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# Asymptotic eigenvalue estimates for a Robin problem with a large parameter

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**Abstract.** Robin problem for the Laplacian in a bounded planar domain with a smooth boundary and a large parameter in the boundary condition is considered. We prove a two-sided three-term asymptotic estimate for the negative eigenvalues.

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## 1. Introduction and the main result

Asymptotic properties of eigenvalues belong among the most often studied problems in the spectral theory. In this paper we are going to discuss an asymptotics of the "attractive" Robin problem for the Laplacian in a bounded domain of $\mathbb{R}^2$ in the situation when the parameter $\beta$ in the boundary condition assumes large values. The problem has a physical motivation; it naturally arises in the study of reaction-diffusion equation where a distributed absorbtion competes with a boundary source – see [4], [5] for details. At the same the question is of mathematical interest. In a recent paper, Levitin and Parnovski [6] investigated the asymptotic behavior of the principal eigenvalue and showed that its leading term is $-c\beta^2$ where $c = 1$ if the domain boundary is smooth and $c > 1$ if it has angles.

A related asymptotic problem is encountered in the theory of leaky quantum graphs [1] where the dynamics is not constrained to a bounded region, instead it is governed by a singular Schrödinger operator with an attractive interaction supported by a manifold or complex of a lower dimension. A particularly close

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analogy occurs in the two-dimensional situation when the interaction support is closed smooth loop; using a combination of bracketing and estimates with separated variables, one is able to derive an asymptotic expansion of negative eigenvalues [3] in which the absolute term with respect to the coupling parameter is given by a one-dimensional Schrödinger operator with a potential determined by the geometry of the problem, specifically the curvature of the loop.

This inspires the question whether the technique used for the singular Schrödinger operators cannot be used also for Robin "billiards" with a smooth boundary. This is the main topic of the present paper. We are going to show that in distinction to the Schrödinger case one does not obtain an asymptotic expansion, but two-sided asymptotic estimates only, which squeeze only when the domain is a circular disc. On the other hand, the estimates hold true not only for the principal eigenvalue, and moreover, they have three terms in the powers of $\beta$ which improves, in particular, the result obtained in [6] for smooth boundaries.

Let us now state the problem properly. We suppose that $\Omega$ be an open, simply connected set in $\mathbb{R}^2$ with a closed $C^4$ Jordan boundary $\partial\Omega = \Gamma : [0, L] \ni s \mapsto (\Gamma_1, \Gamma_2) \in \mathbb{R}^2$ which is parametrized by its arc length; for definiteness we choose the clockwise orientation of the boundary. Let $\gamma : [0, L] \to \mathbb{R}$ be the signed curvature of $\Gamma$, i.e. $\gamma(s) = \Gamma_1''(s)\Gamma_2'(s) - \Gamma_2''(s)\Gamma_1'(s)$. We investigate the spectral boundary-value problem

$$\begin{aligned} -\Delta f &= \lambda f \quad \text{in} \quad \Omega \\ \frac{\partial f}{\partial n} &= \beta f \quad \text{on} \quad \Gamma \end{aligned} \tag{1}$$

with a parameter $\beta > 0$, which will be in the following assumed to be large; the symbol $\frac{\partial}{\partial n}$ in (1) denotes the outward normal derivative. It is straightforward to check that the quadratic form

$$q_\beta[f] = \|\nabla f\|^2_{L^2(\mathbb{R}^2)} - \beta \int_\Gamma |f(x)|^2 \mathrm{d}s \tag{2}$$

with $\mathrm{Dom}(q_\beta) = H^1(\mathbb{R}^2)$ is closed and below bounded; we denote by $H_\beta$ the unique self-adjoint operator associated with it. Our main goal is to study the asymptotic behavior of the negative eigenvalues of $H_\beta$ as parameter $\beta$ tends to infinity. To state the result, we introduce the one-dimensional Schrödinger operator

$$S = -\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{1}{4}\gamma^2(s) \quad \text{in} \quad L^2(0, L) \tag{3}$$

with the domain

$$P = \left\{ f \in H^2(0, L) : \ f(0) = f(L), \ f'(0) = f'(L) \right\}. \tag{4}$$

We use the symbol $\mu_j$ for the $j$-th eigenvalue of $S$ counted with the multiplicity, $j \in \mathbb{N}$, and furthermore, we denote $\gamma^* = \max_{[0,L]} \gamma(s)$ and $\gamma_* = \min_{[0,L]} \gamma(s)$.

Our main result reads then as follows.

**Theorem 1.1.** *Under the stated assumptions, to any fixed integer $n$ there exists a $\beta(n) > 0$ such that the number of negative eigenvalues of $H_\beta$ is not smaller than $n$. For $\beta > \beta(n)$ we denote by $\lambda_n(\beta)$ the $n$-th eigenvalue of $H_\beta$ counted with the multiplicity. Then $\lambda_n(\beta)$ satisfies for $\beta \to \infty$ the asymptotic estimates*

$$-\left(\beta + \frac{\gamma^*}{2}\right)^2 + \mu_j + \mathcal{O}\left(\frac{\log\beta}{\beta}\right) \le \lambda_j(\beta) \le -\left(\beta + \frac{\gamma_*}{2}\right)^2 + \mu_j + \mathcal{O}\left(\frac{\log\beta}{\beta}\right). \quad (5)$$

**Remark 1.2.** It will be clear from the proof that the assumption about simple connectedness of $\Omega$ is done mostly for the sake of simplicity. The result extends easily to multiply connected domains, in general with different parameters at different components of the boundary; each of the components then gives rise to a series of negative eigenvalues tending to $-\infty$ in the limit.

## 2. Proof of Theorem 1.1

Let us first introduce some quadratic forms and operators which we shall need in the argument. To begin with, we need the following result, which is a straightforward modification of Lemma 2.1 of [3], hence we skip the proof.

**Lemma 2.1.** *Let $\Phi$ be the map*

$$[0, L) \times (0, a) \ni (s, u) \mapsto (\Gamma_1(s) + u\Gamma_2'(s), \Gamma_2(s) - u\Gamma_1'(s)) \in \mathbb{R}^2.$$

*Then there exists an $a_1 > 0$ such that the map $\Phi$ is injective for any $a \in (0, a_1]$.*

Choose an $a$ satisfying $0 < a < a_1$, to be specified later, and let $\Sigma_a$ be the strip neighborhood of $\Gamma$ of width $a$,

$$\Sigma_a := \Phi([0, L) \times [0, a)).$$

Then $\Omega \setminus \Sigma_a = \Lambda_a$ is a compact simply connected domain with the boundary $\Gamma_a := \Phi([0, L) \times \{a\})$. We define

$$q^D_{a,\beta}[f] := \|\nabla f\|^2_{\Sigma_a} - \beta \int_\Gamma |f(x)| \mathrm{d}s \quad \text{for } f \in \left\{ f \in H^1(\Sigma_a) : f|_{\Gamma_a} = 0 \right\},$$

$$q^N_{a,\beta}[f] := \|\nabla f\|^2_{\Sigma_a} - \beta \int_\Gamma |f(x)| \mathrm{d}s \quad \text{for } f \in H^1(\Sigma_a),$$

and denote by $L^D_{a,\beta}$ and $L^N_{a,\beta}$ the self-adjoint operators associated with the forms $q^D_{a,\beta}$ and $q^N_{a,\beta}$, respectively. The first key component of the proof is to use the Dirichlet-Neumann bracketing – see [7, Sec. XIII.15, Prop. 4] – imposing additional boundary condition at $\Gamma_a$. This yields

$$(-\Delta^N_{\Lambda_a}) \oplus L^N_{a,\beta} \le H_\beta \le (-\Delta^D_{\Lambda_a}) \oplus L^D_{a,\beta} \quad (6)$$

in $L^2(\Omega) = L^2(\Lambda_a) \oplus L^2(\Sigma_a)$. Since the estimating operators have the direct-sum structure and the first terms in the inequalities (6) referring to the part of $\Omega$ separated from the boundary are positive, in order to estimate the negative eigenvalues of $H_\beta$ it is sufficient to estimate those of $L^D_{a,\beta}$, $L^N_{a,\beta}$.

To this aim we use the second main trick, introducing a "straightening" transformation in the spirit of [2], to produce a pair of operators in $L^2((0,L)\times(0,a))$ that are unitarily equivalent to $L^D_{a,\beta}$ and $L^N_{a,\beta}$, respectively. Specifically, we introduce the following change of variables,

$$f(x_1,x_2) = \frac{1}{(1-u\gamma(s))^{1/2}}\,\varphi(s,u);$$

then it is straightforward to check that for any function $f \in H^2(\Sigma_a)$ we have also $\varphi \in H^2((0,L)\times(0,a))$ and

$$\begin{aligned}\left(|f_{x_1}|^2 + |f_{x_2}|^2\right)\mathrm{d}x_1\mathrm{d}x_2 = \Bigg[&\frac{1}{(1-u\gamma(s))^2}\left|\frac{\partial\varphi}{\partial s}\right|^2 + \left|\frac{\partial\varphi}{\partial u}\right|^2 + \widetilde{V}(s,u)|\varphi|^2 \\ &+\frac{u\gamma'(s)}{2(1-u\gamma(s))^3}\left(\varphi\,\overline{\frac{\partial\varphi}{\partial s}} + \overline{\varphi}\,\frac{\partial\varphi}{\partial s}\right) + \frac{\gamma(s)}{2(1-u\gamma(s))}\left(\varphi\,\overline{\frac{\partial\varphi}{\partial u}} + \overline{\varphi}\,\frac{\partial\varphi}{\partial u}\right)\Bigg]\end{aligned}$$

with

$$\widetilde{V}(s,u) = \frac{\gamma^2(s)}{4(1-u\gamma(s))^2} + \frac{u^2(\gamma'(s))^2}{4(1-u\gamma(s))^4}$$

and

$$\begin{aligned}&\iint_\Omega \left(|f_{x_1}|^2 + |f_{x_2}|^2\right)\mathrm{d}x_1\mathrm{d}x_2 - \beta\int_\Gamma |f(x)|^2\mathrm{d}s \\ &= \int_0^a\!\!\int_0^L \frac{1}{(1-u\gamma(s))^2}\left|\frac{\partial\varphi}{\partial s}\right|^2 \mathrm{d}s\,\mathrm{d}u + \int_0^a\!\!\int_0^L \left|\frac{\partial\varphi}{\partial u}\right|^2 \mathrm{d}s\,\mathrm{d}u + \int_0^a\!\!\int_0^L V(s,u)\,|\varphi|^2\,\mathrm{d}s\,\mathrm{d}u \\ &\quad - \int_0^L \left(\frac{\gamma(s)}{2} + \beta\right)|\varphi(s,0)|^2\mathrm{d}s + \int_0^L \frac{\gamma(s)}{2(1-a\gamma(s))}|\varphi(s,a)|^2\mathrm{d}s,\end{aligned}$$

where

$$\begin{aligned}V(s,u) &= \widetilde{V}(s,u) - \frac{\partial}{\partial s}\left(\frac{u\gamma'(s)}{2(1-u\gamma(s))^3}\right) - \frac{\partial}{\partial u}\left(\frac{\gamma(s)}{2(1-u\gamma(s))}\right) \\ &= \frac{-\gamma^2(s)}{4(1-u\gamma(s))^2} - \frac{u\gamma''(s)}{2(1-u\gamma(s))^3} - \frac{5}{4}\,\frac{u^2(\gamma'(s))^2}{(1-u\gamma(s))^4}.\end{aligned}$$

Armed with these formulæ we can now introduce the two operators in $L^2((0,L)\times(0,a))$ unitarily equivalent to $L^D_{a,\beta}$ and $L^N_{a,\beta}$, respectively. On the domains

$$Q^D_a = \left\{\varphi \in H^1((0,L)\times(0,a)) :\ \varphi(L,.) = \varphi(0,.)\ \text{ on } (0,a),\ \varphi(.,a) = 0\ \text{ on } (0,L)\right\}$$

and

$$Q_a^N = \left\{\varphi \in H^1((0,L)\times(0,a)) : \quad \varphi(L,.) = \varphi(0,.) \text{ on } (0,a)\right\},$$

we define the quadratic forms

$$\begin{aligned} b_{a,\beta}^D[\varphi] = & \int_0^a\int_0^L \frac{1}{(1-u\gamma(s))^2}\left|\frac{\partial\varphi}{\partial s}\right|^2 \mathrm{d}s\,\mathrm{d}u + \int_0^a\int_0^L \left|\frac{\partial\varphi}{\partial u}\right|^2 \mathrm{d}s\,\mathrm{d}u \\ & + \int_0^a\int_0^L V(s,u)\,|\varphi|^2\,\mathrm{d}s\,\mathrm{d}u - \int_0^L \left(\frac{\gamma(s)}{2}+\beta\right)|\varphi(s,0)|^2 \mathrm{d}s \end{aligned}$$

and

$$\begin{aligned} b_{a,\beta}^N[\varphi] = & \int_0^a\int_0^L \frac{1}{(1-u\gamma(s))^2}\left|\frac{\partial\varphi}{\partial s}\right|^2 \mathrm{d}s\,\mathrm{d}u + \int_0^a\int_0^L \left|\frac{\partial\varphi}{\partial u}\right|^2 \mathrm{d}s\,\mathrm{d}u \\ & + \int_0^a\int_0^L V(s,u)\,|\varphi|^2\,\mathrm{d}s\,\mathrm{d}u - \int_0^L \left(\frac{\gamma(s)}{2}+\beta\right)|\varphi(s,0)|^2 \mathrm{d}s \\ & + \int_0^L \frac{\gamma(s)}{2(1-a\gamma(s))}|\varphi(s,a)|^2 \mathrm{d}s, \end{aligned}$$

respectively. It is easy to check the following claim analogous to Lemma 2.2 of [3].

**Lemma 2.2.** *The operators $B_{a,\beta}^D$ and $B_{a,\beta}^N$ associated with above quadratic forms are unitarily equivalent to $L_{a,\beta}^D$ and $L_{a,\beta}^N$, respectively.*

In the next step we estimate $B_{a,\beta}^D$ and $B_{a,\beta}^N$ just introduced by operators with separated variables. We put[1]

$$\begin{aligned} &\gamma_+ = \max_{[0,L]}|\gamma(.)|, \quad \gamma'_+ = \max_{[0,L]}|\gamma'(.)|, \quad \gamma''_+ = \max_{[0,L]}|\gamma''(.)|, \\ &V_+(s) = \frac{-\gamma^2(s)}{4(1+a\gamma_+)^2} + \frac{a\gamma''_+}{2(1-a\gamma_+)^3}\ , \\ &V_-(s) = \frac{-\gamma^2(s)}{4(1-a\gamma_+)^2} - \frac{a\gamma''_+}{2(1-a\gamma_+)^3} - \frac{5}{4}\frac{a^2(\gamma'_+)^2}{(1-a\gamma_+)^4}. \end{aligned}$$

For an $a$ satisfying $0 < a < \gamma_+/2$ and $\varphi$ belonging to $Q_a^D$ and $Q_a^N$, respectively, we

[1] There is a typo in [3]; the second term in the definition of $V_+$ there has to be deleted.

define

$$\widetilde{b}^D_{a,\beta}[\varphi] = (1-a\gamma_+)^{-2} \int_0^a\!\!\int_0^L \left|\frac{\partial\varphi}{\partial s}\right|^2 \mathrm{d}s\,\mathrm{d}u + \int_0^a\!\!\int_0^L \left|\frac{\partial\varphi}{\partial u}\right|^2 \mathrm{d}s\,\mathrm{d}u$$
$$+ \int_0^a\!\!\int_0^L V_+(s)\,|\varphi|^2\,\mathrm{d}s\,\mathrm{d}u - \Big(\frac{\gamma_*}{2}+\beta\Big)\int_0^L |\varphi(s,0)|^2 \mathrm{d}s$$

and

$$\widetilde{b}^N_{a,\beta}[\varphi] = (1+a\gamma_+)^{-2} \int_0^a\!\!\int_0^L \left|\frac{\partial\varphi}{\partial s}\right|^2 \mathrm{d}s\,\mathrm{d}u + \int_0^a\!\!\int_0^L \left|\frac{\partial\varphi}{\partial u}\right|^2 \mathrm{d}s\,\mathrm{d}u$$
$$+ \int_0^a\!\!\int_0^L V_-(s)\,|\varphi|^2\,\mathrm{d}s\,\mathrm{d}u - \left(\frac{\gamma^*}{2}+\beta\right)\int_0^L |\varphi(s,0)|^2 \mathrm{d}s - \frac{\gamma_+}{2(1-a\gamma_+)}\int_0^L |\varphi(s,a)|^2 \mathrm{d}s.$$

Then we have

$$b^D_{a,\beta}[\varphi] \le \widetilde{b}^D_{a,\beta}[\varphi] \quad \text{for } f \in Q^D_a, \tag{7}$$
$$b^N_{a,\beta}[\varphi] \ge \widetilde{b}^N_{a,\beta}[\varphi] \quad \text{for } f \in Q^N_a. \tag{8}$$

Let $\widetilde{H}^D_{a,\beta}$ and $\widetilde{H}^N_{a,\beta}$ be the self-adjoint operators associated with the forms $\widetilde{b}^D_{a,\beta}$ and $\widetilde{b}^N_{a,\beta}$, respectively. By $T^D_{a,\beta}$ we denote the self-adjoint operator associated with the form

$$t^D_{a,\beta}[\varphi] = \int_0^a |\varphi'(u)|^2 \mathrm{d}u - \Big(\frac{\gamma_*}{2}+\beta\Big)\,|\varphi(0)|^2$$

defined on $\{\varphi \in H^1(0,a):\ \varphi(a)=0\}$. Similarly, $T^N_{a,\beta}$ is the self-adjoint operator associated with the form

$$t^N_{a,\beta}(\varphi,\varphi) = \int_0^a |\varphi'(u)|^2 \mathrm{d}u - \left(\frac{\gamma^*}{2}+\beta\right)|\varphi(0)|^2, \quad \varphi \in H^1(0,a).$$

Furthermore, we introduce the operators

$$U^D_a = (1-a\gamma_+)^{-2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right) + V_+(s)\,, \quad U^N_a = (1+a\gamma_+)^{-2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right) + V_-(s)$$

in $L^2(0,L)$, the domain of both of them being $P$ given by (4). Then we have

$$\widetilde{H}^D_{a,\beta} = U^D_a \otimes I + I \otimes T^D_{a,\beta}, \qquad \widetilde{H}^N_{a,\beta} = U^N_a \otimes I + I \otimes T^N_{a,\beta}, \tag{9}$$

and we can estimate contributions from the longitudinal and transverse variables separately. What concerns the former, we denote by $\mu^D_j(a)$, $\mu^N_j(a)$ the $j$-th eigenvalue of $U^D_a$, $U^N_a$, respectively, counted with the multiplicity, and use Lemma 2.3 of [3] which contains the following claim:

**Lemma 2.3.** *There exists a constant $C > 0$ such that the estimates*

$$|\mu_j^D(a) - \mu_j| \le Caj^2 \tag{10}$$

*and*

$$|\mu_j^N(a) - \mu_j| \le Caj^2 \tag{11}$$

*hold for any $j \in \mathbb{N}$ and $0 < a < 1/(2\gamma_+)$. where $C$ is independent on $j, a$.*

We stress that the constant $C$ here is independent of $j$ and $a$. As for the transverse part, let us estimate first the principal eigenvalue of $T_{a,\beta}^D$.

**Lemma 2.4.** *Assume that $a\left(\beta + \frac{\gamma_*}{2}\right) > \dfrac{4}{3}$. Then $T_{a,\beta}^D$ has only one negative eigenvalue which we denote by $\zeta_{a,\beta}^D$. It satisfies the inequalities*

$$-\left(\beta + \frac{\gamma_*}{2}\right)^2 \le \zeta_{a,\beta}^D \le -\left(\beta + \frac{\gamma_*}{2}\right)^2 + 4\left(\beta + \frac{\gamma_*}{2}\right)^2 \mathrm{e}^{-a\left(\beta+\frac{\gamma_*}{2}\right)}.$$

*Proof.* Notice that the domain of the operator is

$$D(T_{a,\beta}^D) = \left\{\varphi \in H^2(0,a) :\ \varphi'(0) = -\left(\frac{\gamma_*}{2} + \beta\right)\varphi(0),\ \varphi(a) = 0\right\}.$$

Assume that $-k^2$ with $k > 0$ is an eigenvalue of $T_{a,\beta}^D$, and let a nonzero $\varphi$ be the corresponding eigenfunction, then we have

(1) $-\varphi''(u) = -k^2\varphi(u)$;

(2) $\varphi'(0) = -\left(\frac{\gamma_*}{2} + \beta\right)\varphi(0)$;

(3) $\varphi(a) = 0$.

In view of the first property, the eigenfunction $\varphi$ is of the form

$$\varphi(u) = A\mathrm{e}^{ku} + B\mathrm{e}^{-ku}.$$

Furthermore, the requirements (2) and (3) yield $kA - kB = \left(-\frac{\gamma_*}{2} - \beta\right)(A + B)$ and $A\mathrm{e}^{ka} + B\mathrm{e}^{-ka} = 0$, respectively. Thus the coefficients $A$, $B$ have to satisfy the equation

$$\begin{pmatrix} \mathrm{e}^{ka} & \mathrm{e}^{-ka} \\ k + \frac{\gamma_*}{2} + \beta & -(k - \frac{\gamma_*}{2} - \beta) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0.$$

Since $(A, B) \ne (0, 0)$, we get

$$\det \begin{pmatrix} \mathrm{e}^{ka} & \mathrm{e}^{-ka} \\ k + \frac{\gamma_*}{2} + \beta & -(k - \frac{\gamma_*}{2} - \beta) \end{pmatrix} = 0$$

which is equivalent to $g_{a,\beta}(k) := 2ak + \log\left(\beta + \frac{\gamma_*}{2} - k\right) - \log\left(\beta + \frac{\gamma_*}{2} + k\right) = 0$. It is easy to see that also the converse is true: if $g_{a,\beta}(k) = 0$, then $-k^2$ is an eigenvalue of $T^D_{a,\beta}$. Let us now show that $g^D_{a,\beta}(.)$ has a unique zero in $\left(0,\beta+\frac{\gamma_*}{2}\right)$. By definition we have $g_{a,\beta}(0)=0$, and since

$$\frac{\mathrm{d}g_{a,\beta}(k)}{\mathrm{d}k}=\frac{2a\left(\beta+\frac{\gamma_*}{2}\right)^2-2\left(\beta+\frac{\gamma_*}{2}\right)-2ak^2}{\left(\beta+\frac{\gamma_*}{2}\right)^2-k^2}$$

we can claim that $g^D_{a,\beta}$ is monotonously increasing in $\left(0,\ \beta+\dfrac{\gamma_*}{2}-\dfrac{1}{a}\right)$ and it is monotonously decreasing in $\left(\beta+\dfrac{\gamma_*}{2}-\dfrac{1}{a}\,,\ \beta+\dfrac{\gamma_*}{2}\right)$. Moreover, we have

$$\lim_{k\to\beta+\frac{\gamma_*}{2}} g^D_{a,\beta}=-\infty\,;$$

this implies that the function $g^D_{a,\beta}$ has a unique zero in $\left(0,\beta+\dfrac{\gamma_*}{2}\right)$. Moreover, since $a(\beta+\dfrac{\gamma_*}{2})>\dfrac{4}{3}$, we have $\sqrt{\left(\beta+\dfrac{\gamma_*}{2}\right)\left(\beta+\dfrac{\gamma_*}{2}-\dfrac{1}{a}\right)}>\dfrac{1}{2}\left(\beta+\dfrac{\gamma_*}{2}\right)$. Consequently, the solution $k$ has the form $k=\beta+\dfrac{\gamma_*}{2}-s$, $0<s<\frac{1}{2}\left(\beta+\frac{\gamma_*}{2}\right)$. Taking into account the relation $g_{a,\beta}(k)=0$, we get

$$\log s=\log(2\beta+\gamma_*-s)-2a\left(\beta+\frac{\gamma_*}{2}-s\right)\le\log(2\beta+\gamma_*)-a\left(\beta+\frac{\gamma_*}{2}\right).$$

Hence we obtain $s\le(2\beta+\gamma_*)\mathrm{e}^{-a(\beta+\gamma_*/2)}$ which concludes the proof. □

Next we estimate the first eigenvalue of $T^N_{a,\beta}$.

**Lemma 2.5.** *Assume that* $\left(\beta+\frac{\gamma^*}{2}\right)>\max\left\{\frac{\gamma_+}{2(1-a\gamma_+)},\ \frac{2\log 5}{3a}\right\}$. *Then* $T^N_{a,\beta}$ *has a unique negative eigenvalue* $\zeta^N_{a,\beta}$, *and moreover, we have*

$$-\left(\beta+\frac{\gamma^*}{2}\right)^2-\frac{45}{4}\left(\beta+\frac{\gamma^*}{2}\right)^2\mathrm{e}^{-a\left(\beta+\frac{\gamma^*}{2}\right)}\le\zeta^N_{a,\beta}\le-\left(\beta+\frac{\gamma^*}{2}\right)^2.$$

*Proof.* The operator domain in this case looks as follows,

$$D(T^N_{a,\beta})=\left\{\varphi\in H^2(0,a):\ \varphi'(0)=-\left(\frac{\gamma^*}{2}+\beta\right)\varphi(0),\ \varphi'(a)=\frac{\gamma_+}{2(1-a\gamma_+)}\varphi(a)\right\}.$$

Assume again that $-k^2$ with $k>0$ is an eigenvalue of $T^N_{a,\beta}$ corresponding to a nonzero eigenfunction $\varphi$. As in the proof of Lemma 2.4 we infer that $-k^2$ is an eigenvalue of $T^N_{a,\beta}$ if and only if[2]

$$\mathrm{e}^{2ka}=\frac{k+\frac{\gamma^*}{2}+\beta}{k-\frac{\gamma^*}{2}-\beta}\ \cdot\ \frac{k+\frac{\gamma_+}{2(1-a\gamma_+)}}{k-\frac{\gamma_+}{2(1-a\gamma_+)}}\ . \tag{12}$$

[2]There is a misplaced exponential in the analogous proof in [3] which does not affect the claim.

Since the left-hand side of the last equation is strictly increasing, and the right-hand side is strictly decreasing for $k > 0$, then the equation (12) has a unique positive solution which lies in fact in the subinterval $\left(\beta + \frac{\gamma^*}{2}, +\infty\right)$.

Next we will show that (12) has no solutions in the interval $k \geq \frac{3}{2}\left(\beta + \frac{\gamma^*}{2}\right)$. Suppose that the opposite is true. As $\frac{\gamma_+}{2(1-a\gamma_+)} < \beta + \frac{\gamma^*}{2}$, we have

$$\frac{k + \frac{\gamma^*}{2} + \beta}{k - \frac{\gamma^*}{2} - \beta} \cdot \frac{k + \frac{\gamma_+}{2(1-a\gamma_+)}}{k - \frac{\gamma_+}{2(1-a\gamma_+)}} \leq \left(\frac{k + \frac{\gamma^*}{2} + \beta}{k - \frac{\gamma^*}{2} - \beta}\right)^2.$$

However, since we assume $k \geq \frac{3}{2}\left(\beta + \frac{\gamma^*}{2}\right)$, this would imply

$$\mathrm{e}^{2ka} \leq \left(\frac{\frac{3}{2}\left(\beta + \frac{\gamma^*}{2}\right) + \frac{\gamma^*}{2} + \beta}{\frac{3}{2}\left(\beta + \frac{\gamma^*}{2}\right) - \frac{\gamma^*}{2} - \beta}\right)^2 = 25.$$

On the other hand, we have $\mathrm{e}^{2ka} \geq \mathrm{e}^{3a\left(\beta + \frac{\gamma^*}{2}\right)} > 25$, so we come to a contradiction. Hence the solution $k$ of (12) is of the form $k = \beta + \frac{\gamma^*}{2} + s$ with $0 < s < \frac{1}{2}\left(\beta + \frac{\gamma^*}{2}\right)$, and using (12) once again we get

$$\mathrm{e}^{2ka} \leq \left(\frac{k + \frac{\gamma^*}{2} + \beta}{k - \frac{\gamma^*}{2} - \beta}\right)^2 \leq \left(\frac{2\beta + \gamma^* + s}{s}\right)^2 \leq \left(\frac{\frac{5}{2}\left(\beta + \frac{\gamma^*}{2}\right)}{s}\right)^2,$$

which further implies

$$s \leq \frac{5}{2}\left(\beta + \frac{\gamma^*}{2}\right)\mathrm{e}^{-ka} = \frac{5}{2}\left(\beta + \frac{\gamma^*}{2}\right)\mathrm{e}^{-a\left(\beta + \frac{\gamma^*}{2}\right) - sa} \leq \frac{5}{2}\left(\beta + \frac{\gamma^*}{2}\right)\mathrm{e}^{-a\left(\beta + \frac{\gamma^*}{2}\right)}.$$

This completes the proof of Lemma 2.5. □

Now we are finally in position to *prove Theorem* 1.1. The argument is pretty much the same as in the proof of Theorem 1 of [3], one has just to replace Propositions 2.4 and 2.5 of that paper by our Lemmata 2.4, 2.5, respectively. □

As we have mentioned in the opening, the estimates of Theorem 1.1 squeeze to produce an exact asymptotic expansion if and only if the curvature is constant, $\gamma^* = \gamma_*$. Let us now look at this case in more detail:

**Example 2.6.** Let $\Omega$ be a disc of radius $R$ centered at the origin. In this case we have

$$\gamma(s) \equiv \gamma = \frac{1}{R} \tag{13}$$

and the eigenvalues $\mu_j$ of the comparison operator $S$ given by (3) can be computed explicitly,

$$\mu_j = \left( -\frac{1}{4} + \left[\frac{j}{2}\right]^2 \right) R^{-2}, \tag{14}$$

where $[y]$ denotes the maximum integer which less or equal to $y$. We introduce the usual polar coordinates,

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} \qquad 0 \le r \le R\,,\ 0 \le \theta < 2\pi\,,$$

writing with an abuse of notation $f(x,y) \equiv f(r,\theta)$. Equations (1) with $\lambda = -k^2$ now read

$$\begin{cases} \dfrac{\partial^2 f}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial f}{\partial r} + \dfrac{1}{r^2}\,\dfrac{\partial^2 f}{\partial \theta^2} = k^2 f, \\[2ex] \dfrac{\partial f}{\partial r}\Big|_{r=R} = \beta f. \end{cases} \tag{15}$$

Solution to the first equation in (15) is conventionally sought in the form

$$f(r,\theta) = \sum_{m\in\mathbb{Z}} c_m I_m(kr)\mathrm{e}^{\mathrm{i}m\theta}. \tag{16}$$

Furthermore, the Hamiltonian commutes with the angular momentum operator, $-i\frac{\partial}{\partial\theta}$ with periodic boundary conditions, hence the two operator have common eigenspaces, and we can consider sequence $\{c_m\}$ with nonzero $c_m$ corresponding to a single values of $|m|$; it goes without saying that the discrete spectrum is twice degenerate except the eigenvalue corresponding to $m = 0$ which is simple. The boundary condition in (15) can be then rewritten as

$$kI_m'(kR) - \beta I_m(kR) = 0. \tag{17}$$

for a fixed $m \in \mathbb{Z}$. To find its solutions, let us change the variables to $X = kR$, $\alpha = \beta R$, in which case the condition (17) reads

$$\frac{XI_m'(X)}{I_m(X)} = \alpha. \tag{18}$$

The function at the left-hand side of (18) is strictly increasing for $k > 0$, hence (18) has a unique solution for any fixed $\alpha$ and $m$. As $\alpha \to +\infty$, so does $X$ in (18), and using the well-known asymptotics of modified Bessel functions, we find

$$\frac{XI_m'(X)}{I_m(X)} = X - \frac{1}{2} + \frac{4m^2-1}{8X} + O(X^{-2}), \quad X \to +\infty.$$

In combination with the spectral condition (18) this yields

$$X = \alpha + \frac{1}{2} - \frac{4m^2-1}{8\alpha} + O(\alpha^{-2}), \quad \alpha \to +\infty.$$

This, in turn, implies the asymptotics for $X^2$, and returning to the original variables $\beta, k$ we find

$$-k^2 = -\left(\beta + \frac{1}{2R}\right)^2 + \left(m^2 - \frac{1}{4}\right) R^{-2} + \mathcal{O}(\beta^{-1}), \quad \beta \to +\infty.$$

This agrees, of course, with the conclusion of Theorem 1.1 according to (13) and (14). At the time it shows that there is not much room for improving the error term in the theorem, because it differs from the one in this explicitly solvable example by the logarithmic factor only.

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