ON BOUNDEDNESS OF THE CURVE GIVEN BY ITS CURVATURE AND TORSION

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Abstract. We consider a curve in the three dimensional Euclidean space and provide sufficient conditions on the curvature and the torsion for the curve to be unbounded.

We also present sufficient conditions on the curvatures for the curve to be bounded in the four dimensional Euclidean space.

1. Introduction

In this short note we concern a smooth curve γ in the standard three dimensional Euclidean space $\mathbb{R}^3$. Let this curve be defined (up to translations and rotations of $\mathbb{R}^3$) by its curvature $\kappa(s)$ and its torsion $\tau(s)$, the argument s is the arc-length parameter. The pair $(\kappa(s),\tau(s))$ is called the intrinsic equation of the curve.

In the sequel we assume that $\kappa,\tau\in C[0,+\infty)$.

To obtain the radius-vector of the curve γ one must solve the system of Frenet-Serret equations:

$$\begin{aligned}\mathbf{v}'(s)&=\kappa(s)\mathbf{n}(s),\\ \mathbf{n}'(s)&=-\kappa(s)\mathbf{v}(s)+\tau(s)\mathbf{b}(s),\\ \mathbf{b}'(s)&=-\tau(s)\mathbf{n}(s).\end{aligned}\tag{1.1}$$

The vectors $\mathbf{v}(s),\mathbf{n}(s),\mathbf{b}(s)$ stand for the Frenet-Serret frame at the point with parameter s. Then the radius-vector of the curve is computed as follows $\mathbf{r}(s)=\int_0^s\mathbf{v}(\xi)d\xi+\mathbf{r}(0)$.

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So we obtain very natural and pretty problem: having the curvature $\kappa(s)$ and the torsion $\tau(s)$ to restore the properties of the curve γ .

For example, which conditions should be imposed on the functions κ, τ so that the curve γ is closed? This is a hard open problem. There may by another question: on which conditions does the curve lie on a sphere? This question is much simpler. Such a type questions have been discussed in [4], [3], [5].

There is sufficient condition for a curve to be unbounded given in terms of curvature only [1]. The corresponding condition in terms of curvature alone is sufficient in a much broader class of spaces, including Hilbert spaces and Riemannian manifolds of nonpositive curvature.

The curve γ is a planar curve if and only if $\tau(s) = 0$ and system (1.1) is integrated explicitly. This case is not very interesting.

In the general case, (1.1) is a linear system of ninth order with matrix depending on s. To describe the properties of γ one must study this system.

In this note we formulate and prove some sufficient conditions for unboundedness of the curve γ.

We also present sufficient conditions on the curvatures for the curve to be bounded in the four dimensional Euclidean space.

2. Main Theorem

We shall say that γ is unbounded if $\sup_{s\ge 0} |\mathbf{r}(s)| = \infty$.

Theorem 2.1. *Suppose there exists a function $\lambda(s)$ such that functions*

$$k(s) = \lambda(s)\kappa(s), \quad t(s) = \lambda(s)\tau(s)$$

are monotone[1] *and belong to $C[0, \infty)$.*

Introduce a function $T(s) = \int_0^s t(\xi)d\xi$.

Suppose also that the following equalities hold

$$\lim_{s\to\infty} T(s) = \infty, \quad \lim_{s\to\infty} \frac{k(s)}{T(s)} = \lim_{s\to\infty} \frac{t(s)}{T(s)} = 0. \tag{2.1}$$

Then the curve γ is unbounded.

The proof of this theorem is contained in Section 4.1.

Putting in this Theorem $\lambda = 1/\tau$, we deduce the following corollary.

[1] i.e. one of these functions, for example $k(s)$ is monotonically increased: $s' < s'' \Rightarrow k(s') \le k(s'')$, $s', s'' \in [0, \infty)$ while another one $t(s)$ is monotonically decreased: $s' < s'' \Rightarrow t(s') \ge t(s'')$, $s', s'' \in [0, \infty)$. The inverse situation is also allowed, or the both functions can be increased or decreased simultaneously .

Corollary 1. *Suppose that a function $\kappa(s)/\tau(s)$ is monotone and*

$$\lim_{s\to\infty}\frac{\kappa(s)}{s\cdot\tau(s)}=0. \tag{2.2}$$

Then the curve γ is unbounded.

Note that the geodesic curvature of the tantrix[2] $\kappa_T(s)$ is equal to $\tau(s)/\kappa(s)$ [3]. So that formula (2.2) can be rewritten as follows

$$\lim_{s\to\infty}\kappa_T(s)s=\infty.$$

Theorem 2.1 is not reduced to Corollary 1. Consider an example. Let the curve γ be given by

$$\kappa(s)=1,\quad \tau(s)=\frac{1}{1+s}.$$

Since $\tau(s)\to 0$ as $s\to\infty$ it may seem that this curve is about a circle with $\kappa(s)=1$. Nevertheless applying Theorem 2.1 with $\lambda=1$ we see that the curve γ is unbounded.

Consider a system which consists of (1.1) together with the equation $\mathbf{r}'(s)=\mathbf{v}(s)$. From the stability theory viewpoint Theorem 2.1 states that under certain conditions this system is unstable.

Since $|\mathbf{r}(s)|=O(s)$ as $s\to\infty$, this instability is too mild to study it by standard methods such as the Lyapunov exponents method.

3. Supplementary Remarks: Bounded Curves in $\mathbb{R}^4$

Actually the above developed technique can be generalized to the curves in any multidimensional Euclidean space $\mathbb{R}^m$. For the case of the odd m we can prove a theorem similar to Theorem 2.1. But for the case when m is even our method allows to obtain sufficient conditions for the curve to be bounded.

In this section we illustrate such an effect. To avoid of big formulas we consider only the case $m=4$.

So let a curve $\gamma\subset\mathbb{R}^4$ be given by its curvatures

$$\kappa_i(s)\in C[0,\infty),\quad i=1,2,3.$$

And let $\mathbf{v}_j(s),\quad j=1,2,3,4$ be the Frenet-Serret frame.

[2] The tangential spherical image of the curve γ is the curve on the unit sphere. This curve has the radius-vector $\mathbf{r}'(s)$.

Then the Frenet-Serret equations are

$$\frac{d}{ds}\begin{pmatrix}\mathbf{v_1}\\ \mathbf{v_2}\\ \mathbf{v_3}\\ \mathbf{v_4}\end{pmatrix}(s)=A(s)\begin{pmatrix}\mathbf{v_1}\\ \mathbf{v_2}\\ \mathbf{v_3}\\ \mathbf{v_4}\end{pmatrix}(s),$$

$$A(s)=\begin{pmatrix}0 & \kappa_1(s) & 0 & 0\\ -\kappa_1(s) & 0 & \kappa_2(s) & 0\\ 0 & -\kappa_2(s) & 0 & \kappa_3(s)\\ 0 & 0 & -\kappa_3(s) & 0\end{pmatrix}$$

Theorem 3.1. *Suppose that the function $\kappa_1(s)\kappa_3(s)$ does not take the zero value. The functions*

$$f_1(s)=\frac{1}{\kappa_1(s)},\quad f_2(s)=\frac{\kappa_2(s)}{\kappa_1(s)\kappa_3(s)}$$

are monotone and

$$\sup_{s\ge 0}|f_i(s)|<\infty,\quad i=1,2.$$

Then the curve γ is bounded.

The proof of this theorem is contained in Section 4.2.

4. Proofs

4.1. **Proof of Theorem 2.1.** Let us write the formula

$$\mathbf{r}(s)=r_1(s)\mathbf{v}(s)+r_2(s)\mathbf{n}(s)+r_3(s)\mathbf{b}(s).$$

Differentiating this formula we obtain

$$\mathbf{v}(s)=r_1'(s)\mathbf{v}(s)+r_2'(s)\mathbf{n}(s)+r_3'(s)\mathbf{b}(s)$$
$$+r_1(s)\mathbf{v}'(s)+r_2(s)\mathbf{n}'(s)+r_3(s)\mathbf{b}'(s).$$

This formula is easily solved for (r_1', r_2', r_3') by using the orthogonality of the coefficient matrix $(\mathbf{v},\mathbf{n},\mathbf{b})$ and the system (1.1). So we obtain

$$r'(s)=\begin{pmatrix}0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{pmatrix}r(s)+\begin{pmatrix}1\\0\\0\end{pmatrix},\qquad r=\begin{pmatrix}r_1\\ r_2\\ r_3\end{pmatrix}. \quad (4.1)$$

The author was informed about system (4.1) by Professor Ya. V. Tatarinov.

Let us multiply both sides of system (4.1) on the left by the row vector $(\lambda(s)(\tau(s),0,\kappa(s))$:

$$t(s)r_1'(s)+k(s)r_3'(s)=t(s).$$

Then we integrate this equality:

$$\int_0^s t(a)r_1'(a)da + \int_0^s k(a)r_3'(a)da = T(s). \tag{4.2}$$

From the Second Mean Value Theorem [2], we know that there is a parameter $\xi \in [0, s]$ such that

$$\begin{aligned}\int_0^s t(a)r_1'(a)da &= t(0)\int_0^\xi r_1'(a)da + t(s)\int_\xi^s r_1'(a)da \\ &= t(0)\big(r_1(\xi) - r_1(0)\big) + t(s)\big(r_1(s) - r_1(\xi)\big)\end{aligned}$$

By the same argument for some $\eta \in [0, s]$ we have

$$\int_0^s k(a)r_3'(a)da = k(0)\big(r_3(\eta) - r_3(0)\big) + k(s)\big(r_3(s) - r_3(\eta)\big).$$

Thus formula (4.2) takes the form

$$\begin{aligned}t(0)\big(r_1(\xi) - r_1(0)\big) + t(s)\big(r_1(s) - r_1(\xi)\big) \\ + k(0)\big(r_3(\eta) - r_3(0)\big) + k(s)\big(r_3(s) - r_3(\eta)\big) = T(s). \end{aligned}\tag{4.3}$$

Since the Frenet-Serret frame is orthonormal we have

$$|\mathbf{r}(s)|^2 = r_1^2(s) + r_2^2(s) + r_3^2(s) = |r(s)|^2.$$

Assume the converse: the curve γ is bounded i.e. $\sup_{s\geq 0} |\mathbf{r}(s)| < \infty$. Then due to conditions (2.1) the left side of formula (4.3) is $o(T(s))$ as $s \to \infty$. This is the contradiction.

The Theorem is proved.

4.2. **Proof of Theorem 3.1.** Let $\mathbf{r}(s)$ be the radius-vector of the curve γ. Then one can write

$$\mathbf{r}(s) = \sum_{i=1}^4 r_i \mathbf{v}_i(s), \quad \mathbf{r}'(s) = \mathbf{v}_1(s).$$

Similarly as in the previous section, by the Frenet-Serret equations this gives

$$r'(s) = A(s)r(s) + \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad r = \begin{pmatrix}r_1\\r_2\\r_3\\r_4\end{pmatrix}.$$

First we multiply this equation by $r'^T(s)A^{-1}(s)$, $\quad(\det A = (\kappa_1\kappa_3)^2)$:

$$r'^T(s)A^{-1}(s)r'(s) = r'^T(s)r(s) + r'^T(s)A^{-1}(s)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}. \tag{4.4}$$

Since A^{-1} is a skew-symmetric matrix we have $r'^T(s)A^{-1}(s)r'(s) = 0$, and some calculation yields

$$r'^T(s)A^{-1}(s)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = r_2'(s)f_1(s) + r_4'(s)f_2(s).$$

Then formula (4.4) takes the form

$$-\frac{1}{2}\Big(|r(s)|^2\Big)' = r_2'(s)f_1(s) + r_4'(s)f_2(s).$$

Integrating this formula we obtain

$$-\frac{1}{2}\Big(|r(s)|^2 - |r(0)|^2\Big) = \int_0^s r_2'(a)f_1(a) + r_4'(a)f_2(a)da.$$

By the same argument which was employed to obtain formula (4.3) it follows that

$$\begin{aligned}&-\frac{1}{2}\Big(|r(s)|^2 - |r(0)|^2\Big) = \\ &f_1(0)\big(r_2(\xi) - r_2(0)\big) + f_1(s)\big(r_2(s) - r_2(\xi)\big) + \\ &f_2(0)\big(r_4(\eta) - r_4(0)\big) + f_2(s)\big(r_4(s) - r_4(\eta)\big),\end{aligned} \tag{4.5}$$

here $\xi, \eta \in [0, s]$.

To proceed with the proof assume the converse. Let the curve γ be unbounded: $\sup_{s\ge0}|r(s)| = \infty$. Take a sequence s_k such that

$$|r(s_k)| = \max_{s\in[0,k]}|r(s)|, \quad k \in \mathbb{N}, \quad s_k \in [0, k].$$

It is easy to see that

$$s_k \to \infty, \quad |r(s)| \le |r(s_k)|, \quad s \in [0, s_k]$$

and $|r(s_k)| \to \infty$ as $k \to \infty$.

Substitute this sequence to formula (4.5):

$$\begin{aligned}&-\frac{1}{2}\Big(|r(s_k)|^2 - |r(0)|^2\Big) = \\ &f_1(0)\big(r_2(\xi_k) - r_2(0)\big) + f_1(s_k)\big(r_2(s_k) - r_2(\xi_k)\big) + \\ &f_2(0)\big(r_4(\eta_k) - r_4(0)\big) + f_2(s_k)\big(r_4(s_k) - r_4(\eta_k)\big),\end{aligned} \tag{4.6}$$

here $\xi_k, \eta_k \in [0, s_k]$ and thus $|r_2(\xi_k)| \le |r(s_k)|, \quad |r_4(\eta_k)| \le |r(s_k)|$.

Due to conditions of the Theorem and the choice of the sequence s_k the right-hand side of formula (4.6) grows not faster than $O(|r(s_k)|)$ as $k \to \infty$. But the left-hand one is of order $-|r(s_k)|^2/2$. This is the contradiction.

The Theorem is proved.

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