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1 Highlights

- Quantum classical correspondence of Dirac equation has been studied.

- a novel prescription for calculating observables has been outlined.

- quantum relativistic behavior of an elctron subjected to super-critical potential has been analysed.

- Few intriguing results obtained in this context are masslessness and near luminal propagation of electron.

On Quantum classical correspondence of Dirac equation

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Abstract

The unphysical results of quantum classical correspondence of Dirac equation have been analyzed. To evaluate observables in Dirac's theory of electron, a *'new prescription'* has been put forward that completely eliminates unphysical phenomena such as *zitterbewegung*. Spin angular momentum of free, non relativistic electron has been found questionable. Ultra relativistic behavior of electron has been studied for remarkable consequences like apparent masslessness and luminal propagation.

Keywords:
Zitterbewegung, intrinsic angular momentum, Dirac equation, Quantum classical correspondence, Heisenberg's equation of motion

1. Introduction

Dirac equation [1] has been extremely successful in explaining various quantum relativistic phenomena of bounded and free electron such as spin-orbit interaction (Darwin term, Thomas precession), correct magnetic moment, duplicity of polarization states and many other aspects that Schrodinger and Pauli-Schrodinger equation were just inadequate to handle. It has also been an important precursor to more fundamental Quantum theory of electromagnetic interaction i.e. Quantum electrodynamics.

However, it would be unfair to state that the current understanding of all aspects of this equation is thoroughly satisfactory. One such issue is

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of reduction of quantum mechanical observables of a given system to their classical counterparts in correspondence limit (Bohr's correspondence principle). While Schrodinger and Pauli-Schrodinger equations, in conjugation with Heisenberg's equation of motion, show remarkable conformity with Bohr's principle by yielding physically plausible results; Dirac equation fails to achieve the same in a relatively straightforward manner. Despite an attempt to resolve this problem, there seems to be no non ambiguous method of extracting observables in Dirac's theory and interpret them in a suitable manner. Unfortunately, in the due course, this deficiency of theory has become posed merely as a mathematical curiosity devoid of any physical content.

We can argue that the need for studying quantum-classical correspondence of Dirac equation (assuming that it can cater to both relativistic and non relativistic domain) has become more and more relevant in the wake of experimental verification of Ultra relativistic behavior electron in graphene and graphene like materials [2]. The reported cases of apparent masslessness and hence linear dispersion of electrons [3, 4, 5] in such systems are few exotic phenomena that can be given a cogent explanation if we have at our disposal, an effective method of calculating observables in relativistic quantum mechanics.

We begin our exposition with an elaborate restatement of the correspondence problem in the context of Dirac equation.

It's known that the very concept of particle velocity is of extremely important consequence in characterizing the dynamics of a particle both in quantum and classical domain. To obtain a velocity operator for a free electron in relativistic quantum mechanics, we consider the commutation relation Eq. (1) of position operator with Dirac Hamiltonian. The operator thus obtained αc is a non-dynamical (independent of field conditions and energy-momentum of particle) matrix operator and its only Eigen value corresponds to the velocity of light; a proposition strictly forbidden by special theory of relativity as a characteristic of a massive particle. Moreover, due to non dynamical nature of it, the entire velocity spectrum of particle seems contracted to a single value.

In comparison, well known non relativistic expression of velocity operator $\hat{p}/m$ consists of momentum operator divided by the mass of the given particle. Clearly, there appears no way to reduce velocity operator αc of Dirac equation to its non-relativistic limit without opting for unitarily altered Newton-Wigner representation [6] of the Dirac hamiltonian. With the help of a complementary, unitary transformation of wave function i.e. Foldy-

Wouthuysen transformation [7] a mildly successful resolution of this issue is possible. Though physically valid, the representation itself is not free of some mathematical contradictions. For example, comparing the velocity operators for an 'at rest particle' in Dirac and Newton-Wigner representation, we have;

$$\frac{[\hat{H}, \hat{x}]}{i\hbar} = \alpha c \tag{1}$$

$$\frac{[\hat{H}_{NW}, \hat{x}]}{i\hbar} = 0 \; ; \tag{2}$$

As we expect a coincidence of velocity operators at rest, in both representations, Eq. (1) and Eq. (2) are contradictory.

Non-reducibility of operators in Dirac's theory to non relativistic limits is at least as puzzling as their unphysical nature. It is an unsettling aspect of Dirac equation as the Hamiltonian itself can easily be reduced to non relativistic Schrodinger and Pauli Hamiltonian.

Further, It's well known that the complete position-time dependence of a free Dirac particle has features of rapid oscillations ($\nu \approx 10^{21} \; hz$) of very small amplitude termed as *zitterbewegung* motion [8]. An accelerating motion in absence of field or perturbation can be counted as one more unphysical result as a classical counterpart is highly unlikely to be observed for it would be in clear violation of Newton's law of inertia.

Usually, this phenomenon is speculated as a fundamental feature of Dirac equation which caters to both +ve and −ve energy Eigen states of particle. It has been argued that the origin of *zitterbewegung* lies in overlapping of these Eigen states [8]. Contrary to theoretical observation however, there have been only a little, direct evidence of *zitterbewegung* like motion [9]gerritsmafool. Furthermore, various attempt to relate this motion with the structure of electron [11, 12, 13, 14, 15] appear to be fraught with difficulties of theoretical nature and do not have much experimental validation either. Therefore, it can be contended that any effort in the aforementioned direction shall partially or totally undermine Bohr's principle.

Here, it's to be noticed that the non-dynamical nature of operators promptly hinders seamless contextualization (reduction to a particular limit) and hence unphysical nature, in Dirac's theory. A prime example is the case of spin angular momentum operator which is a quintessential non dynamical operator entirely independent of all field conditions and immune to relativistic corrections as well. It's not surprising that in the classical dynamics of a free, non

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relativistic electron, there appears to be no analog of spin angular momentum.

In the view of these shortcomings, we shall adopt a paradigm of identifying only dynamical operators as valid operators for evaluating observables in Dirac's theory. Two advantages of such operators are

a) Eigen value spectra of dynamical operators give more accurate description of nature and behavior of particle.
b) These operators are susceptible to relativistic corrections and can be easily reduced to any dynamical limit i.e. (field conditions, momentum energy limit).

Pondering over the relation of operators and Eigen functions in Dirac's theory, a glaring mismatch comes into view. Dirac hamiltonian $\hat{H}_d$ like Schrodinger hamiltonian $\hat{H}_s$ is not a scalar operator but a 4×4 spinor operator in the Hilbert space of spinor wave functions. Therefore, a conjugation of 4×4 operator and 1×1 scalar wave function vis-a-vis calculation of observables and commutation relations, is evidently an invalid approach. All physically unsuitable results pertaining to observables and commutations relations and origin of non dynamical operators can be eventually traced to it. Evaluation of expectation values and commutation relations involving 4×4 component operators and normalized spinor wave functions *'new prescription'*; immediately resolves the long standing problem of obtaining physically significant values of observables in most straightforward way in relativistic quantum mechanics and brings out some intriguing features of quantum relativistic electron. Following 'new prescription', non relativistic and ultra relativistic behavior of electron have been briefly studied in the following sections. The operators emerging in this new paradigm are dynamical operators. Any additional need to modify the definitions of operators [6] and secondary unitary transformations [7] seem only superfluous.

2. A generic wave function and normalization

Before proceeding towards calculations, in order to avoid clutter of notations, we obtain shorthand symbols for some elements of Dirac equation. Eigen vectors of Pauli matrices are denoted by

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$$z_+ = \left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\} \quad z_- = \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\} \quad z_+^\dagger = \left\{ \begin{array}{cc} 1 & 0 \end{array} \right\} \quad z_-^\dagger = \left\{ \begin{array}{cc} 0 & 1 \end{array} \right\}$$

Evidently;

$$z_+^\dagger z_+ = 1 \quad z_-^\dagger z_- = 1 \quad z_-^\dagger z_+ = 0 \quad z_+^\dagger z_- = 0$$

$|\uparrow\rangle$ and $|\downarrow\rangle$ have been used to denote $+ve$ and $-ve$ polarisation states.

Dirac equation has been used explicitly in this form;
$$\hat{H}\,|\psi\rangle = \left[\begin{array}{cc} mc^2 & \sigma.\hat{p}c \\ \sigma.\hat{p}c & -mc^2 \end{array} \right] |\psi\rangle$$

Using these notation, a generic electron wave function $|\psi\rangle$ polarized along $+ve$ z-axis and its conjugate $\langle\psi|$ can be written as

$$|\psi\rangle = \frac{1}{N} \left[\begin{array}{c} z_+ \\ \frac{\sigma.\hat{p}c}{\hat{H}+mc^2} z_+ \end{array} \right] \psi \qquad \langle\psi| = \psi^* \left[\begin{array}{cc} z_+^\dagger & z_+^\dagger \frac{\sigma.\hat{p}c}{\hat{H}+mc^2} \end{array} \right] \frac{1}{N^\dagger}$$

where $\frac{1}{N^\dagger}$ and $\frac{1}{N}$ stand for normalisation operators. We shall analyse two important cases of normalization here.

2.1. Normalizations in non-relativistic limit

$$\langle\psi|\psi\rangle = \psi^* \left[\begin{array}{cc} z_+^\dagger & z_+^\dagger \frac{\sigma.\hat{p}c}{\hat{H}+mc^2} \end{array} \right] \frac{1}{N^\dagger}\frac{1}{N} \left[\begin{array}{c} z_+ \\ \frac{\sigma.\hat{p}c}{\hat{H}+mc^2} z_+ \end{array} \right] \psi$$

In the above expression, momentum energy dependent terms can be neglected on account of their smallness and therefore normalisation operators should not differ appreciably from unity i.e. $\frac{1}{N^\dagger} = \frac{1}{N} = 1$. Implying;

$$\langle\psi|\psi\rangle = \psi^*\psi$$

2.2. Relativistic normalization

$$\langle\psi|\psi\rangle = \psi^* \left[\begin{array}{cc} z_+^\dagger & z_+^\dagger \frac{\sigma.\hat{p}c}{\hat{H}+mc^2} \end{array} \right] \frac{1}{N^\dagger}\frac{1}{N} \left[\begin{array}{c} z_+ \\ \frac{\sigma.\hat{p}c}{\hat{H}+mc^2} z_+ \end{array} \right] \psi \quad = \psi^* \left[\frac{1}{N^\dagger N} + \frac{1}{N^\dagger}\frac{\hat{p}^2 c^2}{(\hat{H}+mc^2)^2}\frac{1}{N} \right] \psi$$

We shall stipulate $\frac{1}{N^\dagger} = \frac{1}{N} = \left[1 + \frac{\hat{p}^2 c^2}{(\hat{H}+mc^2)^2} \right]^{-\frac{1}{2}}$; to obtain normalised wave function.

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3. Expectation values in non relativistic limit

3.1. Velocity expectation value

According to Heisenberg's equation of motion, temporal derivative of position expectation value is given by

$$\left\langle \frac{dx}{dt} \right\rangle = \frac{i}{\hbar} \langle \psi | \left[\hat{H}\hat{x} - \hat{H}\hat{x} \right] | \psi \rangle + \langle \psi | \frac{\partial x}{\partial t} | \psi \rangle$$

Assuming $\hat{x}$ has no explicit time dependence i.e. $\langle \psi | \frac{\partial x}{\partial t} | \psi \rangle = 0$; velocity expectation value for a free electron has following expression

$$\left\langle \frac{dx}{dt} \right\rangle = \frac{i}{\hbar} \langle \psi | \left[\hat{H}\hat{x} - \hat{H}\hat{x} \right] | \psi \rangle$$

Substituting Dirac hamiltonian and non-relativistic wave function in above expression we get;

$$\frac{\hbar}{i} \left\langle \frac{dx}{dt} \right\rangle =$$

$$\psi^* \left[\begin{array}{cc} z_+^\dagger & z_+^\dagger \frac{\sigma.\hat{p}c}{\hat{H}+mc^2} \end{array} \right] \left\{ \left[\begin{array}{cc} mc^2 & \sigma.\hat{p}c \\ \sigma.\hat{p}c & -mc^2 \end{array} \right] \hat{x} - \hat{x} \left[\begin{array}{cc} mc^2 & \sigma.\hat{p}c \\ \sigma.\hat{p}c & -mc^2 \end{array} \right] \right\} \left[\begin{array}{c} z_+ \\ \frac{\sigma.\hat{p}c}{\hat{H}+mc^2} z_+ \end{array} \right] \psi$$

In non relativistic limit, $\hat{H} + mc^2$ can be approximated as $\hat{H} + mc^2 \approx 2mc^2$. Subsequently;

$$\frac{\hbar}{i} \left\langle \frac{dx}{dt} \right\rangle =$$

$$\psi^* \left[\begin{array}{cc} z_+^\dagger & z_+^\dagger \frac{\sigma.\hat{p}c}{2mc^2} \end{array} \right] \left\{ \left[\begin{array}{cc} mc^2\hat{x} & \sigma.\hat{p}c\hat{x} \\ \sigma.\hat{p}c\hat{x} & -mc^2\hat{x} \end{array} \right] - \left[\begin{array}{cc} \hat{x}mc^2 & \hat{x}\sigma.\hat{p}c \\ \hat{x}\sigma.\hat{p}c & -\hat{x}mc^2 \end{array} \right] \right\} \left[\begin{array}{c} z_+ \\ \frac{\sigma.\hat{p}c}{2mc^2} z_+ \end{array} \right] \psi$$

Neglecting the terms involving powers of c; we obtain

$$\frac{\hbar}{i} \left\langle \frac{dx}{dt} \right\rangle =$$

$$\psi^* \left\{ mc^2\hat{x} - \hat{x}mc^2 + \frac{1}{2m}\sigma.\hat{p}\hat{x}\sigma.\hat{p} - \frac{1}{2m}\sigma.\hat{p}\hat{x}\sigma.\hat{p} + \frac{\hat{p}^2}{2m}\hat{x} - \hat{x}\frac{\hat{p}^2}{2m} \right\} \psi$$

Simplifying;

$$\frac{\hbar}{i}\left\langle\frac{dx}{dt}\right\rangle = \psi^*\left\{\frac{\hat{p}^2\hat{x}}{2m} - \frac{\hat{x}\hat{p}^2}{2m}\right\}\psi$$

Thus;

$$\left\langle\frac{dx}{dt}\right\rangle = \psi^*\frac{\hat{p}}{m}\psi$$

The above result coincides with that obtained from Schrdinger's non- relativistic equation. We shall expect orders of relativistic correction by writing down terms involving corresponding powers of c. In this way, not only physically valid result is obtained but a straightforward mechanism to calculate desired relativistic corrections is also identified. The velocity operator derived here, is essentially a dynamical operator.

3.2. Average acceleration of a Dirac electron

To find the average acceleration of electron, we calculate commutation relation of velocity operator with Dirac Hamiltonian. The result obtained here differs from Schrodinger's treatment and appears to be a physically more viable.

We know that;

$$\left\langle\frac{d^2x}{dt^2}\right\rangle = \frac{i}{\hbar}\langle\psi|\left[\hat{H}, \frac{dx}{dt}\right]|\psi\rangle$$

Substituting $\frac{dx}{dt}$ with $\frac{\hat{P}}{m}$ and expanding the above expression.

$$\frac{\hbar}{i}\left\langle\frac{d^2x}{dt^2}\right\rangle =$$

$$\psi^*\begin{bmatrix} z_+^\dagger & z_+^\dagger\frac{\sigma.\hat{p}c}{2mc^2} \end{bmatrix}\left\{\begin{bmatrix} mc^2\frac{\hat{p}}{m} & \sigma.\hat{p}c\frac{\hat{p}}{m} \\ \sigma.\hat{p}c\frac{\hat{p}}{m} & -mc^2\frac{\hat{p}}{m} \end{bmatrix} - \begin{bmatrix} \frac{\hat{p}}{m}mc^2 & \frac{\hat{p}}{m}\sigma.\hat{p}c \\ \frac{\hat{p}}{m}\sigma.\hat{p}c & -\frac{\hat{p}}{m}mc^2 \end{bmatrix}\right\}\begin{bmatrix} z_+ \\ \frac{\sigma.\hat{p}c}{2mc^2}z_+ \end{bmatrix}\psi$$

A bit of simplification and rearrangement yields

$$\frac{\hbar}{i}\left\langle\frac{d^2x}{dt^2}\right\rangle = 0$$

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In absence of any force field, this is certainly a valid outcome. It implies that Newton's laws hold without any conflict in quantum domain. In the light of Section 5.1 and 5.2; it can be concluded that *Zitterbewegung* like motion is entirely absent from the dynamics of a Dirac electron hence experimental results obtained by Gerritsma et. al. [9] is debatable.

3.3. Angular momentum as a constant of motion for free electron

For a free electron, it can be readily shown that the total angular momentum conserved in non relativistic limit is merely $\hat{L}$. Spin angular momentum does not arise in this context. From Heisenberg's equation of motion, we have;

$$\left\langle \frac{dL}{dt} \right\rangle = \frac{i}{\hbar} \langle \psi | [\hat{H}\hat{L} - \hat{H}\hat{L}] | \psi \rangle + \langle \psi | \frac{\partial L}{\partial t} | \psi \rangle$$

In absence of any perturbation, we can set $\langle \psi | \frac{\partial L}{\partial t} | \psi \rangle = 0$; therfore the only requirement for angular momentum to remain constant in time is

$$\left\langle \frac{dL}{dt} \right\rangle = \frac{i}{\hbar} \langle \psi | [\hat{H}\hat{L} - \hat{H}\hat{L}] | \psi \rangle = 0$$

Explicitly;

$$\frac{\hbar}{i} \left\langle \frac{dL}{dt} \right\rangle =$$

$$\psi^* \begin{bmatrix} z_+^\dagger & z_+^\dagger \frac{\sigma.\hat{p}c}{2mc^2} \end{bmatrix} \left\{ \begin{bmatrix} mc^2\hat{L} & \sigma.\hat{p}c\hat{L} \\ \sigma.\hat{p}c\hat{L} & -mc^2\hat{L} \end{bmatrix} - \begin{bmatrix} \hat{L}mc^2 & \hat{L}\sigma.\hat{p}c \\ \hat{L}\sigma.\hat{p}c & -\hat{L}mc^2 \end{bmatrix} \right\} \begin{bmatrix} z_+ \\ \frac{\sigma.\hat{p}c}{2mc^2}z_+ \end{bmatrix} \psi$$

Simplifying and rearranging the above expression; we obtain

$$\frac{\hbar}{i} \left\langle \frac{dL}{dt} \right\rangle = \psi^* \left\{ \frac{\hat{p}^2\hat{L}}{2m} - \frac{\hat{L}\hat{p}^2}{2m} \right\} \psi$$

or;

$$\frac{\hbar}{i} \left\langle \frac{dL}{dt} \right\rangle = 0 \text{ since; } \left[\hat{p}^2, \hat{L} \right] = 0$$

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This result implies that, contrary to the established view, a free, non-relativistic electron does not possess intrinsic angular momentum or spin. Energy Eigen values of both polarization states are equal i.e. $E\left|\uparrow\right\rangle \equiv E\left|\downarrow\right\rangle$. This is precisely the case as described by Schrodinger's scalar wave function. Presence of an external field (read Pauli-Schrodinger spinor wave-function) however, breaks this degeneracy as these states acquire an additional amount of energy[3] proportional to $\sigma.e\hbar/2$.

4. Ultra-relativistic limit

A crucial test of *new prescription* lays in providing satisfactory results in all energy limits including Ultra relativistic limit. Two important cases that define this limit are:

(a) A free particle in extremum of momentum and energy ($\hat{p} \approx \infty \ hatH \approx \infty$). It's a theoretical case that entails supplying an electron with infinite amount of momentum-energy. From special theory of relativity, we expect the electron to move close to the velocity of light.

(b) Application of super critical[4] potential. ($\hat{H} + mc^2 - \hat{p}c \approx e\phi$). Recent developments in physics of single and bi-layer graphene offer possibilities of realizing these conditions. Behavior of electron in super-critical system closely resembles that of a massless quasi particle propagating with near luminal average velocity.

We shall proceed towards calculation of observables after obtaining normalized wave functions in both the case

4.1. Case (a) Extremum of momentum-energy

Recalling Section 4.2 we have

$$\frac{1}{N^\dagger} = \frac{1}{N} = \left[1 + \frac{\hat{p}^2 c^2}{(\hat{H} + mc^2)^2}\right]^{-\frac{1}{2}}$$

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[3]This quantity is merely a non-relativistic approximation.

[4]For extremely small value of momentum ($\hat{p} \approx 0; or \hat{H} + mc^2 \approx e\phi$); it becomes a case of Klein potential

Evaluating the limit using relation $\hat{H} = \sqrt{\hat{p}^2 c^2 + m^2 c^4}$;

$$\lim_{\hat{p} \to \infty} \frac{1}{N^\dagger} = \frac{1}{N} = \left[1 + \frac{\hat{p}^2 c^2}{(\hat{H} + mc^2)^2}\right]^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Therefore the normalized wave function is

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[\begin{array}{c} z_+ \\ \frac{\sigma.\hat{p}c}{\hat{H}+mc^2} z_+ \end{array} \right] \psi$$

The above expression can be further simplified by taking limit in the lower component of wave function. Noticing $\lim_{\hat{p} \to \infty} \frac{\sigma.\hat{p}c}{\hat{H}+mc^2} z_+ = \sigma.z_+$

$$\text{We finally have } |\psi\rangle = \frac{1}{\sqrt{2}} \left[\begin{array}{c} z_+ \\ \sigma.z_+ \end{array} \right] \psi$$

4.2. Case (b) Super-critical potential

Proceeding in a similar fashion, we evaluate normalization operators; given ($e\phi \approx \hat{H} + mc^2 - \hat{p}c$)

$$\frac{1}{N^\dagger} = \frac{1}{N} = \left[1 + \frac{\hat{p}^2 c^2}{(\hat{H} + mc^2 - e\phi)^2}\right]^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Evidently, normalized wave function is $|\psi\rangle = \frac{1}{\sqrt{2}} \left[\begin{array}{c} z_+ \\ \sigma.z_+ \end{array} \right] \psi$ We notice that

the wave function in two seemingly different cases Section 4.1 and 4.2 are same and therefore conclude that the dynamical behavior of both systems will be alike. Various physical characteristics of these systems have been studied using Heisenberg's equation of motion and summarized comprehensively in the following section.

4.3. Velocity expectation value

From Heisenberg's equation of motion;

$$\left\langle \frac{dx}{dt} \right\rangle = \frac{i}{\hbar} \langle\psi| [\hat{H}\hat{x} - \hat{H}\hat{x}] |\psi\rangle$$

Substituting normalized ultra relativistic wave function, we get

$$\left\langle \frac{dx}{dt} \right\rangle = \frac{i}{\hbar} \psi^* \left[\begin{array}{cc} z_+^\dagger & \sigma.z_+^\dagger \end{array} \right] \frac{1}{\sqrt{2}} [\hat{H}\hat{x} - \hat{H}\hat{x}] \frac{1}{\sqrt{2}} \left[\begin{array}{c} z_+ \\ \sigma.z_+ \end{array} \right] \psi$$

Explicitly;

$$\frac{\hbar}{i} \left\langle \frac{dx}{dt} \right\rangle =$$

$$\psi^* \left[\begin{array}{cc} z_+^\dagger & \sigma.z_+^\dagger \end{array} \right] \frac{1}{\sqrt{2}} \left\{ \left[\begin{array}{cc} mc^2\hat{x} & \sigma.\hat{p}c\hat{x} \\ \sigma.\hat{p}c\hat{x} & -mc^2\hat{x} \end{array} \right] - \left[\begin{array}{cc} \hat{x}mc^2 & \hat{x}\sigma.\hat{p}c \\ \hat{x}\sigma.\hat{p}c & -\hat{x}mc^2 \end{array} \right] \right\} \frac{1}{\sqrt{2}} \left[\begin{array}{c} z_+ \\ \sigma.z_+ \end{array} \right] \psi$$

By simplifying above expression, we readily obtain

$$\left\langle \frac{dx}{dt} \right\rangle = \psi^* \, c \, \psi$$

In the context of Case (a) 'extremum of momentum-energy', this result can be termed trivial; its implication being a purely quantum relativistic upper bound on maximum average velocity of a particle.

On the other hand, luminal propagation is a non trivial aspect of quantum mechanical behavior of electrons subjected to super critical potentials.

4.4. Contribution of rest energy to Total energy

Evaluation of rest energy contribution to the total energy of particle in ultra relativistic limit can help assimilate the results obtained in previous section.
Evidently;

$$\langle E_{rest} \rangle = \langle \psi | \left[\begin{array}{cc} mc^2 & 0 \\ 0 & -mc^2 \end{array} \right] | \psi \rangle$$

or;

$$\langle E_{rest} \rangle = \psi^* \left[\begin{array}{cc} z_+^\dagger & \sigma.z_+^\dagger \end{array} \right] \frac{1}{\sqrt{2}} \left[\begin{array}{cc} mc^2 & 0 \\ 0 & -mc^2 \end{array} \right] \frac{1}{\sqrt{2}} \left[\begin{array}{c} z_+^\dagger \\ \sigma.z_+^\dagger \end{array} \right] \psi$$

On simplification;

$$\langle E_{rest} \rangle = 0$$

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Therefore, the observable rest energy of particle vanishes in Ultra relativistic limit. This result truly complements luminal propagation of ultra relativistic electron. Though apparent masslessness is certainly expected in extremum of momentum-energy; electrons in critical potential show this interesting behavior too. From Sections 4.1 and 4.2, it can be easily shown that in a rather idealized case of potential $\phi(x, y, z) \approx 2mc^2/e$ at given (x, y, z); the dispersion relation in the vicinity of given point should be linear i.e. $\omega \approx ck$.

4.5. Conserved angular momentum

An intriguing case that emerges in ultra-relativistic limit is reappearance of intrinsic angular momentum in absence of any field or potential. To find the conserved angular momentum of a ultra-relativistic system; we have

$$\left\langle \frac{dL}{dt} \right\rangle = \frac{i}{\hbar} \langle \psi | [\hat{H}\hat{L} - \hat{H}\hat{L}] | \psi \rangle = 0$$

We substitute ultra relativistic wave functions to obtain;

$$\left\langle \frac{dL}{dt} \right\rangle = \psi^* \begin{bmatrix} z_+^\dagger & \sigma.z_+^\dagger \end{bmatrix} \frac{1}{\sqrt{2}} [\hat{H}\hat{L} - \hat{H}\hat{L}] \frac{1}{\sqrt{2}} \begin{bmatrix} z_+^\dagger \\ \sigma.z_+^\dagger \end{bmatrix} \psi$$

An explicit calculation yields;

$$\left\langle \frac{dL}{dt} \right\rangle = \psi^* [\sigma_x i\hbar c[\hat{p}_z - \hat{p}_y] + \sigma_y i\hbar c[\hat{p}_x - \hat{p}_z] + \sigma_z i\hbar c[\hat{p}_y - \hat{p}_x]]\psi$$

Taking a cue from Dirac's treatment[1], we try $\hat{T} = \hat{L} + \sigma.\hbar/2$ where the total conserved angular momentum is $\hat{T}$ and $\sigma.\hbar/2$ is an additional angular momentum. It is readily shown that $\langle dT/dt \rangle = 0$. This result suggests that unlike the non relativistic case (See Section 3.3), even a free electron in ultra-relativistic limit possesses a non vanishing angular momentum $\sigma.\hbar/2$.

5. Comments and Conclusion

In the view of arguments and results discussed in previous sections, it can be safely assumed that our objective of obtaining an agreeable mechanism for Quantum classical correspondence of Dirac equation has been fulfilled. Results obtained in section 5.1 and 5.2 provide a physical picture that accurately describes a quantum analog of classical point-sized electron. All

of the quantum mechanical properties seem to have much verified classical counterparts. Vanishing of spin angular momentum for a free non relativistic electron is an experimentally verifiable result. Another prediction that seems imminently testable is masslessness and nearly luminal propagation of electron in systems featuring super-critical potential.

6. Acknowledgement

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