}_K(u_j)| \leqslant \kappa\,,$$

so that

$$\mathcal{J}_K(u_j) - \frac{1}{\mu}\langle \mathcal{J}_K'(u_j), u_j\rangle \leqslant \kappa\,(1 + \|u_j\|_{X_0})\,, \tag{3.3}$$

where μ is the parameter given in (1.10).

By (1.9) and integrating it is easily seen that for any $x \in \overline{\Omega}$ and for any $t \in \mathbb{R}$

$$|F(x,t)| \leqslant a_1\,|t| + \frac{a_2}{q}\,|t|^q\,. \tag{3.4}$$

Hence, by (3.4) and again (1.9) we have that for any $j \in \mathbb{N}$

$$\begin{aligned}\Big|\int_{\Omega\cap\{|u_j|\leqslant r\}} &\Big(F(x,u_j(x)) - \frac{1}{\mu}\,f(x,u_j(x))\,u_j(x)\,\Big)\,dx\Big| \\ &\leqslant \Big(a_1 r + \frac{a_2}{q}\,r^q + \frac{a_1}{\mu} r + \frac{a_2}{\mu}\,r^q\Big)\,|\Omega| =: \tilde{\kappa}\,.\end{aligned} \tag{3.5}$$

Thus, thanks to (1.10) and (3.5), we get that

$$\begin{aligned}\mathcal{J}_K(u_j) - \frac{1}{\mu}\langle \mathcal{J}_K'(u_j), u_j\rangle &= \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\,\|u_j\|^2_{X_0} \\ &\qquad - \int_\Omega \Big(F(x,u_j(x)) - \frac{1}{\mu} f(x,u_j(x))\,u_j(x)\Big)\,dx \\ &\geqslant \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\,\|u_j\|^2_{X_0} \\ &\qquad - \int_{\Omega\cap\{|u_j|\leqslant r\}} \Big(F(x,u_j(x)) - \frac{1}{\mu}\,f(x,u_j(x))\,u_j(x)\Big)\,dx \\ &\geqslant \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\,\|u_j\|^2_{X_0} - \tilde{\kappa}\end{aligned} \tag{3.6}$$

for any $j \in \mathbb{N}$.

By (3.3), (3.6) and the fact that $\mu > 2$ we have that

$$\Big(\frac{1}{2} - \frac{1}{\mu}\Big)\,\|u_j\|^2_{X_0} \leqslant \kappa\,(1 + \|u_j\|_{X_0}) + \tilde{\kappa}$$

for any $j \in \mathbb{N}$. Hence, Step 1 is proved.

Step 2. *Up to a subsequence, $\{u_j\}_{j\in\mathbb{N}}$ strongly converges in X_0* . Since $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0 by Step 1 and X_0 is a reflexive space (being a Hilbert space, by [38, Lemma 7]), up to a subsequence, still denoted by $\{u_j\}_{j\in\mathbb{N}}$, there exists $u_\infty \in X_0$ such that

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n} \big(u_j(x)-u_j(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy \to \\ \int_{\mathbb{R}^n\times\mathbb{R}^n} \big(u_\infty(x)-u_\infty(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy \quad \text{for any } \varphi\in X_0\end{aligned} \tag{3.7}$$

as $j\to+\infty$. Moreover, by [40, Lemma 9], up to a subsequence,

$$\begin{aligned} u_j &\to u_\infty \quad \text{in } L^q(\mathbb{R}^n)\\ u_j &\to u_\infty \quad \text{a.e. in } \mathbb{R}^n\end{aligned} \tag{3.8}$$

as $j\to+\infty$ and there exists $\ell\in L^q(\mathbb{R}^n)$ such that

$$|u_j(x)|\leqslant \ell(x) \quad \text{a.e. in } \mathbb{R}^n \quad \text{for any } j\in\mathbb{N} \tag{3.9}$$

(see, for instance, [7, Theorem IV.9]).

By (1.9), (3.7)–(3.9), the fact that the map $t\mapsto f(\cdot,t)$ is continuous in $t\in\mathbb{R}$ and the Dominated Convergence Theorem we get

$$\int_\Omega f(x,u_j(x))u_j(x)\,dx \to \int_\Omega f(x,u_\infty(x))u_\infty(x)\,dx \tag{3.10}$$

and

$$\int_\Omega f(x,u_j(x))u_\infty(x)\,dx \to \int_\Omega f(x,u_\infty(x))u_\infty(x)\,dx \tag{3.11}$$

as $j\to+\infty$. Moreover, by (3.2) and Step 1 we have that

$$0\leftarrow \langle \mathcal{J}_K'(u_j),u_j\rangle = \int_{\mathbb{R}^n\times\mathbb{R}^n}|u_j(x)-u_j(y)|^2K(x-y)\,dx\,dy - \int_\Omega f(x,u_j(x))u_j(x)\,dx$$

so that, by (3.10) we deduce that

$$\int_{\mathbb{R}^n\times\mathbb{R}^n}|u_j(x)-u_j(y)|^2K(x-y)\,dx\,dy \to \int_\Omega f(x,u_\infty(x))u_\infty(x)\,dx \tag{3.12}$$

as $j\to+\infty$. Furthermore, again by (3.2), we get

$$\begin{aligned}0\leftarrow \langle \mathcal{J}_K'(u_j),u_\infty\rangle &= \int_{\mathbb{R}^n\times\mathbb{R}^n} \big(u_j(x)-u_j(y)\big)\big(u_\infty(x)-u_\infty(y)\big)K(x-y)\,dx\,dy\\ &\quad - \int_\Omega f(x,u_j(x))u_\infty(x)\,dx\end{aligned} \tag{3.13}$$

as $j\to+\infty$. By (3.7) with $\varphi=u_\infty$, (3.11) and (3.13) we obtain

$$\int_{\mathbb{R}^n\times\mathbb{R}^n}|u_\infty(x)-u_\infty(y)|^2K(x-y)\,dx\,dy = \int_\Omega f(x,u_\infty(x))u_\infty(x)\,dx\,. \tag{3.14}$$

Thus, (3.12) and (3.14) give that

$$\|u_j\|_{X_0}\to\|u_\infty\|_{X_0}\,, \tag{3.15}$$

as $j\to\infty$.

Finally, it is easy to see that

$$\begin{aligned}\|u_j-u_\infty\|_{X_0}^2 &= \|u_j\|_{X_0}^2+\|u_\infty\|_{X_0}^2 - 2\int_{\mathbb{R}^n\times\mathbb{R}^n}\big(u_j(x)-u_j(y)\big)\big(u_\infty(x)-u_\infty(y)\big)K(x-y)\,dx\,dy\\ &\to 2\|u_\infty\|_{X_0}^2 - 2\int_{\mathbb{R}^n\times\mathbb{R}^n}|u_\infty(x)-u_\infty(y)|^2K(x-y)\,dx\,dy = 0\end{aligned}$$

as $j\to+\infty$, thanks to (3.7) and (3.15). Then, the assertion of Step 2 is proved. This concludes the proof of Proposition 7.

□

3.2. **Nonlinearities under the superlinear assumptions** (1.14) **and** (1.16)**.** In this framework we show that the functional $\mathcal{J}_K$ verifies the Cerami condition. Before proving this fact, we would note that, as a consequence of the assumptions (1.12) and (1.16), the following condition is verified:

$$\begin{aligned} &\text{there exists } \gamma \geqslant 1 \text{ such that for any } x \in \Omega \\ &\mathcal{F}(x,t') \leqslant \gamma \mathcal{F}(x,t) \text{ for any } t,t' \in \mathbb{R} \text{ with } 0<|t'|\leqslant |t|\,, \end{aligned} \tag{3.16}$$

where $\mathcal{F}$ is the function given in (1.17).

Now, we are ready to prove the next result, that is

Proposition 8. *Let $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ be a function verifying conditions* (1.8), (1.9), (1.12), (1.14) *and* (1.16). *Then, $\mathcal{J}_K$ satisfies the Cerami condition at any level $c\in\mathbb{R}$.*

Proof. Let $c \in \mathbb{R}$ and let $\{u_j\}_{j\in\mathbb{N}}$ be a Cerami sequence in X_0, that is let $\{u_j\}_{j\in\mathbb{N}}$ be such that

$$\mathcal{J}_K(u_j)\to c \tag{3.17}$$

and

$$(1+\|u_j\|)\sup\Big\{\big|\langle\,\mathcal{J}_K'(u_j),\varphi\,\rangle\big|\,:\,\varphi\in X_0\,,\|\varphi\|_{X_0}=1\Big\}\to 0 \tag{3.18}$$

as $j\to+\infty$.

First of all, we show that the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0. At this purpose we argue as in the proof of [15, Lemma 2.2]. We assume, by contradiction, that $\{u_j\}_{j\in\mathbb{N}}$ is unbounded in X_0, that is we may suppose that, up to a subsequence (still denoted by $\{u_j\}_{j\in\mathbb{N}}$)

$$\|u_j\|_{X_0}\to+\infty \tag{3.19}$$

as $j\to+\infty$.

Note that, by (3.18) and (3.19), it is easily seen that

$$\sup\Big\{\big|\langle\,\mathcal{J}_K'(u_j),\varphi\,\rangle\big|\,:\,\varphi\in X_0\,,\|\varphi\|_{X_0}=1\Big\}\to 0 \tag{3.20}$$

as $j\to+\infty$.

Now, for any $j\in\mathbb{N}$, let

$$v_j=\frac{u_j}{\|u_j\|_{X_0}}\,. \tag{3.21}$$

Of course, the sequence $\{v_j\}_{j\in\mathbb{N}}$ is bounded in X_0 and so, by [40, Lemma 9], up to a subsequence, we have that there exists $v_\infty\in X_0$ such that

$$\begin{aligned} &v_j\to v_\infty \quad \text{in } L^q(\mathbb{R}^n)\\ &v_j\to v_\infty \quad \text{a.e. in } \mathbb{R}^n \end{aligned} \tag{3.22}$$

as $j\to+\infty$ and there exists $\ell\in L^q(\mathbb{R}^n)$ such that

$$|v_j(x)|\leqslant \ell(x) \quad \text{a.e. in } \mathbb{R}^n \quad \text{for any } j\in\mathbb{N} \tag{3.23}$$

(see [7, Theorem IV.9]). In the sequel we will consider separately the cases when $v_\infty\equiv 0$ and $v_\infty\not\equiv 0$ and we will prove that in both these situations a contradiction occurs.

Firstly, let us suppose that

$$v_\infty\equiv 0\,. \tag{3.24}$$

As in [20], we can say that for any $j\in\mathbb{N}$ there exists $t_j\in[0,1]$ such that

$$\mathcal{J}_K(t_ju_j)=\max_{t\in[0,1]}\mathcal{J}_K(tu_j)\,. \tag{3.25}$$

Since (3.19) holds true, for any $m \in \mathbb{N}$, we can choose $r_m = 2\sqrt{m}$ such that

$$r_m \|u_j\|_{X_0}^{-1} \in (0,1)\,, \tag{3.26}$$

provided j is large enough, say $j > \bar{\jmath}$, with $\bar{\jmath} = \bar{\jmath}(m)$.

By (3.22) and the continuity of the function F, we get that

$$F(x, r_m v_j(x)) \to F(x, r_m v_\infty(x)) \quad \text{a.e. } x \in \Omega \tag{3.27}$$

as $j \to +\infty$ for any $m \in \mathbb{N}$. Moreover, integrating (1.9) and taking into account (3.23), we have that

$$\begin{aligned} |F(x, r_m v_j(x))| &\leqslant a_1\, |r_m v_j(x)| + \frac{a_2}{q}\, |r_m v_j(x)|^q \\ &\leqslant a_1\, r_m \ell(x) + \frac{a_2}{q}\, (r_m \ell(x))^q \in L^1(\Omega)\,, \end{aligned} \tag{3.28}$$

a.e. $x \in \Omega$ and for any $m, j \in \mathbb{N}$. Hence, (3.27), (3.28) and the Dominated Convergence Theorem yield that

$$F(\cdot, r_m v_j(\cdot)) \to F(\cdot, r_m v_\infty(\cdot)) \quad \text{in } L^1(\Omega) \tag{3.29}$$

as $j \to +\infty$ for any $m \in \mathbb{N}$. Since $F(x, 0) = 0$ for any $x \in \overline{\Omega}$ and (3.24) holds true, (3.29) gives that

$$\int_\Omega F(x, r_m v_j(x))\, dx \to 0 \tag{3.30}$$

as $j \to +\infty$ for any $m \in \mathbb{N}$. Thus, (3.25), (3.26) and (3.30) yield

$$\begin{aligned} \mathcal{J}_K(t_j u_j) &\geqslant \mathcal{J}_K(r_m \|u_j\|_{X_0}^{-1} u_j) \\ &= \mathcal{J}_K(r_m v_j) \\ &= \frac{1}{2}\|r_m v_j\|_{X_0}^2 - \int_\Omega F(x, r_m v_j(x))\, dx \\ &= 2m - \int_\Omega F(x, r_m v_j(x))\, dx \geqslant m\,, \end{aligned}$$

provided j is large enough and for any $m \in \mathbb{N}$. From this we deduce that

$$\mathcal{J}_K(t_j u_j) \to +\infty \tag{3.31}$$

as $j \to +\infty$.

Now, note that $\mathcal{J}_K(0) = 0$ and (3.17) holds true. Combining these two facts it is easily seen that $t_j \in (0,1)$ and so, by (3.25), we get that

$$\frac{d}{dt}\Big|_{t=t_j} \mathcal{J}_K(t u_j) = 0$$

for any $j \in \mathbb{N}$. As a consequence of this, we have that

$$\langle \mathcal{J}_K'(t_j u_j), t_j u_j \rangle = t_j \frac{d}{dt}\Big|_{t=t_j} \mathcal{J}_K(t u_j) = 0\,. \tag{3.32}$$

We claim that

$$\limsup_{j \to +\infty} \mathcal{J}_K(t_j u_j) \leqslant \kappa\,, \tag{3.33}$$

for a suitable positive constant κ. Indeed, by (3.32) and using (3.16), we get

$$\begin{aligned}\frac{1}{\gamma}\mathcal{J}_K(t_ju_j) &= \frac{1}{\gamma}\Big(\mathcal{J}_K(t_ju_j) - \frac{1}{2}\langle \mathcal{J}'_K(t_ju_j), t_ju_j\rangle\Big)\\ &= \frac{1}{\gamma}\Big(-\int_\Omega F(x,t_ju_j(x))\,dx + \frac{1}{2}\int_\Omega t_ju_j(x)\,f(x,t_ju_j(x))\,dx\Big)\\ &= \frac{1}{\gamma}\int_\Omega \mathcal{F}(x,t_ju_j(x))\,dx\\ &\leqslant \int_\Omega \mathcal{F}(x,u_j(x))\,dx\\ &= \int_\Omega \Big[\frac{1}{2}\,u_j(x)f(x,u_j(x)) - F(x,u_j(x))\Big]\,dx\\ &= \mathcal{J}_K(u_j) - \frac{1}{2}\langle \mathcal{J}'_K(u_j), u_j\rangle \to c\end{aligned}$$

as $j\to+\infty$, thanks to (3.17) and (3.20). This proves (3.33), which contradicts (3.31). Thus, the sequence $\{u_j\}_{j\in\mathbb{N}}$ has to be bounded in X_0.

Now, suppose that $v_\infty \not\equiv 0$. Then, the set $\Omega' := \{x\in\Omega : v_\infty(x)\neq 0\}$ has positive Lebesgue measure and

$$|u_j(x)| \to +\infty \quad \text{a.e. } x\in\Omega' \tag{3.34}$$

as $j\to+\infty$, thanks to (3.21), (3.22) and the fact that $v_\infty\not\equiv 0$.

By (3.17) and (3.19) it is easy to see that

$$\frac{\mathcal{J}_K(u_j)}{\|u_j\|^2_{X_0}} \to 0\,,$$

that is

$$\frac{1}{2} - \int_{\Omega'}\frac{F(x,u_j(x))}{\|u_j\|^2_{X_0}}\,dx - \int_{\Omega\setminus\Omega'}\frac{F(x,u_j(x))}{\|u_j\|^2_{X_0}}\,dx = o(1) \tag{3.35}$$

as $j\to+\infty$.

Let us consider separately the two integrals in formula (3.35). With respect to the first one, we have that

$$\begin{aligned}\frac{F(x,u_j(x))}{\|u_j\|^2_{X_0}} &= \frac{F(x,u_j(x))}{|u_j(x)|^2}\frac{|u_j(x)|^2}{\|u_j\|^2_{X_0}}\\ &= \frac{F(x,u_j(x))}{|u_j(x)|^2}|v_j(x)|^2 \to +\infty \quad \text{a.e. } x\in\Omega'\end{aligned}$$

as $j\to+\infty$, thanks to (1.14), (3.22), (3.34) and the definition of Ω'. Hence, by using the Fatou lemma, we obtain

$$\int_{\Omega'}\frac{F(x,u_j(x))}{\|u_j\|^2_{X_0}}\,dx \to +\infty \tag{3.36}$$

as $j\to+\infty$.

As for the second integral in (3.35), we claim that

$$\int_{\Omega\setminus\Omega'}\frac{F(x,u_j(x))}{\|u_j\|^2_{X_0}}\,dx \geqslant -\frac{\kappa}{\|u_j\|^2_{X_0}}|\Omega\setminus\Omega'|\,, \tag{3.37}$$

for some positive constant κ. Indeed by (1.14), it follows that

$$\lim_{|t|\to+\infty} F(x,t) = +\infty \ \text{ uniformly for any } x\in\overline{\Omega}\,. \tag{3.38}$$

Hence, by (3.38) there exist two positive constants $\tilde{t}$ and H such that

$$F(x,t) \geqslant H \tag{3.39}$$

for every $x \in \overline{\Omega}$ and $|t| > \tilde{t}$. On the other hand, since F is continuous in $\overline{\Omega} \times \mathbb{R}$, one has

$$F(x,t) \geqslant \min_{(x,t)\in\overline{\Omega}\times[-\tilde{t},\tilde{t}]} F(x,t)\,, \tag{3.40}$$

for every $x \in \overline{\Omega}$ and $|t| \leqslant \tilde{t}$. Note that $\min_{(x,t)\in\overline{\Omega}\times[-\tilde{t},\tilde{t}]} F(x,t) \leqslant 0$, being $F(x,0) = 0$ for any $x \in \overline{\Omega}$. Then, by (3.39) and (3.40) it follows that

$$F(x,t) \geqslant -\kappa \quad \text{for any } \ (x,t) \in \overline{\Omega} \times \mathbb{R} \tag{3.41}$$

for some positive constant κ. Inequality (3.41) immediately yields the claim (3.37).

As a consequence of (3.19) and (3.37) it is easy to see that

$$\lim_{j\to+\infty} \int_{\Omega\setminus\Omega'} \frac{F(x,u_j(x))}{\|u_j\|^2_{X_0}}\,dx \geqslant 0\,, \tag{3.42}$$

(note that this limit exists thanks to (3.35) and (3.36)). All in all, by (3.35), (3.36) and (3.42) we get a contradiction. Thus, the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0.

In order to prove the assertion of Proposition 8 from now on we can argue as in Step 2 of the proof of Proposition 7. □

We would remark that along the proof of Proposition 8 the assumption (1.16) was used (and was crucial) just for proving the inequality (3.33).

3.3. **Nonlinearities verifying the superlinear conditions** (1.14) **and** (1.19)**.** In this setting we need the following lemma, which will be crucial in the proof of the main result of this subsection.

Lemma 9. *If* (1.19) *holds true, then for any $x \in \Omega$, the function $\mathcal{F}(x,t)$ is increasing in $t \geqslant \bar{t}$ and decreasing in $t \leqslant -\bar{t}$, where $\mathcal{F}$ is the function given in* (1.17).

In particular, there exists $C_1 > 0$ such that

$$\mathcal{F}(x,s) \leqslant \mathcal{F}(x,t) + C_1$$

for any $x \in \Omega$ and $0 \leqslant s \leqslant t$ or $t \leqslant s \leqslant 0$.

See [22, Lemma 2.3] for details.

Proposition 10. *Let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a function verifying conditions* (1.8), (1.9), (1.14) *and* (1.19)*. Then, $\mathcal{J}_K$ satisfies the Cerami condition at any level $c \in \mathbb{R}$.*

Proof. We can argue exactly as in the proof of Proposition 8. We just have to modify the proof of inequality (3.33): indeed, for proving it, in Proposition 8 we used condition (1.16) (actually (3.16)), which now is no more assumed.

Here we will show the validity of (3.33) making use of the assumption (1.19) and of Lemma 9. We point out that our notation is the one used in the proof of Proposition 8. In view of Lemma 9 we have that

$$\begin{aligned}
\mathcal{J}_K(t_j u_j) &= \mathcal{J}_K(t_j u_j) - \frac{1}{2}\langle \mathcal{J}_K'(t_j u_j), t_j u_j\rangle \\
&= \int_\Omega \mathcal{F}(x, t_j u_j(x))\,dx \\
&= \int_{\{u_j\geqslant 0\}} \mathcal{F}(x, t_j u_j(x))\,dx + \int_{\{u_j<0\}} \mathcal{F}(x, t_j u_j(x))\,dx \\
&\leqslant \int_{\{u_j\geqslant 0\}} \Big[\mathcal{F}(x, u_j(x)) + C_1\Big] + \int_{\{u_j<0\}} \Big[\mathcal{F}(x, u_j(x)) + C_1\Big] \\
&= \int_\Omega \mathcal{F}(x, u_j(x))\,dx + C_1|\Omega| \\
&= \mathcal{J}_K(u_j) - \frac{1}{2}\,\langle \mathcal{J}_K'(u_j), u_j\rangle + C_1|\Omega| \to c + C_1|\Omega|
\end{aligned}$$

as $j \to +\infty$. This proves (3.33). The proof of Proposition 10 is thus completed. $\square$

4. The proofs of the main results

In this section we give the proofs of the existence of infinitely many solutions for problem (1.3), both when the right-hand side satisfies the Ambrosetti–Rabinowitz condition (see Theorem 1) and when other superlinear assumptions are required (see Theorem 2 and Theorem 3). In both cases the strategy consists in applying the Fountain Theorem of Bartsch (see [6]) to the functional $\mathcal{J}_K$.

Following the notation used in [6, Theorem 2.5] (see also [44]), in the sequel for any $k \in \mathbb{N}$ we put

$$Y_k := \mathrm{span}\{e_1, \dots, e_k\}$$

and

$$Z_k := \overline{\mathrm{span}\{e_k, e_{k+1}, \dots\}}\,.$$

Since Y_k is finite-dimensional, all norms on Y_k are equivalent. Therefore, there exist two positive constants $C_{k,q}$ and $\tilde{C}_{k,q}$, depending on k and q, such that for any $u \in Y_k$

$$C_{k,q}\|u\|_{X_0} \leqslant \|u\|_{L^q(\Omega)} \leqslant \tilde{C}_{k,q}\|u\|_{X_0}\,. \tag{4.1}$$

The Fountain Theorem provides the existence of an unbounded sequence of critical value for a smooth functional, under suitable compactness condition (say, the Palais–Smale condition) and geometric assumptions on it, which, in our framework, read as follows:

(*i*) $a_k := \max\Big\{\mathcal{J}_K(u) : u \in Y_k, \|u\|_{X_0} = r_k\Big\} \leqslant 0$;

(*ii*) $b_k := \inf\Big\{\mathcal{J}_K(u) : u \in Z_k, \|u\|_{X_0} = \gamma_k\Big\} \to \infty$ as $k \to \infty$.

4.1. **Proof of Theorem 1.** In order to perform the proof of Theorem 1, we first need the following result:

Lemma 11. *Let $1 \leqslant q < 2^*$ and, for any $k \in \mathbb{N}$, let*

$$\beta_k := \sup\Big\{\|u\|_{L^q(\Omega)} : u \in Z_k, \|u\|_{X_0} = 1\Big\}\,.$$

Then, $\beta_k \to 0$ as $k \to \infty$.

Proof. By definition of Z_k, we have that $Z_{k+1} \subset Z_k$ and so, as a consequence, $0 < \beta_{k+1} \leqslant \beta_k$ for any $k \in \mathbb{N}$. Hence,

$$\beta_k \to \beta \tag{4.2}$$

as $k \to +\infty$, for some $\beta \geqslant 0$. Moreover, by definition of β_k, for any $k \in \mathbb{N}$ there exists $u_k \in Z_k$ such that

$$\|u_k\|_{X_0} = 1 \quad \text{and} \quad \|u_k\|_{L^q(\Omega)} > \beta_k/2\,. \tag{4.3}$$

Since X_0 is a Hilbert space, and hence a reflexive Banach space, there exist $u_\infty \in X_0$ and a subsequence of u_k (still denoted by u_k) such that $u_k \to u_\infty$ weakly in X_0, that is

$$\langle u_k, \varphi\rangle_{X_0} \to \langle u_\infty, \varphi\rangle_{X_0} \qquad \text{for any } \varphi \in X_0$$

as $k \to +\infty$. Since $\varphi = \sum_{j=1}^{+\infty} c_j e_j$, then

$$\langle u_\infty, \varphi\rangle_{X_0} = \lim_{k\to+\infty}\langle u_k, \varphi\rangle_{X_0} = \lim_{k\to+\infty}\sum_{j=1}^{+\infty} c_j\langle u_k, e_j\rangle_{X_0} = 0\,,$$

thank to the fact that the sequence $\{e_k\}_{k\in\mathbb{N}}$ of eigenfunctions of $-\mathcal{L}_K$ is an orthogonal basis of X_0. Therefore, we deduce that $u_\infty \equiv 0$. Hence, by the Sobolev embedding theorem (see [40, Lemma 9]), we get

$$u_k \to 0 \quad \text{in } L^q(\Omega) \tag{4.4}$$

as $k \to +\infty$. By (4.2), the fact that β is nonnegative, (4.3) and (4.4) we get that $\beta_k \to 0$ as $k \to +\infty$ and this concludes the proof of Lemma 11. □

Proof of Theorem 1. We mimic the proof of [44, Theorem 3.7]. By Proposition 7 we have that $\mathcal{J}_K$ satisfies the Palais–Smale condition, while, by (1.12) we get that $\mathcal{J}_K(-u) = \mathcal{J}_K(u)$ for any $u \in X_0$. In order to apply the Fountain Theorem, it remains to study the geometry of the functional $\mathcal{J}_K$. At this purpose, let us proceed by steps.

Step 1. *For any $k \in \mathbb{N}$ there exists $r_k > 0$ such that*

$$a_k = \max\Big\{\mathcal{J}_K(u) : u \in Y_k, \|u\|_{X_0} = r_k\Big\} \leqslant 0\,.$$

By (1.13) and (4.1) we get that for any $u \in Y_k$

$$\begin{aligned} \mathcal{J}_K(u) &\leqslant \frac{1}{2}\|u\|^2_{X_0} - a_3\|u\|^{\mu}_{L^{\mu}(\Omega)} + a_4|\Omega| \\ &\leqslant \frac{1}{2}\|u\|^2_{X_0} - \hat{C}_{k,\mu}\|u\|^{\mu}_{X_0} + a_4|\Omega| \end{aligned} \tag{4.5}$$

for a suitable positive constant $\hat{C}_{k,\mu}$ depending on k and μ. As a consequence of (4.5), for any $u \in Y_k$ with $\|u\|_{X_0} = r_k$ we get that

$$\mathcal{J}_K(u) \leqslant 0\,,$$

provided $r_k > 0$ is large enough, due to the fact that $\mu > 2$. Thus, Step 1 is proved.

Step 2. *There exists $\gamma_k > 0$ such that*

$$b_k = \inf\Big\{\mathcal{J}_K(u) : u \in Z_k, \|u\|_{X_0} = \gamma_k\Big\} \to +\infty \ \ as \ \ k \to +\infty\,.$$

By (1.9) and integrating, it is easy to see that (3.4) holds true, and so, as a consequence, we get that there exists a constant $C > 0$ such that

$$|F(x,t)| \leqslant C(1+|t|^q) \tag{4.6}$$

for any $x \in \overline{\Omega}$ and $t \in \mathbb{R}$. Then, by (4.6) for any $u \in Z_k \setminus \{0\}$, we obtain

$$\begin{aligned} \mathcal{J}_K(u) &\geqslant \frac{1}{2}\|u\|^2_{X_0} - C\|u\|^q_{L^q(\Omega)} - C|\Omega| \\ &= \frac{1}{2}\|u\|^2_{X_0} - C\Big\|\frac{u}{\|u\|_{X_0}}\Big\|^q_{L^q(\Omega)}\|u\|^q_{X_0} - C|\Omega| \\ &\geqslant \frac{1}{2}\|u\|^2_{X_0} - C\beta_k^q\|u\|^q_{X_0} - C|\Omega| \\ &= \|u\|^2_{X_0}\left(\frac{1}{2} - C\beta_k^q\|u\|^{q-2}_{X_0}\right) - C|\Omega|\,, \end{aligned} \tag{4.7}$$

where β_k is defined as in Lemma 11. Choosing

$$\gamma_k = \big(qC\beta_k^q\big)^{-1/(q-2)}\,,$$

it is easy to see that $\gamma_k \to +\infty$ as $k \to +\infty$, thanks to Lemma 11 and the fact that $q > 2$. As a consequence of this and by (4.7) we get that for any $u \in Z_k$ with $\|u\|_{X_0} = \gamma_k$

$$\mathcal{J}_K(u) \geqslant \|u\|^2_{X_0}\left(\frac{1}{2} - C\beta_k^q\|u\|^{q-2}_{X_0}\right) - C|\Omega| = \left(\frac{1}{2} - \frac{1}{q}\right)\gamma_k^2 - C|\Omega| \to +\infty$$

as $k \to +\infty$. Thus, Step 2 is fulfilled.

The proof of Theorem 1 is complete. □

We notice that the Ambrosetti-Rabinowitz condition (1.10) was used just for proving Step 1 in the verification of the geometric structure of the functional $\mathcal{J}_K$ (actually we used (1.13)). While in the proof of Step 2 the main tools were the assumption (1.9) and the Sobolev embedding theorems (see Lemma 11).

4.2. **Proof of Theorem 2.** By Proposition 8 and (1.12), we have that $\mathcal{J}_K$ satisfies the Cerami condition (and so the Palais–Smale condition) and $\mathcal{J}_K(-u) = \mathcal{J}_K(u)$ for any $u \in X_0$. The verification of the geometric assumption (ii) of the Fountain Theorem follows as in Step 2 in Subsection 4.1. It remains to verify the condition (i). At this purpose we will use the finite-dimension of the linear subspace Y_k and assumption (1.14).

Indeed, for any $k \in \mathbb{N}$, by (1.14) there exists $\delta_k > 0$ such that

$$F(x,t) \geqslant \frac{1}{C_k^2}\,|t|^2 \text{ for any } x \in \overline{\Omega} \text{ and any } t \in \mathbb{R} \text{ with } |t| > \delta_k\,, \tag{4.8}$$

where $C_k := C_{k,2}$, being $C_{k,2}$ the positive constant given in (4.1) with $q = 2$. Moreover, by Weierstrass Theorem, we have that

$$F(x,t) \geqslant m_k := \min_{x\in\overline{\Omega}, |t|\leqslant \delta_k} F(x,t) \text{ for any } x \in \overline{\Omega} \text{ and any } t \in \mathbb{R} \text{ with } |t| \leqslant \delta_k\,. \tag{4.9}$$

Note that $m_k \leqslant 0$, since $F(x,0) = 0$ for any $x \in \overline{\Omega}$. By (4.8) and (4.9), it is easy to see that

$$F(x,t) \geqslant \frac{1}{C_k^2}|t|^2 - B_k \ \text{ for any } (x,t) \in \overline{\Omega} \times \mathbb{R}$$

for a suitable positive constant B_k (say, $B_k \geqslant \delta_k^2/C_k^2 - m_k$).

As a consequence of this and by (4.1), for any $u \in Y_k$ we have

$$\begin{aligned} \mathcal{J}_K(u) &= \frac{1}{2}\|u\|_{X_0}^2 - \int_\Omega F(x,u(x))\,dx \\ &\leqslant \frac{1}{2}\|u\|_{X_0}^2 - \frac{1}{C_k^2}\|u\|_{L^2(\Omega)}^2 + B_k|\Omega| \\ &\leqslant \frac{1}{2}\|u\|_{X_0}^2 - \|u\|_{X_0}^2 + B_k|\Omega| \\ &= -\frac{1}{2}\|u\|_{X_0}^2 + B_k|\Omega|\,, \end{aligned} \tag{4.10}$$

so that, when $\|u\|_{X_0} = r_k$ it follows that

$$\mathcal{J}_K(u) \leqslant 0\,,$$

provided $r_k > 0$ is large enough. This proves that $\mathcal{J}_K$ satisfies condition (i) of the Fountain Theorem and this ends the proof of Theorem 2.

4.3. **Proof of Theorem 3.** The functional $\mathcal{J}_K$ satisfies the Cerami condition by Proposition 10, and so also the Palais-Smale assumption is verified. Moreover, $\mathcal{J}_K(-u) = \mathcal{J}_K(u)$ for any $u \in X_0$, thanks to (1.12).

As for the geometric features of $\mathcal{J}_K$, condition (ii) of the Fountain Theorem follows as in Step 2 in the proof of Theorem 1. While condition (i) can be proved as in the proof of Theorem 2. Hence, the proof of Theorem 3 is complete.

5. Infinitely many solutions for any $\lambda \in \mathbb{R}$

This section is devoted to the existence of infinitely many solutions for problem (1.20). The arguments for proving Theorem 4, Theorem 5 and Theorem 6 are the same of the ones performed in the proofs of Theorem 1, Theorem 2 and Theorem 3, respectively. We just need, in some steps of the proofs, suitable estimates for the term $\lambda\|u\|_{L^2(\Omega)}^2$. In the sequel we will focus on these key steps.

In order to study problem (1.20), we consider its weak formulation, given by

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^n}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dx\,dy-\lambda\int_\Omega u(x)\varphi(x)\,dx \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \displaystyle\int_\Omega f(x,u(x))\varphi(x)dx \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\,\varphi\in X_0 \\ u\in X_0, \end{cases} \tag{5.1}$$

and the energy functional $\mathcal{J}_{K,\lambda}:X_0\to\mathbb{R}$ defined as

$$\begin{aligned} \mathcal{J}_{K,\lambda}(u) := \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}|u(x)-u(y)|^2K(x-y)\,dx\,dy-\frac{\lambda}{2}\int_\Omega u^2(x)\,dx \\ -\int_\Omega F(x,u(x))\,dx\,, \end{aligned} \tag{5.2}$$

where F is the function in (1.11).

5.1. **Palais–Smale condition.** As we did for the functional $\mathcal{J}_K$ associated with problem (1.3), here we will prove that $\mathcal{J}_{K,\lambda}$ verifies the Palais–Smale condition, when the right-hand side in problem (1.20) satisfies the Ambrosetti–Rabinowitz condition (1.10).

If $\lambda<0$, then we can proceed exactly as in the proof of Proposition 7 just replacing $\mathcal{J}_K$ with $\mathcal{J}_{K,\lambda}$ and taking into account in (3.6) that

$$\begin{aligned} \mathcal{J}_{K,\lambda}(u_j)-\frac{1}{\mu}\langle\mathcal{J}'_{K,\lambda}(u_j),u_j\rangle \geqslant & \mathcal{J}_K(u_j)-\frac{1}{\mu}\langle\mathcal{J}'_K(u_j),u_j\rangle \\ = & \left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_j\|^2_{X_0} \\ & -\int_\Omega\Big(F(x,u_j(x))-\frac{1}{\mu}f(x,u_j(x))\,u_j(x)\Big)\,dx\,. \end{aligned}$$

Otherwise, if $\lambda>0$, a more careful analysis is required in proving the boundedness of the Palais–Smale sequences. For reader's convenience, we prefer to give all the details.

Let $c\in\mathbb{R}$ and let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in X_0 such that

$$\mathcal{J}_{K,\lambda}(u_j)\to c \tag{5.3}$$

and

$$\sup\Big\{\big|\langle\,\mathcal{J}'_{K,\lambda}(u_j),\varphi\,\rangle\big|\;:\;\varphi\in X_0\,,\|\varphi\|_{X_0}=1\Big\}\to 0 \tag{5.4}$$

as $j\to+\infty$. Then, for any $j\in\mathbb{N}$ by (5.3) and (5.4) it easily follows that there exists $\kappa>0$ such that

$$\mathcal{J}_{K,\lambda}(u_j)-\frac{1}{\mu}\langle\mathcal{J}'_{K,\lambda}(u_j),u_j\rangle\leqslant\kappa\,(1+\|u_j\|_{X_0})\;. \tag{5.5}$$

Now, let us fix $\sigma\in(2,\mu)$, where $\mu>2$ is given in assumption (1.10). Arguing as in proof of Proposition 7 we get that for any $j\in\mathbb{N}$

$$\Big|\int_{\Omega\cap\{|u_j|\leqslant r\}}\Big(F(x,u_j(x))-\frac{1}{\sigma}\,f(x,u_j(x))\,u_j(x)\Big)\,dx\Big|\leqslant\tilde{\kappa}\,, \tag{5.6}$$

for a suitable $\tilde{\kappa} > 0$. Then, using (1.10), (1.13) and (5.6), we have that for any $j \in \mathbb{N}$

$$\begin{aligned} \mathcal{J}_{K,\lambda}(u_j) - \frac{1}{\sigma}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j\rangle &= \Big(\frac{1}{2}-\frac{1}{\sigma}\Big)\Big(\|u_j\|^2_{X_0} - \lambda\|u_j\|^2_{L^2(\Omega)}\Big) \\ &\quad - \int_\Omega \Big(F(x,u_j(x)) - \frac{1}{\sigma} f(x,u_j(x))\,u_j(x)\Big)\,dx \\ &\geqslant \Big(\frac{1}{2}-\frac{1}{\sigma}\Big)\Big(\|u_j\|^2_{X_0} - \lambda\|u_j\|^2_{L^2(\Omega)}\Big) \\ &\quad + \Big(\frac{\mu}{\sigma}-1\Big)\int_{\Omega\cap\{|u_j|\geqslant r\}} F(x,u_j(x))\,dx \\ &\quad - \int_{\Omega\cap\{|u_j|\leqslant r\}} \Big(F(x,u_j(x)) - \frac{1}{\sigma} f(x,u_j(x))\,u_j(x)\Big)\,dx \\ &\geqslant \Big(\frac{1}{2}-\frac{1}{\sigma}\Big)\Big(\|u_j\|^2_{X_0} - \lambda\|u_j\|^2_{L^2(\Omega)}\Big) \\ &\quad + \Big(\frac{\mu}{\sigma}-1\Big)\int_{\Omega\cap\{|u_j|\geqslant r\}} F(x,u_j(x))\,dx - \tilde{\kappa} \\ &\geqslant \Big(\frac{1}{2}-\frac{1}{\sigma}\Big)\Big(\|u_j\|^2_{X_0} - \lambda\|u_j\|^2_{L^2(\Omega)}\Big) \\ &\quad + a_3\Big(\frac{\mu}{\sigma}-1\Big)\|u_j\|^\mu_{L^\mu(\Omega)} - a_4\Big(1-\frac{\mu}{\sigma}\Big)|\Omega| - \tilde{\kappa}\,. \end{aligned} \tag{5.7}$$

Furthermore, for any $\varepsilon > 0$ the Young inequality (with conjugate exponents $\mu/2 > 1$ and $\mu/(\mu-2)$) yields

$$\|u_j\|^2_{L^2(\Omega)} \leqslant \frac{2\varepsilon}{\mu}\|u_j\|^\mu_{L^\mu(\Omega)} + \frac{\mu-2}{\mu}\varepsilon^{-2/(\mu-2)}|\Omega|\,, \tag{5.8}$$

so that, by (5.7) and (5.8) we deduce that for any $j \in \mathbb{N}$

$$\begin{aligned} \mathcal{J}_{K,\lambda}(u_j) - \frac{1}{\sigma}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j\rangle &\geqslant \Big(\frac{1}{2}-\frac{1}{\sigma}\Big)\|u_j\|^2_{X_0} - \lambda\Big(\frac{1}{2}-\frac{1}{\sigma}\Big)\frac{2\varepsilon}{\mu}\|u_j\|^\mu_{L^\mu(\Omega)} \\ &\quad - \lambda\Big(\frac{1}{2}-\frac{1}{\sigma}\Big)\frac{\mu-2}{\mu}\varepsilon^{-2/(\mu-2)}|\Omega| \\ &\quad + a_3\Big(\frac{\mu}{\sigma}-1\Big)\|u_j\|^\mu_{L^\mu(\Omega)} - a_4\Big(1-\frac{\mu}{\sigma}\Big)|\Omega| - \tilde{\kappa} \\ &= \Big(\frac{1}{2}-\frac{1}{\sigma}\Big)\|u_j\|^2_{X_0} \\ &\quad + \Big[a_3\Big(\frac{\mu}{\sigma}-1\Big) - \lambda\Big(\frac{1}{2}-\frac{1}{\sigma}\Big)\frac{2\varepsilon}{\mu}\Big]\|u_j\|^\mu_{L^\mu(\Omega)} - C_\varepsilon\,, \end{aligned} \tag{5.9}$$

where C_ε is a constant such that $C_\varepsilon \to +\infty$ as $\varepsilon \to 0$, being $\mu > \sigma > 2$.

Now, choosing ε so small that

$$a_3\Big(\frac{\mu}{\sigma}-1\Big) - \lambda\Big(\frac{1}{2}-\frac{1}{\sigma}\Big)\frac{2\varepsilon}{\mu} > 0\,,$$

by (5.9), for any $j \in \mathbb{N}$ we get

$$\mathcal{J}_{K,\lambda}(u_j) - \frac{1}{\sigma}\langle \mathcal{J}'_{K,\lambda}(u_j), u_j\rangle \geqslant \Big(\frac{1}{2}-\frac{1}{\sigma}\Big)\|u_j\|^2_{X_0} - C_\varepsilon\,. \tag{5.10}$$

Combining (5.5) and (5.10) we deduce that for any $j \in \mathbb{N}$

$$\|u_j\|^2_{X_0} \leqslant \kappa_*\,(1 + \|u_j\|_{X_0})$$

for a suitable positive constant κ_*. This proves that the Palais–Smale sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0.

5.2. **Cerami condition.** When replacing the Ambrosetti–Rabinowitz condition (1.10) with assumptions (1.14) and (1.16) or (1.19), we will show that $\mathcal{J}_{K,\lambda}$ verifies the Cerami condition.

Also in this case we just have to prove the boundedness of the Cerami sequences. At this purpose we can argue as in the case when $\lambda = 0$, that is as in Proposition 8, just replacing $\mathcal{J}_K$ with $\mathcal{J}_{K,\lambda}$ and taking into account the following facts (we use the notation of Proposition 8):

- case when $v_\infty \equiv 0$: by (3.22) and (3.24) it is easy to see that
$$\lambda \int_\Omega |r_m v_j(x)|^2\,dx \to 0$$
as $j \to +\infty$;
- case when $v_\infty \not\equiv 0$: relation (3.35) has to be replaced with
$$\frac{1}{2} - \frac{\lambda}{2}\int_\Omega \frac{|u_j(x)|^2}{\|u_j\|^2_{X_0}}\,dx - \int_{\Omega'} \frac{F(x,u_j(x))}{\|u_j\|^2_{X_0}}\,dx - \int_{\Omega\setminus\Omega'} \frac{F(x,u_j(x))}{\|u_j\|^2_{X_0}}\,dx = o(1) \tag{5.11}$$
as $j \to +\infty$.

If $\lambda > 0$, by (5.11) we deduce that
$$\begin{aligned} o(1) &= \frac{1}{2} - \frac{\lambda}{2}\int_\Omega \frac{|u_j(x)|^2}{\|u_j\|^2_{X_0}}\,dx - \int_{\Omega'} \frac{F(x,u_j(x))}{\|u_j\|^2_{X_0}}\,dx - \int_{\Omega\setminus\Omega'} \frac{F(x,u_j(x))}{\|u_j\|^2_{X_0}}\,dx \\ &\leqslant \frac{1}{2} - \int_{\Omega'} \frac{F(x,u_j(x))}{\|u_j\|^2_{X_0}}\,dx - \int_{\Omega\setminus\Omega'} \frac{F(x,u_j(x))}{\|u_j\|^2_{X_0}}\,dx \end{aligned} \tag{5.12}$$
as $j \to +\infty$. By (3.36), (3.42) and (5.12) we get a contradiction.

Otherwise, if $\lambda < 0$, by the variational characterization of the first eigenvalue λ_1 of $-\mathcal{L}_K$ (see [39, Proposition 9]), that is
$$\lambda_1 = \min_{u \in X_0\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(y)|^2 K(x-y)dx\,dy}{\displaystyle\int_\Omega |u(x)|^2\,dx},$$
we get that for any $u \in X_0$
$$\|u\|^2_{L^2(\Omega)} \leqslant \frac{1}{\lambda_1}\|u\|^2_{X_0}. \tag{5.13}$$
Hence, by (5.11) and (5.13), we deduce that
$$\begin{aligned} o(1) &= \frac{1}{2} - \frac{\lambda}{2}\int_\Omega \frac{|u_j(x)|^2}{\|u_j\|^2_{X_0}}\,dx - \int_{\Omega'} \frac{F(x,u_j(x))}{\|u_j\|^2_{X_0}}\,dx - \int_{\Omega\setminus\Omega'} \frac{F(x,u_j(x))}{\|u_j\|^2_{X_0}}\,dx \\ &\leqslant \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right) - \int_{\Omega'} \frac{F(x,u_j(x))}{\|u_j\|^2_{X_0}}\,dx - \int_{\Omega\setminus\Omega'} \frac{F(x,u_j(x))}{\|u_j\|^2_{X_0}}\,dx \end{aligned} \tag{5.14}$$
as $j \to +\infty$. Now, a contradiction occurs by (3.36), (3.42) and (5.14).

In both cases we get that the any Cerami sequence for $\mathcal{J}_{K,\lambda}$ has to be bounded in X_0.

5.3. **Geometric features.** Here we have to verify that the functional $\mathcal{J}_{K,\lambda}$ has the geometric features described in (i) and (ii).

As for (i), when the Ambrosetti–Rabinowitz condition is assumed, we can argue as in the proof of Theorem 1, just replacing $\mathcal{J}_K$ with $\mathcal{J}_{K,\lambda}$ and (4.5) with the following one
$$\begin{aligned} \mathcal{J}_{K,\lambda}(u) &\leqslant \frac{1}{2}\|u\|^2_{X_0} - \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)} - a_3\|u\|^\mu_{L^\mu(\Omega)} + a_4|\Omega| \\ &\leqslant \frac{C_{k,\lambda}}{2}\|u\|^2_{X_0} - \hat{C}_{k,\mu}\|u\|^\mu_{X_0} + a_4|\Omega| \end{aligned} \tag{5.15}$$

for suitable positive constants $C_{k,\lambda}$, depending on k and λ, and $\hat{C}_{k,\mu}$, depending on k and μ. Here we used the fact that in Y_k all the norms are equivalent.

Otherwise, if other superlinear assumptions different from the Ambrosetti–Rabinowitz condition (see (1.14) and (1.16) or (1.19)) are assumed, we can argue as in the proof of Theorem 2, just replacing (4.8) with

$$F(x,t) \geqslant \frac{C_{k,\lambda}}{C_k^2}\,|t|^2 \text{ for any } x \in \overline{\Omega} \text{ and any } t \in \mathbb{R} \text{ with } |t| > \delta_k\,, \tag{5.16}$$

where $C_{k,\lambda}$ is the positive constant such that

$$\|u\|_{X_0}^2 - \lambda\|u\|_{L^2(\Omega)}^2 \leqslant C_{k,\lambda}\|u\|_{X_0}^2 \tag{5.17}$$

for any $u \in Y_k$, and $C_k := C_{k,2}$, being $C_{k,2}$ the positive constant given in (4.1) with $q = 2$. Here, we use again the fact that Y_k is a finite-dimensional space.

By (4.9) and (5.16), it is easy to see that

$$F(x,t) \geqslant \frac{C_{k,\lambda}}{C_k^2}|t|^2 - B_k \text{ for any } (x,t) \in \overline{\Omega} \times \mathbb{R}$$

for a suitable positive constant B_k (say, $B_k \geqslant \delta_k^2 C_{k,\lambda}/C_k^2 - m_k$).

As a consequence of this and by (4.1) and (5.17), for any $u \in Y_k$ we deduce that

$$\begin{aligned} \mathcal{J}_{K,\lambda}(u) &= \frac{1}{2}\|u\|_{X_0}^2 - \frac{\lambda}{2}\|u\|_{L^2(\Omega)}^2 - \int_\Omega F(x,u(x))\,dx \\ &\leqslant \frac{C_{k,\lambda}}{2}\|u\|_{X_0}^2 - \frac{C_{k,\lambda}}{C_k^2}\|u\|_{L^2(\Omega)}^2 + B_k|\Omega| \\ &\leqslant \frac{C_{k,\lambda}}{2}\|u\|_{X_0}^2 - C_{k,\lambda}\|u\|_{X_0}^2 + B_k|\Omega| \\ &= -\frac{C_{k,\lambda}}{2}\|u\|_{X_0}^2 + B_k|\Omega|\,. \end{aligned} \tag{5.18}$$

Thus, the functional $\mathcal{J}_K$ satisfies condition (i) of the Fountain Theorem.

Finally, to prove the geometric condition (ii) of the Fountain Theorem, we can proceed as in the proof of Theorem 1 (see Step 2 in this proof).

If $\lambda < \lambda_1$, then we can argue exactly as in Step 2 of the proof of Theorem 1 just replacing $\mathcal{J}_K$ with $\mathcal{J}_{K,\lambda}$ and taking into account in (4.7) that

$$\begin{aligned} \mathcal{J}_{K,\lambda}(u) &\geqslant \frac{1}{2}\|u\|_{X_0}^2 - \frac{\lambda}{2}\|u\|_{L^2(\Omega)}^2 - C\|u\|_{L^q(\Omega)}^q - C|\Omega| \\ &\geqslant \frac{1}{2}\min\Big\{1, 1-\frac{\lambda}{\lambda_1}\Big\}\|u\|_{X_0}^2 - C\|u\|_{L^q(\Omega)}^q - C|\Omega|\,. \end{aligned}$$

Here we used (5.13).

Otherwise, if $\lambda \geqslant \lambda_1$, due to the fact that the sequence λ_k of the eigenvalues of $-\mathcal{L}_K$ is positive and divergent, we can assume that $\lambda \in [\lambda_{k-1}, \lambda_k)$ for some $k \in \mathbb{N}$, $k \geqslant 2$. Hence, we can proceed again as in Step 2 of the proof of Theorem 1. We just have to take into account the variational characterization of λ_k, given by

$$\lambda_k = \min_{u \in Z_k \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x)-u(y)|^2 K(x-y)\,dx\,dy}{\displaystyle\int_\Omega |u(x)|^2\,dx}\,,$$

and that by (4.6) for any $u \in Z_k \setminus \{0\}$, we have that

$$
\begin{aligned}
\mathcal{J}_{K,\lambda}(u) &\geqslant \frac{1}{2}\|u\|^2_{X_0} - \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)} - C\|u\|^q_{L^q(\Omega)} - C|\Omega| \\
&\geqslant \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\|u\|^2_{X_0} - C\left\|\frac{u}{\|u\|_{X_0}}\right\|^q_{L^q(\Omega)}\|u\|^q_{X_0} - C|\Omega| \\
&\geqslant \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\|u\|^2_{X_0} - C\beta_k^q\|u\|^q_{X_0} - C|\Omega| \\
&= \|u\|^2_{X_0}\left[\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right) - C\beta_k^q\|u\|^{q-2}_{X_0}\right] - C|\Omega|\,,
\end{aligned}
\tag{5.19}
$$

where β_k is defined as in Lemma 11. Choosing

$$
\gamma_k = \left(1 - \frac{\lambda}{\lambda_k}\right)^{1/(q-2)} \left(qC\beta_k^q\right)^{-1/(q-2)},
$$

it is easy to see that $\gamma_k \to +\infty$ as $k \to +\infty$, thanks to Lemma 11, the fact that $q > 2$ and that $\lambda < \lambda_k$. Thus, by the choice of γ_k and (5.19) we get that for any $u \in Z_k$ with $\|u\|_{X_0} = \gamma_k$

$$
\begin{aligned}
\mathcal{J}_K(u) &\geqslant \|u\|^2_{X_0}\left[\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right) - C\beta_k^q\|u\|^{q-2}_{X_0}\right] - C|\Omega| \\
&= \left(\frac{1}{2} - \frac{1}{q}\right)\left(1 - \frac{\lambda}{\lambda_k}\right)\gamma_k^2 - C|\Omega| \to +\infty
\end{aligned}
$$

as $k \to +\infty$. Thus, the geometric condition (ii) is proved.

Combining these steps in proving the Palais–Smale and the Cerami conditions and the geometric structure of the functional $\mathcal{J}_{K,\lambda}$ together with the arguments used in the proofs of Theorem 1, Theorem 2 and Theorem 3, the validity of Theorem 4, Theorem 5 and Theorem 6 easily follows.

References

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] C.O. Alves and S.B. Liu, *On superlinear $p(x)$-Laplacian equations in $\mathbb{R}^N$*, Nonlinear Anal., 73, no. 8, 2566–2579 (2010).

[3] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal., 14, 349–381 (1973).

[4] G. Autuori and P. Pucci, *Elliptic problems involving the fractional Laplacian in $\mathbb{R}^N$*, J. Differential Equations, 25, no. 8, 2340–2362 (2013).

[5] B. Barrios, E. Colorado, R. Servadei and F. Soria, *A critical fractional equation with concave-convex power nonlinearities*, to appear in Ann. Inst. H. Poincaré Anal. Non Linéaire.

[6] T. Bartsch, *Infinitely many solutions of a symmetric Dirichlet problem*, Nonlinear Anal., 20, no. 1, 1205–1216 (1993).

[7] H. Brezis, Analyse fonctionelle. Théorie et applications, *Masson*, Paris (1983).

[8] X. Cabré and J. Tan, *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Adv. Math., 224, 2052–2093 (2010).

[9] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations 32, no. 7-9, 1245–1260 (2007).

[10] L. Caffarelli and L. Silvestre, *Regularity theory for fully nonlinear integro-differential equations*, Comm. Pure Appl. Math. 62, no. 5, 597–638 (2009).

[11] G. Cerami, *An existence criterion for the critical points on unbounded manifolds*, (Italian) Istit. Lombardo Accad. Sci. Lett. Rend. A 112, no. 2, 332–336 (1978).

[12] G. Cerami, *On the existence of eigenvalues for a nonlinear boundary value problem*, (Italian) Ann. Mat. Pura Appl. (4) 124, 161–179 (1980).

[13] D.G. Costa, C.A. Magalhães, *Variational elliptic problems which are nonquadratic at infinity*, Nonlinear Anal. 23, 1401–1412 (1994).

[14] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math., 136, no. 5, 521–573 (2012).

[15] F. Fang and S.B. Liu, *Nontrivial solutions of superlinear p-Laplacian equations*, J. Math. Anal. Appl., 351, no. 1, 138–146 (2009).

[16] M. Ferrara, G. Molica Bisci and B. Zhang, *Existence of weak solutions for non-local fractional problems via Morse theory*, to appear in Discrete and Continuous Dynamical Dystems Series B.

[17] A. Fiscella, G. Molica Bisci and R. Servadei, *Bifurcation and multiplicity results for critical nonlocal fractional Laplacian problems*, preprint, 2014, available at http://www.math.utexas.edu/mp_arc-bin/mpa?yn=14-43.

[18] A. Fiscella, R. Servadei and E. Valdinoci, *A resonance problem for non-local elliptic operators*, Z. Anal. Anwendungen, 32, 411–431 (2013).

[19] A. Fiscella and E. Valdinoci, *A critical Kirchhoff type problem involving a nonlocal operator*, Nonlinear Anal., 94, 156–170 (2014).

[20] L. Jeanjean, *On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer type problem set on* $\mathbb{R}^N$, Proc. Roy. Soc. Edinburgh Sect.A, 129, no. 4, 787–809 (1999).

[21] S.B. Liu, *On ground states of superlinear p-Laplacian equations in* $\mathbb{R}^N$, J. Math. Anal. Appl., 361, no. 81, 48–58 (2010).

[22] S.B. Liu, *On superlinear problems without Ambrosetti-Rabinowitz condition*, Nonlinear Anal., 73, no. 3, 788–795 (2010).

[23] S.B. Liu and S.J. Liu, *Infinitely many solutions for a superlinear elliptic equation*, Acta. Math. Sinica (Chin. Ser.), 46, no. 4, 625–630 (2003).

[24] O. Miyagaki and M. Souto, *Superlinear problems without Ambrosetti and Rabinowitz growth condition*, J. Differential Equations, 245, no. 12, 3628–3638 (2008).

[25] G. Molica Bisci, *Fractional equations with bounded primitive*, Appl. Math. Lett., 27, 53–58 (2014).

[26] G. Molica Bisci, *Sequences of weak solutions for fractional equations*, Math. Res. Lett., 21, no. 2, 1–13 (2014).

[27] G. Molica Bisci and B.A. Pansera, *Three weak solutions for nonlocal fractional equations*, Adv. Nonlinear Stud., 14, 591–601 (2014).

[28] G. Molica Bisci and D. Repovs, *Higher nonlocal problems with bounded potential*, to appear in J. Math. Anal. Appl.

[29] G. Molica Bisci and R. Servadei, *A bifurcation result for non-local fractional equations*, to appear in Anal. Appl., DOI:10.1142/S0219530514500067.

[30] P. Pucci and S. Saldi, *Critical stationary Kirchhoff equations in* $\mathbb{R}^N$ *involving nonlocal operators*, to appear in Rev. Mat. Iberoam.

[31] P.H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, *CBMS Reg. Conf. Ser. Math.*, 65, *American Mathematical Society*, Providence, RI (1986).

[32] X. Ros-Oton and J. Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, to appear in J. Math. Pures Appl.

[33] M. Schechter and W. Zou, *Superlinear problems*, Pacific J. Math., 214, no. 1, 145–160 (2004).

[34] R. Servadei, *The Yamabe equation in a non-local setting*, Adv. Nonlinear Anal. 2, 235–270 (2013).

[35] R. Servadei, *Infinitely many solutions for fractional Laplace equations with subcritical nonlinearity*, Contemp. Math., 595, 317–340 (2013).

[36] R. Servadei, *A critical fractional Laplace equation in the resonant case*, Topol. Methods Nonlinear Anal., 43, no. 1, 251–267 (2014).

[37] R. Servadei and E. Valdinoci, *Lewy-Stampacchia type estimates for variational inequalities driven by (non)local operators*, Rev. Mat. Iberoam. 29, no. 3, 1091–1126 (2013).

[38] R. Servadei and E. Valdinoci, *Mountain Pass solutions for non-local elliptic operators*, J. Math. Anal. Appl., 389, 887–898 (2012).

[39] R. Servadei and E. Valdinoci, *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst. 33, no. 5, 2105–2137 (2013).

[40] R. Servadei and E. Valdinoci, *The Brezis-Nirenberg result for the fractional Laplacian*, to appear in Trans. Amer. Math. Soc.

[41] R. Servadei and E. Valdinoci, *A Brezis-Nirenberg result for non-local critical equations in low dimension*, Commun. Pure Appl. Anal. 12, no. 6, 2445–2464 (2013).

[42] M. Struwe, Variational methods, Applications to nonlinear partial differential equations and Hamiltonian systems, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 3, *Springer Verlag*, Berlin–Heidelberg (1990).

[43] J. Tan, *The Brezis-Nirenberg type problem involving the square root of the Laplacian*, Calc. Var. Partial Differential Equations, 36, no. 1-2, 21–41 (2011).

[44] M. Willem, Minimax Theorems, Progress in Nonlinear Differential Equations and their Applications, 24, Birkhäuser, Boston (1996).

[45] M. Willem and W. Zou, *On a Schrödinger equation with periodic potential and spectrum point zero*, Indiana Univ. Math. J. 52, 109–132 (2003).

[46] W.M. Zou, *Variant fountain theorems and their applications*, Manuscripta Math. 104, 343–358 (2001).

Department of Mathematics, Heilongjiang Institute of Technology, 150050 Harbin, China
E-mail address: zhbinlin@gmail.com

Dipartimento PAU, Università 'Mediterranea' di Reggio Calabria, Via Melissari 24, 89124 Reggio Calabria, Italy
E-mail address: gmolica@unirc.it

Dipartimento di Matematica e Informatica, Università della Calabria, Ponte Pietro Bucci 31 B, 87036 Arcavacata di Rende (Cosenza), Italy
E-mail address: servadei@mat.unical.it