THERMOACOUSTIC TOMOGRAPHY WITH AN ARBITRARY ELLIPTIC OPERATOR

MICHAEL V. KLIBANOV*

Abstract. Thermoacoustic tomography is a term for the inverse problem of determining of one of initial conditions of a hyperbolic equation from boundary measurements. In the past publications both stability estimates and convergent numerical methods for this problem were obtained only under some restrictive conditions imposed on the principal part of the elliptic operator. In this paper logarithmic stability estimates are obatined for an arbitrary variable principal part of that operator. Convergence of the Quasi-Reversibility Method to the exact solution is also established for this case. Both complete and incomplete data collection cases are considered.

This preprint is posted at http://www.ma.utexas.edu/mp_arc/ on August 24, 2012.

1. Introduction. The goal of this paper is to show that logarithmic stability estimates as well as convergent numerical methods for the inverse problem of determining an initial condition in a general hyperbolic PDE of the second order can be obtained without any restrictions on its coefficients, except of some natural ones. In all previous publications on this topic the principal part of the elliptic operator was subjected to some restrictive conditions. Naturally, our stability estimates imply uniqueness. Both complete and incomplete data collection cases are considered. We assume here that the data are given on the infinite time interval $t \in (0, \infty)$. Second and third Remarks 2.1 (section 2) justify this assumption. For brevity, we leave for possible future publications the finest assumptions, like, e.g. the minimal smoothness, etc.

In thermoacoustic tomography (TAT) a short radio frequency pulse is sent in a biological tissue [1, 9]. Some energy is absorbed. It is well known that malignant legions absorb more energy than healthy ones. Then the tissue expands and radiates a pressure wave which is the solution of the following Cauchy problem

$$u_{tt} = c^2(x)\,\Delta u, x \in \mathbb{R}^3, t > 0, \tag{1.1}$$

$$u(x,0) = f(x), u_t(x,0) = 0. \tag{1.2}$$

The function $u(x,t)$ is measured by transducers at certain locations either at the boundary of the medium of interest or outside of this medium. The function $f(x)$ characterizes the absorption of the medium. Hence, if one would know the function $f(x)$, then one would know locations of malignant spots. The inverse problem consists in determining $f(x)$ using those measurements.

Both stability estimates and convergent numerical methods for the problem of determining the initial condition f in (1.2) are currently known only under some restrictive conditions imposed on the coefficient $c(x)$ (subsection 1.2). In addition, except of the case $c(x) \equiv 1$ in [20], those numerical methods are known only for the case of complete data collection, i.e. when boundary measurements are given at the entire boundary of the domain of interest.

First, we apply a well known analog of the Laplace transform to obtain a similar inverse problem for a parabolic PDE. Next, previous results of the author [15, 16] are used. In the complete data case the logarithmic stability estimate follows from [15]. In the case when the data are given on a hyperplane, we significantly modify the proof of Theorem 1 of [16]. More precisely, we prove our logarithmic stability estimate for an integral inequality rather than for the parabolic PDE. We need this generalization to establish convergence rate of our numerical method. Results of both publications [15, 16] were obtained via Carleman estimates. In particular, a quite technical non-standard Carleman estimate was derived in [16], see Lemma 2.1 in section 2. We refer to [30] for another logarithmic stability estimate of the initial condition of a parabolic equation with the self-adjoint operator L in a finite domain. A Carleman estimate was also used in this reference. An interesting feature of [30] is that observations are performed on an internal subdomain for times $t \in (\tau, T)$ where $\tau > 0$. In addition, a numerical method was developed in [30].

*Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA, email: mklibanv@uncc.edu

1.1. Statements of inverse problems. Let $\Omega \subset \{x_1 > 0\}$ be a bounded domain with the boundary $\partial\Omega \in C^3$. Let $T > 0$. Denote

$$Q_T = \Omega \times (0,T), S_T = \partial\Omega \times (0,T), P = \{x_1 = 0\}, P_T = P \times (0,T).$$

Let $k \geq 0$ be an integer and $\alpha \in (0,1)$. Below $C^{k+\alpha}, C^{2k+\alpha,k+\alpha/2}$ are Hölder spaces. Consider the elliptic operator L of the second order with its principal part L_0,

$$Lu = \sum_{i,j=1}^{n} a_{i,j}(x) u_{x_i x_j} + \sum_{j=1}^{n} b_j(x) u_{x_j} + b_0(x) u, x \in \mathbb{R}^n, \tag{1.3}$$

$$L_0 u = \sum_{i,j=1}^{n} a_{i,j}(x) u_{x_i x_j}, \tag{1.4}$$

$$a_{i,j} \in C^{k+\alpha}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n), b_j, b_0 \in C^{k+\alpha}(\mathbb{R}^n), k \geq 2, \alpha \in (0,1), \tag{1.5}$$

$$\mu_1 |\eta|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x) \eta_i \eta_j \leq \mu_2 |\eta|^2, \forall x, \eta \in \mathbb{R}^n; \mu_1, \mu_2 = const. > 0. \tag{1.6}$$

Let the function $f(x)$ be such that

$$f \in C^p(\mathbb{R}^n), p \geq 3, f(x) = 0, x \in \mathbb{R}^n \diagdown \Omega. \tag{1.7}$$

Consider the following Cauchy problem

$$u_{tt} = Lu, x \in \mathbb{R}^n, t \in (0,\infty), \tag{1.8}$$

$$u(x,0) = f(x), u_t(x,0) = 0. \tag{1.9}$$

We use everywhere below the following assumption.

Assumption. We assume that integers $k \geq 2, p \geq 3$ in (1.5), (1.7), coefficients of the operator L and the initial condition f are such that there exists unique solution $u \in C^3(\mathbb{R}^n \times [0,T]), \forall T > 0$ of the problem (1.8), (1.9) satisfying

$$\|u\|_{C^3(\mathbb{R}^n \times [0,T])} \leq B e^{dT}, \forall T > 0, \tag{1.10}$$

where the constants $B = B(L) > 0, d = d(L) > 0$ depend only from the coefficients of the operator L and an upper estimate $\overline{B}$ of the norm $\|f\|_{C^p(\overline{\Omega})} \leq \overline{B}$.

Note that (1.7) in combination with the finite speed of propagation of the solution of problem (1.8), (1.9) guarantee that the function $u(x,t)$ has a finite support $\Psi(T) \subset \mathbb{R}^n, \forall t \in (0,T), \forall T > 0$ [24]. Hence, $C^3(\mathbb{R}^n \times [0,T])$ above is actually the space $C^3\left(\overline{\Psi(T)} \times [0,T]\right)$. Using the classical tool of energy estimates [24], one can easily find non-restrictive sufficient conditions imposed on both coefficients of the operator L and the function f guaranteeing the smoothness $u \in C^3(\mathbb{R}^n \times [0,T]), \forall T > 0$ as well as (1.10). We are not doing this here for brevity. We consider the following two Inverse Problems.

Inverse Problem 1 (IP1). *Suppose that conditions (1.3)-(1.7) and Assumption hold. Let* $u \in C^3(\mathbb{R}^n \times [0,T]), \forall T > 0$ *be the solution of the problem (1.8), (1.9). Assume that the function* $f(x)$ *is unknown. Determine this function, assuming that the following function* $\varphi_1(x,t)$ *is known*

$$u\mid_{S_\infty} = \varphi_1(x,t). \tag{1.11}$$

Inverse Problem 2 (IP2). *Suppose that conditions (1.3)-(1.7) are Assumption hold. Let* $u \in C^3(\mathbb{R}^n \times [0,T]), \forall T > 0$ *be the solution of the problem (1.8), (1.9). Assume that the function* $f(x)$ *is unknown. Determine this function, assuming that the following function* $\varphi_2(x,t)$ *is known*

$$u\mid_{x \in P_\infty} = \varphi_2(x,t). \tag{1.12}$$

IP1 has complete data collection, since the function φ_1 is known at the entire boundary of the domain of interest Ω. On the other hand, IP2 represents a special case of incomplete data collection, since $\Omega \subset \{x_1 > 0\}$.

1.2. Brief overview of published results. TAT has attracted a significant interest in the past several years. We now provide a brief overview of published mathematical results for TAT. We refer to [19] for a review paper. Stability estimates and convergent numerical methods for an arbitrary time independent principal part L_0 in (1.4) were not obtained in the past. Explicit formulas for the reconstruction of the function $f(x)$ for IP1 in the case when in (1.1) $c \equiv 1$ are given in a number of publications, see, e.g. [7, 8, 9, 19, 20]. These formulas lead to some stability estimates as well as to numerical methods with good performances.

Another approach to IP1, IP2 is via analyzing the case when both Dirichlet and Neumann data are given at S_T for IP1 and at P_T for IP2. An elementary, well known and stable procedure of deriving the Neumann condition from the given Dirichlet condition for both IP1 and IP2 is described in subsection 2.1 for the parabolic PDE. A very similar procedure takes place in the hyperbolic case. Consider now IP1. Since a certain norm of the Neumann boundary condition at S_T can be estimated from the above by another norm of the data $\varphi_1(x,t)$ for $(x,t) \in S_T$, then the problem of estimating the initial condition $f(x)$ can be reformulated in a slightly different setting as the Cauchy problem for equation (1.8) with the lateral Dirichlet and Neumann data at S_T. This problem consists in estimating the function $u(x,t)$ inside of the time cylinder Q_T.

We now comment on the Lipschitz stability estimate for that Cauchy problem with lateral data for the particular case when initial conditions are as in (1.9). Consider the even extension of the function $u(x,t)$ with respect to t and do not change notations for brevity, $u(x,-t) := u(x,t), t \in (0,T)$. Let $Q_T^{\pm} = \Omega \times (-T,T), S_T^{\pm} = \partial\Omega \times (-T,T)$. Obviously $\left\| u \mid_{S_T^{\pm}} \right\|_{H^1\left(S_T^{\pm}\right)} = 2 \left\| u \mid_{S_T} \right\|_{H^1(S_T)}$ and $\left\| \partial_\nu u \mid_{S_T^{\pm}} \right\|_{L_2\left(S_T^{\pm}\right)} = 2 \left\| \partial_\nu u \mid_{S_T} \right\|_{L_2(S_T)}$, where ∂_ν means the normal derivative. The Lipschitz stability estimate for the Cauchy problem with the lateral data is

$$\|u\|_{H^1\left(Q_T^{\pm}\right)} \leq C \left[\left\| u \mid_{S_T} \right\|_{H^1(S_T)} + \left\| \partial_\nu u \mid_{S_T} \right\|_{L_2(S_T)} \right] \tag{1.13}$$

with a certain constant $C > 0$ independent on the function u. Hence, the trace theorem implies the Lipschitz stability estimate for the function f with a different constant C,

$$\|u(x,0)\|_{L_2(\Omega)} = \|f\|_{L_2(\Omega)} \leq C \left[\left\| u \mid_{S_T} \right\|_{H^1(S_T)} + \left\| \partial_\nu u \mid_{S_T} \right\|_{L_2(S_T)} \right].$$

Estimate (1.13) is important in the control theory, since it is used for proofs of exact controllability theorems. For the first time, estimate (1.13) was proved in 1986 in [31] for equation (1.1) with $c \equiv 1$ with the aim of applying to the control theory. However, the method of multipliers, which was proposed in [31], cannot handle neither variable lower order terms of the operator L nor a variable coefficient $c(x)$. On the other hand, Carleman estimates are not sensitive to lower order terms of PDE operators and also can handle the case of a variable coefficient $c(x)$.

For the first time, the idea of using Carleman estimates for obtaining (1.13) was realized in [12]. In this reference (1.13) was proved for the case of the hyperbolic equation (1.8) with $L = \Delta +$ (variable lower order terms). Next, the result of [12] was extended in [11, 14] to a more general case of the hyperbolic inequality

$$|u_{tt} - \Delta u| \leq A \left[|\nabla u| + |u_t| + |u| + |f| \right] \text{ in } Q_T, \tag{1.14}$$

where $A = const. > 0$ and $f \in L_2(Q_T)$. Although in publications [11, 12, 14] $c \equiv 1$, it is clear from them that the key idea is in applying the Carleman estimate, while a specific form of the principal part of the hyperbolic operator should be such that the Carleman estimate would be valid. This thought is reflected in the proof of Theorem 3.4.8 of the book [10]. Thus, the Lipschitz stability estimate (1.13) for the variable coefficient $c(x)$ was obtained in section 2.4 of the book [17] as well as in [5]. In particular, in [17] the hyperbolic inequality (1.14) was considered, in which $|u_{tt} - \Delta u|$ was replaced with $\left| c^{-2}(x) u_{tt} - \Delta u \right|$. The idea of [11] was used in the control theory in, e.g. [26, 27].

In the case of parabolic and elliptic operators, Carleman estimates are known for rather arbitrary variable principal parts [10, 17, 29]. On the other hand, it is well known that in the hyperbolic case the Carleman estimate can be effectively analytically verified for a generic operator $\partial_t^2 - L_0$ only if $L_0 = c^2(x)\Delta$, and a condition like

$$\left(x - x_0, \nabla\left(c^{-2}(x)\right)\right) \geq 0, \forall x \in \overline{\Omega} \tag{1.15}$$

holds. In (1.15) x_0 is a certain point and $(,)$ is the scalar product in $\mathbb{R}^n$. This is the reason why the above mentioned Lipschitz stability estimates were established only using assumptions like the one in (1.15). Clearly, (1.15) holds for $c \equiv const. \neq 0$. See, e.g. Theorem 1.10.2 in [4] for the proof of the Carleman estimate with condition (1.15). A more general case of condition (1.15) can be found in Theorem 3.4.1 of [10]. The second way of proving Lipschitz stability estimates is via imposing some conditions of the Riemannian geometry on coefficients of the operator L_0 [3, 25, 33, 34, 35]. Publications [25, 33, 34] use Carleman estimates. In particular, the case of a hyperbolic inequality was considered in [33]. Unlike (1.15), conditions of the Riemannian geometry cannot be effectively analytically verified for an operator L_0 with generic coefficients, e.g. $L_0 = c^2(x)\Delta$. A slight variation of (1.15) guarantees the non-trapping condition, see formula (3.24) in [32]. Uniqueness theorems for TAT were also obtained in [1, 9, 35] for the case (1.1), (1.2).

In addition, to the Lipschitz stability, the Quasi-Reversibility Method (QRM) for the above mentioned Cauchy problem with the lateral data was developed in [12] and numerically tested in [5, 13, 18]. We refer to [28] for the originating work on QRM. The convergence of the QRM solution to the exact solution was proven on the basis of the above Lipschitz stability results. Numerical testing has consistently demonstrated a high degree of robustness. In particular, accurate results were obtained in [18] with up to 50% noise in the data. Some other numerical methods were proposed in [1, 35]. Convergence of all numerical methods mentioned in this paragraph was proven only for the complete data collection case of IP1 with $L_0 = c^2(x)\Delta$ and under some restrictive conditions imposed on the function $c(x)$.

2. Logarithmic Stability.

2.1. Transformation. First, we consider the following well known Laplace-like transformation [17, 29], which transforms the hyperbolic Cauchy problem in a similar parabolic Cauchy problem,

$$\mathcal{L}g = \overline{g}(t) = \frac{1}{\sqrt{\pi t}} \int_0^\infty \exp\left(-\frac{\tau^2}{4t}\right) g(\tau)\, d\tau. \tag{2.1}$$

The transformation (2.1) is an analog of the Laplace transform, and it is one-to-one. It is valid for, e.g. all functions $g \in C[0,\infty)$ which satisfy $|g(t)| \leq A_g e^{k_g t}$, where A_g and k_g are positive constants depending on g. It follows from (1.10) that the solution $u(x,t)$ of the problem (1.8), (1.9) satisfies this condition together with its derivatives up to the third order. Obviously

$$\frac{\partial}{\partial t}\left[\frac{1}{\sqrt{\pi t}} \exp\left(-\frac{\tau^2}{4t}\right)\right] = \frac{\partial^2}{\partial \tau^2}\left[\frac{1}{\sqrt{\pi t}} \exp\left(-\frac{\tau^2}{4t}\right)\right].$$

Hence,

$$\mathcal{L}(g'') = \overline{g}'(t), \forall g \in C^2[0,\infty) \text{ such that } g'(0) = 0. \tag{2.2}$$

Changing variables in (2.1) $\tau \Leftrightarrow z, \tau/2\sqrt{t} := z$, we obtain $\lim_{t\to 0^+} \overline{g}(t) = g(0)$. Denote $v := \mathcal{L}u$. It follows from (1.10) and (2.2) that

$$v \in C^{2+\alpha,1+\alpha/2}(\mathbb{R}^n \times [0,T]), \forall \alpha \in (0,1), \forall T > 0. \tag{2.3}$$

By (1.8), (1.9) and (2.3) the function $v(x,t)$ is the solution of the following parabolic Cauchy problem

$$v_t = Lv, x \in \mathbb{R}^n, t > 0, \tag{2.4}$$

$$v(x,0) = f(x). \tag{2.5}$$

We refer here to the well known uniqueness result for the solution $v \in C^{2+\alpha,1+\alpha/2}(\mathbb{R}^n \times [0,T]), \forall T > 0$ of the problem (2.4), (2.5) [23].

Below we work only with the function v. As to this function, we set everywhere below $T := 1$ for the sake of definiteness. Denote

$$\mathcal{L}\varphi_1 = \overline{\varphi}_1(x,t) = v\mid_{S_1}, \ \mathcal{L}\varphi_2 = \overline{\varphi}_2(x,t) = v\mid_{P_1}. \tag{2.6}$$

Then

$$\overline{\varphi}_1 \in C^{2+\alpha,1+\alpha/2}\left(\overline{S}_1\right), \overline{\varphi}_2 \in C^{2+\alpha,1+\alpha/2}\left(\overline{P}_1\right). \tag{2.7}$$

Let

$$\overline{\psi}_1(x,t) = \partial_\nu v\mid_{S_1}, \overline{\psi}_2(x,t) = \partial_{x_1} v\mid_{P_1}. \tag{2.8}$$

By Theorem 5.2 of Chapter IV of [23], (1.10) and (2.6)-(2.8) there exist numbers $C(\Omega,L), C(P,L) > 0$ depending only on listed parameters such that

$$\left\|\overline{\psi}_1\right\|_{C^{1+\alpha,\alpha/2}\left(\overline{S}_1\right)} \leq C(\Omega,L)\left\|\overline{\varphi}_1\right\|_{C^{2+\alpha,1+\alpha/2}\left(\overline{S}_1\right)}, \tag{2.9}$$

$$\left\|\overline{\psi}_2\right\|_{C^{1+\alpha,\alpha/2}\left(\overline{P}_1\right)} \leq C(P,L)\left\|\overline{\varphi}_2\right\|_{C^{2+\alpha,1+\alpha/2}\left(\overline{P}_1\right)}. \tag{2.10}$$

We now describe an elementary and well known procedure of finding the normal derivative of the function v either at S_1 (in the case of IP1) or at P_1 (in the case of IP2). In the case of IP1 we solve the initial boundary value problem for equation (2.4) for $(x,t) \in (\mathbb{R}^n \diagdown \Omega) \times (0,1)$ with the zero initial condition in $\mathbb{R}^n \diagdown \Omega$ (because of (1.7)) and the Dirichlet boundary condition $v\mid_{S_1} = \overline{\varphi}_1$. Then we uniquely find the normal derivative $\partial_\nu v\mid_{S_1} = \overline{\psi}_1$. Similarly, in the case of IP2, we uniquely find the Neumann boundary condition $\partial_{x_1} v\mid_{P_1} = \overline{\psi}_2$. Estimates (2.9), (2.10) ensure the stability of this procedure.

Therefore, each problem IP1, IP2 is replaced with a problem for the parabolic PDE (2.4) with the lateral Cauchy data. These data are given at S_1 for IP1 and at P_1 for IP2. Uniqueness of the solution of each of these parabolic inverse problems follows from standard theorems about uniqueness of the continuation of solutions of parabolic PDEs with the data at the lateral surface [10, 17, 29].

In stability estimates one is usually interested to see how the solution varies for a small variation of the input data. Therefore, following (1.10), (1.11) and (1.12), we assume that in the case of IP1

$$\|\varphi_1\|_{C^3\left(\overline{S}_T\right)} \leq \delta e^{dT}, \forall T > 0, \tag{2.11}$$

and in the case of IP2

$$\|\varphi_2\|_{C^3\left(\overline{P}_T\right)} \leq \delta e^{dT}, \forall T > 0, \tag{2.12}$$

where $\delta \in (0,1)$ is a sufficiently small number. Note that it is not necessary that $\delta = B$, where B is the number from (1.10). Indeed, while the number B in (1.10) is not assumed to be sufficiently small and is involved in the estimate of the norm $\|u\|_{C^3(\mathbb{R}^n \times [0,T])}, \forall T > 0$ in the entire space, the number δ is a part of the estimate of the norm of the boundary data for either of above inverse problems. Using (2.1), (2.2) and (2.6)-(2.12), we obtain

$$\left\|\overline{\varphi}_1\right\|_{C^{2+\alpha,1+\alpha/2}\left(\overline{S}_1\right)} + \left\|\overline{\psi}_1\right\|_{C^{1+\alpha,\alpha/2}\left(\overline{S}_1\right)} \leq C(\Omega,L,d)\,\delta, \tag{2.13}$$

$$\left\|\overline{\varphi}_2\right\|_{C^{2+\alpha,1+\alpha/2}\left(\overline{P}_1\right)} + \left\|\overline{\psi}_2\right\|_{C^{1+\alpha,\alpha/2}\left(\overline{P}_1\right)} \leq C(P,L,d)\,\delta, \tag{2.14}$$

where constants $C(\Omega,L,d), C(P,L,d) > 0$ depend only on listed parameters. It follows from (2.13) that with a different constant $\overline{C} := \overline{C}(\Omega,L,d) > 0$

$$\left\|\overline{\varphi}_1\right\|_{H^1(S_1)} + \left\|\overline{\psi}_1\right\|_{L_2(S_1)} \leq \overline{C}\delta. \tag{2.15}$$

Remarks 2.1.

1. The number δ can be viewed as an upper estimate of the level of error in the data φ_1, φ_2. Hence, Theorems 2.1, 2.2 below address the question of estimating variations of the solution f of either IP1 or IP2 via the upper estimate of the level of error in the data.

2. Since the kernel of the transform $\mathcal{L}$ decays rapidly with $\tau \to \infty$, then the condition $t \in (0,\infty)$ in (1.11), (1.12) is not a serious restriction from the applied standpoint. In addition, if having the data in (1.11), (1.12) only on a finite time interval $t \in (0,T)$ and knowing an upper estimate of a norm of the function f in (1.9), one can estimate the error in the integral (2.1) when integrating over $\tau \in (T,\infty)$. This error will be small if either T is large or t is small in (2.1), (2.4). Next, this error can be incorporated in the stability estimates of theorems of this section.

3. Another argument about $t \in (0,\infty)$ comes from the recent experience of the author of working with time resolved real data for wave processes [4]. The author has learned that almost all time resolved experimental data for wave processes in non-attenuating media are highly oscillatory due to some unknown processes in measurement devices, see graphs of those data in these references. Because of high oscillations, these data are not governed by a hyperbolic PDE even for the case of the free space, where the wave equation is supposed to work (see the graphs of experimental data in chapters 5 and 6 of [4]). Therefore, the first step to make the inverse algorithm work was to preprocess the experimental data via a new data preprocessing procedure. This procedure uses only a small portion of the real data and immerses it in a specially processed data for the uniform medium. Since the case of the uniform medium can be solved analytically, then there is no problem to know the immersed data for all $t \in (0,\infty)$. Since accurate imaging results were obtained in [4] for the case of blind experimental data, then that data preprocessing procedure was unbiased.

2.2. Logarithmic stability estimate for Inverse Problem 1. To prove convergence of the QRM (Theorem 3.1), it is convenient to consider a parabolic inequality in the integral form, which is more general than equation (2.4). Consider the function $w \in C^{2,1}\left(\overline{Q}_1\right)$ satisfying the following inequality

$$\int\limits_{Q_1} \left(w_t - Lw\right)^2 dxdt \leq K^2, K = const. \geq 0. \tag{2.16}$$

Theorem 2.1. *Let conditions (1.3)-(1.6) be fulfilled. Let the function $w \in C^{2,1}\left(\overline{Q}_1\right)$ satisfies inequality (2.16). Denote*

$$\begin{aligned} g(x) = w(x,0), \beta_0(x,t) = w\mid_{S_1}, \beta_1(x,t) = \partial_\nu w\mid_{S_1}, \\ F = \|\beta_0\|_{H^1(S_1)} + \|\beta_1\|_{L_2(S_1)} + K. \end{aligned} \tag{2.17}$$

Assume that an upper bound $C_1 = const. > 0$ for the norm $\|\nabla g\|_{L_2(\Omega)}$ is known,

$$\left(\sum_{i=1}^n \|g_{x_i}\|_{L_2(\Omega)}^2\right)^{1/2} := \|\nabla g\|_{L_2(\Omega)} \leq C_1. \tag{2.18}$$

Then there exist a constant $M = M(L,\Omega) > 0$ and a sufficiently small number $\delta_0 = \delta_0(L,\Omega,C_1) \in (0,1)$, both dependent only on listed parameters, such that if $F \in (0,\delta_0)$, then the following logarithmic stability estimate is valid

$$\|g\|_{L_2(\Omega)} \leq \frac{MC_1}{\sqrt{\ln(F^{-1})}}. \tag{2.19}$$

In particular, in the case of IP1, let Assumption holds and (1.7), (2.11) be valid. Suppose that the number δ in (2.11) is so small that $\overline{C}\delta \in (0,\delta_0)$, where $\overline{C} = \overline{C}(\Omega,L,d) > 0$ is the number in (2.15). Also, assume that the upper bound C_1 of the norm $\|\nabla f\|_{L_2(\Omega)}$ is given,

$$\|\nabla f\|_{L_2(\Omega)} \leq C_1. \tag{2.20}$$

Then

$$\|f\|_{L_2(\Omega)} \leq \frac{MC_1}{\sqrt{\ln\left[\left(\overline{C}\delta\right)^{-1}\right]}}. \tag{2.21}$$

Proof. In this proof $M = M(L, \Omega) > 0$ denotes a generic positive constant depending only on L, Ω. First, we prove (2.19). Let $\varkappa \in (0,1)$ be an arbitrary number. Then it follows from Theorem 2 of [15] that there exists a constant $r = r(L, \Omega, \varkappa) \in (0,1)$ such that

$$\|g\|_{L_2(\Omega)} \leq \frac{MC_1}{\varkappa\sqrt{\ln\left[(rF)^{-1}\right]}} + M\left(\frac{1}{r}\right)^{\varkappa} F^{1-\varkappa}, \tag{2.22}$$

as long as $F \in (0,1)$. We can fix $\varkappa$ via, e.g. setting $\varkappa := 1/2$. It is clear therefore that there exists a sufficiently small number $\delta_0 = \delta_0(L, \Omega, C_1) \in (0,1)$ such that if $F \in (0, \delta_0)$, then (2.22) implies (2.19).

We now prove (2.21). It follows from (2.4) that (2.16) holds for the function $w := v$ with $K = 0$. As it was shown above, (2.15) follows from (2.11). Hence, using (2.15) and (2.17), we obtain $F \leq \overline{C}\delta$. Hence, (2.21) follows from (2.19). □

Remark 2.2. Estimates (2.19), (2.21) are the so-called "conditional stability estimates", which is often the case in ill-posed problems [4, 29, 36]. For another example we refer to Hölder stability estimates for solutions of some ill-posed problems for PDEs, see, e.g. [10, 17, 29]. The knowledge of the upper bound C_1 for the gradient in (2.18), (2.20) corresponds well with the Tikhonov concept of compact sets as sets of "admissible" solutions of ill-posed problems [2, 4, 6, 29, 36]. Indeed, since by (1.7) $f\mid_{\partial\Omega}= 0$, then $\|f\|_{L_2(\Omega)} \leq \overline{R}\,\|\nabla f\|_{L_2(\Omega)} \leq \overline{R}C_1$, where the constant $\overline{R} > 0$ depends only on the domain Ω. Thus, in this case the function f belongs to a compact set in $L_2(\Omega)$, and this set is determined by the constant C_1.

2.3. Logarithmic stability estimate for Inverse Problem 2. The logarithmic stability estimate of the paper [16] in the infinite domain was obtained for the case of the pointwise inequality

$$|v_t - L_0 v| \leq A(|\nabla v| + |v|), A = const. > 0, \tag{2.23}$$

where the operator L_0 is defined in (1.4). However, to prove convergence of the numerical method of section 3, we need to estimate the initial condition for the case of the integral inequality, like the one in (2.16). The Carleman estimate of [16] is not a standard one. Indeed, unlike the standard Carleman estimate for the parabolic operator [4, 17, 29], the integration domain of [16] is a part of the strip $\{|t-\varepsilon| < \tau\varepsilon, \tau \in (0,1)\}$, and that Carleman estimate does not break when $\varepsilon \to 0^+$.

There are two main differences between Theorem 2.2 (below) and Theorem 1 of [16]. First, we work now with the integral inequality instead of the pointwise inequality (2.23) of [16]. Second, it is assumed in [16] that the inequality (2.23) is valid in $\Theta \times (0,T)$, where $\Theta \subseteq \mathbb{R}^n$ is an unbounded domain. It is also assumed that the Dirichlet boundary condition $v\mid_{\partial\Theta\times(0,T)}= 0$. Unlike this, Theorem 2.2 does not use the assumption about the knowledge of this Dirichlet boundary condition.

Denote $\overline{x} = (x_2, ..., x_n)$. Below we specify numbers $1/4, 1/2, 3/4$ for brevity only. In fact, some other numbers, respectively $\eta_1 < \eta_2 < \eta_3 < 1$ from the interval $(0,1)$ can be used. Changing variables $(x', t') = (\sqrt{c}x, dt)$ with an appropriate constant $c > 0$ and keeping the same notations for new variables for brevity, we obtain that

$$\Omega \subset \left\{x_1 + |\overline{x}|^2 < \frac{1}{4}, x_1 > 0\right\}. \tag{2.24}$$

Let $\varepsilon \in (0,1)$ be a sufficiently small number. Consider the following functions $\psi(x,t), \varphi(x,t)$,

$$\psi(x,t) = x_1 + |\overline{x}|^2 + \frac{(t-\varepsilon)^2}{\varepsilon^2} + \frac{1}{4}, \tag{2.25}$$

$$\varphi(x,t) = \exp\left(\frac{\psi^{-\nu}}{\varepsilon}\right), \tag{2.26}$$

where $\nu > 1$ is a large parameter which will be defined later. The function $\varphi(x,t)$ is the Carleman Weight Function (CWF) in the Carleman estimate of Lemma 3.1. The main difference between $\varphi(x,t)$ in (2.26) and the standard CWF for the parabolic operator [4, 17, 29] is that the small parameter ε is involved in both functions $\psi(x,t)$ and $\varphi(x,t)$. Denote

$$G_{3/4} = \left\{ (x,t) : \psi(x,t) < \frac{3}{4}, x_1 > 0 \right\}, \tag{2.27}$$

$$G_{1/2} = \left\{ (x,t) : \psi(x,t) < \frac{1}{2}, x_1 > 0 \right\}. \tag{2.28}$$

Using (2.24)-(2.28), we obtain

$$G_{1/2} \subset G_{3/4}, \varphi^2(x,t) \geq \exp\left[\frac{2^{\nu+1}}{\varepsilon}\right] \text{ in } G_{1/2}, \tag{2.29}$$

$$G_{3/4} \subset \left\{ |t-\varepsilon| < \frac{\varepsilon}{\sqrt{2}} \right\} \subset \{t \in (0,1)\}, \tag{2.30}$$

$$\Omega \subset RG_{1/2} \subset RG_{3/4}, \tag{2.31}$$

$$\partial G_{3/4} = \partial_1 G_{3/4} \cup \partial_2 G_{3/4}, \partial_1 G_{3/4} = \{x_1 = 0\} \cap \overline{G}_{3/4}, \partial_2 G_{3/4} = \left\{ \psi(x,t) = \frac{3}{4}, x_1 > 0 \right\}. \tag{2.32}$$

In (2.31) $RG_{1/2}$ and $RG_{3/4}$ are orthogonal projections of domain $G_{1/2}$ and $G_{3/4}$ respectively on the hyperplane $\{t=0\}$. The same notation RH is kept below for the projection of any other domain $H \subset [\mathbb{R}^n \times (0,1)]$ on the hyperplane $\{t=0\}$. Denote

$$\Phi = \left\{ (x,t) : x_1 \in (0,1), (x_2, x_3, ..., x_n) \in (-1,1)^{n-1}, t \in (0,1) \right\}, \tag{2.33}$$

$$\partial_1 \Phi = \overline{\Phi} \cap P = \left\{ (x,t) : x_1 = 0, \overline{x} \in (-1,1)^{n-1}, t \in (0,1) \right\}. \tag{2.34}$$

By (2.27), (2.30) and (2.32)-(2.34)

$$\partial_1 G_{3/4} \subset \partial_1 \Phi. \tag{2.35}$$

Recall that (2.12) implies (2.14). Hence, assuming that (2.12) holds and using (2.34), we derive, similarly with the above derivation of (2.15) from (2.11), (2.13), that there exists a constant $\widetilde{C} = \widetilde{C}(P, L, \Phi, b) > 0$ such that

$$\|\overline{\varphi}_2\|_{H^1(\partial_1\Phi)} + \left\|\overline{\psi}_2\right\|_{L_2(\partial_1\Phi)} \leq \widetilde{C}\delta. \tag{2.36}$$

Everywhere below $C = C\left(L_0, RG_{3/4}\right) > 0$ and $M_1 = M_1(L, \Phi) > 0$ denote different positive constants depending only on listed parameters. The following lemma is follows immediately from Theorem 2 of [16] and (2.32).

Lemma 2.1. *Let coefficients of the operator L_0 in (1.4) satisfy conditions (1.5), (1.6). Then there exist a sufficiently large constant $\nu_0 = \nu_0\left(L_0, RG_{3/4}\right) > 1$ and a sufficiently small number $\varepsilon_0 = \varepsilon_0\left(L_0, RG_{3/4}\right) \in (0,1)$, both dependent only on L_0 and $PG_{3/4}$, such that the following Carleman estimate holds*

$$\begin{aligned} &\frac{C\nu^3}{\varepsilon^3}\exp\left(\frac{2\cdot 4^\nu}{\varepsilon}\right)\int\limits_{\partial_1 G_{3/4}}\left(u^2 + |\nabla u|^2 + u_t^2\right)d\overline{x}dt \\ &+\frac{C\nu^3}{\varepsilon^3}\left(\frac{4}{3}\right)^{2\nu}\exp\left[\frac{2}{\varepsilon}\left(\frac{4}{3}\right)^\nu\right]\int\limits_{\partial_2 G_{3/4}}\left(u^2 + |\nabla u|^2 + u_t^2\right)d\sigma + \int\limits_{G_{3/4}}\left(u_t - L_0 u\right)^2\varphi^2(x,t)\,dxdt \\ \geq\, & C\int\limits_{G_{3/4}}\left(\frac{\nu}{\varepsilon}|\nabla u|^2 + \frac{\nu^4}{\varepsilon^3}\psi^{-2\nu}u^2\right)\varphi^2(x,t)\,dxdt, \forall \nu \geq \nu_0, \forall \varepsilon \in (0,\varepsilon_0), \forall u \in C^{2,1}\left(\overline{G}_{3/4}\right). \end{aligned}$$

By (2.30) Lemma 2.1 provides the Carleman estimate in the narrow strip $\left\{|t-\varepsilon|<\varepsilon/\sqrt{2}\right\}$. At the same time, it is also important in numerical studies of the QRM to estimate its solution in a not narrow strip. This can be done via the standard Carleman estimate. Therefore, we introduce now notations, which are similar with (2.25)-(2.32), except that a narrow strip with respect to t is not used. Let

$$\theta(x,t)=x_1+|\overline{x}|^2+\left(t-\frac{1}{2}\right)^2+\frac{1}{4}, \tag{2.37}$$

$$\xi(x,t)=\exp\left(\lambda\theta^{-\nu}\right), \tag{2.38}$$

where $\lambda>1$ is a large parameter which is chosen later. Denote

$$D_{3/4}=\left\{(x,t):\theta(x,t)<\frac{3}{4},x_1>0\right\}, \tag{2.39}$$

$$D_{1/2}=\left\{(x,t):\theta(x,t)<\frac{1}{2},x_1>0\right\}, \tag{2.40}$$

$$\partial D_{3/4}=\partial_1 D_{3/4}\cup\partial_2 D_{3/4},\partial_1 D_{3/4}=\{x_1=0\}\cap\overline{D}_{3/4},\partial_2 D_{3/4}=\left\{\psi(x,t)=\frac{3}{4},x_1>0\right\}. \tag{2.41}$$

Using (2.24), (2.31), (2.34), (2.37) and (2.39)-(2.41), we obtain

$$\Omega\subset RD_{3/4}=RG_{3/4}, \tag{2.42}$$

$$D_{3/4}\subset\left\{\left|t-\frac{1}{2}\right|<\frac{1}{2}\right\}\subset\{t\in(0,1)\}, \tag{2.43}$$

$$D_{3/4}\subset\Phi. \tag{2.44}$$

Lemma 2.2 follows from the Carleman estimate for the parabolic operator of Lemma 3 of §1 of Chapter 4 of the book [29] as well as from (2.42).

Lemma 2.2. *Let coefficients of the operator L_0 in (1.4) satisfy conditions (1.5), (1.6). Then there exist sufficiently large constants $\nu_0=\nu_0\left(L_0,RG_{3/4}\right)>1,\lambda_0=\lambda_0\left(L_0,RG_{3/4}\right)>1$ such that the following Carleman estimate holds*

$$C\lambda^3\nu^3\exp\left(2\lambda\cdot 4^{\nu}\right)\int\limits_{\partial_1 D_{3/4}}\left(u^2+|\nabla u|^2+u_t^2\right)d\overline{x}dt$$

$$+C\lambda^3\nu^3\left(\frac{4}{3}\right)^{2\nu}\exp\left[2\lambda\left(\frac{4}{3}\right)^{\nu}\right]\int\limits_{\partial_2 D_{3/4}}\left(u^2+|\nabla u|^2+u_t^2\right)d\sigma+\int\limits_{D_{3/4}}\left(u_t-L_0u\right)^2\xi^2(x,t)\,dxdt$$

$$\geq C\int\limits_{D_{3/4}}\left(\lambda\nu|\nabla u|^2+\lambda^3\nu^4\psi^{-2\nu}u^2\right)\xi^2(x,t)\,dxdt,\forall\nu\geq\nu_0,\forall\lambda\geq\lambda_0,\forall u\in C^{2,1}\left(\overline{D}_{3/4}\right).$$

Theorem 2.2. *Let conditions (1.3)-(1.6) and (2.24) be valid. Suppose that the function $w\in C^{2,1}\left(\overline{Q}\right)$ satisfies the following integral inequality*

$$\int\limits_{Q}\left(w_t-Lw\right)^2dxdt\leq K^2,K=const.\geq 0. \tag{2.45}$$

Let

$$\beta_0(x,t)=w\mid_{\partial_1\Phi},\beta_1(x,t)=\partial_{x_1}w\mid_{\partial_1\Phi},g(x)=w(x,0),x\in\Omega.$$

Denote

$$F=\|\beta_0\|_{H^1(\partial_1\Phi)}+\|\beta_1\|_{L_2(\partial_1\Phi)}+K. \tag{2.46}$$

Assume that an upper bound $C_2 = const. > 0$ for the norm $\|w\|_{C^1(\overline{Q})}$ is known,

$$\|w\|_{C^1(\overline{\Phi})} \leq C_2. \tag{2.47}$$

Then there exists a sufficiently small number $\delta_0 = \delta_0\left(L, RG_{3/4}\right) \in (0,1)$ dependent only on listed parameters, such that if $F \in (0, \delta_0)$, then the following logarithmic stability estimate is valid

$$\|g\|_{L_2(\Omega)} \leq \frac{M_1 C_2}{\sqrt{\ln\left(F^{-1}\right)}}, \tag{2.48}$$

In particular, in the case of IP2, let Assumption holds and (1.7) be valid. Assume that, in addition to the above, (2.12) is valid, and the number δ in (2.12) is so small that $\widetilde{C}\delta \in (0, \delta_0)$, where $\widetilde{C} = \widetilde{C}(P, L, \Phi, d) > 0$ is the number from (2.36). Also, assume that for a certain $\alpha \in (0,1)$ the upper bound C_3 of the norm $\|f\|_{C^{2+\alpha}(\overline{\Omega})}$ is given, i.e. $\|f\|_{C^{2+\alpha}(\overline{\Omega})} \leq C_3$. Then

$$\|f\|_{L_2(\Omega)} \leq \frac{M_1 C_3}{\sqrt{\ln\left[\left(\widetilde{C}\delta\right)^{-1}\right]}}. \tag{2.49}$$

In addition, there exists a number $\rho = \rho\left(L_0, RG_{3/4}\right) \in (0, 1/2)$ such that if $F \in (0, \delta_0)$, then the following Hölder stability estimate is valid

$$\|w\|_{H^{1,0}\left(D_{1/2}\right)} \leq M_1 C_2 F^{\rho}. \tag{2.50}$$

Proof. In this proof $\varepsilon_0 = \varepsilon_0\left(L, RG_{3/4}\right) \in (0,1)$ denotes different sufficiently small numbers associated with Lemma 3.1. By (2.24), (2.28), (2.31) and (2.33)

$$\Omega \subset RG_{1/2} \subset R\Phi. \tag{2.51}$$

Using (2.25), (2.26) and (2.45), we obtain

$$\int\limits_{G_{3/4}} (w_t - Lw)^2 \varphi^2 dxdt \leq K^2 \exp\left(\frac{2 \cdot 4^{\nu}}{\varepsilon}\right), \forall \varepsilon \in (0, \varepsilon_0). \tag{2.52}$$

On the other hand, using Lemma 3.1, we obtain for all $\nu \geq \nu_0, \varepsilon \in (0, \varepsilon_0)$

$$\begin{aligned} \int\limits_{G_{3/4}} (w_t - Lw)^2 \varphi^2 dxdt &\geq \int\limits_{G_{3/4}} (w_t - L_0 w)^2 \varphi^2 dxdt - M_1 \int\limits_{G_{3/4}} \left(|w|^2 + w^2\right) \varphi^2 dxdt \\ &\geq C \int\limits_{G_{3/4}} \left(\frac{\nu}{\varepsilon} |\nabla \overline{w}|^2 + \frac{\nu^4}{\varepsilon^3} \psi^{-2\nu} \overline{w}^2\right) \varphi^2(x,t)\, dxdt - M_1 \int\limits_{G_{3/4}} \left(|w|^2 + w^2\right) \varphi^2 dxdt \\ &\quad - \frac{C\nu^3}{\varepsilon^3} \exp\left(\frac{2 \cdot 4^{\nu}}{\varepsilon}\right) \left(\|\beta_0\|^2_{H^1(\partial_1 \Phi)} + \|\beta_1\|^2_{L_2(\partial_1 \Phi)}\right) \\ &\quad - \frac{C\nu^3}{\varepsilon^3} \left(\frac{4}{3}\right)^{2\nu} \exp\left[\frac{2}{\varepsilon}\left(\frac{4}{3}\right)^{\nu}\right] \int\limits_{\partial_2 G_{3/4}} \left(w^2 + |\nabla w|^2 + w_t^2\right) d\sigma. \end{aligned} \tag{2.53}$$

Fix a number $\nu \geq \nu_0\left(L_0, RG_{3/4}\right) > 1$ such that

$$\left(\frac{5}{6}\right)^{\nu} < \frac{1}{2}. \tag{2.54}$$

If necessary, decrease $\varepsilon_0=\varepsilon_0\left(L,RG_{3/4}\right)\in(0,1)$, so that $M_1<C\nu/\left(2\varepsilon\right),\forall\varepsilon\in(0,\varepsilon_0)$. Then (2.53) leads to the following estimate for all $\varepsilon\in(0,\varepsilon_0)$

$$\int\limits_{G_{3/4}}\left(w_t-Lw\right)^2\varphi^2dxdt\geq\frac{C}{\varepsilon}\int\limits_{G_{3/4}}\left(\left|\nabla w\right|^2+w^2\right)\varphi^2\left(x,t\right)dxdt$$
$$-\frac{C}{\varepsilon^3}\exp\left(\frac{2\cdot 4^\nu}{\varepsilon}\right)\left(\left\|\beta_0\right\|^2_{H^1(\partial_1\Phi)}+\left\|\beta_1\right\|^2_{L_2(\partial_1\Phi)}\right)$$
$$-\frac{C}{\varepsilon^3}\exp\left[\frac{2}{\varepsilon}\left(\frac{4}{3}\right)^\nu\right]\int\limits_{\partial_2G_{3/4}}\left(w^2+\left|\nabla w\right|^2+w_t^2\right)d\sigma.$$

Combining this with (2.46), (2.47) and (2.52) and decreasing ε_0, if necessary, we obtain

$$\int\limits_{G_{3/4}}\left(\left|\nabla w\right|^2+w^2\right)\varphi^2\left(x,t\right)dxdt\leq C\exp\left(\frac{2\cdot 5^\nu}{\varepsilon}\right)F^2+CC_2^2\exp\left[\frac{2}{\varepsilon}\left(\frac{5}{3}\right)^\nu\right],\forall\varepsilon\in(0,\varepsilon_0). \tag{2.55}$$

On the other hand, by (2.29)

$$\int\limits_{G_{3/4}}\left(\left|\nabla w\right|^2+w^2\right)\varphi^2\left(x,t\right)dxdt\geq\int\limits_{G_{1/2}}\left(\left|\nabla w\right|^2+w^2\right)\varphi^2\left(x,t\right)dxdt\geq\exp\left[\frac{2^{\nu+1}}{\varepsilon}\right]\int\limits_{G_{1/2}}\left(\left|\nabla w\right|^2+w^2\right)dxdt,\forall\varepsilon\in(0,\varepsilon_0).$$

Combining this with (2.55), we obtain

$$\int\limits_{G_{1/2}}\left(\left|\nabla w\right|^2+w^2\right)dxdt\leq C\exp\left(\frac{2\cdot 5^\nu}{\varepsilon}\right)F^2+CC_2^2\exp\left[-\frac{2^{\nu+1}}{\varepsilon}\left(1-\left(\frac{5}{6}\right)^\nu\right)\right],\forall\varepsilon\in(0,\varepsilon_0).$$

Hence, using (2.54), we obtain

$$\int\limits_{G_{1/2}}\left(\left|\nabla w\right|^2+w^2\right)dxdt\leq C\exp\left(\frac{2\cdot 5^\nu}{\varepsilon}\right)F^2+CC_2^2\exp\left(-\frac{2^\nu}{\varepsilon}\right),\forall\varepsilon\in(0,\varepsilon_0). \tag{2.56}$$

By (2.25) and (2.28) $G_{1/2}\subset\{t\in(\varepsilon/2,3\varepsilon/2)\}$. Hence, the mean value theorem, (2.51) and (2.56) imply that there exists a number $t^*\in(\varepsilon/2,3\varepsilon/2)$ such that for all $\varepsilon\in(0,\varepsilon_0)$

$$\left\|w\left(x,t^*\right)\right\|^2_{L_2(\Omega)}\leq\frac{1}{\varepsilon}\left\|w\left(x,t^*\right)\right\|^2_{H^1\left(RG_{1/2}\right)}\leq C\exp\left(\frac{2\cdot 5^\nu}{\varepsilon}\right)F^2+CC_2^2\exp\left(-\frac{2^\nu}{\varepsilon}\right). \tag{2.57}$$

We have

$$w\left(x,t^*\right)=g\left(x\right)+\int\limits_0^t w_t\left(x,\tau\right)d\tau.$$

Hence, using (2.47), we obtain

$$\left\|w\left(x,t^*\right)\right\|^2_{L_2(\Omega)}\geq\left\|g\right\|^2_{L_2(\Omega)}-\varepsilon\left\|w_t\right\|^2_{L_2(\Phi)}\geq\left\|g\right\|^2_{L_2(\Omega)}-M_1C_2^2\varepsilon.$$

Combining this with (2.56), we obtain

$$\left\|g\right\|^2_{L_2(\Omega)}\leq M_1C_2^2\varepsilon+M_1\exp\left(\frac{2\cdot 5^\nu}{\varepsilon}\right)F^2+M_1C_2^2\exp\left(-\frac{2^\nu}{\varepsilon}\right),\forall\varepsilon\in(0,\varepsilon_0). \tag{2.58}$$

Choose $\varepsilon = \varepsilon(F)$ such that

$$\exp\left(\frac{2\cdot 5^{\nu}}{\varepsilon}\right)F^{2} = \exp\left(-\frac{2^{\nu}}{\varepsilon}\right). \tag{2.59}$$

Hence,

$$\varepsilon = \frac{1}{\ln\left(F^{-2/a}\right)}, a = 2\cdot 5^{\nu} + 2^{\nu}. \tag{2.60}$$

To ensure that ε is sufficiently small, i.e. $\varepsilon \in (0, \varepsilon_0)$, we need to choose F so small that

$$0 < F < \exp\left(-\frac{2}{\varepsilon_0}\right).$$

Hence, we choose $\delta_0 = \delta_0\left(L, RG_{3/4}\right) = \exp(-2/\varepsilon_0)$. Hence, (2.58), (2.59) and lead to

$$\|g\|^2_{L_2(\Omega)} \leq \frac{M_1 C_2^2}{\ln\left(F^{-2/a}\right)} + M_1\left(1 + C_2^2\right)\left(F^{2^{\nu+1}}\right)^{1/a} = \frac{M_1 C_2^2 a}{2\ln\left(F^{-1}\right)} + M_1\left(1 + C_2^2\right)\left(F^{2^{\nu+1}}\right)^{1/a}, \tag{2.61}$$

as long as $F \in (0, \delta_0)$. Decreasing, if necessary δ_0, we obtain (2.48) from (2.61).

We now prove (2.49). Let the function $v \in C^{2+\alpha, 1+\alpha/2}\left(\mathbb{R}^n \times [0, T]\right), \forall T > 0$ be the solution of the problem (2.4), (2.5). Recall that by (1.7) $f(x) = 0$ in $\mathbb{R}^n \diagdown \Omega$. It follows from the formula (14.6) of §14 of Chapter 4 of the book [23] as well as from (2.33) that

$$\|v\|_{C^1\left(\overline{\Phi}\right)} \leq \|v\|_{C^{2+\alpha,1+\alpha/2}\left(\overline{\Phi}\right)} \leq M_1 \|f\|_{C^{2+\alpha}\left(\overline{\Omega}\right)} \leq M_1 C_3. \tag{2.62}$$

By (2.4) $K = 0$ in (2.45). Next, since (2.36) follows from (2.12), we obtain for the new number F in (2.46)

$$F := \|\overline{\varphi}_2\|_{H^1(\partial_1\Phi)} + \left\|\overline{\psi}_2\right\|_{L_2(\partial_1\Phi)} \leq \widetilde{C}\delta. \tag{2.63}$$

Thus, (2.47) and (2.48) imply that (2.49) follows from (2.62) and (2.63).

Finally, we prove (2.50). Using (2.37)-(2.44) as well as Lemma 2.2, we obtain similarly with (2.56)

$$\int\limits_{D_{1/2}} \left(|\nabla w|^2 + w^2\right) dxdt \leq C\exp\left(2\lambda\cdot 5^{\nu}\right)F^2 + CC_2^2\exp\left(-2^{\nu+1}\lambda\right), \forall\lambda \geq \lambda_1,$$

where $\lambda_1 = \lambda_1\left(L, RG_{3/4}\right) > 1$ is a sufficiently large number. Hence, we obtain similarly with (2.59) and (2.61)

$$\int\limits_{D_{1/2}} \left(|\nabla w|^2 + w^2\right) dxdt \leq M_1 C_2^2 F^{2\rho}. \square$$

3. The Quasi-Reversibility Method (QRM). We construct the QRM only for the more difficult case of IP2. The case of IP1 is similar, and it can be derived from [15]. Also, we work in this section only in 3-d, keeping the same notations as above. The construction in the 2-d case is similar. Since it was described in subsection 2.1 how to stably obtain the Neumann boundary condition in the parabolic case for both IP1 and IP2, we assume now that we have both Dirichlet and Neumann boundary conditions at $\partial_1\Phi$,

$$v\mid_{\partial_1\Phi} = \overline{\varphi}_2(x,t), \partial_{x_1} v\mid_{\partial_1\Phi} = \overline{\psi}_2(x,t). \tag{3.1}$$

The QRM means in our case the minimization of the following Tikhonov functional

$$J_{\gamma}(v) = \|v_t - Lv\|^2_{L_2(\Phi)} + \gamma\|v\|^2_{H^4(\Phi)}, \tag{3.2}$$

subject to the boundary conditions (3.1). In (3.2) $\gamma > 0$ is the regularization parameter, which should be chosen in accordance with the level of the error in the data.

The requirement $v \in H^4(\Phi)$ is an over-smoothness. This condition is imposed to ensure that $v \in C^1(\overline{\Phi})$: because of (2.47) and the embedding theorem. However, the author's numerical experience with the QRM has consistently demonstrated that one can significantly relax the required smoothness in practical computation, see [18, 21, 22] and chapter 6 of [4]. This is likely because one is not using an overly small grid step size in finite differences when computing via the QRM. Hence, one effectively works with a finite dimensional space with not too many dimensions. This means that one can rely in this case on the equivalence of all norms in finite dimensional spaces. Thus, most likely one can replace in real computations $\gamma \|v\|^2_{H^4(\Phi)}$ with $\gamma \|v\|^2_{H^{2,1}(\Phi)}$.

While (3.2) is good for computations, to prove convergence of the QRM, we need to have zero boundary conditions at $\partial_1\Phi$. Assume that both functions $\overline{\varphi}_2, \overline{\psi}_2 \in H^{2,1}(\partial_1\Phi)$. Denote

$$\begin{aligned} r(x,t) &= \overline{\varphi}_2(x,t) + x_1\overline{\psi}_2(x,t) = \overline{\varphi}_2(\overline{x},t) + x_1\overline{\psi}_2(\overline{x},t), \\ \widehat{v}(x,t) &= v(x,t) - r(x,t), p(x,t) = -(r_t - Lr)(x,t), \\ \widehat{f}(x) &= \widehat{v}(x,0) = f(x) - r(x,0). \end{aligned}$$

Using (2.4), (2.5) and (3.1), we obtain

$$\widehat{v}_t - L\widehat{v} = p(x,t), (x,t) \in \Phi, \tag{3.3}$$

$$\widehat{v} \mid_{\partial_1\Phi} = 0, \widehat{v}_{x_1} \mid_{\partial_1\Phi} = 0. \tag{3.4}$$

Thus, we have obtained Inverse Problem 3.

Inverse Problem 3 (IP3). *Find the function $\widehat{f}(x)$ for $x \in \Omega$ from conditions (3.3), (3.4).*

To solve IP3 via the QRM, we minimize the following analog of the functional (3.2)

$$\widehat{J}_\gamma(\widehat{v}) = \|\widehat{v}_t - L\widehat{v} - p\|^2_{L_2(\Phi)} + \gamma\|\widehat{v}\|^2_{H^4(\Phi)}, \widehat{v} \in H^4_0(\Phi), \tag{3.5}$$

$$H^4_0(\Phi) := \left\{u \in H^4(\Phi) : u\mid_{\partial_1\Phi} = u_{x_1}\mid_{\partial_1\Phi} = 0\right\}.$$

Let $(,)$ and $[,]$ be scalar products in $L_2(\Phi)$ and $H^4(\Phi)$ respectively. Let the function $u_\gamma \in H^4_0(\Phi)$ be a minimizer of the functional (3.5). Then the variational principle implies that

$$(\partial_t u_\gamma - Lu, \partial_t w - Lw) + \gamma[u,w] = (p, w_t - Lw), \forall w \in H^4_0(\Phi). \tag{3.6}$$

Lemma 3.1 follows immediately from the Riesz theorem and (3.6).

Lemma 3.1. *For every function $p \in L_2(\Phi)$ and every $\gamma > 0$ there exists unique minimizer $u_\gamma = u_\gamma(p) \in H^4_0(\Phi)$ of the functional (3.5). Furthermore the following estimate holds*

$$\|u_\gamma\|_{H^4(\Phi)} \leq \frac{M_1}{\sqrt{\gamma}}\|p\|_{L_2(\Phi)}.$$

The idea now is that if $u_\gamma(x,t) \in H^4_0(\Phi)$ is the minimizer mentioned in Lemma 3.1, then the approximate solution of IP3 is

$$\widehat{f}_\gamma(x) = u_\gamma(x,0). \tag{3.7}$$

The question of convergence of minimizers of $\widehat{J}_\gamma$ to the exact solution is more difficult than the existence question of Lemma 3.1. To address the question of convergence, we need to introduce the exact solution as well as the error in the data, just as this is always done in the regularization theory [2, 4, 36]. We assume that there exists an "ideal" noiseless data $p^* \in L_2(\Phi)$. We also assume that there exists the ideal noiseless solution $\widehat{v}^* \in H^4_0(\Phi)$ of the following problem

$$\widehat{v}^*_t - L\widehat{v}^* = p^*(x,t), (x,t) \in \Phi, \tag{3.8}$$

$$\widehat{v}^* \mid_{\partial_1\Phi} = 0, \widehat{v}^*_{x_1} \mid_{\partial_1\Phi} = 0. \tag{3.9}$$

Let $\omega \in (0,1)$ be a small number, which we regard as the level of the error in the data. We assume that

$$\|p - p^*\|_{L_2(\Phi)} \leq \omega. \tag{3.10}$$

Remark 3.1. For brevity, we work in this section with the parabolic IP3. Still, Theorem 3.1 can be easily linked with the original hyperbolic IP2. Indeed, to ensure that $p \in L_2(\Phi)$,we need $\overline{\varphi}_2, \overline{\psi}_2 \in H^{2,1}(\partial_1 \Phi)$. While Assumption implies (2.7), which, in turn guarantees that $\overline{\varphi}_2 \in H^{2,1}(\partial_1 \Phi)$, there is no guarantee that $\overline{\psi}_2 \in H^{2,1}(\partial_1 \Phi)$ (see (2.10)). To ensure the latter, we should replace in Assumption $C^3(\mathbb{R}^n \times [0,T])$ with $C^5(\mathbb{R}^n \times [0,T])$. In this case (2.10) would be replaced with $\left\|\overline{\psi}_2\right\|_{C^{2+\alpha,1+\alpha/2}\left(\overline{P}_1\right)} \leq C(P,L)\left\|\overline{\varphi}_2\right\|_{C^{4+\alpha,2+\alpha/2}\left(\overline{P}_1\right)}$. The latter means, in turn that the comparison of the functions p with the exact function p^* in (3.10) would be replaced with the comparison of the approximate and exact data φ_2 and φ_2^* of IP2. This can be done via a routine procedure by replacing (2.12) with $\|\varphi_2 - \varphi_2^*\|_{C^5\left(\overline{P}_T\right)} \leq \delta e^{bT}, \forall T > 0$. In this case we would have in (3.10) $\omega = \omega(\delta)$.

Theorem 3.1 establishes the convergence rate of the QRM. Note that an upper estimate of the exact solution is often assumed to be known in the regularization theory, also see Remark 2.2.

Theorem 3.1. *Let conditions (2.24), (3.3), (3.4) and (3.10) be satisfied and the regularization parameter γ in (3.5) is chosen such that $\gamma = \gamma(\omega) = \omega \in (0,1)$. Let the function $u_\gamma \in H_0^4(\Phi)$ be the unique minimizer of the functional (3.5), which is guaranteed by Lemma 3.1. Let the upper estimate $Y = const. > 0$ for the exact solution $\widehat{v}^* \in H_0^4(\Phi)$ be known, $\|\widehat{v}^*\|_{H^4(\Phi)} \leq Y$. Then there exists a sufficiently small number $\omega_0 = \omega_0(L,\Phi) \in (0,1)$ such that if ω is so small that $\omega\sqrt{(Y^2+1)} \in (0,\omega_0)$, then the following logarithmic convergence rate takes place*

$$\left\|\widehat{f}^* - f_{\gamma(\omega)}\right\|_{L_2(\Omega)} \leq \frac{M_1 Y}{\sqrt{\ln(\omega^{-1})}}, \tag{3.11}$$

where the function $f_{\gamma(\eta)}(x)$ is defined in (3.7) and $\widehat{f}^*(x) = \widehat{v}^*(x,0)$. In addition, for every $\omega \in (0,\omega_0)$ there exists a number $\rho = \rho(L,\Phi) \in (0,1/2)$ such that the following convergence rate takes place

$$\left\|\widehat{v}^* - u_{\gamma(\eta)}\right\|_{H^{1,0}\left(D_{1/2}\right)} \leq M_1 Y \omega^\rho. \tag{3.12}$$

Proof. It follows from (3.8) and (3.9) that the function $\widehat{v}^*$ satisfies the following analog of (3.6)

$$(\widehat{v}_t^* - L\widehat{v}^*, w_t - Lw) + \gamma[\widehat{v}^*, w] = (p, w_t - Lw) + \gamma[\widehat{v}^*, w], \forall w \in H_0^4(\Phi). \tag{3.13}$$

Let $\widetilde{v} = u_\gamma - \widehat{v}^* \in H_0^4(\Phi)$ and $\widetilde{p} = p - p^* \in L_2(\Phi)$. Subtracting (3.13) from (3.6), we obtain

$$(\widetilde{v}_t - L\widetilde{v}, w_t - Lw) + \gamma[\widetilde{v}, w] = (\widetilde{p}, w_t - Lw) - \gamma[\widehat{v}^*, w], \forall w \in H_0^4(\Phi).$$

Setting here $w := \widetilde{v}$ and using Cauchy-Schwarz inequality and (3.10), we obtain

$$\int_{\Phi_1} (\widetilde{v}_t - L\widetilde{v})^2 \, dxdt + \gamma\|\widetilde{v}\|_{H^4(\Phi)}^2 \leq \omega^2 + \gamma\|\widehat{v}^*\|_{H^4(\Phi)}^2 \leq \omega^2 + \gamma Y^2. \tag{3.14}$$

Since $\gamma(\omega) = \omega \in (0,1)$, then (3.14) implies that $\|\widetilde{v}\|_{H^4(\Phi)} \leq Y + 1$. Hence, using again (3.14) as well as embedding theorem, we obtain with the constant $c = c(\Phi) > 0$ depending only on the domain Φ

$$\|\widetilde{v}\|_{C^1\left(\overline{\Phi}\right)} \leq cY, \tag{3.15}$$

$$\int_{\Phi} (\widetilde{v}_t - L\widetilde{v})^2 \, dxdt \leq \left(Y^2 + 1\right)\omega^2, \tag{3.16}$$

We now apply Theorem 2.2. Comparing (3.16) and (3.15) with (2.45) and (2.47) respectively, we set

$$K := F := \omega\sqrt{(Y^2+1)}, C_2 := cY. \tag{3.17}$$

Therefore, (3.11) and (3.12) follow from (3.17), (2.48) and (2.50). $\square$

Acknowledgment

This research was supported by US Army Research Laboratory and US Army Research Office grant W911NF-11-1-0399.

REFERENCES

[1] M. Agranovsky and P. Kuchment, Uniqueness of reconstruction and an inversion procedure for thermoacoustic and phtoacoustic tomography with variable sound speed, *Inverse Problems*, 23, 2089-2102, 2007.

[2] A.B. Bakushinskii and M.Yu. Kokurin, *Iterative Methods for Approximate Solutions of Inverse Problems*, Springer, New York, 2004.

[3] C. Bardos, G. Lebeau and J. Rauch, Sharp sufficient conditions for observation control and stabilization of waves from the boundary, *SIAM J. Contr. Opt.*, 30, 1024-1065, 1992.

[4] L. Beilina and M.V. Klibanov, *Approximate Global Convergence and Adaptivity for Coefficient Inverse Problems*, Springer, New York, 2012.

[5] C. Clason and M.V. Klibanov, The quasi-reversibility method for thermoacoustic tomography in a heterogeneous medim, *SIAM J. Sci. Comp.*, 30, 1-23, 2007.

[6] H.W. Engl, M. Hanke and A. Neubauer, *Regularization of Inverse Problems*, Kluwer Academic Publishers, Boston, 2000.

[7] D. Finch, S.K Patch and Rakesh, Determining a function from its mean values over a family of spheres, SIAM *J. Math. Anal.*, 35, 1213-1240, 2004.

[8] D. Finch, M. Haltmeier and Rakesh, Inversion of spherical means and the wave equation in even dimensions, *SIAM J. Appl. Math.*, 68, 392-412, 2007.

[9] D. Finch and Rakesh, Recovering a function from its spherical mean values in two and three dimensions, *Photoacoustic Imaging and Spectroscopy*, CRC Press, Boca Raton, Florida, 2009.

[10] V. Isakov, *Inverse Problems for Partial Differential Equations*, Second Edition, Springer, New York, 2006.

[11] M. Kazemi and M.V. Klibanov, Stability estimates for ill-posed Cauchy problem involving hyperbolic equation and inequalities, *Applicable Analysis*, 50, 93-102, 1993.

[12] M.V. Klibanov and J. Malinsky, Newton-Kantorovich method for 3-dimensional potential inverse scattering problem and stability for the hyperbolic Cauchy problem with time dependent data, *Inverse Problems*, 7, 577-596, 1991.

[13] M.V. Klibanov and Rakesh, Numerical solution of a timelike Cauchy problem for the wave equation, *Math. Meth. in Appl. Sci.*, 15, 559-570, 1992.

[14] M.V. Klibanov, Lipschitz stability for hyperbolic inequalities in octants with the lateral Caucby data and refocusing in time reversal, *J. Inverse and Ill-Posed Problems*, 13, 353-363, 2005.

[15] M.V. Klibanov, Estimates of initial conditions of parabolic equations and inequalities via lateral Cauchy data, *Inverse Problems*, 22, 495-514, 2006.

[16] M.V. Klibanov and A.V. Tikhonravov, Estimates of initial conditions of parabolic equations and inequalities in infinite domains via lateral Cauchy data, *Journal of Differential Equations*, 237, 198-224, 2007.

[17] M.V. Klibanov and A. Timonov, *Carleman Estimates for Coefficient Inverse Problems and Numerical Applications*, VSP, Utrecht, The Netherlands, 2004.

[18] M.V. Klibanov, A.V. Kuzhuget, S.I. Kabanikhin and D.V. Nechaev, A new version of the quasi-reversibility method for the thermoacoustic tomography and a coefficient inverse problem, *Applicable Analysis*, 87, 1227-1254, 2008.

[19] P. Kuchment and L. Kunyansky, Mathematics of thermoacoustic tomography, *European J. Applied Mathematics*, 19, 191-224, 2008.

[20] L. Kunyansky, Thermoacoustic tomography with detectors on an open curve: an efficient reconstruction algorithm, *Inverse Problems*, 24, 055021, 2008.

[21] A.V. Kuzhuget, N. Pantong and M.V. Klibanov, A globally convergent numerical method for a coefficient inverse problem with backscattering data, *Methods and Applications of Analysis,* 18, 47-68, 2011.

[22] A.V. Kuzhuget, L. Beilina and M.V. Klibanov, Approximate global convergence and quasireversibility of a coefficient inverse problem with backscattering data, *Journal of Mathematical Sciences*, 181, 19-49, 2012.

[23] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, AMS, Providence, R.I., 1968.

[24] O.A. Ladyzhenskaya, *Boundary Value Problems of Mathematical Physics*, Springer, New York, 1985.

[25] I. Lasiecka, R. Triggiani and X. Zhang, Inverse/observability estimates for second order hyperbolic equations with variable coefficients, *J. Math. Anal. Appl.*, 235, 13-57, 1999.

[26] I. Lasiecka, R. Triggiani and X. Zhang, Global uniqueness, observability and stabilization of non-conservative Schrödinger equations via pointwise Carleman estimates. Part I: $H^1(\Omega)-$estimates, *J. of Inverse and Ill-Posed Problems*, 12, 1-81, 2004.

[27] I. Lasiecka, R. Triggiani and X. Zhang, Global uniqueness, observability and stabilization of non-conservative Schrödinger equations via pointwise Carleman estimates. Part II: $L_2(\Omega)-$estimates, *J. of Inverse and Ill-Posed Problems*, 12, 182-231, 2004.

[28] R. Lattes and J.-L. Lions, *The Method of Quasireversibility: Applications to Partial Differential Equations,* Elsevier, New York, 1969.

[29] M.M. Lavrentiev, V.G. Romanov and S.P. Shishatskii, *Ill-Posed Problems of Mathematical Physics and Analysis*, AMS, Providence, R.I., 1986.

[30] J. Li, M. Yamamoto and J. Zou, Conditional stability and numerical reconstruction of initial temperature, *Communications on Pure and Applied Analysis*, 8, 361-382, 2009.
[31] Lop Fat Ho, Observabilité frontierè de l'equation des ondes, *C.R. Acad. Sc. Paris*, t. 302, Ser. I, No. 12, 443-446, 1986.
[32] V.G. Romanov, *Inverse Problems of Mathematical Physics*, VNU Press, Utrecht, The Netherlands, 1986.
[33] V.G. Romanov, Estimates of a solution to a differential inequality related to a second order hyperbolic operator an Cauchy data on a timelike surface, *Doklady Mathematics*, 73, 51-53, 2006.
[34] V.G. Romanov, Stability estimates in inverse problems for hyperbolic equations, *Milan J. Math.*, 74, 357-385, 2006.
[35] P. Stefanov and G. Uhlmann, Thermoacoustic tomography with variable sound speed, *Inverse Problems*, 25, 075011, 2009.
[36] A.N. Tikhonov, A.V. Goncharsky, V.V. Stepanov and A.G. Yagola, *Numerical Methods for the Solution of Ill-Posed Problems*, Kluwer, London, 1995.