Type transition of simple random walks on randomly directed regular lattices[1]

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Abstract: Simple random walks on a partially directed version of $\mathbb{Z}^2$ are considered. More precisely, vertical edges between neighbouring vertices of $\mathbb{Z}^2$ can be traversed in both directions (they are undirected) while horizontal edges are one-way. The horizontal orientation is prescribed by a random perturbation of a periodic function, the perturbation probability decays according to a power law in the absolute value of the ordinate. We study the type of the simple random walk, i.e. its being recurrent or transient, and show that there exists a critical value of the decay power, above which the walk is almost surely recurrent and below which is almost surely transient.

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1 Introduction

1.1 Motivations

We term, as usual, simple random walk on a connected (finitely or infinitely) denumerable graph the vertex-valued Markov chain jumping from vertex v to any vertex v', adjacent to v, with uniform probability on the set of neighbours. We say the lattice is undirected when the adjacency matrix of the graph is symmetric. Simple random walks on connected and undirected graphs are irreducible Markov chains; therefore the probability that such a walk visits any particular vertex is strictly positive. There is a closely interplay between the combinatorial exploration of the graph and the asymptotic behaviour of the random walk.

Although general graphs are merely one-dimensional simplicial complexes, regular undirected graphs are very often interpreted as the Cayley graphs of finitely generated groups Γ. Among them the simplest examples are provided by the family of d-dimensional lattices (Abelian groups) $\mathbb{Z}^d$, for some d; they admit the presentation $\langle S \rangle$, where $S = \{e_1, -e_1, \ldots, e_d, -e_d\}$ is the *symmetric* set composed from the standard basis of $\mathbb{R}^d$ and their inverses, i.e. a finite set of generators of $\mathbb{Z}^d$. Simple random walks on $\mathbb{Z}^d$, for $d = 1, 2$, or 3, were introduced and studied by Pólya [19]; in that seminal paper, he solves the *type problem* of the simple random walk on $\mathbb{Z}^d$. Namely he shows that the walk is recurrent (returns almost surely infinitely often to its starting point) in $d \leq 2$ and is transient (returns almost surely only a finite number of times to its starting point) in $d = 3$ (and later shown for all $d \geq 3$). The connection of undirected graphs with Cayley graphs of groups has been extended to non-commutative groups, leading to a theory of random walks interconnected with algebraic and geometric properties of the underlying groups and their amenability properties. Properties such as the rate of growth of the size of balls in the underlying group determine the type of the random walk.

Another characteristic of simple random walks on undirected graphs is their reversibility. Roughly, reversibility means that observing the evolution of the Markov chain in the normal flow of time is statistically indistinguishable from the evolution with reverted arrow of time flow. Reversibility is closely connected with the existence of an invariant measure (not necessarily a probability) verifying the condition of detailed balance[2] on the set of ver-

[2]In most textbooks, reversibility is connected with the existence of an invariant prob-

tices and with the possibility of establishing a close analogy (a bijection as a matter of fact) between all probabilistic quantities pertinent to the random walk and corresponding currents and voltages on a network of resistors having the same adjacency matrix as the graph and whose edge conductance is determined by the stochastic matrix and the invariant measure (see [12] and the monograph [8] for a more pedagogical review of the topic). Existence of current flows of finite energy induced by a unit voltage difference between a vertex and infinity is equivalent to a random walk of transient type. In that way, random walks become interconnected with harmonic analysis and potential theory.

Finally, another interesting feature of undirected graphs is the spectrum of the discrete Laplacian; isoperimetric inequalities and Cheeger's bound provide lower bounds on the spectrum of the Laplacian leading to criteria of transience of the random walk [7].

All the aforementioned techniques fail when the underlying graph is directed (the corresponding simple random walk can never be a reversible Markov chain). Although random walks on partially directed lattices have been introduced long-time ago to study the hydrodynamic dispersion of a tracer particle in a porous medium [15] very little was known on them beyond some computer simulation heuristics [21] and estimates of the persistence of random walkers studied in [13]. Therefore, it arose as a surprise for us that so little was rigourously known when we first considered simple random walks on partially directed 2-dimensional lattices in [2, 3]. In those papers, we determined the type of simple random walks on lattices obtained from $\mathbb{Z}^2$ by keeping vertical edges bi-directional while horizontal edges become one-way. Depending on how the horizontal allowed direction to the left or the right is determined we obtain dramatically different behaviour, namely:

- if the direction to the left or the right is chosen by the parity of the ordinate[3], then the random walk remains recurrent;

ability measure. We follow here the convention (of [24] or [9] for instance) consisting to use the term reversibility in the more general situation where the invariant measure is not necessarily of finite mass. The thus extended notion of reversibility is sometimes called *local reversibility* in the literature. As a matter of fact, if μ is an invariant measure (not necessarily a probability), then $c(x, y) = \mu(x)P(x, y)$ is called the conductance between x and y. Detailed balance $\mu(x)P(x, y) = \mu(y)P(x, y)$ implies that $c(x, y) = c(y, x)$ i.e. electric flow can be reverted locally as is the case in an electrical circuit with passive elements only. Finiteness of the total mass of μ is not necessary for this analogy to hold.

[3] This is precisely the model considered in [15].

- if the whole upper half plane is composed by eastward lines while the lower half-plane by westward lines, the random walk is transient;
- when the direction of the horizontal lines is chosen by tossing a honest coin, then the random walk is transient for almost every choice of the orientation.

This result triggered several developments by various authors. In [10], the choice of the orientation is made by means of a correlated sequence or by a dynamical system; in both cases, provided that some variance condition holds, almost sure transience is established and additionally a functional limit theorem is obtained. In [17], the case of orientations chosen according to a stationary sequence is treated. In [18], our results of [2, 3] are used to study corner percolation on $\mathbb{Z}^2$. In [4], the Martin boundary of these walks has been studied for the models that are transient and proved to be trivial, i.e. the only positive harmonic functions for the Markov kernel of these walks are the constants. In [6] a model where the horizontal directions are chosen according to an arbitrary (deterministic or random) sequence but the probability of performing a horizontal or vertical move is not determined by the degree but by a sequence of non-degenerate random variables is considered and shown to be a.s. transient.

It is worth noting that all the previous directed lattices are regular in the sense that both the inward and the outward degrees are constant (and equal to 3) all over the lattice. Therefore, the dramatic change of type is due only to the directedness. However, the type result was always either recurrent or transient. Not a single example was known where the type could be controlled by some continuous parameter so that a transition from recurrence to transience could be observed by fine tuning this parameter. The present paper provides such an example improving the insight we have on those non reversible random walks. Let us mention also that beyond their theoretical interest (a short list of problems remaining open in the context of such random walks is given in the conclusion section), directed random walks are much more natural models of propagation on large networks like internet than reversible ones. As a matter of fact, lattice directedness can be seen as discretisation of the notion of causality [16, 14]. Therefore, advances in the theoretical understanding will have numerous implications in applied domains.

1.2 Notation and definitions

A *directed* graph[4] $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1, r, s)$ is the quadruple of a denumerable set $\mathbb{G}^0$ of vertices, a denumerable set $\mathbb{G}^1$ of directed edges, and a pair of *range* and a *source* functions, denoted respectively r and s, i.e. mappings $r, s : \mathbb{G}^1 \to \mathbb{G}^0$. In the sequel, we only consider graphs without loops (i.e. not containing edges $\alpha \in \mathbb{G}^1$ such that $r(\alpha) = s(\alpha)$) and without multiple edges (i.e. if α and β are edges verifying simultaneously $s(\alpha) = s(\beta)$ and $r(\alpha) = r(\beta)$ then $\alpha = \beta$, in other words, the compound map $(s, r) : \mathbb{G}^1 \to \mathbb{G}^0 \times \mathbb{G}^0$ is injective). With these restrictions in force, $\mathbb{G}^1$ can be identified with a particular subset of $\mathbb{G}^0 \times \mathbb{G}^0$ and the functions r and s become superfluous because they are trivial i.e. $s((\mathbf{u}, \mathbf{v})) = \mathbf{u}$ and $r((\mathbf{u}, \mathbf{v})) = \mathbf{v}$). The corresponding directed graph is then termed *simple*. All the graphs we consider in this paper will be simple without explicitly stating so.

We can therefore define, for each vertex $\mathbf{v} \in \mathbb{G}^0$, its *inwards degree* $d^+_{\mathbf{v}} = \mathsf{card}\{a \in \mathbb{G}^1 : r(a) = \mathbf{v}\}$ and its *outwards degree* $d^-_{\mathbf{v}} = \mathsf{card}\{a \in \mathbb{G}^1 : s(a) = \mathbf{v}\}$. All the graphs we consider are *transitive* in the sense that for any two distinct vertices $\mathbf{u}, \mathbf{v} \in \mathbb{G}^0$, there is a *finite* sequence $\alpha = (\alpha_1, \ldots, \alpha_k)$ of *composable edges* $\alpha_i \in \mathbb{G}^1$, for $i = 1, \ldots, k$, $k \in \mathbb{N}$, with $s(\alpha_1) = \mathbf{u}$ and $r(\alpha_k) = \mathbf{v}$, such that $r(\alpha_i) = s(\alpha_{i+1}) \in \mathbb{G}^0, \forall i = 1, \ldots, k-1$. The above sequence α is called a path of length $k = |\alpha|$ from $\mathbf{u}$ to $\mathbf{v}$, the set of all paths of lenght k is denoted[5] by $\mathbb{G}^k$. Finite transitivity implies in particular the *no sink condition*: $d^-_{\mathbf{v}} \geq 1$ for all $\mathbf{v} \in \mathbb{G}^0$. We always consider graphs that are genuinely directed in the sense that there exist vertices $\mathbf{u}$ and $\mathbf{v}$ with $(\mathbf{u}, \mathbf{v}) \in \mathbb{G}^1$ but $(\mathbf{v}, \mathbf{u}) \notin \mathbb{G}^1$.

Definition 1.1. [Simple random walk on a directed graph] Let $\mathbb{G}$ be a directed graph. A *simple random walk* on $\mathbb{G}$ is a $\mathbb{G}^0$-valued Markov chain

[4] Although often used interchangeably in common language, *directedness* and *orientation* denote distinct notions in graph theory: directedness is a property encoded into the set $\mathbb{G}^1$ of allowed edges; orientation is an assignement of plus or minus sign to every edge (viewed as the set — not the ordered pair — of its endpoints). On defining a map $\iota : \mathbb{G}^0 \times \mathbb{G}^0 \to \mathbb{G}^0 \times \mathbb{G}^0$ by $\mathbb{G}^0 \times \mathbb{G}^0 \ni (\mathbf{u}, \mathbf{v}) \mapsto \iota((\mathbf{u}, \mathbf{v})) = (\mathbf{v}, \mathbf{u}) \in \mathbb{G}^0 \times \mathbb{G}^0$ (this map reverts the order of the pair), we observe that for an oriented but undirected graph, the image of $\mathbb{G}^1$ by ι can be identified with $\mathbb{G}^1$; for a directed graph, the image of $\mathbb{G}^1$ can contain elements in $\mathbb{G}^0 \times \mathbb{G}^0 \setminus \mathbb{G}^1$. In both cases ι is involutive. An undirected graph can be viewed as a directed one such that if $\alpha := (\mathbf{u}, \mathbf{v}) \in \mathbb{G}^1$ then $\iota(\alpha) = (\mathbf{v}, \mathbf{u}) \in \mathbb{G}^1$, i.e. the set of edges $\mathbb{G}^1$ is a *symmetric* subset of the Cartesian product $\mathbb{G}^0 \times \mathbb{G}^0$.

[5] Notice that $\mathbb{G}^k$ is the set of paths composed from k composable edges, in general *strictly* contained into the Cartesian product $\times_{l=1}^{k} \mathbb{G}^1$.

$(\mathbf{M}_n)_{n\in\mathbb{N}}$ with transition probability matrix $\mathbf{P}$ having as matrix elements

$$P(\mathbf{u},\mathbf{v}) = \mathbb{P}(\mathbf{M}_{n+1}=\mathbf{v}|\mathbf{M}_n=\mathbf{u}) = \begin{cases} \frac{1}{d_{\mathbf{u}}^-} & \text{if} \quad (\mathbf{u},\mathbf{v}) \in \mathbb{G}^1 \\ 0 & \text{otherwise.} \end{cases}$$

Remark: When the underlying graph is genuinely directed, the Markov chain $(\mathbf{M}_n)_{n\in\mathbb{N}}$ *cannot be reversible*. Therefore, all the powerful techniques based on the analogy with electrical circuits (see [8, 22] for instance) do not apply. The failure of this criterion is based on the following observation: for an undirected graph we have $d_{\mathbf{v}}^- = d_{\mathbf{v}}$ for all $\mathbf{v}$ and for $f \in \ell^2(\mathbb{G}^0)$ the Markov operator of the simple random walk $\mathbb{E}\left(f(\mathbf{M}_{n+1}) - f(\mathbf{M}_n) \mid M_n = \mathbf{u}\right) = \sum_{\mathbf{v}} P(\mathbf{u},\mathbf{v}) f(\mathbf{v}) - f(\mathbf{u}) = \frac{1}{d_{\mathbf{u}}^-} \sum_{\mathbf{v}\in t(s^{-1}(\mathbf{u}))} f(\mathbf{v}) - f(\mathbf{u}) = \frac{1}{d_{\mathbf{u}}} \Delta f(\mathbf{u})$ is immediately expressible in terms of the *Laplace-Beltrami operator* Δ. Now, choosing an orientation on the graph, we can express $\Delta = -D^* D$ where $D : \ell^2(\mathbb{G}^0) \to \ell^2(\mathbb{G}^1)$ is the *Dirac* operator, defined by $Df(\alpha) = f(s(\alpha)) - f(t(\alpha))$ and $D^* : \ell^2(\mathbb{G}^1) \to \mathbb{G}^0$ is its adjoint (for the Hilbert scalar product) defined by $D^*\phi(\mathbf{v}) = \sum_{\alpha\in s^{-1}(\mathbf{v})} \phi(\alpha) - \sum_{\alpha\in t^{-1}(\mathbf{v})} \phi(\alpha)$. For directed graphs, the Markov operator is expressible merely as $\frac{1}{d_{\mathbf{u}}^-} \sum_{\alpha\in s^{-1}(\mathbf{u})} Df(\alpha)$ but the Laplace-Beltrami operator is not defined on this lattice.

All the graphs that we shall consider in this paper are two-dimensional lattices, i.e. $\mathbb{G}^0 = \mathbb{Z}^2$ and $\mathbb{G}^1$ is a subset of the set of nearest neighbours in $\mathbb{Z}^2$. We often write $\mathbb{G}^0 = \mathbb{G}_1^0 \times \mathbb{G}_2^0$, with $\mathbb{G}_1^0$ and $\mathbb{G}_2^0$ isomorphic to $\mathbb{Z}$ when we wish to specify horizontal and vertical directions.

Let $\boldsymbol{\epsilon} = (\epsilon_y)_{y\in\mathbb{G}_2^0}$ be a $\{-1,1\}$-valued sequence of variables assigned to each ordinate. The sequence $\boldsymbol{\epsilon}$ can be deterministic or random as it will be specified later.

Definition 1.2. [Two-dimensional $\boldsymbol{\epsilon}$-horizontally directed lattice] Let $\mathbb{G}^0 = \mathbb{G}_1^0 \times \mathbb{G}_2^0 = \mathbb{Z}^2$, with $\mathbb{G}_1^0$ and $\mathbb{G}_2^0$ isomorphic to $\mathbb{Z}$ and $\boldsymbol{\epsilon} = (\epsilon_y)_{y\in\mathbb{G}_2^0}$ be a sequence of $\{-1,1\}$-valued variables assigned to each ordinate. We call *two-dimensional $\boldsymbol{\epsilon}$-horizontally directed lattice* $\mathbb{G} = \mathbb{G}(\mathbb{G}^0, \boldsymbol{\epsilon})$, the directed graph with vertex set $\mathbb{G}^0 = \mathbb{Z}^2$ and edge set $\mathbb{G}^1$ defined by the condition $(\mathbf{u},\mathbf{v}) \in \mathbb{G}^1$ if, and only if, $\mathbf{u}$ and $\mathbf{v}$ are distinct vertices satisfying one of the following conditions:

1. either $v_1 = u_1$ and $v_2 = u_2 \pm 1$,
2. or $v_2 = u_2$ and $v_1 = u_1 + \epsilon_{u_2}$.

Remark: Notice that the $\boldsymbol{\epsilon}$-horizontally directed lattice is regular; this means that the vertex degrees (both inwards and outwards) are constant $d_{\mathbf{v}}^{-} = d_{\mathbf{v}}^{+} = d = 3$, $\forall \mathbf{v} \in \mathbb{G}^0$. The vertical directions of the graph are both-ways; the horizontal directions are one-way, the sign of ϵ_y determining whether the horizontal line at level y is left- or right-going.

Several $\boldsymbol{\epsilon}$-horizontally directed lattices have been introduced in [2], where the following theorem has been established.

Theorem 1.3 ([2]). *Let $\mathbb{G}^0 = \mathbb{Z}^2$ and consider an $\boldsymbol{\epsilon}$-horizontally directed lattice in dimension 2.*

1. *If the lattice is alternatively directed, i.e. $\epsilon_y = (-1)^y$, for $y \in \mathbb{G}_2^0 \sim \mathbb{Z}$, then the simple random walk on it is recurrent.*
2. *If the lattice has directed half-planes i.e.*

$$\epsilon_y = \begin{cases} 1 & \text{if } \ y \geq 0 \\ -1 & \text{if } \ y < 0, \end{cases}$$

then the simple random walk on it is transient.

3. *If $\boldsymbol{\epsilon} := (\epsilon_y)_{y \in \mathbb{G}_2^0}$ is a sequence of $\{-1, 1\}$-valued random variables, independent and identically distributed with uniform probability, the simple random walk on it is transient for almost all possible choices of the horizontal directions.*

Notice that the above simple random walks are defined on topologically non-trivial directed graphs in the sense that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{y=-N}^{N} \epsilon_y = 0.$$

For the two first cases, this is shown by a simple calculation and for the third case this is an almost sure statement stemming from the independence of the sequence $\boldsymbol{\epsilon}$. The above condition guarantees that transience is not a trivial consequence of a non-zero drift but an intrinsic property of the walk in spite of its jumps being statistically symmetric.

1.3 Results

In this paper, we consider a different model. Again the lattice is a two-dimensional $\boldsymbol{\epsilon}$-horizontally directed lattice. The difference is that the horizontal directions are given by a decaying random perturbation of the periodic situtation. More precisely we have the following

Definition 1.4. Let $f : \mathbb{G}_2^0 \to \{-1, 1\}$ be a Q-periodic function with some even integer $Q \geq 2$ verifying $\sum_{y=1}^{Q} f(y) = 0$ and $\boldsymbol{\rho} = (\rho_y)_{y \in \mathbb{G}_2^0}$ a Rademacher sequence of independent and identically distributed $\{-1, 1\}$-valued random variables. Let $\boldsymbol{\lambda} = (\lambda_y)_{y \in \mathbb{G}_2^0}$ be a $\{0, 1\}$-valued sequence of independent random variables and suppose there exist constants β (and c) such that $\mathbb{P}(\lambda_y = 1) = \frac{c}{|y|^\beta}$ for large $|y|$. We define the horizontal orientations $\boldsymbol{\epsilon} = (\epsilon_y)_{y \in \mathbb{G}_2^0}$ through $\epsilon_y = (1 - \lambda_y) f(y) + \lambda_y \rho_y$. Then the $\boldsymbol{\epsilon}$-directed lattice defined above is termed a **randomly horizontally directed lattice with randomness decaying in power** β.

Theorem 1.5. *Consider the two-dimensional $\boldsymbol{\epsilon}$-randomly horizontally directed lattice with randomness decaying in power β.*

1. *If $\beta < 1$ then the simple random walk is transient for almost all realisations of the sequence (λ_y, ρ_y).*
2. *If $\beta > 1$ then the simple random walk is recurrent for almost all realisations of the sequence (λ_y, ρ_y).*

2 Technical preliminaries

Since the general framework developed in [2] is still useful here, we only recall here the basic facts. It is always possible to choose a sufficiently large abstract probability space $(\Omega, \mathcal{A}, \mathbb{P})$ on which are defined all the sequences of random variables we shall use, namely $(\rho_y), (\lambda_y)$, etc. and, in particular the Markov chain $(\mathbf{M}_n)_{n \in \mathbb{N}}$ itself. When the initial probability of the chain is μ then, obviously $\mathbb{P} := \mathbb{P}_\mu$ i.e. depends on μ. The idea of the proof is to consider the components of the stochastic process $(\mathbf{M}_n)_{n \in \mathbb{N}}$ termed respectively vertical skeleton and horizontal component at precisely chosen instants.

Definition 2.1. Let $(\psi_n)_{n \in \mathbb{N}^*}$ be a sequence of independent, identically distributed, $\{-1,1\}$-valued symmetric Bernoulli variables and

$$Y_n = \sum_{k=1}^{n} \psi_k, \quad n = 1, 2, \ldots$$

with $Y_0 \equiv 0$, the simple $\mathbb{G}_2^0$-valued symmetric one-dimensional random walk. We call the process $(Y_n)_{n\in\mathbb{N}}$ the *vertical skeleton*. We denote by

$$\eta_n(A) = \sum_{k=0}^{n} \mathbb{1}_{\{Y_k \in A\}}, \quad n \in \mathbb{N}, A \subseteq \mathbb{G}_2^0$$

the corresponding *occupation measure* of the set A up to time n. More generally, consider for $0 \le m < n$ the occupation measure of the set A between times m and n defined by $\eta_{m,n}(A) = \sum_{k=m}^{n} \mathbb{1}_{\{Y_k \in A\}}$. For $y \in \mathbb{G}_2^0$, we use the simplified notation $\eta_n(y)$ (resp. $\eta_{m,n}(y)$) for $\eta_n(\{y\})$ (resp. $\eta_{m,n}(\{y\})$).

Definition 2.2. Suppose the vertical skeleton and the environments of the orientations are given. Let $(\xi_n^{(y)})_{n\in\mathbb{N}^*, y\in\mathbb{G}_2^0}$ be a doubly infinite sequence of independent identically distributed $\mathbb{N}$-valued geometric random variables of parametres $p = 1/3$ and $q = 1 - p$. Let $(\eta_n(y))$ be the occupation times of the vertical skeleton. We call *horizontally embedded* random walk the process $(X_n)_{n\in\mathbb{N}}$ with

$$X_n = \sum_{y\in\mathbb{G}_2^0} \epsilon_y \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}, \quad n \in \mathbb{N}.$$

Remark: The significance of the random variable X_n is the horizontal displacement after $n-1$ vertical moves of the skeleton (Y_l). Notice that the random walk (X_n) has unbounded (although integrable) increments. As a matter of fact, they are signed integer-valued geometric random variables.

Lemma 2.3 ([2]). *Let*

$$T_n = n + \sum_{y\in\mathbb{G}_2^0} \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}$$

be the instant just after the random walk $(\mathbf{M}_k)$ *has performed its* n^{th} *vertical move (with the convention that the sum* $\sum_i$ *vanishes whenever* $\eta_{n-1}(y) = 0$.*) Then*

$$\mathbf{M}_{T_n} = (X_n, Y_n).$$

Define $\sigma_0 = 0$ and recursively, for $n = 1, 2, \ldots$, $\sigma_n = \inf\{k \geq \sigma_{n-1} : Y_k = 0\} > \sigma_{n-1}$, the n^{th} return to the origin for the vertical skeleton. Then obviously, $\mathbf{M}_{T_{\sigma_n}} = (X_{\sigma_n}, 0)$. To study the recurrence or the transience of $(\mathbf{M}_k)$, we must study how often $\mathbf{M}_k = (0,0)$. Now, $\mathbf{M}_k = (0,0)$ if and only if $X_k = 0$ and $Y_k = 0$. Since (Y_k) is a simple random walk, the event $\{Y_k = 0\}$ is realised only at the instants σ_n, $n = 0, 1, 2, \ldots$.

Recall that all random variables are defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$; introduce the following sub-σ-algebras:

$$
\begin{aligned}
\mathcal{H} &= \sigma(\xi_i^{(y)}, i \in \mathbb{N}, y \in \mathbb{G}_2^0), \\
\mathcal{G} &= \sigma(\rho_y, \lambda_y y \in \mathbb{G}_2^0), \\
\mathcal{F}_n &= \sigma(\psi_i, i = 1, \ldots, n),
\end{aligned}
$$

with $\mathcal{F} \equiv \mathcal{F}_\infty$.

Lemma 2.4 ([2]).

$$\sum_{l=0}^{\infty} \mathbb{P}(\mathbf{M}_l = (0,0) | \mathcal{F} \vee \mathcal{G}) = \sum_{n=0}^{\infty} \mathbb{P}(I(X_{\sigma_n}, \epsilon_0 \xi_0) \ni 0 | \mathcal{F} \vee \mathcal{G}),$$

where, for $x \in \mathbb{Z}$, $z \in \mathbb{N}$, and $\epsilon = \pm 1$, $I(x, \epsilon z) = \{x, \ldots, x + z\}$ if $\epsilon = +1$ and $\{x - z, \ldots, x\}$ if $\epsilon = -1$.

Remark: The recurrence/transience properties of the random walk $(\mathbf{M}_l)$ on the two-dimensional directed lattice are essentially given by the recurrence/transience properties of the embedded random walk (X_{σ_n}) which is an one-dimensional random walk with unbounded jumps in a random scenery. Notice however that this situation is fundamentally different from the random walk in a random scenery studied in [11]. Therefore, although all the subsequent estimates for recurrence/transience of the process can be carried on by using the right hand side expression of the formula in lemma 2.4, some can be simplified if we take advantage of the following

Lemma 2.5 ([2]).
1. *If $\sum_{n=0}^{\infty} \mathbb{P}_0(X_{\sigma_n} = 0 | \mathcal{F} \vee \mathcal{G}) = \infty$ then $\sum_{l=0}^{\infty} \mathbb{P}(\mathbf{M}_l = (0,0) | \mathcal{F} \vee \mathcal{G}) = \infty$.*
2. *If $(X_{\sigma_n})_{n \in \mathbb{N}}$ is transient then $(M_n)_{n \in \mathbb{N}}$ is also transient.*

Let ξ be a geometric random variable equidistributed with $\xi_i^{(y)}$. Denote

$$\chi(\theta) = \mathbb{E} \exp(i\theta\xi) = \frac{p}{1 - q\exp(i\theta)} = r(\theta)\exp(i\alpha(\theta)), \quad \theta \in [-\pi, \pi]$$

its characteristic function, where

$$r(\theta) = |\chi(\theta)| = \frac{p}{\sqrt{p^2 + 2q(1-\cos\theta)}} = r(-\theta)$$

and

$$\alpha(\theta) = \arctan\frac{q\sin\theta}{1-q\cos\theta} = -\alpha(-\theta).$$

Notice that $r(\theta) < 1$ for $\theta \in [-\pi, \pi] \setminus \{0\}$. Recall that we denote $\mathcal{F} = \sigma(\psi_i, i \in \mathbb{N})$ and $\mathcal{G} = \sigma(\rho_y, \lambda_y, y \in \mathbb{G}_2^0)$. Then

$$\begin{aligned}
\mathbb{E}\exp(i\theta X_n) &= \mathbb{E}\left(\mathbb{E}(\exp(i\theta X_n)|\mathcal{F}\vee\mathcal{G})\right) \\
&= \mathbb{E}\left(\mathbb{E}(\exp(i\theta \sum_{y\in\mathbb{G}_2^0} \epsilon_y \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}|\mathcal{F}\vee\mathcal{G})\right) \\
&= \mathbb{E}\left(\prod_{y\in\mathbb{G}_2^0} \chi(\theta\epsilon_y)^{\eta_{n-1}(y)}\right).
\end{aligned}$$

3 Proof of transience

Introduce, as was the case in [2], constants $\delta_i > 0$ for $i = 1, 2, 3$ and for $n \in \mathbb{N}$ the sequence of events $A_n = A_{n,1} \cap A_{n,2}$ and B_n defined by

$$\begin{aligned}
A_{n,1} &= \left\{\omega \in \Omega : \max_{0\le k\le 2n} |Y_k| < n^{\frac{1}{2}+\delta_1}\right\}, \\
A_{n,2} &= \left\{\omega \in \Omega : \max_{y\in\mathbb{G}_2^0} \eta_{2n-1}(y) < n^{\frac{1}{2}+\delta_2}\right\}, \\
B_n &= \left\{\omega \in A_n : \left|\sum_{y\in\mathbb{G}_2^0} \epsilon_y \eta_{2n-1}(y)\right| > n^{\frac{1}{2}+\delta_3}\right\};
\end{aligned}$$

the range of possible values for δ_i, $i = 1, 2, 3$, will be chosen later (see end of the proof of proposition 3.3). Obviously $A_{n,1}, A_{n,2}$ and hence A_n belong to $\mathcal{F}_{2n}$; moreover $B_n \subseteq A_n$ and $B_n \in \mathcal{F}_{2n} \vee \mathcal{G}$. We denote in the sequel generically $d_{n,i} = n^{\frac{1}{2}+\delta_i}$, for $i = 1, 2, 3$.

Since $B_n \subseteq A_n$ and both sets are $\mathcal{F}_{2n} \vee \mathcal{G}$-measurable, decomposing the unity as

$$1 = \mathbb{1}_{B_n} + \mathbb{1}_{A_n\setminus B_n} + \mathbb{1}_{A_n^c},$$

we have

$$\begin{aligned}\mathbb{P}(X_{2n}=0;Y_{2n}=0|\mathcal{F}\vee\mathcal{G}) &= 1\!\!1_{B_n}1\!\!1_{\{Y_{2n}=0\}}\mathbb{P}(X_{2n}=0|\mathcal{F}\vee\mathcal{G})\\ &\quad +1\!\!1_{A_n\setminus B_n}1\!\!1_{\{Y_{2n}=0\}}\mathbb{P}(X_{2n}=0|\mathcal{F}\vee\mathcal{G})\\ &\quad +1\!\!1_{A_n^c}1\!\!1_{\{Y_{2n}=0\}}\mathbb{P}(X_{2n}=0|\mathcal{F}\vee\mathcal{G}),\end{aligned}$$

and taking expectations on both sides of the equality, we get

$$p_n = p_{n,1}+p_{n,2}+p_{n,3},$$

where

$$\begin{aligned}p_n &= \mathbb{P}(X_{2n}=0;Y_{2n}=0)\\ p_{n,1} &= \mathbb{P}(X_{2n}=0;Y_{2n}=0;B_n)\\ p_{n,2} &= \mathbb{P}(X_{2n}=0;Y_{2n}=0;A_n\setminus B_n)\\ p_{n,3} &= \mathbb{P}(X_{2n}=0;Y_{2n}=0;A_n^c).\end{aligned}$$

By repeating verbatim the reasoning in [2], we get

Proposition 3.1. *For large n, there exist $\delta>0$ and $\delta'>0$ and $c>0$ and $c'>0$ such that*

$$p_{n,1}=\mathcal{O}(\exp(-cn^{\delta})) \quad \textit{and} \quad p_{n,3}=\mathcal{O}(\exp(-c'n^{\delta'})).$$

Consequently $\sum_{n\in\mathbb{N}}(p_{n,1}+p_{n,3})<\infty$. The proof will be complete if we show that $\sum_{n\in\mathbb{N}}p_{n,2}<\infty$.

Recall that we have

$$X_{2n}=\sum_{y\in\mathbb{G}_2^0}\epsilon_y\sum_{i=1}^{\eta_{2n-1}(y)}\xi_i^{(y)}=\sum_{k=1}^{2n}\epsilon_{Y_k}\xi_k.$$

Introduce the random variables:

$$\begin{aligned}N_+ &= \sum_{k=1}^{2n}1\!\!1_{\{\epsilon_{Y_k}=1\}}\\ N_- &= \sum_{k=1}^{2n}1\!\!1_{\{\epsilon_{Y_k}=-1\}}\\ \Delta_n &= N_+-N_- = \sum_{y\in\mathbb{G}_2^0}\epsilon_y\eta_{2n-1}(y).\end{aligned}$$

Lemma 3.2. *On the set $A_n \setminus B_n$, we have*

$$\mathbb{P}(X_{2n} = 0 | \mathcal{F} \vee \mathcal{G}) = \mathcal{O}(\sqrt{\frac{\ln n}{n}}).$$

Proof. Use the $\mathcal{F}\vee\mathcal{G}$-measurability of the variables $(\epsilon_y)_{y\in\mathbb{G}_2^0}$ and $(\eta_n(y))_{y\in\mathbb{G}_2^0, n\in\mathbb{N}}$ to express the conditional characteristic function of the variable X_{2n} as follows:

$$\chi_1(\theta) = \mathbb{E}(\exp(i\theta X_{2n})|\mathcal{F} \vee \mathcal{G}) = \prod_{y\in\mathbb{G}_2^0} \chi(\theta\epsilon_y)^{\eta_{2n-1}(y)}.$$

Hence,

$$\mathbb{P}(X_{2n} = 0|\mathcal{F} \vee \mathcal{G}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_1(\theta) d\theta.$$

Now use the decomposition of χ into a the modulus part, $r(\theta)$ — that is an even function of θ — and the angular part of $\alpha(\theta)$ and the fact that there is a constant $K < 1$ such that for $\theta \in [-\pi, -\pi/2] \cup [\pi/2, \pi]$ we can bound $r(\theta) < K$ to majorise

$$\mathbb{P}(X_{2n} = 0|\mathcal{F} \vee \mathcal{G}) \le \frac{1}{\pi} \int_0^{\pi/2} r(\theta)^{2n} d\theta + \mathcal{O}(K^n).$$

Fix $a_n = \sqrt{\frac{\ln n}{n}}$ and split the above integral over $[0, \pi/2] = [0, a_n] \cup [a_n, \pi/2]$. For the first part, we majorise the integrand by 1, so that

$$\int_0^{a_n} r(\theta)^{2n} d\theta \le a_n.$$

For the second part, use the majorisation $r(\theta) \le \exp(-\frac{3}{8}\theta^2)$ valid for $\theta \in]0, \pi/2]$ to estimate

$$\frac{1}{\pi} \int_{a_n}^{\pi/2} r(\theta)^{2n} d\theta = \mathcal{O}(n^{-3/4}).$$

Since the estimate of the first part dominates, the result follows. □

It remains to estimate $p_{n,2}$.

Let $(a_k)_{k\in\mathbb{N}}$ be a complex sequence such that its generating function $A(t) := \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$ is well defined in a neighbourhood of 0. Then the generating function K of its **semi-invariants (cumulants)** is defined by $A(t) = \exp(K(t))$ with $K(t) = \sum_{k\ge1} \kappa_k \frac{t^k}{k!}$. Let Z be a random variable; if Z has exponential moments, we can use the previous formula for $A(t) = \mathbb{E}\exp(tZ)$, or $a_k = \mathbb{E}Z^k$; otherwise K is always defined (formally) for $A(t) = \mathbb{E}\exp(itZ)$.

Proposition 3.3. *For all $\delta_5 > 0$, and for large n*

$$\mathbb{P}(A_n \setminus B_n | \mathcal{F}) = \mathcal{O}(n^{-\frac{1}{4}+\delta_5}).$$

Proof. The required probability is an estimate, on the event A_n, of the conditional probability $\mathbb{P}(|\sum_{y\in\mathbb{G}_2^0} \zeta_y| \leq d_{n,3}|\mathcal{F})$, where we denote $\zeta_y = \epsilon_y \eta_{2n-1}(y)$. Extend the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to carry an auxilliary variable G assumed to be centred Gaussian with variance $d_{n,3}^2$, (conditionally on $\mathcal{F}$) independent of the ζ_y's. Since both G is a symmetric random variable and $[-d_{n,3}, d_{n,3}]$ is a symmetric set around 0, then by Anderson's inequality, there exists a positive constant c such that

$$\mathbb{P}(|\sum_{y\in\mathbb{G}_2^0} \zeta_y| \leq d_{n,3}|\mathcal{F}) \leq c\mathbb{P}(|\sum_{y\in\mathbb{G}_2^0} \zeta_y + G| \leq d_{n,3}|\mathcal{F}).$$

Let

$$\chi_2(t) = \mathbb{E}(\exp(it\sum_y \zeta_y)|\mathcal{F}) = \prod_y A_y(t),$$

where $A_y(t) = \mathbb{E}\left(\exp(it\zeta_y|\mathcal{F}\right)$, and

$$\chi_3(t) = \mathbb{E}(\exp(itG)|\mathcal{F}) = \exp(-t^2 d_{n,3}^2/2).$$

Therefore,

$$\mathbb{E}(\exp(it(\sum_y \zeta_y + G))|\mathcal{F}) = \chi_2(t)\chi_3(t),$$

and using the Plancherel's formula,

$$\mathbb{P}(|\sum_{y\in\mathbb{G}_2^0} \zeta_y + G| \leq d_{n,3}|\mathcal{F}) = \frac{d_{n,3}}{\pi}\int \frac{\sin(td_{n,3})}{td_{n,3}}\chi_2(t)\chi_3(t)dt \leq Cd_{n,3}I,$$

where

$$I = \int |\chi_2(t)| \exp(-t^2 d_{n,3}^2/2)dt.$$

Fix $b_n = \frac{n^{\delta_4}}{d_{n,3}}$, for some $\delta_4 > 0$ and split the integral defining I into $I_1 + I_2$, the first part being for $|t| \leq b_n$ and the second for $|t| > b_n$.

We have

$$\begin{aligned} I_2 &\leq C\int_{|t|>b_n} \exp(-t^2 d_{n,3}^2/2)\frac{dt}{2\pi} \\ &= \frac{C}{d_{n,3}}\int_{|s|>n^{\delta_4}} \exp(-s^2/2)\frac{ds}{2\pi} \\ &\leq 2\frac{C}{d_{n,3}}\frac{1}{n^{\delta_4}}\frac{\exp(-n^{2\delta_4}/2)}{2\pi}, \end{aligned}$$

because the probability that a centred normal random variable of variance 1, whose density is denoted ϕ, exceeds a threshold $x > 0$ is majorised by $\frac{\phi(x)}{x}$.

For I_1 we get,

$$I_1 \leq \int_{|t|\leq b_n} \prod_y |A_y(t)| dt.$$

Now, conditionally on $\mathcal{F}$, the random variable ζ_y has exponential moments. Using cumulant expansion, we write $A_y(t) = \exp(K_y(t))$ and determine easily that for large $|y|$, we get the estimates

$$\begin{aligned}
\kappa_1(y) &= \mathbb{E}(i\epsilon_y \eta_{2n-1}(y)|\mathcal{F}) = if(y)\eta_{2n-1}(y)(1 - \frac{c}{|y|^\beta}) \\
\kappa_2(y) &= -\eta^2_{2n-1}(y) + \eta^2_{2n-1}(y)(1 - \frac{c}{|y|^\beta})^2 = -2c\eta^2_{2n-1}(y)\frac{1}{|y|^\beta} + \mathcal{O}(\frac{1}{|y|^{2\beta}}).
\end{aligned}$$

Therefore,

$$|\chi_2(t)| \leq \prod_y \exp\left(-\frac{t^2}{4}\eta^2_{2n-1}(y)\frac{c}{|y|^\beta}\right).$$

Now, define $\pi_n(y) = \frac{\eta_{2n-1}(y)}{2n}$; obviously $\sum_y \pi_n(y) = 1$, establishing that $(\pi_n(y))_y$ is a probability measure on $\mathbb{G}^0_2$. Therefore, applying Hölder's inequality we obtain $I_1 \leq \prod'_y J_n(y)^{\pi_n(y)}$, where $\prod'_y$ means that the product runs over those y such that $\eta_{2n-1}(y) \neq 0$ and

$$\begin{aligned}
J_n(y) &= \int_{|t|\leq b_n} \exp\left(-\frac{t^2}{4}\eta^2_{2n-1}(y)\frac{c}{|y|^\beta}\frac{1}{\pi_n(y)}\right) \\
&= \int_{|t|\leq b_n} \exp\left(-\frac{t^2}{2}n\eta_{2n-1}(y)\frac{c}{|y|^\beta}\right) dt \\
&= \sqrt{\frac{2\pi|y|^\beta}{cn\eta_{2n-1}(y)}} \int_{|v|\leq b_n\sqrt{\frac{cn\eta_{2n-1}(y)}{2|y|^\beta}}} \exp(-v^2/2)\frac{dv}{2\pi} \\
&\leq \sqrt{\frac{4\pi}{c}} \exp\left(-\log 2n - \frac{1}{2}\log \pi_n(y) + \frac{\beta}{2}\log|y|\right).
\end{aligned}$$

We conclude that

$$I_1 \leq {\prod_y}' J_n(y)^{\pi_n(y)} = \sqrt{\frac{2\pi}{c}} \exp\left(-\log 2n + \frac{1}{2}H(\pi_n) + \frac{\beta}{2}\sum_y \pi_n(y)\log|y|\right),$$

where $H(\pi_n)$ is the entropy of the probability measure π_n, reading

$$H(\pi_n) := -\sum_y \pi_n(y)\log \pi_n(y) \leq \log \mathsf{card} C_n,$$

where $C_n := \mathsf{supp}\,\pi_n \leq n^{\frac{1}{2}+\delta}$. We conclude that we can always chose the parameters δ_1 and δ_3 such that, for every $\beta < 1$ there exists a parameter $\delta_\beta > 0$ such that

$$d_{n,3} I_1 \leq C n^{-\delta_\beta}.$$

□

Corollary 3.4.

$$\sum_{n\in\mathbb{N}} p_{n,2} < \infty.$$

Proof. Recalling that for the standard random walk $\mathbb{P}(Y_{2n} = 0) = \mathcal{O}(n^{-1/2})$ and from the estimates obtained in 3.2 and 3.3, we have

$$\begin{aligned} p_{n,2} &= \mathbb{P}(X_{2n} = 0; Y_{2n} = 0; A_n \setminus B_n) \\ &= \mathbb{E}(\mathbb{E}\left(\mathbb{1}_{Y_{2n}=0}\left[\mathbb{E}(\mathbb{1}_{A_n\setminus B_n}\mathbb{P}(X_{2n} = 0|\mathcal{F}\vee\mathcal{G})|\mathcal{F})\right]\right)) \\ &= \mathcal{O}(n^{-1/2} n^{-\delta_\beta}\sqrt{\frac{\ln n}{n}}) \\ &= \mathcal{O}(n^{-(1+\delta_\beta)}\ln n), \end{aligned}$$

proving thus the summability of $p_{n,2}$. □

We can now complete the

Proof the statement on transience of the theorem 1.5: The transience is a simple consequence of the previous propositions. As a matter of fact $p_n = p_{n,1} + p_{n,2} + p_{n,3}$ is summable because the partial probabilities $p_{n,i}$, for $i = 1, 2, 3$ are all shown to be summable. □

4 Proof of recurrence

We define additionally the following sequence of random times:

$$\tau_0 \equiv 0 \quad \text{and} \quad \tau_{n+1} = \inf\{k : k > \tau_n, |Y_k - Y_{\tau_n}| = Q\} \quad \text{for} \quad n \geq 0.$$

The random variables $(\tau_{n+1} - \tau_n)_{n\geq 0}$ are independent and for all n the variable $\tau_{n+1} - \tau_n$ has the same distribution (under $\mathbb{P}_0$) as τ_1. It is easy to show further (see proposition 1.13.4 of the textbook [1] for instance) that these random variables have exponential moments i.e. $\mathbb{E}_0(\exp(\alpha\tau_1)) < \infty$ for $|\alpha|$ sufficiently small.

Let $\mathbb{Z}_Q = \mathbb{Z}/Q\mathbb{Z} = \{0, 1, \dots, Q-1\}$ with integer addition replaced by addition modulo Q and for any $y \in \mathbb{Z}$ denote by $\overline{y} = y \mod Q \in \mathbb{Z}_Q$. Consistently, we define $\overline{Y}_n = Y_n \mod Q$.

Lemma 4.1. *Define for $n \geq 1$ and $\overline{y} \in \mathbb{Z}_Q$,*

$$N_n(\overline{y}) = \sum_{k=\tau_{n-1}}^{\tau_n - 1} \mathbb{1}_{\overline{y}}(\overline{Y}_k).$$

Then

$$\mathbb{E}_0 N_1(\overline{y}) = \frac{1}{2}\mathbb{E}_0\left(N_1(\overline{y}) \mid Y_{\tau_1} = Q\right) + \frac{1}{2}\mathbb{E}_0\left(N_1(\overline{y}) \mid Y_{\tau_1} = -Q\right) = \frac{\mathbb{E}_0\tau_1}{Q}.$$

Proof. Since the random walk (Y_n) is symmetric, the probability of exiting the strip of width Q by up-crossing is the the same as for a down-crossing. This remark establishes the leftmost equality of the statement.

To prove the rightmost equality, let $g : \mathbb{Z} \to \mathbb{R}$ be a bounded function and denote by $S_n[g] = \sum_{k=0}^{n-1} g(Y_k)$. On defining $W_n[g] = \sum_{k=\tau_n}^{\tau_{n+1}-1} g(Y_k)$ and $R_n = \max\{k : \tau_k \leq n\}$, we have the decomposition:

$$S_n[g] = \sum_{k=0}^{R_n} W_k[g] - \sum_{k=n}^{\tau_{R_n+1}-1} g(Y_k).$$

Since $\tau_{R_n+1} - n \leq \tau_{R_n+1} - \tau_{R_n}$ and the latter random variable is distributed as τ_1 under $\mathbb{P}_0$, we have, thanks to the boundedness of g, that $\frac{1}{n}\left|\sum_{k=n}^{\tau_{R_n+1}-1} g(Y_k)\right| \leq \frac{\tau_{R_{n+1}} - \tau_{R_n}}{n} \sup_{y\in\mathbb{Z}} |g(z)|$ and since $\tau_{R_{n+1}} - \tau_{R_n} \stackrel{\mathrm{d}}{=} \tau_1$, the remainder term tends to 0 in probability.

It remains to estimate $\frac{S_n[g]}{n}$ by $\frac{R_n}{n}\frac{1}{R_n}\sum_{k=1}^{R_n} W_k[g]$. Obviously $R_n \to \infty$ and, by the renewal theorem (see p. 221 of [1] for instance), $\frac{R_n}{n} \to \frac{1}{\mathbb{E}_0\tau_1}$ a.s. Fix any $\overline{y} \in \mathbb{Z}_Q$ and choose $g(z) := \mathbb{1}_{\{\overline{y}\}}(z \mod Q)$. For this g, we have $S_n[g] = \overline{\eta}_n(\overline{y})$, where $\overline{\eta}_n(\overline{y}) = \sum_{k=0}^{n-1} \mathbb{1}_{\{\overline{y}\}}(\overline{Y}_k)$. But $(\overline{Y}_k)$ is a simple random walk on the finite set $\mathbb{Z}_Q$ therefore admits a unique invariant probability $\overline{\pi}(\overline{y}) = \frac{1}{Q}$. By the ergodic theorem for Markov chains, we have $\frac{S_n[g]}{n} \to \frac{1}{Q}$ a.s.

Additionally, for this choice of g, the sequence $(W_k[g])_{k\in\mathbb{N}}$ are independent random variables, identically distributed as $N_1(\overline{y})$. We conclude by applying the law of large numbers to the ratio $\frac{1}{R_n}\sum_{k=1}^{R_n} W_k[g]$. □

Lemma 4.2. *Fix $K > 0$. For every $\delta > 0$ there exists a constant $c = c(\delta) > 0$ such that for all n,*

$$\mathbb{P}_0\left(\eta_{2n}([-K,K]) > c\sqrt{n} \mid Y_{2n} = 0\right) < \delta.$$

Proof. Write

$$\begin{aligned}\mathbb{E}_0\left(\eta_{2n}([-K,K]) \mid Y_{2n} = 0\right) &= \frac{\sum_{y=-K}^{K}\sum_{k=0}^{2n}\mathbb{P}_0(Y_k = y; Y_{2n} = 0)}{\mathbb{P}_0(Y_{2n} = 0)} \\ &= \frac{\sum_{y=-K}^{K}\sum_{k=0}^{2n}P^k(0,y)P^{2n-k}(0,-y)}{P^{2n}(0,0)}.\end{aligned}$$

There exist constants c_1, c_2, and c_3 such that $P^{2n}(0,0) \sim \frac{c_1}{\sqrt{2n}}$ and $P^l(0,z) \leq \frac{c_2}{\sqrt{l}}$. Comparing now $\frac{\sum_{k=0}^{2n}P^k(0,y)P^{2n-k}(0,-y)}{P^{2n}(0,0)}$ with $\int_0^{2n}\sqrt{\frac{2n}{t(2n-t)}}dt = \pi\sqrt{2n}$, we get

$$\mathbb{E}_0\left(\eta_{2n}([-K,K]) \mid Y_{2n} = 0\right) \leq c_4\sqrt{n}.$$

We conclude by conditional Markov inequality, on choosing $c = c_4/\delta$. $\square$

To prove recurrence, it is enough to show $\sum_{k\in\mathbb{N}}\mathbb{P}_0\left(X_{\sigma_k} = 0, Y_{\sigma_k} = 0 \mid \mathcal{G}\right) = \infty$. If $\beta > 1$ then $\sum_y \mathbb{P}(\lambda_y = 1) < \infty$; hence there is almost surely a finite number of y such that $\lambda_y = 1$, by Borel-Cantelli theorem, i.e. the $\mathcal{G}$-measurable random variable $l(\omega) = \max\{|y| : \lambda_y = 1\}/Q$ is almost surely finite. Fix an integer $L \geq l(\omega) + 1$, and introduce the random sets:

$$\begin{aligned}F_{L,2n}(\omega) &= \left\{k : 0 \leq k \leq 2n-1; |Y_{\tau_k}| \leq LQ; |Y_{\tau_{k+1}}| \leq LQ\right\} \\ G_{L,2n}(\omega) &= \left\{k : 0 \leq k \leq 2n-1; |Y_{\tau_k}| \geq LQ; |Y_{\tau_{k+1}}| \geq LQ\right\},\end{aligned}$$

defined on the event $\{\sigma_1 = \tau_{2n}\}$.

We shall further decompose the latter set into

$$\begin{aligned}G^+_{L,2n}(\omega) &= \left\{k \in G_{L,2n} : Y_{\tau_{k+1}} = Y_{\tau_k} + Q\right\}, \\ G^-_{L,2n}(\omega) &= \left\{k \in G_{L,2n} : Y_{\tau_{k+1}} = Y_{\tau_k} - Q\right\},\end{aligned}$$

corresponding respectively to up-crossing and down-crossing excursions of the strips of width Q.

Denote by $\mathsf{Adm}(2n)$ the set of *admissible paths* $\mathbf{z} = (z_0, z_1, \ldots, z_{2n-1}, z_{2n}) \in \mathbb{Z}^{2n+1}$ satisfying $|z_{i+1} - z_i| = Q$ for $i = 0, \ldots 2n-1$, $z_0 = z_{2n} = 0$, and $|z_i| > 0$

for $i = 1, \ldots, 2n-1$. For any $\mathbf{z} \in \mathsf{Adm}(2n)$, we denote $C[\mathbf{z}] := C[\mathbf{z}, \omega]$ the random cylinder set

$$C[\mathbf{z}] = \left\{Y_0 = z_0 = 0, Y_{\tau_1} = z_1, \ldots, Y_{\tau_{2n-1}} = z_{2n-1}, Y_{\tau_{2n}} = z_{2n} = 0\right\} \in \mathcal{F}.$$

Denote by $\theta_k = X_{\tau_{k+1}} - X_{\tau_k}$, for $k \in \{0, \ldots, 2n-1\}$, and observe that

$$X_{\tau_{2n}} = \sum_{k=0}^{2n-1} \theta_k = \sum_{k \in F_{L,2n}} \theta_k + \sum_{k \in G^+_{L,2n}} \theta_k + \sum_{k \in G^-_{L,2n}} \theta_k,$$

the three sums appearing in the above decomposition referring to disjoint excursions. On the set $C[\mathbf{z}]$ with $\mathbf{z} \in \mathsf{Adm}(2n)$ and since $Y_0 = Y_{\tau_{2n}} = 0$, for every $k \in G^+_{L,2n}$, there exists a $k' \in G^-_{L,2n}$, and a $s \in \{0, \ldots, 2n-1\}$, with $k' \neq k$ verifying simultaneously

$$\begin{aligned} z_{s+1} &= z_s + Q \\ Y_{\tau_k} &= Y_{\tau_{k'+1}} = z_s \\ Y_{\tau_{k'}} &= Y_{\tau_{k+1}} = z_{s+1}, \end{aligned}$$

i.e. while k corresponds to an up-crossing of the strip $[z_s, z_{s+1}]$, the excursion corresponding to k' down-crosses the **same** strip. In case the same strip is up-crossed by several excursions, the index k' is not unambiguously determined. Nevertheless, we can always lift the degeneracy so that the mapping $G^+_{L,2n} \ni k \mapsto d(k) = k' \in G^-_{L,2n}$ becomes a bijection. Therefore

$$\sum_{k \in G^+_{L,2n}} \theta_k + \sum_{k \in G^-_{L,2n}} \theta_k = \sum_{k \in G^+_{L,2n}} (\theta_k + \theta_{d(k)}).$$

Proposition 4.3 (Extended reflection principle). *For every $k \in G_{L,2n^+}$ and every $\mathbf{z} \in \mathsf{Adm}(2n)$,*

$$a := \mathbb{E}_0 \left(\theta_k + \theta_{d(k)} \middle| C[\mathbf{z}]\right) = 0.$$

Proof. Let $\mathbf{z}$ be an arbitrary admissible path. Since k corresponds to an up-crossing denote by $z := z_s$ and $z + Q = z_{s+1}$ the bottom and top levels of the crossed strip. Since $\mathbf{z} \in \mathsf{Adm}(2n)$, there are random times $\tau_1, \ldots, \tau_{2n}$ that are **compatible** with $\mathbf{z}$. We have

$$\begin{aligned} Y_{\tau_k} &= Y_{\tau_{d(k)+1}} = z \\ Y_{\tau_{k+1}} &= Y_{\tau_{d(k)}} = z + Q. \end{aligned}$$

Therefore $T = \tau_{k+1} - \tau_k$ is an up-crossing time of the strip while $T' = \tau_{d(k+1)} - \tau_{d(k)}$ is a down-crossing time of the same strip. We shall construct a new admissible path having T' as an up-crossing time and T as a down-crossing time of the strip while the occupation times of the sites in the strip (modulo Q) remain unchanged. The figure 1 illustrates the construction. Consider the path Y, between the times τ_k and $\tau_{k+1} = \tau_k + T$; it crosses the

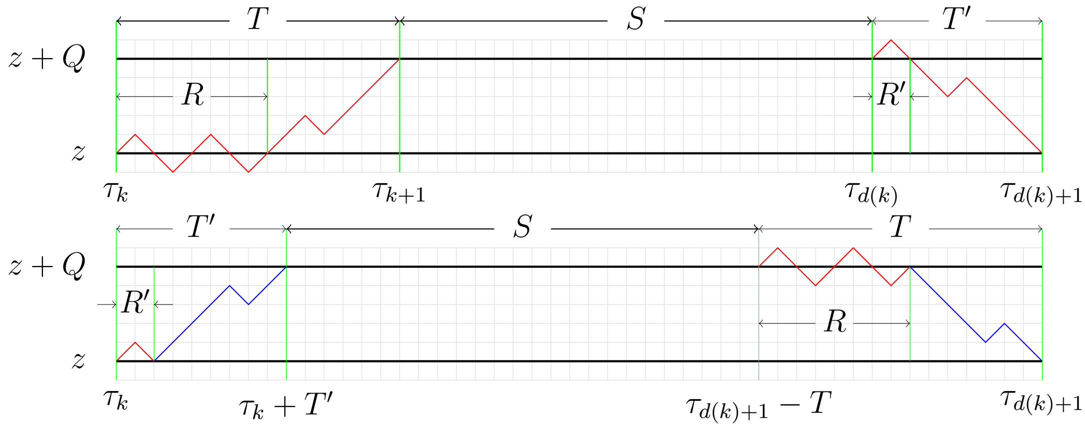


Figure 1: Illustration of the extended reflexion principle. The top figure depicts a detail of the up-crossing excursion, occurring between times τ_k and $\tau_{k+1} = \tau_k + T$, and of the down-crossing excursion, occurring between times and $\tau_{d(k)}$ and $\tau_{d(k)+1} = \tau_{d(k)} + T'$. The bottom figure depicts the details of a new admissible path bijectively obtained by parallel transporting, time reverting and swapping pieces of the the previous path as explained in the text.

strip from its bottom z to the top $z + Q$. Define $R := \max\{t : \tau_k \leq t < \tau_{k+1}, Y_t = Y_{\tau_k} = z\} - \tau_k$. Between times τ_k and $\tau_k + R$, the path Y wanders around the bottom level z. For times t such that $\tau_k + R < t < \tau_{k+1}$, the path remains strictly confined within the (interior of the) strip. We shall define a new path $Z_{[\tau_{d(k)+1}-T, \tau_{d(k)+1}]}$ between the times $\tau_{d(k)+1} - T$ and $\tau_{d(k)+1}$ as follows:

$$Z_t = \begin{cases} Y_{t-S} + Q & \text{for} \quad \tau_{d(k)+1} - T \leq t \leq \tau_{d(k)+1} - T + R \\ Y_{\tau_{d(k)+1}-t-S} & \text{for} \quad \tau_{d(k)+1} - T + R \leq t \leq \tau_{d(k)+1}. \end{cases}$$

Therefore, the first part of the path is parallel transported from level z to level $z + Q$ while the second part is time reversed. By construction, the path $Z_{[\tau_{d(k)+1}-T, \tau_{d(k)+1}]}$ down-crosses the strip and is in bijection with $Y_{[\tau_{d(k)+1}-T, \tau_{d(k)+1}]}$. The same construction can be performed to transform the down-crossing path $Y_{[\tau_{d(k)},\tau_{d(k)+1}]}$ into an up-crossing one $Z_{[\tau_k,\tau_k+T']}$ (see figure 1). Since the time spans T and T' were admissible on the top figure, they remain admissible on the bottom figure. The path Z can be extended outside the considered excursions by defining it as coinciding with Y elsewhere. To distinguish between the occupation times associated with paths Y and Z continue denoting by η the occupation time for Y and introduce the symbol κ to denote the occupation time for Z. Introduce finally the symbols $\overline{\eta}$ and $\overline{\kappa}$ to denote the occupation times for $\overline{Y}$ and $\overline{Z}$ respectively.

The two paths Y and Z arise with the same probability and for all integers $y \in \mathbb{Z}$, by construction of the process Z, we have

$$\overline{\eta}_{\tau_k,\tau_{k+1}-1}(\overline{y}) = \sum_{t=\tau_k}^{\tau_k+T-1} \mathbb{1}_y(\overline{Y}_t) = \sum_{t=\tau_{d(k)+1}-T}^{\tau_{d(k)+1}-1} \mathbb{1}_y(\overline{Z}_t) = \overline{\kappa}_{\tau_{d(k)+1}-T,\tau_{d(k)+1}-1}(\overline{y}).$$

We are now in position to complete the proof of the proposition.

$$\begin{aligned} a &= \mathbb{E}_0\left(\sum_{y\in\mathbb{Z}} f(y)\left[\sum_{i=0}^{\eta_{\tau_k,\tau_{k+1}-1}(y)} \xi_i^y + \sum_{i=0}^{\eta_{\tau_{d(k)},\tau_{d(k)+1}-1}(y)} \xi'^y_i\right]\middle| C[\mathbf{z}]\right) \\ &= \mathbb{E}(\xi_0^0)\sum_{y=z-Q+1}^{z+2Q-1} f(\overline{y})\mathbb{E}_z\left(\eta_{T-1}(y)|Y_T = z+Q; C[\mathbf{z}]\right)\mathbb{P}_z(Y_T = z+Q|C[\mathbf{z}]) \\ &\quad +\mathbb{E}(\xi_0^0)\sum_{y=z-Q+1}^{z+2Q-1} f(\overline{y})\mathbb{E}_{z+Q}\left(\eta_{T'-1}(y)|Y_{T'} = z; C[\mathbf{z}]\right)\mathbb{P}_{z+Q}(Y_{T'} = z|C[\mathbf{z}]) \\ &= \mathbb{E}(\xi_0^0)(b_1+b_2). \end{aligned}$$

Consider the first sum arising in the penultimate line of the previous formula

$$\begin{aligned}
b_1 &= \sum_{y=z-Q+1}^{z+2Q-1} f(\overline{y})\mathbb{E}_z\left(\eta_{T-1}(y)|Y_T = z+Q; C[\mathbf{z}]\right)\mathbb{P}_z(Y_T = z+Q|C[\mathbf{z}]) \\
&= \sum_{y=\mathbb{Z}_Q} f(\overline{y})\mathbb{E}_z\left(\overline{\eta}_{T-1}(\overline{y})\big|Y_T = z+Q\right)\mathbb{P}_z(Y_T = z+Q|C[\mathbf{z}]) \\
&= \frac{1}{2}\sum_{y=\mathbb{Z}_Q} f(\overline{y})\Big[\mathbb{E}_z\left(\overline{\eta}_{T-1}(\overline{y})\big|Y_T = z+Q\right)\mathbb{P}_z(Y_T = z+Q|C[\mathbf{z}]) \\
&\qquad\qquad \mathbb{E}_{z+Q}\left(\overline{\kappa}_{T-1}(\overline{y})|Z_T = z\right)\mathbb{P}_{z+Q}(Z_T = z|C[\mathbf{z}])\Big] \\
&= \sum_{y\in\mathbb{Z}_Q} f(\overline{y})\mathbb{P}_0(N_1(\overline{y}))\mathbb{P}_0(Y_T = Q|C[\mathbf{z}]) \\
&= 0,
\end{aligned}$$

where we used lemma 4.1 and the centering condition $\sum_{\overline{y}\in\mathbb{Z}_Q} f(\overline{y}) = 0$ to conclude. With similar arguments, we establish that the term $b_2 = 0$ as well, so that finally $a = 0$. □

Proof of the recurrence statement of theorem 1.5:

For any $\delta \in]0,1[$, and $c = c(\delta)$ as in lemma 4.2, we have from this very same lemma that $\mathbb{P}_0\left(\mathsf{card}F_{L,2n} \leq c(\delta)\sqrt{n}\right) \geq 1-\delta$. Fix some constant c and define

$$\mathsf{ConsAdm}(L,2n,c) = \left\{\mathbf{z} \in \mathsf{Adm}(2n) : \mathsf{card}\{k : 0 \leq k < 2n, |z_k| \leq LQ; |z_{k+1}| \leq LQ\} \leq c\sqrt{n}\right\}$$

the set of *constrained admissible paths*. Then obviously, omitting the ω dependence: $\{\mathsf{card}F_{L,2n} \leq c\sqrt{n}\} = \cup_{\mathbf{z}\in\mathsf{ConsAdm}(L,2n,c)} C[\mathbf{z}]$. On the event $\{\sigma_1 = \tau_{2n}\}$ the condition $Y_{2n} = 0$ is satisfied, hence

$$\begin{aligned}
\mathbb{P}_0\left(X_{\tau_{2n}} = 0; Y_{\tau_{2n}} = 0 \mid \mathcal{G}\right) &\geq \mathbb{P}_0\left(X_{\tau_{2n}} = 0; Y_{\tau_{2n}} = 0; \mathsf{card}F_{L,2n} \leq c\sqrt{n} \mid \mathcal{G}\right) \\
&= \sum_{\mathbf{z}\in\mathsf{ConsAdm}(L,2n,c)} \mathbb{P}_0\left(\{X_{\tau_{2n}} = 0\} \cap C[\mathbf{z}] \mid \mathcal{G}\right) \\
&= \sum_{\mathbf{z}\in\mathsf{ConsAdm}(L,2n,c)} \mathbb{P}_0\left(X_{\tau_{2n}} = 0 \mid \mathcal{G}, C[\mathbf{z}]\right)\mathbb{P}_0\left(C[\mathbf{z}] \mid \mathcal{G}\right).
\end{aligned}$$

Denote, as before, $\theta_k = X_{\tau_{k+1}} - X_{\tau_k}$ for $k \in \{0,\ldots,2n-1\}$ and recall that $X_{\tau_{2n}} = \sum_{k=0}^{2n-1}\theta_k = \sum_{k\in F_{L,2n}}\theta_k + \sum_{k\in G^+_{L,2n}}(\theta_k + \theta_{d(k)})$. Now, for any

$\mathbf{z} \in \mathsf{ConsAdm}(L, 2n, c)$,

$$\begin{aligned}
\mathbb{P}_0\left(X_{\tau_{2n}} = 0 \mid \mathcal{G}, C[\mathbf{z}]\right) &= \sum_{m\in\mathbb{Z}} \mathbb{P}_0\left(\sum_{k\in F_{L,2n}} \theta_k = m; \sum_{k\in G^+_{L,2n}} (\theta_k + \theta_{d(k)}) = -m \,\middle|\, \mathcal{G}, C[\mathbf{z}]\right)\\
&\geq \sum_{|m|\leq c\sqrt{n}} \mathbb{P}_0\left(\sum_{k\in F_{L,2n}} \theta_k = m; \sum_{k\in G^+_{L,2n}} (\theta_k + \theta_{d(k)}) = -m \,\middle|\, \mathcal{G}, C[\mathbf{z}]\right)\\
&= \sum_{|m|\leq c\sqrt{n}} \mathbb{P}_0\left(\sum_{k\in F_{L,2n}} \theta_k = m \,\middle|\, \mathcal{G}, C[\mathbf{z}]\right)\\
&\qquad \times \mathbb{P}_0\left(\sum_{k\in G^+_{L,2n}} (\theta_k + \theta_{d(k)}) = -m \,\middle|\, C[\mathbf{z}]\right).
\end{aligned}$$

The joint probability factors into the terms appearing in the last line because occurs because the $\mathcal{F}$-measurable random sets $G_{L,2n}$ and $F_{L,2n}$ are disjoint, hence the terms in $F_{L,2n}$ and $G_{L,2n}$ refer to different excursions of the random walk Y. Independence follows as a consequence of the strong Markov property. Additionally, for $k \in G^+_{L,2n}$, the $\mathcal{G}$-measurable components of the random variables entering in the sum $\sum_{k\in G^+_{L,2n}}(\theta_k + \theta_{d(k)})$ are trivial (i.e. constants); therefore $\mathbb{E}((\theta_k + \theta_{d(k)})|\mathcal{G}, C[\mathbf{z}]) = \mathbb{E}((\theta_k + \theta_{d(k)})|C[\mathbf{z}])$.

By the proposition 4.3, we know that $\mathbb{E}((\theta_k + \theta_{d(k)})|C[\mathbf{z}])$; moreover, the sequence of random variables $(\theta_k + \theta_{d(k)})_{k\in G^+_{L,2n}}$ have the same conditional law (under $C[\mathbf{z}]$) for all $k \in G^+_{L,2n}$. Finally, we can majorise the conditional variance of these random variables as follows:

$$\begin{aligned}
\sigma^2 &= \mathbb{E}_0((\theta_k + \theta_{d(k)})^2|C[\mathbf{z}])\\
&= \frac{1}{2}\sum_{\epsilon\in\{-1,1\}} \mathbb{E}_z\left(\left[\sum_y \sum_{i=0}^{\eta_{\tau_1+\tau_2-1}(y)} \xi_i^y\right]^2 \middle| Y_{\tau_1} = z + \epsilon Q, Y_{\tau_1+\tau_2} = z\right)\\
&\leq \mathbb{E}(\tau_1+\tau_2)\mathbb{E}((\xi_0^0)^2) + \mathbb{E}[(\tau_1+\tau_2)^2][\mathbb{E}(\xi_0^0)]^2\\
&< \infty.
\end{aligned}$$

Therefore, we are in the situation of applicability of the local central limit theorem (see proposition 52.15, p. 706 of [20] for instance), reading for

$|m| \leq c\sqrt{n}$,

$$\mathbb{P}_0 \left(\sum_{k \in G^+_{L,2n}} (\theta_k + \theta_{d(k)}) = -m \,\middle|\, C[\mathbf{z}] \right) \geq \frac{c_{11}}{\sqrt{\mathsf{card} G^+_{L,2n}}} \exp\left(-\frac{c^2 n}{2\sigma^2 \mathsf{card} G^+_{L,2n}} \right).$$

Now, on $C[\mathbf{z}]$, $2n \geq 2\mathsf{card}G^+_{L,2n} \geq 2n - c\sqrt{n}$. Hence, $\mathbb{P}_0 \left(\sum_{k \in G^+_{L,2n}} \theta_k = -m \,\middle|\, C[\mathbf{z}] \right) \geq \frac{c_{12}}{\sqrt{n}}$, uniformly in $\mathbf{z}$.

Summarising, and using the lemma 4.2,

$$\begin{aligned} \mathbb{P}_0 \left(X_{\tau_{2n}} = 0, Y_{\tau_{2n}} = 0 \mid \mathcal{G} \right) &\geq \frac{c_9}{\sqrt{n}} \sum_{\mathbf{z} \in \mathsf{ConsAdm}(L,2n,c)} \mathbb{P}_0 \left(C[\mathbf{z}] \right) \\ &= \frac{c_9}{\sqrt{n}} \mathbb{P}_0 \left(F_{L,2n} \right) \\ &\geq \frac{c_{10}}{n}. \end{aligned}$$

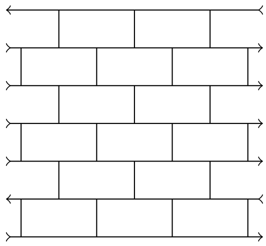
This concludes the proof of the recurrence. □

5 Conclusion, open problems, and further developments

As was apparent in the course of the proof of recurrence, the condition $\beta := \beta_0 > 1$ can be improved. For instance, we can show that if the decay is of the form $\frac{c}{|y| \ln^{\beta_1} |y|}$, with $\beta_1 > 1$ or $\frac{c}{|y| \ln |y| \ln \ln^{\beta_2} |y|}$, with $\beta_2 > 1$, etc., then the random walk is still recurrent. As a matter of fact, the walk is recurrent provided that there exists an arbitrarily large integer l such that the decay is of the form $\frac{c}{|y| \ln |y| \cdots \ln_{l-1} |y| \ln_l^{\beta_l} |y|}$ for some $\beta_l > 1$ (arbitrarily close to 1), where $\ln_l$ is the l-times iterated logarithm. Nevertheless, our methods do not allow the treatment of the really critical case $\beta_0 = 1$.

We make however the conjecture that the random walk is recurrent even when there are infinitely many defects on the orientations of the horizontal lines provided they are sparse, i.e. their density is zero.

Another interesting question is what happens in more general lattices, like the hexagonal. Since hexagonal lattice can be deformed to be presented as below, we can define random horizontal orientations an ask what will be the type of the walk in random environment.



Here the vertical and horizontal components of the random walk no longer factor out completely as was the case in the square lattice.

As stated in the introductory section, regular (undirected) lattices correspond to Cayley graphs of finitely generated groups Γ. More precisely, let Γ be a finitely generated group, not necessarily Abelian, and S_Γ a finite symmetric set of generators of Γ. Then the **Cayley graph** is the infinite graph $\mathsf{Cayley}(\Gamma, S_\Gamma) = (\mathbb{G}^0, \mathbb{G}^1, s, t)$ with $\mathbb{G}^0 = \Gamma$ and $(u, v) \in \mathbb{G}^1 \Leftrightarrow u^{-1}v \in S_\Gamma$. This graph is necessarily undirected and the most prominent examples are the Abelian graphs $\mathbb{Z}^d$ with some integer $d \geq 1$, the homogeneous tree with d free generators $\mathbb{F}_d$, etc. The construction of the graph can be seen as a recursive nested construction $(\mathbb{G}^0_n)_{n\in\mathbb{N}}$ with $\mathbb{G}^0_n \subset \mathbb{G}^0_{n+1}$ for all $n \geq 1$: let $\gamma_0 \in$ be some fixed element of Γ, for instance the neutral element, identified as a particular vertex of the graph, and assume $\mathbb{G}^0_0 = \{\gamma_0\}$, be the germ set. Then adjacent vertices are adjoined to get the recursive sequence of sets $\mathbb{G}^0_{n+1} = \{\gamma s : \gamma \in \mathbb{G}^0_n, s \in S\}$. Now, this construction can be generalised by introducing a **selection mapping** $F : \Gamma \times S \to \{0, 1\}$; new vertices of the form γs, with $s \in S$, adjacent to γ can be added, solely[6] if $F(\gamma, s) = 1$. The generated combinatorial object is not any longer a group but merely a groupoid or a semi-groupoid. These constructions occur in a multitude of applications: Penrose lattices obtained from the cut-and-project method enter into the above groupoid category (diffusive properties [23] or type problem [5] of random walks on Penrose quasi-crystals), directed lattices considered in this paper into the semi-groupoid one, random graphs are also of the groupoid or semi-groupoid class. The algebraic object supporting these (semi)-groupoids are C^*-algebras. Therefore, there are interesting counterparts, not yet fully exploited, of the graphs we consider here and various natural objects like Penrose lattices, Cuntz-Krieger algebras, wavelet cascades, quantum channels, etc. Several of those extensions towards semi-groupoids and C^*-algebras are currently under investigation.

[6]The Cayley graph case corresponds to the trivial function $F \equiv 1$.

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