On the existence of stationary solutions for some systems of non-Fredholm integro-differential equations

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Abstract. We prove the existence of stationary solutions for certain systems of reaction-diffusion type equations in the corresponding H^2 spaces. Our method relies on the fixed point theorem when the elliptic problem involves second order differential operators with and without Fredholm property.

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1 Introduction

We recall that a linear operator L, which acts from a Banach space E into another Banach space F possesses the Fredholm property when its image is closed, the dimension of its kernel and the codimension of its image are finite. Consequently, the equation $Lu = f$ is solvable if and only if $\phi_k(f) = 0$ for a finite number of functionals ϕ_k from the dual space F^*. Such properties of Fredholm operators are broadly used in various methods of linear and nonlinear analysis.

Elliptic problems considered in bounded domains with a sufficiently smooth boundary satisfy the Fredholm property when the ellipticity condition, proper ellipticity and Lopatinskii conditions are fulfilled (see e.g. [1], [11], [13]), which is the main result of the theory of linear elliptic problems. When dealing with unbounded domains, these conditions may not be sufficient and the Fredholm property may not be satisfied. For instance, for the Laplace operator, $Lu = \Delta u$ considered in $\mathbb{R}^d$ Fredholm property does not hold when the problem is studied either in Hölder spaces, such that $L : \mathbb{C}^{2+\alpha}(\mathbb{R}^d) \to \mathbb{C}^{\alpha}(\mathbb{R}^d)$ or in Sobolev spaces, $L : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.

For linear elliptic problems studied in unbounded domains the Fredholm property is fulfilled if and only if, in addition to the conditions stated above, limiting operators are invertible (see [14]). In certain simple cases, limiting operators can be constructed explicitly. For instance, when

$$Lu = a(x)u'' + b(x)u' + c(x)u, \quad x \in \mathbb{R},$$

with the coefficients of the operator having limits at infinity,

$$a_{\pm} = \lim_{x \to \pm\infty} a(x), \quad b_{\pm} = \lim_{x \to \pm\infty} b(x), \quad c_{\pm} = \lim_{x \to \pm\infty} c(x),$$

the limiting operators are given by

$$L_{\pm}u = a_{\pm}u'' + b_{\pm}u' + c_{\pm}u.$$

Due to the fact that the coefficients here are constants, the essential spectrum of the operator, which is the set of complex numbers λ for which the operator $L - \lambda$ does not possess the Fredholm property, can be found explicitly via the standard Fourier transform, such that

$$\lambda_{\pm}(\xi) = -a_{\pm}\xi^2 + b_{\pm}i\xi + c_{\pm}, \quad \xi \in \mathbb{R}.$$

The limiting operators are invertible if and only if the origin is not contained in the essential spectrum.

For general elliptic problems the analogous assertions are valid. The Fredholm property is fulfilled when the essential spectrum does not contain the origin or when the limiting operators are invertible. Such conditions may not be written explicitly.

For non-Fredholm operators we may not apply the standard solvability conditions and in a general case solvability conditions are unknown. However, solvability conditions were obtained recently for certain classes of operators. For instance, consider the following equation

$$Lu \equiv \Delta u + au = f \tag{1.1}$$

in $\mathbb{R}^d$, $d \in \mathbb{N}$ with a positive constant a. Here the operator L and its limiting operators coincide. The corresponding homogeneous equation admits a nontrivial bounded solution, such that the Fredholm property is not fulfilled. Due to the fact that the differential operator involved in (1.1) has constant coefficients, we are able to find the solution explicitly by applying the standard Fourier transform. In [25] we obtained the following solvability conditions. Let $f(x) \in L^2(\mathbb{R}^d)$ and $xf(x) \in L^1(\mathbb{R}^d)$. Then problem (1.1) admits a unique solution in $H^2(\mathbb{R}^d)$ if and only if

$$\left(f(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}} \right)_{L^2(\mathbb{R}^d)} = 0, \quad p \in S^d_{\sqrt{a}} \quad a.e.$$

Here and further down S^d_r stands for the sphere in $\mathbb{R}^d$ of radius r centered at the origin. Hence, although the Fredholm property is not fulfilled for the operator, we are able to formulate solvability conditions in a similar way. Note that this similarity is only formal due to the fact that the range of the operator is not closed. In the situation when the operator involves a scalar potential, such that

$$Lu \equiv \Delta u + b(x)u = f,$$

we cannot apply the standard Fourier transform directly. However, solvability conditions in three dimensions can be derived by means of the spectral and the scattering theory of Schrödinger type operators (see [16]). As in the constant coefficient case, solvability conditions are expressed in terms of orthogonality to solutions of the adjoint homogeneous problem. We obtain solvability conditions for several other examples of non Fredholm linear elliptic operators (see [14] – [22], [25]).

Solvability conditions are crucial in the analysis of nonlinear elliptic equations. When non-Fredholm operators are involved, in spite of some progress in studies of linear problems, nonlinear non-Fredholm operators were analyzed only in few examples (see [5]– [7], [23], [25]). Obviously, this situation can be explained by the fact that the majority of methods of linear and nonlinear analysis rely on the Fredholm property. In the present work we study certain systems of N nonlinear integro-differential reaction-diffusion type equations, for which the Fredholm property may not be satisfied:

$$\frac{\partial u_k}{\partial t} = \Delta u_k + \int_\Omega G_k(x-y)F_k(u_1(y,t), u_2(y,t), ..., u_N(y,t), y)dy + a_k u_k, \ 1 \le k \le N. \quad (1.2)$$

Here $\{a_k\}_{k=1}^N$ are nonnegative, $\Omega \subseteq \mathbb{R}^d$, $d = 1, 2, 3$ are the more physically relevant dimensions. In population dynamics the integro-differential problems are used to describe biological systems with intra-specific competition and nonlocal consumption of resources (see e.g. [2], [4], [8], [24]). The stability issues for the travelling fronts of reaction- diffusion type equations with the essential spectrum of the linearized operator crossing the imaginary axis were also addressed in [3] and [9]. Note that the single equation of (1.2) type has been studied in [23]. Reaction-diffusion type equations in which in the diffusion term the Laplacian is replaced by the nonlocal operator with an integral kernel were treated in [12].

The nonlinear terms of system (1.2) will satisfy the following regularity requirements.

Assumption 1. *Functions $F_k(u, x) : \mathbb{R}^N \times \Omega \to \mathbb{R}, \ 1 \le k \le N$ are such that*

$$\sqrt{\sum_{k=1}^N F_k^2(u, x)} \le k|u|_{\mathbb{R}^N} + h(x) \quad for \quad u \in \mathbb{R}^N, \ x \in \Omega, \quad (1.3)$$

with a constant $k > 0$ and $h(x) : \Omega \to \mathbb{R}^+$, $h(x) \in L^2(\Omega)$. Moreover, they are Lipschitz continuous functions, such that

$$\sqrt{\sum_{k=1}^N (F_k(u^{(1)}, x) - F_k(u^{(2)}, x))^2} \le l|u^{(1)} - u^{(2)}|_{\mathbb{R}^N} \quad for \quad any \quad u^{(1),(2)} \in \mathbb{R}^N, \quad x \in \Omega, \quad (1.4)$$

where a constant $l > 0$.

Here and further down we use the notations for a vector $u := (u_1, u_2, ..., u_N) \in \mathbb{R}^N$ and its norm $|u|_{\mathbb{R}^N} := \sqrt{\sum_{k=1}^N u_k^2}$. Apparently, the stationary solutions of system (1.2), if any exist, will solve the system of nonlocal elliptic equations

$$\Delta u_k + \int_\Omega G_k(x-y)F_k(u_1(y), u_2(y), ..., u_N(y), y)dy + a_k u_k = 0, \ a_k \ge 0, \ 1 \le k \le N.$$

For the technical purposes we consider the auxiliary semi-linear problem

$$-\Delta u_k - a_k u_k = \int_\Omega G_k(x-y)F_k(v_1(y), v_2(y), ..., v_N(y), y)dy, \ 1 \le k \le N. \tag{1.5}$$

Let us designate $(f_1(x), f_2(x))_{L^2(\Omega)} := \int_\Omega f_1(x)\bar{f}_2(x)dx$, with a slight abuse of notations in the case when these functions do not belong to $L^2(\Omega)$, like for instance those used in the orthogonality conditions of the assumption below. Indeed, if $f_1(x) \in L^1(\Omega)$ and $f_2(x)$ is bounded there, then the integral over Ω mentioned above is well defined. We begin the article with the studies of the whole space case, such that $\Omega = \mathbb{R}^d$ and the corresponding Sobolev space is equipped with the norm

$$\|u\|^2_{H^2(\mathbb{R}^d, \ \mathbb{R}^N)} := \sum_{k=1}^N \|u_k\|^2_{H^2(\mathbb{R}^d)} = \sum_{k=1}^N \{\|u_k\|^2_{L^2(\mathbb{R}^d)} + \|\Delta u_k\|^2_{L^2(\mathbb{R}^d)}\},$$

where $u(x) : \mathbb{R}^d \to \mathbb{R}^N$. The primary obstacle in solving problem (1.5) is that operators $-\Delta - a_k : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \ a_k \ge 0$ do not satisfy the Fredholm property. The analogous situations in linear problems, which can be self- adjoint or non self-adjoint containing non Fredholm second, fourth and sixth order differential operators or even systems of equations including non Fredholm operators have been studied actively in recent years (see [16]-[22]). We manage to prove that system of equations (1.5) defines a map $T_a : H^2(\mathbb{R}^d, \ \mathbb{R}^N) \to H^2(\mathbb{R}^d, \ \mathbb{R}^N), \ a_k \ge 0, \ 1 \le k \le N$, which is a strict contraction under specified technical conditions. The notation S_r^d will designate the sphere of radius r in $\mathbb{R}^d$ centered at the origin. We make the following assumption on the integral kernels involved in the nonlocal parts of system (1.5).

Assumption 2. *Let* $G_k(x) : \mathbb{R}^d \to \mathbb{R}, \ G_k(x) \in L^1(\mathbb{R}^d), \ 1 \le k \le N, \ 1 \le d \le 3$ *and* $1 \le m \le N-1, \ m \in \mathbb{N}$ *with* $N \ge 2$.

I) Let $a_k > 0, \ 1 \le k \le m$, *assume that* $xG_k(x) \in L^1(\mathbb{R}^d)$ *and*

$$\left(G_k(x), \frac{e^{\pm i\sqrt{a_k}x}}{\sqrt{2\pi}}\right)_{L^2(\mathbb{R})} = 0 \ when \ d = 1. \tag{1.6}$$

$$\left(G_k(x), \frac{e^{ipx}}{(2\pi)^{\frac{d}{2}}}\right)_{L^2(\mathbb{R}^d)} = 0 \ for \ p \in S^d_{\sqrt{a_k}} \ a.e. \ when \ d = 2, 3. \tag{1.7}$$

II) Let $a_k = 0,\ m+1 \le k \le N,$ assume that $x^2 G_k(x) \in L^1(\mathbb{R}^d)$ and

$$(G_k(x), 1)_{L^2(\mathbb{R}^d)} = 0 \text{ and } (G_k(x), x_s)_{L^2(\mathbb{R}^d)} = 0,\ 1 \le s \le d. \tag{1.8}$$

We will use the hat symbol here and further down to denote the standard Fourier transform, such that

$$\widehat{G_k}(p) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} G_k(x) e^{-ipx} dx,\ p \in \mathbb{R}^d.$$

Hence

$$\|\widehat{G_k}(p)\|_{L^\infty(\mathbb{R}^d)} \le \frac{1}{(2\pi)^{\frac{d}{2}}} \|G_k\|_{L^1(\mathbb{R}^d)}.$$

We define the following auxiliary quantities

$$M_k := \max\left\{ \left\| \frac{\widehat{G_k}(p)}{p^2 - a_k} \right\|_{L^\infty(\mathbb{R}^d)},\ \left\| \frac{p^2 \widehat{G_k}(p)}{p^2 - a_k} \right\|_{L^\infty(\mathbb{R}^d)} \right\},\ 1 \le k \le m. \tag{1.9}$$

$$M_k := \max\left\{ \left\| \frac{\widehat{G_k}(p)}{p^2} \right\|_{L^\infty(\mathbb{R}^d)},\ \left\| \widehat{G_k}(p) \right\|_{L^\infty(\mathbb{R}^d)} \right\},\ m+1 \le k \le N. \tag{1.10}$$

Note that expressions (1.9) and (1.10) are finite by means of Lemma A1 in one dimension and Lemma A2 for $d = 2, 3$ of [23] under our Assumption 2. Hence let us define

$$M := \max M_k,\ 1 \le k \le N, \tag{1.11}$$

where M_k are given by (1.9) and (1.10). We have the following statement.

Theorem 3. *Let $\Omega = \mathbb{R}^d,\ d = 1, 2, 3,$ Assumptions 1 and 2 hold and $\sqrt{2}(2\pi)^{\frac{d}{2}} Ml < 1$.*

Then the map $T_a v = u$ on $H^2(\mathbb{R}^d,\ \mathbb{R}^N)$ defined by the system of equations (1.5) has a unique fixed point $v_a(x) : \mathbb{R}^d \to \mathbb{R}^N$, which is the only stationary solution of problem (1.2) in $H^2(\mathbb{R}^d,\ \mathbb{R}^N)$. This fixed point $v_a(x)$ is nontrivial provided the intersection of supports of the Fourier transforms of functions $supp\widehat{F_k(0,x)}(p) \cap supp\widehat{G_k}(p)$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$ for some $1 \le k \le N$.

Then we turn our attention to the studies of the analogical system of equations on the interval $\Omega = I := [0,\ 2\pi]$ with periodic boundary conditions for the solution vector function and its first derivative. Let us assume the following about the integral kernels present in the nonlocal parts of problem (1.5) in this case.

Assumption 4. *Let $G_k(x) : I \to \mathbb{R},\ G_k(x) \in L^1(I)$ with $G_k(0) = G_k(2\pi),\ 1 \le k \le N,$ where $N \ge 3$ and $1 \le m < q \le N-1,\ m, q \in \mathbb{N}$.*

I) Let $a_k > 0$ and $a_k \ne n^2,\ n \in \mathbb{Z}$ for $1 \le k \le m$.

II) Let $a_k = n_k^2,\ n_k \in \mathbb{N}$ and

$$\left(G_k(x), \frac{e^{\pm i n_k x}}{\sqrt{2\pi}} \right)_{L^2(I)} = 0 \text{ for } m+1 \le k \le q. \tag{1.12}$$

III) Let $a_k = 0$ *and*

$$(G_k(x), 1)_{L^2(I)} = 0 \ for \ q+1 \le k \le N. \tag{1.13}$$

Let $F_k(u, 0) = F_k(u, 2\pi)$ *for* $u \in \mathbb{R}^N$ *and* $k = 1, ..., N$.

Let us introduce the Fourier transform for functions on the $[0, 2\pi]$ interval as

$$G_{k,\ n} := \int_0^{2\pi} G_k(x) \frac{e^{-inx}}{\sqrt{2\pi}} dx, \ \ n \in \mathbb{Z}$$

and define the following expressions

$$P_k := \max\left\{ \left\| \frac{G_{k,\ n}}{n^2 - a_k} \right\|_{l^\infty}, \left\| \frac{n^2 G_{k,\ n}}{n^2 - a_k} \right\|_{l^\infty} \right\}, \ \ 1 \le k \le m. \tag{1.14}$$

$$P_k := \max\left\{ \left\| \frac{G_{k,\ n}}{n^2 - n_k^2} \right\|_{l^\infty}, \left\| \frac{n^2 G_{k,\ n}}{n^2 - n_k^2} \right\|_{l^\infty} \right\}, \ \ m+1 \le k \le q. \tag{1.15}$$

$$P_k := \max\left\{ \left\| \frac{G_{k,\ n}}{n^2} \right\|_{l^\infty}, \left\| G_{k,\ n} \right\|_{l^\infty} \right\}, \ \ q+1 \le k \le N. \tag{1.16}$$

By means of Lemma A3 of [23] under Assumption 4 the quantities given by (1.14), (1.15) and (1.16) are finite, which enables us to define

$$P := \max P_k, \ \ 1 \le k \le N,$$

where P_k are stated in formulas (1.14), (1.15) and (1.16). For the studies of the existence of stationary solutions of our problem we use the corresponding functional space

$$H^2(I) = \{v(x) : I \to \mathbb{R} \mid v(x), v''(x) \in L^2(I), \quad v(0) = v(2\pi), \quad v'(0) = v'(2\pi)\},$$

aiming at $u_k(x) \in H^2(I), \ 1 \le k \le m$. Then we introduce the following auxiliary constrained subspaces

$$H_k^2(I) := \left\{ v \in H^2(I) \mid \left(v(x), \frac{e^{\pm i n_k x}}{\sqrt{2\pi}} \right)_{L^2(I)} = 0 \right\}, \ \ n_k \in \mathbb{N}, \ m+1 \le k \le q,$$

with the goal of having $u_k(x) \in H_k^2(I), \ m+1 \le k \le q$. And, finally

$$H_0^2(I) = \{v \in H^2(I) \mid (v(x), 1)_{L^2(I)} = 0\}, \ \ q+1 \le k \le N.$$

Our goal is to have $u_k(x) \in H_0^2(I), \ q+1 \le k \le N$. The constrained subspaces defined above are Hilbert spaces as well (see e.g. Chapter 2.1 of [10]). The resulting space used for establishing the existence of solutions $u(x) : I \to \mathbb{R}^N$ of problem (1.5) will be the direct sum of the spaces mentioned above, namely

$$H_c^2(I, \ \mathbb{R}^N) := \oplus_{k=1}^m H^2(I) \oplus_{k=m+1}^q H_k^2(I) \oplus_{k=q+1}^N H_0^2(I),$$

such that the corresponding Sobolev norm is given by

$$\|u\|^2_{H^2_c(I,\ \mathbb{R}^N)} := \sum_{k=1}^{N} \{\|u_k\|^2_{L^2(I)} + \|u''_k\|^2_{L^2(I)}\},$$

where $u(x): I \to \mathbb{R}^N$. We show that the system of equations (1.5) in this case defines a map on the space mentioned above, which will be a strict contraction under given assumptions.

Theorem 5. *Let $\Omega = I$, Assumptions 1 and 4 hold and $2\sqrt{\pi}Pl < 1$.*
Then the map $\tau_a v = u$ on $H^2_c(I, \mathbb{R}^N)$ defined by the system of equations (1.5) possesses a unique fixed point $v_a(x): I \to \mathbb{R}^N$, the only stationary solution of problem (1.2) in $H^2_c(I, \mathbb{R}^N)$. This fixed point $v_a(x)$ is nontrivial provided the Fourier coefficients $G_{k,\ n}F_k(0,x)_n \neq 0$ for some $k = 1, ..., N$ and some $n \in \mathbb{Z}$.

Note that the constrained subspaces $H^2_k(I)$ and $H^2_0(I)$ involved in the direct sum of spaces $H^2_c(I, \mathbb{R}^N)$ are such that the operators

$$-\frac{d^2}{dx^2} - n_k^2 : H^2_k(I) \to L^2(I) \ \textit{and} \ -\frac{d^2}{dx^2} : H^2_0(I) \to L^2(I)$$

having the Fredholm property, possess trivial kernels.

Finally, we turn our attention to the studies of our system of equations in the layer domain, which is the product of the two spaces, such that one is the I interval with periodic boundary conditions as in the previous part of the article and another is the whole space of dimension either one or two, namely $\Omega = I \times \mathbb{R}^d = [0, 2\pi] \times \mathbb{R}^d$, $d = 1, 2$ and $x = (x_1, x_\perp)$, where $x_1 \in I$ and $x_\perp \in \mathbb{R}^d$. The total Laplace operator in this context will be $\Delta := \frac{\partial^2}{\partial x_1^2} + \sum_{s=1}^{d} \frac{\partial^2}{\partial x_{\perp,\ s}^2}$. The corresponding Sobolev space for our problem will be $H^2(\Omega,\ \mathbb{R}^N)$ of vector functions $u(x): \Omega \to \mathbb{R}^N$, such that for $k = 1, ..., N$

$$u_k(x), \Delta u_k(x) \in L^2(\Omega),\ u_k(0, x_\perp) = u_k(2\pi, x_\perp),\ \frac{\partial u_k}{\partial x_1}(0, x_\perp) = \frac{\partial u_k}{\partial x_1}(2\pi, x_\perp),$$

where $x_\perp \in \mathbb{R}^d$ *a.e.* It is equipped with the norm

$$\|u\|^2_{H^2(\Omega,\ \mathbb{R}^N)} = \sum_{k=1}^{N} \{\|u_k\|^2_{L^2(\Omega)} + \|\Delta u_k\|^2_{L^2(\Omega)}\}.$$

Analogously to the whole space case discussed in Theorem 3, the operators $-\Delta - a_k$: $H^2(\Omega) \to L^2(\Omega)$ for $a_k \geq 0$ do not have the Fredholm property. We prove that problem (1.5) in this case defines a map $t_a : H^2(\Omega,\ \mathbb{R}^N) \to H^2(\Omega,\ \mathbb{R}^N)$, which is a strict contraction under the appropriate technical assumptions given below.

Assumption 6. *Let* $G_k(x) : \Omega \to \mathbb{R}$, $G_k(x) \in L^1(\Omega)$, $G_k(0, x_\perp) = G_k(2\pi, x_\perp)$ *and* $F_k(u, 0, x_\perp) = F_k(u, 2\pi, x_\perp)$ *for* $x_\perp \in \mathbb{R}^d$ *a.e.,* $u \in \mathbb{R}^N$, $d = 1, 2$ *and* $k = 1, ..., N$. *Let* $N \geq 3$ *and* $1 \leq m < q \leq N - 1$ *with* $m, q \in \mathbb{N}$.

I) Assume for $1 \leq k \leq m$ *we have* $n_k^2 < a_k < (n_k + 1)^2$, $n_k \in \mathbb{Z}^+ = \mathbb{N} \cup \{0\}$, $x_\perp G_k(x) \in L^1(\Omega)$ *and*

$$\left(G_k(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{\pm i\sqrt{a_k - n^2}x_\perp}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \ |n| \leq n_k \ for \ d = 1, \tag{1.17}$$

$$\left(G_k(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{ipx_\perp}}{2\pi}\right)_{L^2(\Omega)} = 0, \ p \in S^2_{\sqrt{a_k - n^2}} \ a.e., \ |n| \leq n_k \ for \ d = 2. \tag{1.18}$$

II) Assume for $m + 1 \leq k \leq q$ *we have* $a_k = n_k^2$, $n_k \in \mathbb{N}$, $x_\perp^2 G_k(x) \in L^1(\Omega)$ *and*

$$\left(G_k(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{\pm i\sqrt{n_k^2 - n^2}x_\perp}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \ |n| \leq n_k - 1 \ for \ d = 1, \tag{1.19}$$

$$\left(G_k(x_1, x_\perp), \frac{e^{inx_1}}{\sqrt{2\pi}} \frac{e^{ipx_\perp}}{2\pi}\right)_{L^2(\Omega)} = 0, \ p \in S^2_{\sqrt{n_k^2 - n^2}} \ a.e., \ |n| \leq n_k - 1 \ for \ d = 2, \tag{1.20}$$

$$\left(G_k(x_1, x_\perp), \frac{e^{\pm in_k x_1}}{\sqrt{2\pi}}\right)_{L^2(\Omega)} = 0, \ \left(G_k(x_1, x_\perp), \frac{e^{\pm in_k x_1}}{\sqrt{2\pi}} x_{\perp, \ s}\right)_{L^2(\Omega)} = 0, \ 1 \leq s \leq d. \tag{1.21}$$

III) Assume for $q + 1 \leq k \leq N$ *we have* $a_k = 0$, $x_\perp^2 G_k(x) \in L^1(\Omega)$ *and*

$$(G_k(x), 1)_{L^2(\Omega)} = 0, \ (G_k(x), x_{\perp, \ s})_{L^2(\Omega)} = 0, \ 1 \leq s \leq d. \tag{1.22}$$

We will use the Fourier transform for functions on such a product of spaces, such that

$$\widehat{G}_{k, \ n}(p) := \frac{1}{(2\pi)^{\frac{d+1}{2}}} \int_{\mathbb{R}^d} dx_\perp e^{-ipx_\perp} \int_0^{2\pi} G_k(x_1, x_\perp) e^{-inx_1} dx_1, \ p \in \mathbb{R}^d, \ n \in \mathbb{Z}, \ k = 1, ..., N.$$

Hence

$$\|\widehat{G}_{k, \ n}(p)\|_{L^\infty_{n,p}} := \sup_{\{p \in \mathbb{R}^d, \ n \in \mathbb{Z}\}} |\widehat{G}_{k, \ n}(p)| \leq \frac{1}{(2\pi)^{\frac{d+1}{2}}} \|G_k\|_{L^1(\Omega)}.$$

Let us introduce the following quantities

$$R_k := \max\left\{\left\|\frac{\widehat{G}_{k, \ n}(p)}{p^2 + n^2 - a_k}\right\|_{L^\infty_{n,p}}, \ \left\|\frac{(p^2 + n^2)\widehat{G}_{k, \ n}(p)}{p^2 + n^2 - a_k}\right\|_{L^\infty_{n,p}}\right\}, \ k = 1, ..., m. \tag{1.23}$$

$$R_k := \max\left\{\left\|\frac{\widehat{G}_{k, \ n}(p)}{p^2 + n^2 - n_k^2}\right\|_{L^\infty_{n,p}}, \ \left\|\frac{(p^2 + n^2)\widehat{G}_{k, \ n}(p)}{p^2 + n^2 - n_k^2}\right\|_{L^\infty_{n,p}}\right\}, \ k = m + 1, ..., q. \tag{1.24}$$

$$R_k := \max\left\{\left\|\frac{\widehat{G}_{k, \ n}(p)}{p^2 + n^2}\right\|_{L^\infty_{n,p}}, \ \left\|\widehat{G}_{k, \ n}(p)\right\|_{L^\infty_{n,p}}\right\}, \ k = q + 1, ..., N. \tag{1.25}$$

Assumption 6 along with Lemmas A4, A5 and A6 of [23] imply that the expressions given by (1.23), (1.24) and (1.25) are finite. Hence we can define

$$R := \max R_k, \ k = 1, ..., N,$$

where R_k are given in (1.23), (1.24) and (1.25). Our final statement is as follows.

Theorem 7. *Let* $\Omega = I \times \mathbb{R}^d, \ d = 1, 2,$ *Assumptions 1 and 6 hold and* $\sqrt{2}(2\pi)^{\frac{d+1}{2}} Rl < 1$. *Then the map* $t_a v = u$ *on* $H^2(\Omega, \ \mathbb{R}^N)$, *which is defined by the system of equations (1.5) possesses a unique fixed point* $v_a(x) : \Omega \to \mathbb{R}^N$, *which is the only stationary solution of system (1.2) in* $H^2(\Omega, \ \mathbb{R}^N)$. *This fixed point* $v_a(x)$ *is nontrivial provided that the intersection of supports of the Fourier images of functions* $supp\widehat{F_k(0, x)}_n(p) \cap supp\widehat{G}_{k, \ n}(p)$ *is a set of nonzero Lebesgue measure in* $\mathbb{R}^d$ *for some* $k = 1, ..., N$ *and some* $n \in \mathbb{Z}$.

Note that the maps considered in the theorems above are applied to real valued vector functions by means of the assumptions on $F_k(u, x)$ and $G_k(x), \ k = 1, ..., N$ present in the nonlocal terms of (1.5).

2 The Problem in the Whole Space

Proof of Theorem 3. First we suppose that in the case of $\Omega = \mathbb{R}^d, \ d = 1, 2, 3$ there exists $v(x) \in H^2(\mathbb{R}^d, \ \mathbb{R}^N)$ such that problem (1.5) possesses two solutions $u^{(1),(2)}(x) \in H^2(\mathbb{R}^d, \ \mathbb{R}^N)$. Hence the difference vector function $w(x) := u^{(1)}(x) - u^{(2)}(x) \in H^2(\mathbb{R}^d, \ \mathbb{R}^N)$ solves the homogeneous system of equations

$$-\Delta w_k = a_k w_k, \ 1 \le k \le N.$$

Since the negative Laplacian does not possess any nontrivial eigenfunctions belonging to $L^2(\mathbb{R}^d)$, we have $w_k(x) = 0$ a.e. in $\mathbb{R}^d$ for $k = 1, ..., N$.

Consider an arbitrary vector function $v(x) \in H^2(\mathbb{R}^d, \ \mathbb{R}^N)$ and apply the standard Fourier transform to both sides of system (1.5). This yields

$$\widehat{u_k}(p) = (2\pi)^{\frac{d}{2}} \frac{\widehat{G_k}(p)\widehat{f_k}(p)}{p^2 - a_k}, \ k = 1, ..., N. \tag{2.26}$$

Here $\widehat{f_k}(p)$ stands for the Fourier image of $F_k(v(x), x)$. We have the trivial estimates using expressions (1.9) and (1.10)

$$|\widehat{u_k}(p)| \le (2\pi)^{\frac{d}{2}} M_k |\widehat{f_k}(p)| \quad \textit{and} \quad |p^2 \widehat{u_k}(p)| \le (2\pi)^{\frac{d}{2}} M_k |\widehat{f_k}(p)|, \ k = 1, ..., N.$$

This implies the upper bound for the norm

$$\|u\|^2_{H^2(\mathbb{R}^d, \ \mathbb{R}^N)} \le 2(2\pi)^d \sum_{k=1}^N M_k^2 \|F_k(v(x), x)\|^2_{L^2(\mathbb{R}^d)} < \infty$$

due to inequality (1.3) of Assumption 1. Hence for any $v(x) \in H^2(\mathbb{R}^d,\ \mathbb{R}^N)$ there exists a unique vector function $u(x) \in H^2(\mathbb{R}^d,\ \mathbb{R}^N)$, which solves problem (1.5) and its Fourier image is given by (2.26), such that the map $T_a : H^2(\mathbb{R}^d,\ \mathbb{R}^N) \to H^2(\mathbb{R}^d,\ \mathbb{R}^N)$ is well defined.

Therefore, we can choose arbitrary $v^{(1),(2)}(x) \in H^2(\mathbb{R}^d,\ \mathbb{R}^N)$ and obtain their images under the map $u^{(1),(2)} = T_a v^{(1),(2)} \in H^2(\mathbb{R}^d,\ \mathbb{R}^N)$ and arrive easily at the bounds for $k = 1, ..., N$

$$\left|\widehat{u_k^{(1)}}(p) - \widehat{u_k^{(2)}}(p)\right| \leq (2\pi)^{\frac{d}{2}} M \left|\widehat{f_k^{(1)}}(p) - \widehat{f_k^{(2)}}(p)\right|,$$

$$\left|p^2\widehat{u_k^{(1)}}(p) - p^2\widehat{u_k^{(2)}}(p)\right| \leq (2\pi)^{\frac{d}{2}} M \left|\widehat{f_k^{(1)}}(p) - \widehat{f_k^{(2)}}(p)\right|.$$

Here $\widehat{f_k^{(1),(2)}}(p)$ denote the Fourier transforms of $F_k(v^{(1),(2)}(x), x)$. This implies the bound on the corresponding norm of the difference of vector functions

$$\|u^{(1)} - u^{(2)}\|^2_{H^2(\mathbb{R}^d,\ \mathbb{R}^N)} \leq 2(2\pi)^d M^2 \sum_{k=1}^{N} \|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|^2_{L^2(\mathbb{R}^d)}.$$

Due to the Sobolev embedding theorem for $k = 1, ..., N$ we have $v_k^{(1),(2)}(x) \in H^2(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d),\ 1 \leq d \leq 3$. Inequality (1.4) easily implies

$$\|T_a v^{(1)} - T_a v^{(2)}\|_{H^2(\mathbb{R}^d,\ \mathbb{R}^N)} \leq \sqrt{2}(2\pi)^{\frac{d}{2}} M l \|v^{(1)} - v^{(2)}\|_{H^2(\mathbb{R}^d,\ \mathbb{R}^N)}.$$

The constant in the right side of this estimate is less than one according to the assumption of the theorem. Hence, the Fixed Point Theorem yields the existence of a unique vector function $v_a(x) \in H^2(\mathbb{R}^d, \mathbb{R}^N)$, such that $T_a v_a = v_a$. This is the only stationary solution of problem (1.2) in $H^2(\mathbb{R}^d,\ \mathbb{R}^N)$. Finally, let us suppose that $v_a(x) = 0$ a.e. in $\mathbb{R}^d$. This will imply the contradiction to our assumption that for some $k = 1, ..., N$ the Fourier transforms of $G_k(x)$ and $F_k(0, x)$ do not vanish simultaneously on some set of nonzero Lebesgue measure in $\mathbb{R}^d$. ■

3 The Problem on the $[0, 2\pi]$ Interval

Proof of Theorem 5. Let us first suppose that for $v(x) \in H_c^2(I, \mathbb{R}^N)$ there exist two solutions $u^{(1),(2)}(x) \in H_c^2(I, \mathbb{R}^N)$ of problem (1.5) with $\Omega = I$. Then the difference vector function $w(x) := u^{(1)}(x) - u^{(2)}(x) \in H_c^2(I, \mathbb{R}^N)$ will satisfy the equation

$$-w_k'' = a_k w_k,\ k = 1, ..., N.$$

But according to Assumption 4, we have $a_k \neq n^2,\ n \in \mathbb{Z}$ when $k = 1, ..., m$ and consequently, they are not the eigenvalues of the operator $-\dfrac{d^2}{dx^2}$ on $L^2(I)$ with periodic boundary conditions. Hence, $w_k(x)$ vanishes a.e. in I when $k = 1, ..., m$. For $k = m+1, ..., q$ the values of

a_k coincide with the eigenvalues of the negative Laplacian operator with periodic boundary conditions on the $[0, 2\pi]$ interval but w_k belong to the constraned subspaces $H_k^2(I)$. Thus, $w_k = 0$ a.e. in I for $k = m+1, ..., q$ due to their orthogonality to the eigenfunctions $\frac{e^{\pm in_k x}}{\sqrt{2\pi}}$. By means of Assumption 4 the constants a_k vanish when $k = q+1, ..., N$. But w_k belong to the constrained subspace $H_0^2(I)$ of functions orthogonal to the zero mode of $-\frac{d^2}{dx^2}$ on $L^2(I)$ with periodic boundary conditions. Thus, $w_k(x)$ vanishes a.e. in I when $k = q+1, ..., N$ as well.

Let us assume that $v(x) \in H_c^2(I, \mathbb{R}^N)$ is arbitrary. We apply the Fourier transform to both sides of the system of equations (1.5) considered on the interval $[0, 2\pi]$ and obtain

$$u_{k,\ n} = \sqrt{2\pi}\frac{G_{k,\ n}f_{k,\ n}}{n^2 - a_k}, \quad n \in \mathbb{Z}, \tag{3.1}$$

where $f_{k,\ n} := F_k(v(x), x)_n$. Evidently, the Fourier coefficients of the second derivatives are given by

$$(-u_k'')_n = \sqrt{2\pi}\frac{n^2 G_{k,\ n}f_{k,\ n}}{n^2 - a_k}, \quad n \in \mathbb{Z}.$$

We easily arrive at the upper bound

$$\|u\|^2_{H_c^2(I,\ \mathbb{R}^N)} = \sum_{k=1}^N \left\{ \sum_{n=-\infty}^{\infty} |u_{k,\ n}|^2 + \sum_{n=-\infty}^{\infty} |n^2 u_{k,\ n}|^2 \right\} \le 4\pi \sum_{k=1}^N P_k^2 \|F_k(v(x), x)\|^2_{L^2(I)} < \infty,$$

which follows from inequality (1.3) of Assumption 1. Therefore, for an arbitrarily chosen vector function $v(x) \in H_c^2(I,\ \mathbb{R}^N)$ there exists a unique $u(x) \in H_c^2(I,\ \mathbb{R}^N)$, which solves the system of equations (1.5) and its Fourier coefficients are given by formula (3.1), such that the map $\tau_a : H_c^2(I,\ \mathbb{R}^N) \to H_c^2(I,\ \mathbb{R}^N)$ is well defined. Note that orthogonality conditions (1.12) and (1.13) along with (3.1) imply that for $k = m+1, ..., q$ components $u_k(x)$ are orthogonal to Fourier harmonics $\frac{e^{\pm in_k x}}{\sqrt{2\pi}}$ in $L^2(I)$ and for $k = q+1, ..., N$ functions $u_k(x)$ are orthogonal to 1 in $L^2(I)$, since the corresponding Fourier coeffients can be made equal to zero.

Then we consider arbitary vector functions $v^{(1),(2)}(x) \in H_c^2(I,\ \mathbb{R}^N)$, such that their images under the map defined above are $u^{(1),(2)} = \tau_a v^{(1),(2)} \in H_c^2(I,\ \mathbb{R}^N)$ and obtain easily the estimate

$$\|u^{(1)} - u^{(2)}\|^2_{H_c^2(I,\ \mathbb{R}^N)} = \sum_{k=1}^N \left\{ \sum_{n=-\infty}^{\infty} |u^{(1)}{}_{k,\ n} - u^{(2)}{}_{k,\ n}|^2 + \sum_{n=-\infty}^{\infty} |n^2(u^{(1)}{}_{k,\ n} - u^{(2)}{}_{k,\ n})|^2 \right\} \le$$

$$\le 4\pi \sum_{k=1}^N P_k^2 \|F_k(v^{(1)}(x), x) - F_k(v^{(2)}(x), x)\|^2_{L^2(I)}.$$

Clearly, via the Sobolev embedding theorem $v_k^{(1),(2)}(x) \in H^2(I) \subset L^\infty(I)$ for $k = 1, ..., N$. Using (1.4) we easily arrive at

$$\|\tau_a v^{(1)} - \tau_a v^{(2)}\|_{H_c^2(I,\ \mathbb{R}^N)} \leq 2\sqrt{\pi} Pl \|v^{(1)} - v^{(2)}\|_{H_c^2(I,\ \mathbb{R}^N)}.$$

The constant in the right side of this inequality is less than one by the assumption of the theorem. Hence, the Fixed Point Theorem yields the existence and uniqueness of a vector function $v_a(x) \in H_c^2(I,\ \mathbb{R}^N)$, which satisfies $\tau_a v_a = v_a$. This is the only stationary solution of the system of equations (1.2) in $H_c^2(I,\ \mathbb{R}^N)$. Finally, let us suppose that $v_a(x)$ vanishes a.e. in the interval I. This will imply the contradiction to our assumption that the Fourier coefficients $G_{k,\ n} F_k(0, x)_n \neq 0$ for some $k = 1, ..., N$ and some $n \in \mathbb{Z}$. ∎

4 The Problem in the Layer Domain

Proof of Theorem 7. Let us suppose that there exists $v(x) \in H^2(\Omega,\ \mathbb{R}^N)$ generating $u^{(1),(2)}(x) \in H^2(\Omega,\ \mathbb{R}^N)$, which solve system (1.5). Then the difference of these vector functions $w(x) := u^{(1)}(x) - u^{(2)}(x) \in H^2(\Omega,\ \mathbb{R}^N)$ will satisfy the homogeneous system of equations

$$-\Delta w_k = a_k w_k,\ \ k = 1, ..., N.$$

Let us apply the partial Fourier transform to this system, which yields

$$-\Delta_\perp w_{k,\ n}(x_\perp) = (a_k - n^2) w_{k,\ n}(x_\perp),\ \ k = 1, ..., N,\ \ n \in \mathbb{Z},$$

where $w_{k,\ n}(x_\perp) := \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^{2\pi} w_k(x_1, x_\perp) e^{-inx_1} dx_1$. Evidently

$$\|w_k\|^2_{L^2(\Omega)} = \sum_{n=-\infty}^{\infty} \|w_{k,\ n}\|^2_{L^2(\mathbb{R}^d)}.$$

Hence $w_{k,\ n}(x_\perp) \in L^2(\mathbb{R}^d),\ k = 1, ..., N,\ n \in \mathbb{Z}$. But the negative transversal Laplacian operator $-\Delta_\perp$ considered on $L^2(\mathbb{R}^d)$ does not possess any nontrivial square integrable eigenfunctions. Therefore, $w(x) = 0$ a.e. in Ω.

Consider an arbitrary vector function $v(x) \in H^2(\Omega,\ \mathbb{R}^N)$ and apply the Fourier transform to both sides of the system of equations (1.5). We arrive at

$$\widehat{u}_{k,\ n}(p) = (2\pi)^{\frac{d+1}{2}} \frac{\widehat{G}_{k,\ n}(p) \widehat{f}_{k,\ n}(p)}{p^2 + n^2 - a_k},\ \ k = 1, ..., N, \quad n \in \mathbb{Z}, \quad p \in \mathbb{R}^d, \quad d = 1, 2, \tag{4.1}$$

where $\widehat{f}_{k,\ n}(p)$ denotes the Fourier image of $F_k(v(x), x)$. Evidently, for the above mentioned values of $k,\ n$ and p we have the bounds in terms of the quantities given by (1.23), (1.24) and (1.25) as

$$|\widehat{u}_{k,\ n}(p)| \leq (2\pi)^{\frac{d+1}{2}} R_k |\widehat{f}_{k,\ n}(p)| \quad \textit{and} \quad |(p^2 + n^2)\widehat{u}_{k,\ n}(p)| \leq (2\pi)^{\frac{d+1}{2}} R_k |\widehat{f}_{k,\ n}(p)|.$$

Hence using (1.3) of Assumption 1 we obtain

$$\|u\|^2_{H^2(\Omega,\ \mathbb{R}^N)} = \sum_{k=1}^{N}\left\{\sum_{n=-\infty}^{\infty}\int_{\mathbb{R}^d}|\widehat{u}_{k,\ n}(p)|^2dp + \sum_{n=-\infty}^{\infty}\int_{\mathbb{R}^d}|(p^2+n^2)\widehat{u}_{k,\ n}(p)|^2dp\right\} \leq$$

$$\leq 2(2\pi)^{d+1}\sum_{k=1}^{N}{R_k}^2\|F_k(v(x),x)\|^2_{L^2(\Omega)} < \infty.$$

Therefore, for any vector function $v(x) \in H^2(\Omega,\ \mathbb{R}^N)$ there exists a unique $u(x) \in H^2(\Omega,\ \mathbb{R}^N)$ which satisfies the system of equations (1.5) and its Fourier transform is given by formula (4.1). Thus, the map $t_a : H^2(\Omega,\ \mathbb{R}^N) \to H^2(\Omega,\ \mathbb{R}^N)$ is well defined.

Let us consider two arbitrary functions $v^{(1),(2)} \in H^2(\Omega,\ \mathbb{R}^N)$ such that their images under the map discussed above are $u^{(1),(2)} = t_a v^{(1),(2)} \in H^2(\Omega,\ \mathbb{R}^N)$. Thus

$$\|u^{(1)} - u^{(2)}\|^2_{H^2(\Omega,\ \mathbb{R}^N)} = \sum_{k=1}^{N}\sum_{n=-\infty}^{\infty}\int_{\mathbb{R}^d}dp\Big\{|\widehat{u^{(1)}}_{k,\ n}(p) - \widehat{u^{(2)}}_{k,\ n}(p)|^2 +$$

$$+|(p^2+n^2)(\widehat{u^{(1)}}_{k,\ n}(p) - \widehat{u^{(2)}}_{k,\ n}(p))|^2\Big\} \leq 2(2\pi)^{d+1}R^2\sum_{k=1}^{N}\|F_k(v^{(1)}(x),x) - F_k(v^{(2)}(x),x)\|^2_{L^2(\Omega)}.$$

Evidently, due to the Sobolev embedding theorem $v_k^{(1),(2)}(x) \in H^2(\Omega) \subset L^\infty(\Omega)$ for $k = 1, ..., N$. By means of (1.4) we easily obtain the inequality

$$\|t_a v^{(1)} - t_a v^{(2)}\|_{H^2(\Omega,\ \mathbb{R}^N)} \leq \sqrt{2}(2\pi)^{\frac{d+1}{2}}Rl\|v^{(1)} - v^{(2)}\|_{H^2(\Omega,\ \mathbb{R}^N)},$$

such that the constant in its right side is less than one according to our assumption. Hence, the Fixed Point Theorem implies the existence and uniqueness of a function $v_a(x) \in H^2(\Omega,\ \mathbb{R}^N)$, for which $t_a v_a = v_a$ holds. This is the only stationary solution of the system of equations (1.2) in $H^2(\Omega,\ \mathbb{R}^N)$. Let us suppose that the function $v_a(x)$ vanishes a.e. in Ω. This will imply the contradiction to the assumption of the theorem that there exists $k = 1, ..., N$ and $n \in \mathbb{Z}$, such that $supp\widehat{F_k(0,x)}_n(p) \cap supp\widehat{G}_{k,\ n}(p)$ is a set of nonzero Lebesgue measure in $\mathbb{R}^d$. ■

References

[1] M.S Agranovich: *Elliptic boundary problems,* Encyclopaedia Math. Sci., vol 79, Partial Differential Equations, IX, Springer, Berlin, 1997, pp. 1–144

[2] N. Apreutesei, N. Bessonov, V. Volpert, V. Vougalter: *Spatial Structures and Generalized Travelling Waves for an Integro- Differential Equation,* Discrete Contin. Dyn. Syst. Ser. B **13, 3** (2010), 537–557

[3] M. Beck, A. Ghazaryan, B. Sandstede: *Nonlinear convective stability of travelling fronts near Turing and Hopf instabilities,* J. Differential Equations **246, 11** (2009), 4371–4390

[4] H. Berestycki, G. Nadin, B. Perthame, L. Ryzhik: *The non-local Fisher-KPP equation: traveling waves and steady states,* Nonlinearity **22, 12** (2009), 2813–2844

[5] A. Ducrot, M. Marion and V. Volpert: *Systemes de réaction-diffusion sans propriété de Fredholm,* CRAS, **340, 9** (2005), 659–664

[6] A. Ducrot, M. Marion and V. Volpert: *Reaction-diffusion problems with non Fredholm operators,* Advances Diff. Equations , **13, 11-12** (2008), 1151–1192

[7] A. Ducrot, M. Marion and V. Volpert. Reaction-diffusion waves (with the Lewis number different from 1). Publibook, Paris, 2009.

[8] S. Genieys, V. Volpert, P. Auger: *Pattern and waves for a model in population dynamics with nonlocal consumption of resources,* Math. Model. Nat. Phenom. **1, 1** (2006), 63–80

[9] A. Ghazaryan, B. Sandstede: *Nonlinear convective instability of Turing-unstable fronts near onset: a case study,* SIAM J. Appl. Dyn. Syst. **6, 2** (2007), 319–347

[10] P.D. Hislop, I.M. Sigal: *Introduction to spectral theory with applications to Schrödinger operators.* Springer, 1996

[11] J.L. Lions, E. Magenes: *Problemes aux limites non homogenes et applications.* Volume 1. Dunod, Paris, 1968.

[12] W. Shen, A. Zhang: *Spreading speeds for monostable equations with nonlocal dispersal in space periodic habitats,* J. Differential Equations, **249, 4** (2010), 747–795

[13] L.R. Volevich: *Solubility of boundary problems for general elliptic systems,* Mat. Sbor., **68** (1965), 373–416; English translation: Amer. Math. Soc. Transl., **67** (1968), Ser. 2, 182–225.

[14] V. Volpert: *Elliptic partial differential equations.* Volume I. Fredholm theory of elliptic problems in unbounded domains. Birkhäuser, 2011.

[15] V. Volpert, B. Kazmierczak, M. Massot, Z.Peradzynski: *Solvability conditions for elliptic problems with non-Fredholm operators,* Appl. Math., **29, 2** (2002), 219–238

[16] V. Vougalter, V. Volpert: *Solvability conditions for some non Fredholm operators,* Proc. Edinb. Math. Soc. (2), **54, 1** (2011), 249–271

[17] V. Vougalter, V. Volpert: *On the solvability conditions for some non Fredholm operators,* Int. J. Pure Appl. Math., **60, 2** (2010), 169–191

[18] V. Vougalter, V. Volpert: *On the solvability conditions for the diffusion equation with convection terms,* Commun. Pure Appl. Anal., **11, 1** (2012), 365–373

[19] V. Vougalter, V. Volpert: *Solvability relations for some non Fredholm operators,* Int. Electron. J. Pure Appl.Math., **2, 1** (2010), 75–83

[20] V. Volpert, V. Vougalter: *On the solvability conditions for a linearized Cahn-Hilliard equation,* Rend. Istit. Mat. Univ. Trieste, **43** (2011), 1–9

[21] V. Vougalter, V. Volpert: *Solvability conditions for some systems with non Fredholm operators,* Int. Electron. J. Pure Appl.Math., **2, 3** (2010), 183–187

[22] V. Vougalter, V. Volpert: *Solvability conditions for a linearized Cahn-Hilliard equation of sixth order,* Math. Model. Nat. Phenom., **7, 2** (2012), 146–154

[23] V. Vougalter, V. Volpert: *On the existence of stationary solutions for some non-Fredholm integro-differential equations,* Doc. Math., **16** (2011), 561–580

[24] V. Volpert, V. Vougalter: *Emergence and propagation of patterns in nonlocal reaction-diffusion equations arising in the theory of speciation.* Preprint 2011.

[25] V. Vougalter, V. Volpert: *Solvability conditions for some linear and nonlinear non-Fredholm elliptic problems.* To appear in Anal. Math. Phys.