MULTIPLICITY OF SOLUTIONS FOR NON-LOCAL ELLIPTIC EQUATIONS DRIVEN BY FRACTIONAL LAPLACIAN

XIFENG SU AND YUANHONG WEI

ABSTRACT. We consider the semi-linear elliptic PDEs driven by the fractional Laplacian:

$$\begin{cases} (-\Delta)^s u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \backslash \Omega. \end{cases}$$

By the Mountain Pass Theorem and some other nonlinear analysis methods, the existence and multiplicity of non-trivial solutions for the above equation are established. The validity of the Palais-Smale condition without Ambrosetti-Rabinowitz condition for non-local elliptic equations is proved. Two non-trivial solutions are given under some weak hypotheses. Non-local elliptic equations with concave-convex nonlinearities are also studied, and existence of at least six solutions are obtained.

Moreover, a global result of Ambrosetti-Brezis-Cerami type is given, which shows that the effect of the parameter λ in the nonlinear term changes considerably the nonexistence, existence and multiplicity of solutions.

Keywords: Non-local operator; variational method; fractional Laplacian; Nehari manifold; multiplicity

1. INTRODUCTION

The fractional Laplacian $-(-\Delta)^s$ is a classical linear integro-differential operator of order $2s$ which gives the standard Laplacian when $s = 1$.

A range of powers of particular interest is $s \in (0,1)$ and we can write the operator (up to normalization factors) as

$$-(-\Delta)^s u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy, \;\; x \in \mathbb{R}^n, \;\; u \in \mathcal{S}(\mathbb{R}^n), \tag{1.1}$$

where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^n$.

The fractional Laplacian and non-local operators of elliptic type arises in both pure mathematical research and concrete applications, such as the thin obstacle problem [24, 8], minimal surfaces [6, 7], phase transitions [25], crystal dislocation [13], Markov processes [15] and fractional quantum mechanics [17]. See [11] and references therein for an elementary introduction to the literature. In [14], it is pointed out that, being the generator of the symmetric α-stable ($0 < \alpha < 1$) processes (Lévy flights in some of the physical literature), fractional Laplacians are widely used to model systems of stochastic dynamics with applications in operation research, queuing theory, mathematical finance and risk estimation. In contrast to the Brownian motion ($\alpha = 1$), which can be taken as the limiting model of the random walk in which the test particles are assumed to jump to one of the nearest neighbor sites, we can take the stable process as the limiting model of such a random walk in which the test particles are assumed to jump to any other sites with power law decay in probability (see also [30]). Elliptic equations with fractional Laplacian also studied by many other authors, see [4, 5, 28, 29, 19, 12, 9] and references therein.

The purpose of this paper is to study the superlinear elliptic boundary problems driven by the non-local operator:

$$(P) \qquad \begin{cases} \mathcal{L}_K u + f(x,u) = 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases}$$

where

$$\mathcal{L}_K u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \Big(u(x+y) + u(x-y) - 2u(x)\Big)K(y)dy, \qquad x \in \mathbb{R}^n.$$

Note that if $K(x) = |x|^{-(n+2s)}$, $\mathcal{L}_K$ is the fractional Laplacian $-(-\Delta)^s$ in (1.1).

In this paper, $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ is a function satisfying the following properties [23]:

- $\gamma K \in L^1(\mathbb{R})$ with $\gamma(x) = \min\{|x|^2, 1\}$;
- there exists $\lambda > 0$ such that $K(x) \geqslant \lambda |x|^{-(n+2s)}$ for any $x \in \mathbb{R}^n \setminus \{0\}$;
- $K(x) = K(-x)$ for any $x \in \mathbb{R}^n \setminus \{0\}$.

It is quite standard to transform the problem to find the solutions of (P) into the problem of finding the critical points of an associated energy functional on some appropriate space (see Section 2). In [23], the authors establish the existence of non-trivial solution for (P) by the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [3].

In this paper, we will improve some of the existence results for (P) and establish some results about multiplicity of solutions. To be precise, we first derive the existence theorem for (P) without Ambrosetti-Rabinowitz condition of [23]. We obtain the boundedness of Palais-Smale sequence under some weak assumptions, then the existence of solution follows. We refer [16, 20, 18] for such generalizations in the standard Laplacian case.

Following the ideas of [1] which deals with the standard Laplacian case, we consider the fractional Laplacian concave-convex nonlinear problem

$$(P)_\lambda \qquad \begin{cases} \mathcal{L}_K u + \lambda h(x)|u|^{p-2}u + g(x,u) = 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases}$$

where $1 < p < 2$, $\lambda > 0$ is a real parameter. For $\lambda > 0$ small enough, we obtain four non-trivial solutions by Mountain Pass Theorem and Ekeland's variational principle. Furthermore, the existence of multiple solutions (six solutions) is established under some assumptions. Our methods to obtain the fifth solution follows the ideas developed in [2] for Laplacian operator. In [2], assuming that $g(x,u) = g(u)$ is of class C^1 and $G(u) = \int_0^u g(s)ds$ has the following form

$$G(s) = \frac{1}{\alpha}|s|^\alpha + o(|s|^\alpha), \quad 2 < \alpha < 2^*, \quad \text{at} \quad s = 0, s = \infty,$$

six solutions are obtained for elliptic equations with the Laplacian operator. We prove a fractional Laplacian version under some weak assumptions, especially no C^1 constraint imposed on g. In this case, Nehari manifold is not a C^1 manifold anymore and the arguments for the existence of the sixth solution in [2] can not be used to deal with this problem. To overcome this difficult, we introduce some new method of Nehari manifold originally from [27]. Finally, a global results of Ambrosetti-Brezis-Cerami type is also considered, which is motivated by [1] for the elliptic problem with the standard Laplacian. Our result is concerned with the existence, nonexistence and multiplicity of solutions depending on the parameter λ. We show that the combined effects of a sublinear and a superlinear term change considerably the structure of the solution set.

For completeness, we will also consider the particular case at the end of this paper when the functional $\mathscr{F}_\lambda$ in (4.1) is even in u. Thus, we can make use of the Lusternik-Schnirelman theory to find infinitely many pairs of critical points.

This paper is organized as follows. In Section 2, we introduce some preliminary facts and assumptions. In Section 3, we give the proof of validity of the Palais-Smale condition and the existence of nontrivial solutions for problem (P). Sections 4-6 are devoted to the multiplicity of solutions of problem $(P)_\lambda$. Five solutions are obtained in Section 4 under some weak hypotheses, and then, under some slightly strong conditions, the existence of six solutions is established in Section 5, by virtue of the methods of Nehari manifold. Finally, we also consider a result of Ambrosetti-Brezis-Cerami type in the last section, in which we give a global result about existence, nonexistence and multiplicity of solutions and more information about the solutions depending on the parameter λ.

2. Some preliminary facts

Let Ω be an open set in $\mathbb{R}^n$. The usual norm in $L^p(\Omega)$ will be denoted by $|\cdot|_p$. For $s \in (0,1)$, we denote the classical factional Sobolev space

$$H^s(\Omega) \equiv \left\{ u \in L^2(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{n+2s}{2}}} \in L^2(\Omega \times \Omega) \right\} \tag{2.1}$$

with the Gagliardo norm

$$\|u\|_{H^s(\Omega)} = |u|_2 + \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{\frac{1}{2}}. \tag{2.2}$$

Due to non-localness of the fractional Laplacian, we will consider the space $(X_0, \|\cdot\|_{X_0})$ defined as follows, rather than the classical fractional Sobolev space. Note that the norm $\|\cdot\|_{X_0}$ involves the interaction between Ω and $\mathbb{R}^n \setminus \Omega$ which is introduced in [23]. Denote

$$Q = \mathbb{R}^{2n} \setminus ((\mathbb{R}^n \setminus \Omega) \times (\mathbb{R}^n \setminus \Omega)).$$

Note that $Q \supsetneqq \Omega \times \Omega$. We define

$$X \equiv \left\{ u : \mathbb{R}^n \to \mathbb{R} \text{ is Lebesgue measurable} : \ u|_\Omega \in L^2(\Omega);\ (u(x)-u(y))\sqrt{K(x-y)} \in L^2(Q) \right\}$$

with the norm

$$\|u\|_X = \|u\|_{L^2(\Omega)} + \left(\int_Q |u(x) - u(y)|^2 K(x-y)dxdy\right)^{\frac{1}{2}}. \tag{2.3}$$

We consider the space

$$X_0 \equiv \{\, u \in X : \; u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}$$

with the norm

$$\|u\|_{X_0} = \left(\int_Q |u(x) - u(y)|^2 K(x-y)dxdy\right)^{\frac{1}{2}} \tag{2.4}$$

and therefore $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space. (See [23] for the proof.) We denote $\langle , \rangle_{X_0}$ the inner product on X_0 induced by the norm $\|\cdot\|_{X_0}$.

We say that $u \in X_0$ is a weak solution of problem (P), if u satisfies

$$\int_{\mathbb{R}^{2n}} (u(x) - u(y))(\phi(x) - \phi(y))K(x-y)dxdy = \int_\Omega f(x, u(x))\phi(x)dx, \tag{2.5}$$

for all $\phi \in X_0$.

The fact that u is a weak solution is equivalent to being a critical point of the functional

$$\mathscr{F}(u) = \frac{1}{2}\int_Q |u(x) - u(y)|^2 K(x-y)dxdy - \int_\Omega F(x, u(x))dx, \tag{2.6}$$

where $F(x, u) = \int_0^u f(x, s)ds$.

By the property of F, it is easy to check that $\mathscr{F} \in C^1(X_0, \mathbb{R})$ and

$$\langle \mathscr{F}'(u), \phi\rangle = \int_{\mathbb{R}^{2n}} (u(x) - u(y))(\phi(x) - \phi(y))K(x-y)dxdy - \int_\Omega f(x, u(x))\phi(x)dx$$

for any $\phi \in X_0$. In this paper, we are interested in establishing the existence of the non-trivial critical points of $\mathscr{F}$.

In particular, when $K(x) = |x|^{-(n+2s)}$, $\mathcal{L}_K$ is fractional Laplacian $-(-\Delta)^s$, (2.5) is the weak formulation (see [11] for more details) of fractional elliptic equation

$$\begin{cases} (-\Delta)^s u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \backslash \Omega \end{cases} \tag{2.7}$$

and the corresponding functional is given by

$$\mathscr{F}(u) = \frac{1}{2}\int_Q \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}dxdy - \int_\Omega F(x, u(x))dx, \qquad u \in X_0.$$

In order to carry out the nonlinear analysis, we investigate some key facts of X_0.

Note that $C_0^2(\Omega) \subseteq X_0$, $X \subseteq H^s(\Omega)$ and $X_0 \subseteq H^s(\mathbb{R}^n)$ (see [21]). In fact, we have the following embedding theorem.

Lemma 2.1. *The embedding $X_0 \hookrightarrow L^{2^*}(\Omega)$ is continuous where $2^* = \frac{2n}{n-2s}$.*

To apply the Mountain Pass Theorem, we need to assume that the nonlinearity f, a Carathéodory function, satisfies the following conditions:

(H1) There exist $a_1, a_2 > 0$ and $q \in (2, 2^*)$ such that

$$|f(x,t)| \leqslant a_1 + a_2|t|^{q-1} \qquad \text{a.e. } x \in \Omega,\ t \in \mathbb{R};$$

(H2) $\lim_{t\to 0} \frac{f(x,t)}{t} = 0$ uniformly for a.e. $x \in \Omega$.

(H3) $\lim_{|t|\to\infty} \frac{F(x,t)}{t^2} = +\infty$ uniformly for a.e. $x \in \Omega$.

(H4) there exists $T_0 > 0$ such that $\frac{f(x,t)}{|t|}$ is increasing in t when $|t| > T_0$, for all $x \in \Omega$.

We point out that the condition (H4) can also be replaced by the following weaker assumption

(H4)* Denote $H(x,s) = sf(x,s) - 2F(x,s)$. There exists $C_* > 0$ such that

$$H(x,t) \leqslant H(x,s) + C_*$$

for all $0 < t < s$ or $s < t < 0$, $\forall x \in \Omega$.

Remark 2.2. *Note that these conditions on f can be seen as a generalization of the ones in* [23]*. For instance, if we take*

$$F(x,t) = t^2 \log(1 + |t|),$$

it is easy to check that f satisfies our assumptions here but cannot be dealt with by assumptions in [23].

Remark 2.3. *It is not difficult to check that the condition (H4) is equivalent to the following condition (see* [18]*):*

(H4)$_*$ *$H(x,s)$ is increasing in $s \geqslant s_0$ and decreasing in $s \leqslant -s_0$ for all $x \in \Omega$.*

Hence, (H4) implies (H4)*.

3. Palais-Smale condition and existence of solutions

Theorem 3.1. *Assume that (H1)-(H4) hold. Then, problem (P) has at least one non-trivial solution.*

Theorem 3.2. *Assume that (H1)-(H3), (H4)* hold. Then, problem (P) has at least one non-trivial solution.*

Now we prove that the functionals $\mathscr{F}$ has the mountain pass geometry.

Lemma 3.3. *Under the assumption (H3), $\mathscr{F}$ are unbounded from below.*

Proof. (H3) implies that, for all $M > 0$ there exists $C_M > 0$ such that

$$F(x,s) \geqslant Ms^2 - C_M, \quad \forall x \in \Omega, \forall s > 0. \tag{3.1}$$

As in [23], we fix $\phi \in X_0$ with $\phi \geqslant 0$ a.e. in $\mathbb{R}^n$. This choice can be obtained by taking the positive part of any $\phi \in X_0$, which belongs to X_0, thanks to [21].

From (3.1) we obtain

$$\begin{aligned}\mathscr{F}(t\phi) &= \frac{1}{2}\int_Q |t\phi(x)-t\phi(y)|^2K(x-y)dxdy - \int_\Omega F(x,t\phi(x))dx\\ &\leqslant \frac{t^2}{2}\|\phi\|_{X_0}^2 - \int_\Omega Mt^2\phi^2dx + \int_\Omega C_Mdx\\ &= t^2\left(\frac{1}{2}\|\phi\|_{X_0}^2 - M\int_\Omega \phi^2dx\right) + C_M|\Omega|,\end{aligned}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Let $M = \dfrac{\|\phi\|_{X_0}^2}{2\int_\Omega \phi^2dx} + 1$. Then

$$\lim_{t\to+\infty}\mathscr{F}(t\phi) = -\infty. \tag{3.2}$$

□

Lemma 3.4. *Assume that (H1) and (H2) hold. Then there exist $\rho, R > 0$ such that $\mathscr{F}(u) \geqslant R$, if $\|u\|_{X_0} = \rho$.*

Proof. (H1) and (H2) imply that for any given $\epsilon > 0$, there exists $c_\epsilon > 0$ such that

$$F(x,s) \leqslant \epsilon s^2 + c_\epsilon s^q, \quad \text{a.e. } x\in\Omega,\ \forall s>0. \tag{3.3}$$

Combining (3.3) and Hölder inequality, we have

$$\begin{aligned}\mathscr{F}(u) &\geqslant \frac{1}{2}\|u\|_{X_0}^2 - \epsilon\int_\Omega |u|^2dx - c_\epsilon\int_\Omega |u|^qdx\\ &\geqslant \frac{1}{2}\|u\|_{X_0}^2 - \epsilon|u|_2^2 - c_\epsilon|u|_q^q\\ &\geqslant \frac{1}{2}\|u\|_{X_0}^2 - \epsilon|\Omega|^{(2^*-2)/2^*}|u|_{2^*}^2 - c_\epsilon|\Omega|^{(2^*-q)/2^*}|u|_{2^*}^q\\ &\geqslant (\frac{1}{2} - \epsilon c_0|\Omega|^{(2^*-2)/2^*})\|u\|_{X_0}^2 - c_0^{q/2}c_\epsilon|\Omega|^{(2^*-q)/2^*}\|u\|_{X_0}^q,\end{aligned}$$

where c_0 is a positive constant, thanks to the Lemma 2.1 and the fact that Ω is bounded. Taking $\epsilon c_0|\Omega|^{(2^*-2)/2^*} \leqslant \frac{1}{4}$ and choosing $\|u\|_{X_0} = \rho > 0$ small enough, we can find $R > 0$ such that $\mathscr{F}(u) \geqslant R$ when $\|u\|_{X_0} = \rho$. □

Now, we prove that every Palais-Smale sequence of $\mathscr{F}$ is relatively compact.

We recall that a sequence $\{u_j\} \subset X_0$ is said to be a Palais-Smale sequence of functional $\mathscr{F}$ provided that $\mathscr{F}(u_j)$ is bounded and $\mathscr{F}'(u_j) \to 0$ in X_0^*.

Lemma 3.5. *Suppose that (H1), (H3) and (H4) hold. Then every Palais-Smale sequence of $\mathscr{F}$ has a converge subsequence in X_0.*

Proof. We show that every Palais-Smale sequence of $\mathscr{F}$ is bounded in X_0 and Lemma 3.5 follows easily from a standard argument (for instance, Proposition 12 of [23]). Assume that $\{u_j\} \subset X_0$ is a Palais-Smale sequence of $\mathscr{F}$, i.e.,

$$\mathscr{F}(u_j) \to c,\ \langle\mathscr{F}'(u_j),\varphi\rangle \to 0, \quad \forall\varphi\in X_0. \tag{3.4}$$

We suppose, by contradiction, that up to a subsequence, still denoted by u_j,

$$\|u_j\|_{X_0} \to +\infty \quad \text{as } j\to+\infty.$$

Set $\omega_j := \dfrac{u_j}{\|u_j\|_{X_0}}$. Then

$$\|\omega_j\|_{X_0} = 1. \tag{3.5}$$

Passing to a subsequence, we may assume that there exists $\omega \in X_0$ such that

$$\begin{aligned} \omega_j &\rightharpoonup \omega, \quad \text{weakly in } X_0, \quad j \to +\infty, \\ \omega_j &\to \omega, \quad \text{strongly in } L^2(\Omega), \ j \to +\infty, \\ \omega_j(x) &\to \omega(x), \quad \text{a.e. in } \Omega, \ j \to +\infty. \end{aligned}$$

We claim that $\omega(x) \equiv 0$ a.e. in $\mathbb{R}^n$. It suffices to show $\omega(x) \equiv 0$ a.e. in Ω. In fact, we denote $\Omega^* := \{x \in \Omega, \ \omega(x) \neq 0\}$. If $\Omega^* \neq \emptyset$, then for $x \in \Omega^*$, $|u_j(x)| \to +\infty$ as $j \to +\infty$. By (H3) we have

$$\lim_{j\to+\infty} \frac{F(x, u_j(x))}{(u_j(x))^2} (\omega_j(x))^2 = +\infty. \tag{3.6}$$

The Fatou's Lemma and (3.4) imply

$$\begin{aligned} \int_\Omega \lim_{j\to+\infty} \frac{F(x,u_j(x))}{(u_j(x))^2}(\omega_j(x))^2 dx &= \int_\Omega \lim_{j\to+\infty} \frac{F(x,u_j(x))}{(u_j(x))^2} \frac{(u_j(x))^2}{\|u_j\|^2_{X_0}} dx \\ &\leqslant \liminf_{j\to+\infty} \frac{1}{\|u_j\|^2_{X_0}} \int_\Omega F(x,u_j(x))dx \\ &= \lim_{j\to\infty} \frac{1}{\|u_j\|^2_{X_0}} \left(\frac{1}{2}\|u_j\|^2_{X_0} - \mathscr{F}(u_j) \right) \\ &= \frac{1}{2}. \end{aligned} \tag{3.7}$$

Hence Ω^* has zero measure. Consequently, $\omega(x) \equiv 0$ a.e. in Ω.

As in [16], we take $t_j \in [0, 1]$ such that

$$\mathscr{F}(t_j u_j) = \max_{t\in[0,1]} \mathscr{F}(t u_j),$$

which implies that

$$\frac{d}{dt}\mathscr{F}(tu_j)\Big|_{t=t_j} = t_j \|u_j\|^2_{X_0} - \int_\Omega f(x, t_j u_j) \cdot u_j dx = 0. \tag{3.8}$$

Since

$$\langle \mathscr{F}'(t_j u_j), t_j u_j \rangle = t_j^2 \|u_j\|^2_{X_0} - \int_\Omega f(x, t_j u_j) \cdot t_j u_j dx,$$

together with (3.8), it follows that

$$\langle \mathscr{F}'(t_j u_j), t_j u_j \rangle = t_j \cdot \frac{d}{dt}\mathscr{F}(tu_j)\Big|_{t=t_j} = 0.$$

Hence, by (H4)*, we obtain

$$
\begin{aligned}
2\mathscr{F}(tu_j) &\leqslant 2\mathscr{F}(t_ju_j) - \langle \mathscr{F}'(t_ju_j), t_ju_j\rangle \\
&= \int_\Omega (t_ju_j \cdot f(x, t_ju_j) - 2F(x, t_ju_j))dx \\
&\leqslant \int_\Omega (u_j \cdot f(x, u_j) - 2F(x, u_j) + C_*)dx \qquad (3.9)\\
&= 2\mathscr{F}(u_j) - \langle \mathscr{F}'(u_j), u_j\rangle + C_*|\Omega| \\
&\to 2c + C_*|\Omega|.
\end{aligned}
$$

On the other hand, for all $k > 0$,

$$2\mathscr{F}(k\omega_j) = k^2 - 2\int_\Omega F(x, k\omega_j)dx = k^2 + o(1),$$

which contradicts (3.9) for k and j large enough. This completes the proof. □

Proof of Theorem 3.2. Since the functional $\mathscr{F}$ has the mountain pass geometry and satisfies the Palais-Smale condition, the Mountain Pass Theorem (see [3]) gives that there exists a critical point $u \in X_0$. Moreover, $\mathscr{F}(u) \geqslant R > 0 = \mathscr{F}(0)$, so u is a non-trivial solution. □

Similar to [23], one can determine the sign of the Mountain Pass type solutions. Indeed, we have the following corollary which is useful to construct the multiple solutions of Problem (P) with the concave-convex nonlinearity.

Corollary 3.6. *Let all the assumptions of Theorem 3.1 (Theorem 3.2) be satisfied. Then, Problem (P) admits a non-negative solution $u_+ \in X_0$ and a non-positive solution $u_- \in X_0$ that are of Mountain-Pass type and that are non-trivial.*

Proof. Consider the following problem

$$
\begin{cases} \mathcal{L}_K u + f^+(x, u) = 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{3.10}
$$

where

$$
f^+(x, t) = \begin{cases} f(x, t), & t \geqslant 0, \\ 0, & t < 0. \end{cases}
$$

Define the corresponding functional $\mathscr{F}^+ : X_0 \to \mathbb{R}$ as follow:

$$\mathscr{F}^+(u) = \frac{1}{2}\|u\|^2_{X_0} - \int_\Omega F^+(x, u)dx, \quad u \in X_0,$$

where $F^+(x, u) = \int_0^u f^+(x, s)ds$. Obviously, $\mathscr{F}^+ \in C^1(X_0, \mathbb{R})$ and f^+ satisfy all the conditions of Theorem 3.1 (Theorem 3.2). Let u_+ be a non-trivial critical point of $\mathscr{F}^+$, which implies that u_+ is a weak solution of (3.10). It is known by [23] that $u_+ \geqslant 0$ a.e. in $\mathbb{R}^n$. Thus u_+ is also a non-trivial solution of problem (P) and $\mathscr{F}(u_+) = \mathscr{F}^+(u_+)$.

Similarly, we can define

$$
f^-(x, t) = \begin{cases} f(x, t), & t \leqslant 0, \\ 0, & t > 0 \end{cases} \tag{3.11}
$$

and

$$\mathscr{F}^-(u) = \frac{1}{2}\|u\|_{X_0}^2 - \int_\Omega F^-(x,u)dx, \quad u \in X_0,$$

where $F^-(x,u) = \int_0^u f^-(x,s)ds$. We also get a non-trivial solution $u_- \leqslant 0$ in $\mathbb{R}^n$ which is a critical point of $\mathscr{F}^-$, so it is a non-trivial solution of problem (P) with $\mathscr{F}(u_-) = \mathscr{F}^-(u_-)$. □

Corollary 3.7. *Let all the assumptions of Theorem 3.1 (Theorem 3.2) be satisfied. Then, the fractional elliptic equation (2.7) admits a non-negative solution $u_+ \in X_0$ and a non-positive solution $u_- \in X_0$ that are of Mountain-Pass type and that are non-trivial.*

4. Concave-convex nonlinearity

In this section, we discuss the multiplicity of solutions of the problem

$$(P)_\lambda \qquad \begin{cases} \mathcal{L}_K u + \lambda h(x)|u|^{p-2}u + g(x,u) = 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases}$$

where $1 < p < 2$, $\lambda \geqslant 0$ is a parameter, $h \in L^\infty(\Omega)$, $h(x) \geqslant 0$, $h(x) \not\equiv 0$, and $g(x,s)$ is a continuous function on $\overline{\Omega} \times \mathbb{R}$.

In this case, u being a weak solution of problem $(P)_\lambda$ is equivalent to u being a critical point of the Euler-Lagrange functional

$$\mathscr{F}_\lambda(u) = \frac{1}{2}\|u\|_{X_0}^2 - \frac{\lambda}{p}\int_\Omega h(x)|u|^p dx - \int_\Omega G(x,u)dx, \tag{4.1}$$

where $G(x,u) = \int_0^u g(x,s)ds$.

Theorem 4.1. *Assume g satisfies (H1)-(H4) (or (H1)-(H3), (H4)*). Then there exists $\lambda^* > 0$ such that for $\lambda \in (0,\lambda^*)$, problem $(P)_\lambda$ has at least four non-trivial solutions: u_+, u_-, v_+, and v_-, satisfying $u_+ > 0$, $u_- < 0$, $v_+ > 0$, $v_- < 0$, and $\mathscr{F}_\lambda(u_\pm) > 0 > \mathscr{F}_\lambda(v_\pm)$.*

Let $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$. We now define functional $\mathscr{F}_\lambda^\pm : X_0 \to \mathbb{R}$ as follows:

$$\mathscr{F}_\lambda^\pm(u) = \frac{1}{2}\int_Q |u(x)-u(y)|^2 K(x-y)dxdy - \frac{\lambda}{p}\int_\Omega h(x)|u^\pm|^p dx - \int_\Omega G^\pm(x,u)dx,$$

where $G^\pm(x,u) = \int_0^u g^\pm(x,s)ds$. We give the following lemmas which will be used to prove Theorem 4.1.

Lemma 4.2. *$\mathscr{F}_\lambda^+$ and $\mathscr{F}_\lambda^-$ are unbounded from below.*

Proof. The proof is similar to the proof of Lemma 3.3. □

Lemma 4.3. *For $\lambda > 0$ small enough, there exist $\rho, R > 0$ such that $\mathscr{F}_\lambda^\pm(u) \geqslant R$, if $\|u\|_{X_0} = \rho$.*

Proof. (H1), (H2) imply that for all given $\epsilon > 0$, there exists $c_\epsilon > 0$, such that (3.3) holds. Combining (3.3), Hölder inequality and Lemma 2.1, we have

$$\begin{aligned}
\mathscr{F}_\lambda^\pm(u) &\geqslant \frac{1}{2}\|u\|_{X_0}^2 - \frac{\lambda}{p}|h|_\infty|u^\pm|_p^p - \epsilon|u^\pm|_2^2 - c_\epsilon|u^\pm|_q^q \\
&\geqslant \frac{1}{2}\|u\|_{X_0}^2 - \frac{\lambda}{p}|\Omega|^{(2^*-p)/2^*}|h|_\infty|u^\pm|_{2^*}^p - \epsilon|\Omega|^{(2^*-2)/2^*}|u^\pm|_{2^*}^2 - c_\epsilon|\Omega|^{(2^*-q)/2^*}|u^\pm|_{2^*}^q \\
&\geqslant (\frac{1}{2} - \epsilon c_0|\Omega|^{(2^*-2)/2^*})\|u^\pm\|_{X_0}^2 - \lambda K\|u^\pm\|_{X_0}^p - c_q\|u^\pm\|_{X_0}^q \\
&= \|u^\pm\|_{X_0}^2(A - \lambda K\|u^\pm\|_{X_0}^{p-2} - c_q\|u^\pm\|_{X_0}^{q-2}),
\end{aligned}$$

where K, c_q, c_0 are positive constants and $A = \frac{1}{2} - \epsilon c_0|\Omega|^{(2^*-2)/2^*}$. Taking ϵ small enough we get that the constant $A > 0$. Let

$$Q(t) = \lambda K t^{p-2} + c_q t^{q-2}.$$

We claim that there exists t_0 such that

$$Q(t_0) < A.$$

Indeed,

$$Q'(t) = \lambda K(p-2)t^{p-3} + c_q(q-2)t^{q-3}.$$

Setting

$$Q'(t) = 0$$

we know

$$t_0 = (\frac{\lambda K(2-p)}{c_q(q-2)})^{\frac{1}{q-p}}.$$

Obviously, $Q(t)$ has a minimum at $t = t_0$. Let

$$\beta = \frac{K(2-p)}{c_q(q-2)}, \quad \bar{p} = \frac{p-2}{q-p}, \quad \bar{q} = \frac{q-2}{q-p}.$$

Substituting t_0 in $Q(t)$ we have

$$Q(t_0) < A, \quad 0 < \lambda < \lambda^*,$$

where $\lambda^* = (\frac{A}{K\beta^{\bar{p}} + c_q\beta^{\bar{q}}})^{1/\bar{q}}$. Taking $\rho = t_0$ we complete the proof. □

Lemma 4.4. *Suppose that g satisfies (H1), (H3) and (H4). Then $\mathscr{F}_\lambda^\pm$ satisfies the Palais-Smale condition.*

Proof. Since $f(x,u) = \lambda h(x)|u|^{p-2}u + g(x,u)$, g satisfies (H1), (H3), we know f satisfies (H1),(H3). Moreover, $1 < p < 2$ and g satisfies (H4), which imply f satisfies (H4) for large enough $|u|$ by Lemma 3.5. we know that $\{u_n\}$ is bounded in X_0. Then, a standard argument shows that $\{u_n\}$ converges strongly and $\mathscr{F}_\lambda$ satisfies the Palais-Smale condition. Verifying the Palais-Smale condition for $\mathscr{F}_\lambda^\pm$ is similar to the verification for $\mathscr{F}_\lambda$. □

Proof of Theorem 4.1. For $\mathscr{F}_\lambda^\pm$, we first show the existence of local minimum $v_\pm$, with $\mathscr{F}_\lambda^\pm(v_\pm) < 0$. For ρ given in Lemma 4.3, we set

$$\overline{B}(\rho) = \{u \in X_0, \|u\|_{X_0} \leqslant \rho\}, \quad \partial B(\rho) = \{u \in X_0, \|u\|_{X_0} = \rho\}.$$

Then $\overline{B}(\rho)$ is a complete metric space with the distance

$$\mathrm{dist}(u,v) = \|u-v\|_{X_0}, \quad \forall\, u, v \in \overline{B}(\rho).$$

By Lemma 4.3, we know for $0 < \lambda < \lambda^*$,

$$\mathscr{F}_\lambda^\pm(u)|_{\partial B_\rho} \geqslant R > 0.$$

Moreover, it is easy to see that $\mathscr{F}_\lambda^\pm \in C^1(\overline{B}(\rho), \mathbb{R})$, hence $\mathscr{F}_\lambda^\pm$ is lower semi-continuous and bounded from below on $\overline{B}(\rho)$. Let

$$c_1^\pm = \inf\{\mathscr{F}_\lambda^\pm(u), u \in \overline{B}(\rho)\}.$$

Taking $\bar{v}_\pm \in C_c^\infty(\Omega)$, with $\bar{v}_+ > 0$ ($\bar{v}_- < 0$). From (H2) we know that for any $\epsilon > 0$, there exists $T > 0$ such that for $0 < t < T$, $|G^\pm(x, t\bar{v}_\pm)| \leqslant \epsilon t^2$. Then,

$$\begin{aligned}
\mathscr{F}_\lambda^\pm(t\bar{v}_\pm) &= \frac{t^2}{2}\|\bar{v}_\pm\|_{X_0}^2 - \frac{\lambda t^p}{p}\int_\Omega h(x)(\bar{v}_\pm)^p dx - \int_\Omega G^\pm(x, t\bar{v}_\pm)dx \\
&\leqslant \frac{t^2}{2}\|\bar{v}_\pm\|_{X_0}^2 - \frac{\lambda t^p}{p}\int_\Omega h(x)\bar{v}_\pm^p dx + \epsilon|\Omega|t^2 \\
&< 0,
\end{aligned} \tag{4.2}$$

for both $t > 0$, ϵ small enough, since $1 < p < 2$. Hence, $c_1^\pm < 0$.

By Ekeland's variational principle [31], for any $k > 1$, there exists u_k such that

$$c_1^\pm \leqslant \mathscr{F}_\lambda^\pm(u_k) \leqslant c_1^\pm + \frac{1}{k}, \tag{4.3}$$

$$\mathscr{F}_\lambda^\pm(w) \geqslant \mathscr{F}_\lambda^\pm(u_k) - \frac{1}{k}\|u_k - w\|_{X_0}, \quad \forall\, w \in \overline{B}(\rho). \tag{4.4}$$

Then $\|u_k\|_{X_0} < \rho$ for k large enough. Otherwise, if $\|u_k\|_{X_0} = \rho$ for infinitely many k, without loss of generality, we may assume that for all $k \geqslant 1$, $\|u_k\|_{X_0} = \rho$. Then from Lemma 4.3 it follows $0 < R \leqslant \mathscr{F}_\lambda^\pm(u_k) \leqslant c_1^\pm + \dfrac{1}{k} < 0$, for k large enough, which is a contradiction.

Now we prove that $\nabla\mathscr{F}_\lambda^\pm(u_k) \to 0$ in X_0^*. In fact, for any $u \in X_0$ with $\|u\|_{X_0} = 1$, let $w_k = u_k + tu$. Then for a fixed $k > 1$, we know $\|w_k\|_{X_0} \leqslant \|u_k\|_{X_0} + t < \rho$, for $t > 0$ small enough. So, (4.4) implies

$$\mathscr{F}_\lambda^\pm(u_k + tu) \geqslant \mathscr{F}_\lambda^\pm(u_k) - \frac{t}{k}\|u\|_{X_0} = \mathscr{F}_\lambda^\pm(u_k) - \frac{t}{k}.$$

Thus,

$$\frac{\mathscr{F}_\lambda^\pm(u_k + tu) - \mathscr{F}_\lambda^\pm(u_k)}{t} \geqslant -\frac{1}{k}.$$

Setting $t \to 0^+$, we derive that

$$\left|\langle \nabla\mathscr{F}_\lambda^\pm(u_k), u\rangle\right| \leqslant \frac{1}{k},$$

for any $u \in X_0$, with $\|u\|_{X_0} = 1$. So, $\nabla\mathscr{F}_\lambda^\pm(u_k) \to 0$ and (4.3) gives $\mathscr{F}_\lambda^\pm(u_k) \to c_1^\pm$. Hence, it follows from Lemma 4.4 that there exists $v_\pm \in X_0$ such that $\nabla\mathscr{F}_\lambda^\pm(v_\pm) = 0$. $v_\pm$ is a weak solution of problem $(P)_\lambda$ and $\mathscr{F}_\lambda(v_\pm) < 0$. A standard argument shows $v_+ \geqslant 0$ and $v_- \leqslant 0$ a.e. in $\mathbb{R}^n$ (see the proof of Corollary 3.6). Using a similar argument to the proof of Theorem 3.2, we know there exist two non-trivial solutions $u_+ \geqslant 0$ and $u_- \leqslant 0$ of Mountain Pass type, satisfying $\mathscr{F}_\lambda^\pm(u_\pm) \geqslant R > 0$. □

Now we give more information about the multiplicity of the solutions of problem $(P)_\lambda$. More precisely, we give the following multiplicity result about five solutions, which improves Theorem 4.1.

Theorem 4.5. *Assume g satisfies (H1)-(H4) (or (H1)-(H3), (H4)*). $h \in L^\infty(\Omega)$ with $h \geqslant h_0$, where h_0 is a positive constant. Then there exists $\lambda^* > 0$ such that for $\lambda \in (0, \lambda^*)$, problem $(P)_\lambda$ has at least five nontrivial solutions: u_+, u_-, v_+, v_-, and v_3, satisfying $u_+ > 0$, $u_- < 0$, $v_+ > 0$, $v_- < 0$, $\mathscr{F}_\lambda(u_\pm) > 0 > \mathscr{F}_\lambda(v_\pm)$ and $\mathscr{F}_\lambda(v_3) < 0$.*

Remark 4.6. *In* [2]*, the existence of a critical point with negative energy is given, which is different from v_+, v_-. The nonlinearity g in* [2] *is assumed to be convex and*

$$G(s) = \frac{1}{\alpha}|s|^\alpha + o(|s|^\alpha), \quad \text{at} \quad s = 0, s = \infty.$$

Following the idea of [2]*, we here give a similar version of the results of* [2] *with the fractional Laplacian, under some weak assumptions.*

Proof. First of all, by an analogous argument as in the proof of Theorem 4.1, the existence of u_+, u_-, v_+ and v_- follows. We need only to show the existence of v_3 with $\mathscr{F}_\lambda(v_3) < 0$. We assume that there exists no solution such that $\mathscr{F}_\lambda(v_3) < 0$ except $v_\pm$. Note that, according to the proof of Theorem 4.1, v_+ and v_- are local minima of $\mathscr{F}_\lambda$. We can assume that v_+ and v_- are isolated local minima. Let us denote by b_λ the Mountain Pass critical level of $\mathscr{F}_\lambda$ with base points v_+, v_-:

$$b_\lambda = \inf_{\psi\in\Psi} \max_{t\in[0,1]} \mathscr{F}_\lambda(\psi(t)),$$

where

$$\Psi = \{\psi \in C([0,1], X_0) : \ \psi(0) = v_+, \psi(1) = v_-\}.$$

We will prove that $b_\lambda < 0$ if λ is small enough. To this end, we consider

$$\mathscr{F}_\lambda(tv_\pm) = \frac{t^2}{2}\|v_\pm\|_{X_0}^2 - \frac{\lambda t^p}{p}\int_\Omega h(x)|v_\pm|^p dx - \int_\Omega G^\pm(x, tv_\pm)dx.$$

We claim that there exists $\delta > 0$ such that

$$\mathscr{F}_\lambda(tv_\pm) < 0, \forall t \in (0,1), \forall 0 < \lambda < \delta. \tag{4.5}$$

If not, we have $t_0 \in (0,1)$, such that $\mathscr{F}_\lambda(t_0 v_\pm) \geqslant 0$ for λ small enough. Since (H1) holds, by a similar way as (4.2) we know $\mathscr{F}_\lambda(tv_\pm) < 0$ for $t > 0$ small enough. Let $\rho_0 = t_0\|v_\pm\|_{X_0}$ and $c_*^\pm = \inf\{\mathscr{F}_\lambda^\pm(u), u \in \overline{B}(\rho_0)\}$. Using a standard argument as in the proof of Theorem 4.1, we obtain a solution $v_\pm^*$ such that $\mathscr{F}_\lambda(v_\pm^*) < 0$, a contradiction. Hence, (4.5) holds.

Now let us consider the 2-dimensional plane Π_2 containing the straight lines tv_- and tv_+ (if v_- and v_+ are proportional, take any 2-dimensional plane containing them), and take $v \in \Pi_2$ with $\|v\|_{X_0} = \varepsilon$. Note that for such v one has $|v|_p = c_p\varepsilon$, and $|v|_{2^*} = c_*\varepsilon$ for some constants $c_p, c_* > 0$. Then we get

$$\mathscr{F}_\lambda(v) \leqslant \frac{\varepsilon^2}{2} - \frac{\lambda}{p}c_p^p h_0\varepsilon^p - c_*^{2^*}\varepsilon^{2^*},$$

where c_0 is a constant. For small ε,

$$\mathscr{F}_\lambda(v) < 0. \tag{4.6}$$

Consider the path $\bar{\gamma}$ obtained gluing together the segments $\{tv_- : \varepsilon\|v_-\|_{X_0}^{-1} \leqslant t \leqslant 1\}$, $\{tv_+ : \varepsilon\|v_+\|_{X_0}^{-1} \leqslant t \leqslant 1\}$ and the arc $\{v \in \Pi_2 : \|v\|_{X_0} = \varepsilon\}$. From (4.5) and (4.6) it follows that

$$b_\lambda \leqslant \max_{v\in\bar{\gamma}} \mathscr{F}_\lambda(v) < 0,$$

which verifies the claim. Since the Palais-Smale condition holds because of Lemma 4.4, the level $\{\mathscr{F}_\lambda(v) = b_\lambda\}$ carries a critical point v_3 of $\mathscr{F}_\lambda$, and v_3 is different from $v_\pm$. ◻

5. The sixth solution come from Nehari manifold

In this section, to use the Nehari manifold, we replace the assumption (H4) by a slightly strong condition:

(H4)** $\frac{f(x,t)}{|t|}$ is strictly increasing in t on $(-\infty, 0)$ and $(0, +\infty)$.

The main result about multiplicity of solutions reads as follows.

Theorem 5.1. *Assume g satisfies (H1)-(H3), (H4)**, and all the other assumptions of Theorem 4.5 hold. Then, for $\lambda > 0$ small enough, there exists a solution u_3, which is different from the above five solutions given in Theorem 4.5. Moreover, we have $\mathscr{F}_\lambda(u_3) > 0$.*

Remark 5.2. *In* [2]*, the authors consider $g \in C^1(\mathbb{R}, \mathbb{R})$ with the following assumptions:*

(G1) $g(s)s \geqslant \alpha G(s) \geqslant 0$, $\forall s \in \mathbb{R}$, *with* $2 < \alpha < 2^*$;
(G2) $g'(s)s^2 \geqslant \alpha g(s)s$, $\forall s \in \mathbb{R}$;
(G3) $g'(s)s^2 \leqslant c_1|s|^\alpha$, $\forall s \in \mathbb{R}$ $(c_1 > 0)$.

*By using the methods of Nehari manifold, the sixth solution is obtained. It is known that under above condition $g \in C^1(\mathbb{R}, \mathbb{R})$, Nehari manifold is a C^1-submanifold of X_0. However, under (H4)**, the Nehari manifold is not a C^1-submanifold. Moreover, note that (H4)** does not need the differentiability of $g(x, u)$ with respect to u. That is, $f(x, u)$ may be not a C^1 function with respect to u. The methods in* [2] *can not be applied to deal with this problem. To overcome this difficulty, we use some recent arguments developed by Szulkin and Weth* [27, 26].

Let $S := S_1(0) = \{u \in X_0, \|u\|_{X_0} = 1\}$. Since X_0 is a Hilbert space, S is a C^1 submanifold of X_0 and the tangent space of S at w is

$$T_w S = \{z \in X_0 : \langle w, z\rangle_{X_0} = 0\}.$$

For $u \in S$, we set

$$\Gamma_\lambda^\pm(u) := \langle \nabla \mathscr{F}_\lambda^\pm(u), u\rangle = \|u\|_{X_0}^2 - \lambda \int_\Omega h(x)|u^\pm|^p dx - \int_\Omega g(x, u^\pm)u^\pm dx.$$

Let $u > 0$, $t > 0$. We consider the equation

$$\Gamma_\lambda^+(tu) = 0.$$

Thus,

$$t^2 - \lambda t^p \int_\Omega h(x)|u|^p dx - \int_\Omega g(x, tu)tu dx = 0. \tag{5.1}$$

Since for any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that

$$g(x, tu)tu \leqslant \epsilon t^2 u^2 + c_\epsilon t^q u^q,$$

we get

$$\begin{aligned}
& t^2 - \lambda t^p \int_\Omega h(x)|u|^p dx - \int_\Omega g(x, tu)tu dx \\
\geqslant & t^2 - \epsilon t^2 |u|_2^2 - c_\epsilon t^q |u|_q^q - \lambda t^p |h|_\infty |u|_p^p \\
\geqslant & \frac{1}{2}t^2 - c_q t^q - \lambda c_p t^p \\
= & t^2\left(\frac{1}{2} - c_q t^{q-2} - \lambda c_p t^{p-2}\right),
\end{aligned} \tag{5.2}$$

for ϵ small enough and both c_q, c_p are constants. Set $\gamma(t) := \frac{1}{2} - c_q t^{q-2} - \lambda c_p t^{p-2}$. Let $\gamma'(t) = 0$ we obtain $t_1 = (\frac{\lambda c_p(2-p)}{(q-2)c_q})^{1/(q-p)}$. One can verify that $\gamma(t_1) > 0$ for λ small enough. Then, there exists $\lambda_0 > 0$ such that the equation $\gamma(t) = 0$ has precisely two solutions σ_λ, τ_λ, such that $0 < \sigma_\lambda < t_1 < \tau_\lambda$, for all $\lambda \in (0, \lambda_0)$. Hence, a comparison argument shows that (5.1) has two solutions, t_λ^*, t_{λ_*}, such that $0 < t_{\lambda_*} \leqslant \sigma_\lambda < t_1 < \tau_\lambda \leqslant t_\lambda^*$, for all $\lambda \in (0, \lambda_0)$ and $u \in S$, $u > 0$. A similar argument can be carried out for Γ_λ^-.

For $\lambda \in (0, \lambda_0)$, we define the Nehari manifold

$$\mathscr{N}_\lambda^\pm := \left\{u \in X_0 : \Gamma_\lambda^\pm(u) = 0, \quad \|u\|_{X_0} \geqslant t_1\right\}. \tag{5.3}$$

Lemma 5.3. *For each $u \in S$, denote $\alpha_u^\pm(s) := \mathscr{F}_\lambda^\pm(su)$. Then under assumptions of Theorem 5.1 and λ small enough, there exists $s_u^\pm$ such that $(\alpha_u^\pm)'(s) > 0$ for $t_1 < s < s_u^\pm$ and $(\alpha_u^\pm)'(s) < 0$ for $s > s_u^\pm$.*

Proof. From the above, we know that for all $\lambda \in (0, \lambda_0)$, (5.1) has two solutions, t_λ^*, t_{λ_*}, such that $t_{\lambda_*} < t_1 < t_\lambda^*$. Then $\alpha_u^\pm(s)$ is increasing in $(t_{\lambda_*}, t_\lambda^*)$, and decreasing in $(0, t_{\lambda_*}) \cup (t_\lambda^*, +\infty)$. (H3) implies that $\alpha_u^\pm(s) \to -\infty$, as $s \to +\infty$. Hence, the conclusion follows by the definition of $\mathscr{N}_\lambda^\pm$. □

Remark 5.4. *Note that $(\alpha_u^\pm)'(s_u^\pm) = (\mathscr{F}_\lambda^\pm)'(s_u^\pm u)u = 0$. Hence on the ray $s \mapsto su$, $s > 0$, $s_u^\pm u$ is the unique point which intersects $\mathscr{N}_\lambda^\pm$.*

Lemma 5.5. *Under assumptions of Theorem 5.1, for each compact subset $\mathscr{W} \subset S$ there exists a constant $C_{\mathscr{W}}$ such that $s_u^\pm \leqslant C_{\mathscr{W}}$ for all $u^\pm \in \mathscr{W}$.*

Proof. From the definition of Nehari manifold we know $s_u^{\pm} > t_1$ for all $u \in S$, λ small enough. Let $\mathscr{W} \subset S$ be a compact set and sequence $\{u_n\} \subset \mathscr{W}$. We know passing to a subsequence, $u_n \to u_0 \neq 0$. Let

$$I_\lambda^{\pm}(u) = \frac{\lambda}{p}\int_\Omega h(x)|u^{\pm}|^p dx + \int_\Omega G^{\pm}(x,u)dx.$$

Then we have $\mathscr{F}_\lambda^{\pm}(u) = \frac{1}{2}\|u\|_{X_0}^2 - I_\lambda^{\pm}(u)$. We claim that if $s_n^{\pm} \to +\infty$ as $n \to +\infty$, then $I_\lambda^{\pm}(s_n^{\pm}u_n)/(s_n^{\pm})^2 \to +\infty$. Since $|s_n^{\pm}u_n(x)| \to +\infty$ if $u(x) \neq 0$, Fatou's lemma yields

$$\frac{I_\lambda^{\pm}(s_n^{\pm}u_n)}{(s_n^{\pm})^2} \geqslant \int_\Omega \frac{G^{\pm}(x, s_n^{\pm}u_n)}{(s_n^{\pm}u_n)^2}u_n^2 \to +\infty, \quad n \to +\infty.$$

If passing to a subsequence, $s_{u_n}^{\pm} \to +\infty$, we know

$$\mathscr{F}_\lambda^{\pm}(s_{u_n}^{\pm}u_n) = \frac{1}{2}(s_{u_n}^{\pm})^2 - I_\lambda^{\pm}(s_{u_n}^{\pm}u_n) = (s_{u_n}^{\pm})^2\Big(\frac{1}{2} - \frac{I_\lambda^{\pm}(s_{u_n}^{\pm}u_n)}{(s_{u_n}^{\pm})^2}\Big) = -\infty,$$

a contradiction. Hence, there exists a constant $C_{\mathscr{W}}$ such that $s_{u_n}^{\pm} \leqslant C_{\mathscr{W}}$. □

Define the mapping $\widehat{m}_\lambda^{\pm} : X_0 \setminus \{0\} \to \mathscr{N}_\lambda^{\pm}$ and $m_\lambda^{\pm} : S \to \mathscr{N}_\lambda^{\pm}$ by

$$\widehat{m}_\lambda^{\pm}(w) := s_w^{\pm}w \quad \text{and} \quad m_\lambda^{\pm} := \widehat{m}|_S. \tag{5.4}$$

Lemma 5.6. *The mapping $\widehat{m}_\lambda^{\pm}$ is continuous, the mapping $m_\lambda^{\pm}$ is a homeomorphism between S and $\mathscr{N}_\lambda^{\pm}$, and the inverse of $m_\lambda^{\pm}$ is given by $(m_\lambda^{\pm})^{-1}(u) = u/\|u\|_{X_0}$.*

Proof. Suppose $w_n \to w \neq 0$. Since $\widehat{m}_\lambda^{\pm}(tw) = \widehat{m}_\lambda^{\pm}(w)$ for each $t > 0$, we may assume $w_n \in S$. $\widehat{m}_\lambda^{\pm}(w_n) = s_n^{\pm}w_n$. From Lemma 5.5 and the definition of Nehari manifold we know $\{s_n^{\pm}\}$ is bounded and bounded away from 0, so $s_n^{\pm} \to \bar{s}^{\pm} > 0$. Since $\mathscr{N}_\lambda^{\pm}$ is closed and $\widehat{m}_\lambda^{\pm}(w_n) \to \bar{s}^{\pm}w$, $\bar{s}^{\pm}w \in \mathscr{N}_\lambda^{\pm}$. Hence $\bar{s}^{\pm}w = s_w^{\pm}w = \widehat{m}_\lambda^{\pm}(w)$. So $\widehat{m}_\lambda^{\pm}$ is continuous. Then, it follows $m_\lambda^{\pm}$ is a homeomorphism between S and $\mathscr{N}_\lambda^{\pm}$. □

We consider the functionals $\widehat{\Psi}_\lambda^{\pm} : X_0 \setminus \{0\} \to \mathbb{R}$ and $\Psi_\lambda^{\pm} : S \to \mathbb{R}$ defined by

$$\widehat{\Psi}_\lambda^{\pm}(w) := \mathscr{F}_\lambda^{\pm}(\widehat{m}_\lambda^{\pm}(w)) \quad \text{and} \quad \Psi_\lambda^{\pm} := \widehat{\Psi}_\lambda^{\pm}|_S.$$

Although we do not claim that $\mathscr{N}_\lambda^{\pm}$ is a C^1 manifold, we shall show that $\widehat{\Psi}_\lambda^{\pm}$ is of class C^1 and there is a one-to-one correspondence between critical points of $\Psi_\lambda^{\pm}$ and non-trivial critical points of $\mathscr{F}_\lambda^{\pm}$ with $\|u\|_{X_0} \geqslant t_1$, for λ small enough.

Lemma 5.7. $\widehat{\Psi}_\lambda^{\pm} \in C^1(X_0 \setminus \{0\}, \mathbb{R})$ *and*

$$(\widehat{\Psi}_\lambda^{\pm})'(w)z = \frac{\|\widehat{m}_\lambda^{\pm}(w)\|_{X_0}}{\|w\|_{X_0}}(\mathscr{F}_\lambda^{\pm})'(\widehat{m}_\lambda^{\pm}(w))z \quad \textit{for all} \quad w, z \in X_0, w \neq 0.$$

Proof. Let $w \in X_0 \setminus \{0\}$, $z \in X_0$. By the mean value theorem, we obtain

$$\begin{aligned}
\widehat{\Psi}_\lambda^{\pm}(w+tz) - \widehat{\Psi}_\lambda^{\pm}(w) &= \mathscr{F}_\lambda^{\pm}(s_{w+tz}^{\pm}(w+tz)) - \mathscr{F}_\lambda^{\pm}(s_w^{\pm}w) \\
&\leqslant \mathscr{F}_\lambda^{\pm}(s_{w+tz}^{\pm}(w+tz)) - \mathscr{F}_\lambda^{\pm}(s_{w+tz}^{\pm}w) \\
&= (\mathscr{F}_\lambda^{\pm})'(s_{w+tz}^{\pm}(w+\tau_t tz))s_{w+tz}^{\pm}tz,
\end{aligned}$$

where $|t|$ is small enough and $\tau_t \in (0,1)$. Similarly,

$$\begin{aligned}\widehat{\Psi}_\lambda^\pm(w+tz) - \widehat{\Psi}_\lambda^\pm(w) &\geqslant \mathscr{F}_\lambda^\pm(s_w^\pm(w+tz)) - \mathscr{F}_\lambda^\pm(s_w^\pm w)\\ &= (\mathscr{F}_\lambda^\pm)'(s_w^\pm(w+\eta_t tz))s_w^\pm tz, \qquad \text{for } \eta_t \in (0,1).\end{aligned}$$

Since the mapping $w \mapsto s_w^\pm$ is continuous according to Lemma 5.6, we know

$$\lim_{t\to 0}\frac{\widehat{\Psi}_\lambda^\pm(w+tz) - \widehat{\Psi}_\lambda^\pm(w)}{t} = s_w^\pm(\mathscr{F}_\lambda^\pm)'(s_w^\pm w)z = \frac{\|\widehat{m}_\lambda^\pm(w)\|_{X_0}}{\|w\|_{X_0}}(\mathscr{F}_\lambda^\pm)'(\widehat{m}_\lambda^\pm(w))z.$$

Hence, the Gâteaux derivative of $\widehat{\Psi}_\lambda^\pm$ is bounded linear in z and continuous in w. Therefore, $\widehat{\Psi}_\lambda^\pm$ is of class C^1. □

Lemma 5.8. *The following holds:*

(a). *$\Psi_\lambda^\pm \in C^1(S,\mathbb{R})$ and*

$$(\Psi_\lambda^\pm)'(w)z = \|m_\lambda^\pm(w)\|_{X_0}(\mathscr{F}_\lambda^\pm)'(m_\lambda^\pm(w))z, \quad \textit{for all} \quad z \in T_wS.$$

(b). *If $\{w_n\}$ is a Palais-Smale sequence for $\Psi_\lambda^\pm$, then $\{m_\lambda^\pm(w_n)\}$ is a Palais-Smale sequence for $\mathscr{F}_\lambda^\pm$. If $\{u_n\} \subset \mathscr{N}_\lambda^\pm$ is a bounded Palais-Smale sequence for $\Psi_\lambda^\pm$, then $\{m_\lambda^\pm(u_n)\}$ is a bounded Palais-Smale sequence for $\mathscr{F}_\lambda^\pm$.*

(c). *If w is a critical point of $\Psi_\lambda^\pm$, then $m_\lambda^\pm(w)$ is a non-trivial critical point of $\mathscr{F}_\lambda^\pm$. If $m_\lambda^\pm(w)$ is a critical point of $\mathscr{F}_\lambda^\pm$ with $\|m_\lambda^\pm(w)\|_{X_0} \geqslant t_1$, then w is a non-trivial critical point of $\Psi_\lambda^\pm$. Moreover, the corresponding values of $\Psi_\lambda^\pm$ and $\mathscr{F}_\lambda^\pm$ coincide and $\inf\limits_S \Psi_\lambda^\pm = \inf\limits_{\mathscr{N}_\lambda^\pm}\mathscr{F}_\lambda^\pm$.*

(d). *If $\mathscr{F}_\lambda^\pm$ is even, then so is $\Psi_\lambda^\pm$.*

Proof. (a) follows from Lemma 5.7.

(b). We note that $X_0 = T_wS \oplus \mathbb{R}w$ for every $w \in S$, and the projection $X_0 \to T_wS$, $z + tw \mapsto z$ has uniformly bounded norm with respect to $w \in S$. Denote $J(w)z = \langle w, z\rangle_{X_0}$. J is bounded on bounded sets and $J(w)(z+tw) = t$. Then, $|t| \leqslant C\|z+tw\|_{X_0}$. Therefore, $\|z\|_{X_0} \leqslant |t| + \|z+tw\|_{X_0} \leqslant (C+1)\|z+tw\|_{X_0}$, for all $w \in S, z \in T_wS$ and $t \in \mathbb{R}$. Moreover, by (a) we have

$$\|(\Psi_\lambda^\pm)'(w)\|_{X_0} = \sup_{z\in T_wS,\|z\|_{X_0}=1}(\Psi_\lambda^\pm)'(w)z = \|u^\pm\|_{X_0}\sup_{z\in T_wS,\|z\|_{X_0}=1}(\mathscr{F}_\lambda^\pm)'(u^\pm)z, \tag{5.5}$$

with $u^\pm = m_\lambda^\pm(w)$. Since $(\mathscr{F}_\lambda^\pm)'(u^\pm)w = (\mathscr{F}_\lambda^\pm)'(u^\pm)u^\pm/\|u^\pm\|_{X_0} = 0$, we conclude using ($a$) again that

$$\begin{aligned}\|(\Psi_\lambda^\pm)'(w)\|_{X_0} &\leqslant \|u^\pm\|_{X_0}\,\|(\mathscr{F}_\lambda^\pm)'(u^\pm)\|_{X_0}\\ &= \|u^\pm\|_{X_0}\sup_{z\in T_w(S),z+tw\neq 0}\frac{(\mathscr{F}_\lambda^\pm)'(u^\pm)(z+tw)}{\|z+tw\|_{X_0}}\\ &\leqslant (C+1)\|u^\pm\|_{X_0}\sup_{z\in T_w(S)\setminus\{0\}}\frac{(\mathscr{F}_\lambda^\pm)'(u^\pm)(z)}{\|z\|_{X_0}}\\ &= (C+1)\|(\Psi_\lambda^\pm)'(w)\|_{X_0}.\end{aligned}$$

Since $u^\pm \in \mathscr{N}_\lambda^\pm$ is bounded away from 0, together with the fact that $\mathscr{F}_\lambda^\pm(u^\pm) = \Psi_\lambda^\pm(w)$, we obtain ($b$).

(c). From (5.5), $(\Psi_\lambda^\pm)'(w) = 0$ if and only if $(\mathscr{F}_\lambda^\pm)'(m_\lambda^\pm(w)) = 0$, the conclusion (c) follows.

(d). If $\mathscr{F}_\lambda^\pm$ is even, then $s_w^\pm = s_{-w}^\pm$. Hence $\widehat{m}_\lambda^\pm(-w) = -\widehat{m}_\lambda^\pm(w)$ and the conclusion follows from the definition of $\Psi_\lambda^\pm$. □

Remark 5.9. *We note that the following minimax characterization holds:*

$$c_\lambda^\pm = \inf_{u\in\mathscr{N}_\lambda^\pm} \mathscr{F}_\lambda^\pm(u) = \inf_{w\in X_0\setminus\{0\}} \max_{s>0} \mathscr{F}_\lambda^\pm(sw) = \inf_{w\in S} \max_{s>0} \mathscr{F}_\lambda^\pm(sw).$$

Lemma 5.10. *The following holds:*

(a). *If $\{u_n\} \subset \mathscr{N}_\lambda^\pm$ is a sequence such that $\sup\limits_{n\in N} \mathscr{F}_\lambda^\pm(u_n) < +\infty$, then passing to a subsequence, we have $u_n \rightharpoonup u \neq 0$ as $n \to +\infty$, and there is $s_u^\pm > 0$ such that $s_u^\pm u \in \mathscr{N}_\lambda^\pm$ and $\mathscr{F}_\lambda^\pm(s_u^\pm u_n) \leqslant \liminf\limits_{n\to+\infty} \mathscr{F}_\lambda^\pm(u_n)$.*

(b). *$\mathscr{F}_\lambda^\pm|_{\mathscr{N}_\lambda^\pm}$ is coercive, i.e., $\mathscr{F}_\lambda^\pm(u_n) \to +\infty$ as $u_n \in \mathscr{N}_\lambda^\pm$, $\|u_n\|_{X_0} \to +\infty$.*

(c). *$\mathscr{F}_\lambda^\pm$ satisfies the Palais-Smale condition on $\mathscr{N}_\lambda^\pm$.*

Proof. Let $\{u_n\} \subset \mathscr{N}_\lambda^\pm$ be a sequence such that $\mathscr{F}_\lambda^\pm(u_n) \leqslant d < \infty$ for all n. We first claim that $\{u_n\}$ is bounded. Otherwise $\|u_n\|_{X_0} \to +\infty$ and $v_n := u_n/\|u_n\|_{X_0} \rightharpoonup v$ in X_0. We know $v \neq 0$. Then from (H3) we get

$$0 \leqslant \frac{\mathscr{F}_\lambda^\pm(u_n)}{\|u_n\|_{X_0}^2} \leqslant \frac{1}{2} - \int_\Omega \frac{G(x,u_n)}{u_n^2}\frac{u_n^2}{\|u_n\|_{X_0}^2} \to -\infty,$$

as $n \to +\infty$, a contradiction. Hence, $\{u_n\}$ is bounded and $u_n \rightharpoonup u$. The definition of Nehari manifold shows that $u \neq 0$. Then,

$$\mathscr{F}_\lambda^\pm(s_u^\pm u) \leqslant \liminf_{n\to+\infty} \mathscr{F}_\lambda^\pm(s_u^\pm u_n) \leqslant \liminf_{n\to+\infty} \mathscr{F}_\lambda^\pm(u_n),$$

since $u_n \in \mathscr{N}_\lambda^\pm$. Hence (a) is proved and (b) follows. Let $\{u_n\} \subset \mathscr{N}_\lambda^\pm$ be a Palais-Smale sequence. By (a), $\{u_n\}$ is bounded and $u_n \rightharpoonup u$. Then a standard argument shows that $\mathscr{F}_\lambda^\pm$ satisfies the Palais-Smale condition. □

Remark 5.11. *We can also set*

$$\Gamma_\lambda(u) := \langle \nabla\mathscr{F}_\lambda(u), u\rangle = \|u\|_{X_0}^2 - \lambda\int_\Omega h(x)|u|^p dx - \int_\Omega g(x,u)u dx.$$

Then, for $\lambda \in (0,\lambda_0)$, we can also define the Nehari manifold

$$\mathscr{N}_\lambda := \Big\{u \in X_0 : \Gamma_\lambda(u) = 0, \quad \|u\|_{X_0} \geqslant t_1\Big\}. \tag{5.6}$$

By a similar argument as above, we could establish the corresponding results for $\mathscr{N}_\lambda$. We point out that Lemma 5.3, Lemma 5.5–5.8, and Lemma 5.10 also hold for $\mathscr{N}_\lambda^\pm$ replaced by $\mathscr{N}_\lambda$ and $\mathscr{F}_\lambda^\pm$ replaced by $\mathscr{F}_\lambda$.

Proof of Theorem 5.1. Let $\{w_n^\pm\}$ be a minimizing sequence of $\Psi_\lambda^\pm$. By Ekeland's variational principle we may assume $(\Psi_\lambda^\pm)'(w_n^\pm) \to 0$. By Lemma 5.6, we know $u_n^\pm := m_\lambda^\pm(w_n^\pm) \in \mathscr{N}_\lambda^\pm$, and $\{u_n^\pm\}$ is a Palais-Smale sequence for $\mathscr{F}_\lambda^\pm$ according to Lemma 5.8. Then, Lemma 5.10 implies, passing to a subsequence, $u_n^\pm \to u^\pm$ and $w_n^\pm \to (m_\lambda^\pm)^{-1}(u) = w^\pm$. Hence, $w^\pm$ is a minimizer for $\Psi_\lambda^\pm$ and $u^\pm$ is a critical point of $\mathscr{F}_\lambda^\pm$ since Lemma 5.8 holds. Moreover, from Lemma 4.3 and the definition of Nehari manifold, we know that for λ small enough, $\mathscr{F}^{\pm}_{\lambda}(u^{\pm}) \geqslant R > 0$, which implies that $u^{\pm}$ is non-trivial. From above we assume that u^{+}, u^{-} such that

$$\mathscr{F}^{+}_{\lambda}(u^{+}) = \inf_{u\in\mathscr{N}^{+}_{\lambda}} \mathscr{F}^{+}_{\lambda}(u), \quad \mathscr{F}^{-}_{\lambda}(u^{-}) = \inf_{u\in\mathscr{N}^{-}_{\lambda}} \mathscr{F}^{+}_{\lambda}(u).$$

Obviously, $\|u^{+} - u^{-}\|_{X_0} > 0$, so they are different. We assume, without loss of generality,

$$\mathscr{F}_{\lambda}(u^{+}) \geqslant \mathscr{F}_{\lambda}(u^{-}).$$

We claim that u^{+} and u^{-} are local minima of $\mathscr{F}_{\lambda}$ on $\mathscr{N}_{\lambda}$. Otherwise, we take

$$\mathscr{F}_{\lambda}(u_0) = \inf_{u\in\mathscr{N}_{\lambda}} \mathscr{F}_{\lambda}(u).$$

Then

$$\mathscr{F}_{\lambda}(u_0) \leqslant \mathscr{F}_{\lambda}(u^{-}) \leqslant \mathscr{F}_{\lambda}(u^{+}).$$

If $u_0 \neq u^{+}, u^{-}$, we know that u_0 is a solution, which finishes the proof of the theorem. If $u_0 = u^{+}$, by $\mathscr{F}_{\lambda}(u_0) = \mathscr{F}_{\lambda}(u^{-}) = \mathscr{F}_{\lambda}(u^{+})$ we prove the claim. If $u_0 = u^{-}$,

$$\mathscr{F}_{\lambda}(u^{+}) > \mathscr{F}_{\lambda}(u^{-}),$$

by taking $B_{\varepsilon_n} := B_{\varepsilon_n}(u^{-}) = \{u \in X_0, \|u - u^{-}\|_{X_0} < \varepsilon_n\}$, we know that for every $u^{*} \in \mathscr{N}_{\lambda} \cap B_{\varepsilon_n} \setminus \{u^{-}\}$, ε_n small enough,

$$\mathscr{F}_{\lambda}(u^{-}) < \mathscr{F}_{\lambda}(u^{*}) < \mathscr{F}_{\lambda}(u^{+}).$$

If u^{+} and u^{-} are not proportional, let $\Pi(u^{+}, u^{-})$ be the 2-dimensional plane containing the straight lines tu^{+} and tu^{-}. Take $\phi_n \in \Pi(u^{+}, u^{-}) \cap \mathscr{N}_{\lambda} \cap B_{\varepsilon_n} \setminus \{u^{-}\}$, $\phi_n = \phi^{+}_n + \phi^{-}_n = \delta_n u^{+} + (1 - \delta^{*}_n)u^{-}$, where δ, δ^{*}_n small enough are related to ε_n. We know that

$$\mathscr{F}_{\lambda}(u^{-}) < \mathscr{F}_{\lambda}(\phi_n) < \mathscr{F}_{\lambda}(u^{+}).$$

From (4.2), we know that for δ_n small enough, $\mathscr{F}_{\lambda}(\delta_n u^{+}) < 0$. Note that $\mathscr{F}_{\lambda}(\phi_n) = \mathscr{F}_{\lambda}(\phi^{+}_n) + \mathscr{F}_{\lambda}(\phi^{-}_n)$, and from the definition of Nehari manifold, $u^{\pm}$ is the unique point in the ray $u^{\pm} \mapsto tu^{\pm}$, $t > 0$. Hence,

$$\begin{aligned}
\mathscr{F}_{\lambda}(\phi_n) &= \mathscr{F}_{\lambda}(\delta_n u^{+}) + \mathscr{F}_{\lambda}((1 - \delta^{*}_n)u^{-}) \\
&< \mathscr{F}_{\lambda}((1 - \delta^{*}_n)u^{-}) \\
&< \mathscr{F}_{\lambda}(u^{-}) \\
&< \mathscr{F}_{\lambda}(\phi_n),
\end{aligned}$$

a contradiction. If u^{+} and u^{-} are proportional, the definition of Nehari manifold implies that $u^{+} = -su^{-}$, $s > 0$. From (4.2), we know that for δ_n small enough,

$$\begin{aligned}
\mathscr{F}_{\lambda}(u^{-}) &= \mathscr{F}_{\lambda}(\delta_n u^{+}) + \mathscr{F}_{\lambda}((1 + s\delta_n)u^{-}) \\
&< \mathscr{F}_{\lambda}((1 + s\delta_n)u^{-}) \\
&< \mathscr{F}_{\lambda}(u^{-}),
\end{aligned}$$

also a contradiction, which implies the claim. Define

$$\Upsilon = \{\gamma \in C([0, 1], \mathscr{N}_{\lambda}) : \gamma(0) = u_{+}, \gamma(1) = u_{-}\},$$

and

$$c_\lambda^* = \inf_{\gamma\in\Upsilon}\max_{t\in[0,1]} \mathscr{F}_\lambda(\gamma(t)).$$

The definition of $\mathscr{N}_\lambda$, together with Lemma 4.3, shows that $c_\lambda^* > 0$. Since u_+ and u_- are local minima of $\mathscr{F}_\lambda$ on $\mathscr{N}_\lambda$, $\mathscr{F}_\lambda$ satisfies the Mountain Pass geometry. Then Lemma 5.10 implies c_λ^* is a critical value for $\mathscr{F}_\lambda$ on $\mathscr{N}_\lambda$ and there is a critical point v_3 on $\mathscr{N}_\lambda$, which is different from u_+, u_-. By Lemma 5.8, we know that v_3 is actually a solution of $(P)_\lambda$ with $\mathscr{F}_\lambda(v_3) > 0$. □

6. Global results of Ambrosetti-Brezis-Cerami type

Theorem 4.1 and Theorem 4.5 are all local, since λ has to be small enough. A global result of Ambrosetti-Brezis-Cerami type is also given in the following theorem, in which we show that the combined effects of sublinear and superlinear terms change considerably the structure of the solution set.

For convenience we only consider positive solutions.

Theorem 6.1. Assume g satisfies (H1)-(H4) (or (H1)-(H3), (H4)*). $h \in L^\infty(\Omega)$ with $h \geqslant h_0$, where h_0 is a positive constant. Then there exists $\Lambda > 0$ such that

1. for $\lambda \in (0,\Lambda)$, problem $(P)_\lambda$ has at least two positive solutions: u_λ and v_λ, $u_\lambda \not\equiv v_\lambda$ satisfying $v_\lambda \leqslant u_\lambda$, $\mathscr{F}_\lambda(v_\lambda) < 0 < \mathscr{F}_\lambda(u_\lambda)$. Moreover, v_λ is a minimal solution and is non-decreasing with respect to λ;
2. for $\lambda = \Lambda$ problem $(P)_\lambda$ has at least one positive solution;
3. for all $\lambda > \Lambda$ problem $(P)_\lambda$ has no positive solution.

Let us define $\Lambda = \sup\{\lambda > 0 : (P)_\lambda \text{ has a solution}\}$.

Lemma 6.2. $0 < \Lambda < \infty$.

Proof. From Theorem 4.1 it follows that $(P)_\lambda$ has at least two positive solutions whenever $\lambda \in (0,\lambda^*)$ and thus $\Lambda \geqslant \lambda^* > 0$. We first consider the non-local operator eigenvalue problem

$$\begin{cases} -\mathcal{L}_K u = \lambda u, & \text{in } \Omega,\\ u = 0, & \text{in } \mathbb{R}^n\backslash\Omega. \end{cases} \tag{6.1}$$

The first eigenvalue of problem (6.1) is defined by

$$\lambda_1 := \inf_{u\in X_0\backslash\{0\}} \frac{\int_{\mathbb{R}^{2n}}(u(x)-u(y))^2K(x-y)dxdy}{\int_\Omega u^2dx}$$

and the corresponding eigenfunctions of the eigenvalue λ_1 is denoted by φ_1 (see [22]). It is known that $\lambda_1 > 0$ is simple and φ_1 is non-negative.

Let $\overline{\lambda}$ be such that

$$\overline{\lambda}h(x)t^{p-1} + g(x,t) > \lambda_1 t, \ \forall\, t > 0.$$

If λ is such that $(P)_\lambda$ has a non-negative solution u, multiplying $(P)_\lambda$ by φ_1 and integrating over $\mathbb{R}^n$ we find

$$\lambda_1\int_\Omega u\varphi_1 dx = \lambda\int_\Omega h(x)u^{p-1}\varphi_1 dx + \int_\Omega g(x,u)\varphi_1 dx.$$

This implies $\lambda \leqslant \overline{\lambda}$. □

Denote $f(x,t) = \lambda h(x)|u|^{p-2}u + g(x,u)$. Now, we give the definition of the super(sub)-solutions which we will use to prove Theorem 6.4. Let $f(x,s)$ be a Carathéodory function on $\Omega \times \mathbb{R}$ with the property that for any $s_0 > 0$, there exists a constant A such that $|f(x,s)| \leqslant A$ for a.e. $x \in \Omega$ and all $s \in [-s_0, s_0]$. A function $u \in X_0 \bigcap L^\infty(\Omega)$ is called a (weak) sub-solution of the problem $(P)_\lambda$ if

$$\int_{\mathbb{R}^{2n}} (u(x)-u(y))(\phi(x)-\phi(y))K(x-y)dxdy \leqslant \int_\Omega f(x,u(x))\phi(x)dx, \tag{6.2}$$

for all $\phi \in C_0^\infty(\Omega)$, $\phi \geqslant 0$. A super-solution is defined by reversing the inequality sign.

Lemma 6.3. Assume that $\underline{u}$ and $\overline{u}$ are respectively sub-solutions and super-solutions for $(P)_\lambda$. Consider the associated functional

$$\mathscr{F}(u) := \frac{1}{2}\|u\|_{X_0}^2 - \int_\Omega F(x,u)dx.$$

Let $M := \{u \in X_0 : \underline{u} \leqslant u \leqslant \overline{u} \text{ a.e. in } \mathbb{R}^n\}$. Then the infimum of $\mathscr{F}$ on M is achieved at some u and u is a solution of $(P)_\lambda$.

Proof. The proof is adapted from [10] which deals with the p-Laplacian elliptic equation. By coercivity and weak lower semicontinuity, the infimum of $\mathscr{F}$ on M is achieved at some u. Let $\varphi \in C_0^\infty(\Omega)$, $\varepsilon > 0$, and define

$$v_\varepsilon := \min\left\{\overline{u}, \max\{\underline{u}, u+\varepsilon\varphi\}\right\} = u + \varepsilon\varphi - \varphi^\varepsilon + \varphi_\varepsilon,$$

where $\varphi^\varepsilon := \max\{0, u+\varepsilon\varphi - \overline{u}\}$ and $\varphi_\varepsilon := -\min\{0, u+\varepsilon\varphi-\underline{u}\}$. Since u minimizes $\mathscr{F}$ on M, it follows $\langle \nabla\Phi(u), v_\varepsilon - u\rangle \geqslant 0$, which gives

$$\langle \nabla\mathscr{F}(u), \varphi\rangle \geqslant (\langle \nabla\mathscr{F}(u), \varphi^\varepsilon\rangle - \langle \nabla\mathscr{F}(u), \varphi_\varepsilon\rangle)/\varepsilon. \tag{6.3}$$

Since $\overline{u}$ is a super-solution, one also has

$$\begin{aligned}
\langle \nabla\mathscr{F}(u), \varphi^\varepsilon\rangle &\geqslant \langle \nabla\mathscr{F}(u) - \nabla\mathscr{F}(\overline{u}), \varphi^\varepsilon\rangle \\
&= \int_{\mathbb{R}^{2n}} \Big((u(x)-\overline{u}(x)) - (u(y)-\overline{u}(y))\Big)\Big(\varphi^\varepsilon(x) - \varphi^\varepsilon(y)\Big)K(x-y)dxdy \\
&- \int_\Omega (f(x,u) - f(x,\overline{u}))\varphi^\varepsilon dx \\
&\geqslant \varepsilon \int_{\Omega_\varepsilon} \Big((u-\overline{u})(x) - (u-\overline{u})(y)\Big)\Big(\varphi(x)-\varphi(y)\Big)dx \\
&- \varepsilon \int_{\Omega_\varepsilon} |f(x,u) - f(x,\overline{u})||\varphi| dx,
\end{aligned}$$

where $\Omega_\varepsilon := \{x \in \Omega : u(x) + \varepsilon\varphi(x) \geqslant \overline{u}(x) > u(x)\}$. Since $|\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0$, it implies $\langle \nabla\Phi(u), \varphi^\varepsilon\rangle \geqslant o(\varepsilon)$ as $\varepsilon \to 0$. Similarly $\langle \nabla\Phi(u), \varphi_\varepsilon\rangle \leqslant o(\varepsilon)$, and by (6.3), $\langle \nabla\Phi(u), \varphi\rangle \geqslant 0$. Replacing φ by $-\varphi$, we conclude that u solves $(P)_\lambda$. □

Proof of Theorem 6.4. 1. We first show that for all $\lambda \in (0, \Lambda)$, $(P)_\lambda$ has a positive solution v_λ with $I_\lambda(v_\lambda) < 0$. Let $0 < \lambda < \Lambda$ and take $\tilde{\lambda}$ with $\lambda < \tilde{\lambda} < \Lambda$ such that $(P)_{\tilde{\lambda}}$ has a positive solution $\tilde{v}$. Then,

$$\begin{aligned} &\int_{\mathbb{R}^{2n}} \Big(\tilde{v}(x) - \tilde{v}(y)\Big)\Big(\varphi(x) - \varphi(y)\Big)K(x-y)dxdy \\ =\ & \tilde{\lambda} \int_\Omega h(x)|\tilde{v}|^{p-2}\tilde{v}\varphi dx + \int_\Omega g(x, \tilde{v})\varphi dx \\ \geqslant\ & \lambda \int_\Omega h(x)|\tilde{v}|^{p-2}\tilde{v}\varphi dx + \int_\Omega g(x, \tilde{v})\varphi dx \end{aligned}$$

for all $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$. This implies that $\tilde{v}$ is a super solution for $(P)_\lambda$. Let $v_\varepsilon = \varepsilon\varphi_*$, $\varepsilon > 0$, where φ_* is choosing by the following way: If $\tilde{v} > 0$ a.e. in Ω, take $\varphi_* = \varphi_1(\Omega) \geqslant 0$ is the first eigenvalue function of the operator $-\mathcal{L}_K$ in domain Ω as in (6.1), otherwise, assume $\Omega_0 := \{x \in \Omega, \overline{v}(x) = 0\}$, we set $\varphi_* = \varphi_1(\Omega \setminus \Omega_0) \geqslant 0$ which is the first eigenvalue function of the operator $-\mathcal{L}_K$ in domain $\Omega \setminus \Omega_0$. From (H2), there exists ε small enough such that

$$\lambda_1 \varepsilon\varphi_* \leqslant \lambda h(x)\varepsilon^{p-1}\varphi_*^{p-1} + g(x, \varepsilon\varphi_*).$$

Taking $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$, we can easily obtain

$$\begin{aligned} &\int_{\mathbb{R}^{2n}} \Big(v_\varepsilon(x) - v_\varepsilon(y)\Big)\Big(\varphi(x) - \varphi(y)\Big)K(x-y)dxdy \\ \leqslant\ & \int_\Omega \lambda_1 v_\varepsilon \varphi dx \\ \leqslant\ & \lambda \int_\Omega h(x)|v_\varepsilon|^{p-2}v_\varepsilon\varphi dx + \int_\Omega g(x, v_\varepsilon)\varphi dx, \end{aligned}$$

which shows that v_ε is a sub-solution of $(P)_\lambda$. Taking ε small enough such that $\varepsilon\varphi_* \leqslant \tilde{v}$. From Lemma 6.3 we know $(P)_\lambda$ has a non-negative solution v_λ. Moreover the minimization property provided by Lemma 6.3 leads to $\mathscr{F}_\lambda(v_\lambda) \leqslant \mathscr{F}_\lambda(\varepsilon\varphi_*)$. By (H2), $\mathscr{F}_\lambda(\varepsilon\varphi_*) < 0$ for ε small enough. Assume that v_λ is an isolated local minimum. From Lemma 4.4 we can get another solution u_λ by the Mountain Pass Theorem. It remains to prove that $v_\lambda \leqslant v_{\lambda_*}$ whenever $\lambda < \lambda_*$. Indeed, if $\lambda < \lambda_*$ then v_{λ_*} is a super-solution of $(P)_\lambda$. For $\varepsilon > 0$ small, $\varepsilon\varphi_*$ is a sub-solution of $(P)_\lambda$ and $\varepsilon\varphi_* < v_{\lambda_*}$, then $(P)_\lambda$ has a positive solution v with $v \leqslant v_{\lambda_*}$. As v_λ is the minimal solution of $(P)_\lambda$, we have $v_\lambda \leqslant v \leqslant v_{\lambda_*}$.

2. Let $\{\mu_n\}$ be an increasing sequence such that $\mu_n \to \Lambda$. and v_n be a positive solution of $(P)_{\mu_n}$ with $\mathscr{F}_{\mu_n}(v_n) < 0$. Then, for any $\phi \in X_0$,

$$\begin{aligned} &\int_{\mathbb{R}^{2n}} (v_n(x) - v_n(y))(\phi(x) - \phi(y))K(x-y)dxdy \\ =&\mu_n \int_\Omega h(x)|v_n(x)|^{p-2}v_n(x)\phi(x)dx + \int_\Omega g(x, v_n(x))\phi(x)dx. \end{aligned} \tag{6.4}$$

We first show that sequence $\{v_n\}$ is bounded in X_0. We suppose, by contradiction, that up to a subsequence, still denoted by v_n,

$$\|v_n\|_{X_0} \to +\infty \quad \text{as } n \to +\infty.$$

Set $\omega_n := v_n/\|v_n\|_{X_0}$. Then $\|\omega_n\|_{X_0} = 1$. Passing to a subsequence, we may assume that there exists $\omega \in X_0$ such that

$$\begin{aligned}\omega_n &\rightharpoonup \omega, \quad \text{weakly in } X_0, \quad n \to +\infty,\\ \omega_n &\to \omega, \quad \text{strongly in } L^2(\Omega), \quad n \to +\infty,\\ \omega_n(x) &\to \omega(x), \quad \text{a.e. in } \Omega, \quad n \to +\infty.\end{aligned}$$

We claim that $\omega(x) \equiv 0$ a.e. in $\mathbb{R}^n$. It suffices to show $\omega(x) \equiv 0$ a.e. in Ω. (6.4) implies

$$\begin{aligned}&\int_{\mathbb{R}^{2n}} (\omega_n(x) - \omega_n(y))(\phi(x) - \phi(y))K(x-y)dxdy\\ =&\frac{\mu_n}{\|v_n\|_{X_0}^{2-p}} \int_\Omega h(x)|\omega_n(x)|^{p-2}\omega_n(x)\phi(x)dx + \int_\Omega \frac{g(x, v_n(x))}{v_n(x)}\omega_n(x)\phi(x)dx.\end{aligned}$$

Hence,

$$\int_\Omega \frac{g(x, v_n(x))}{v_n(x)}\omega_n^2(x)dx = 1 + o(1).$$

We denote $\Omega^* := \{x \in \Omega,\ \omega(x) \neq 0\}$. Then for $x \in \Omega^*$, $|u_n(x)| \to +\infty$ as $n \to +\infty$. By (H3) we have

$$\lim_{n\to+\infty} \frac{g(x, v_n(x))}{v_n(x)}\omega_n(x)^2 = +\infty.$$

The Fatou's Lemma implies

$$\int_{\Omega^*} \lim_{n\to+\infty} \frac{g(x, v_n(x))}{v_n(x)}\omega_n(x)^2 \leqslant \lim_{n\to+\infty} \int_{\Omega^*} \frac{g(x, v_n(x))}{v_n(x)}\omega_n^2(x)dx \leqslant 1 + o(1).$$

Hence Ω^* has zero measure. Consequently, $\omega(x) \equiv 0$ a.e. in Ω. On the one hand, since $\|v_n\|_{X_0} \to +\infty$, for some $k > 0$, by a standard argument similar to (4.2) we know

$$\mathscr{F}_{\mu_n}\Big(k\frac{v_n}{\|v_n\|_{X_0}}\Big) \leqslant \mathscr{F}_{\mu_n}(0) \leqslant 0, \tag{6.5}$$

as $n \to +\infty$. On the other hand, for all $k > 0$,

$$2\mathscr{F}_{\mu_n}(k\omega_n) = k^2 - \frac{2\mu_n}{p}k^p \int_\Omega h(x)|\omega_n|^p dx - 2\int_\Omega G(x, k\omega_n)dx = k^2 + o(1),$$

which contradicts (6.5) for k and n large enough. Therefore, $\{v_n\}$ is bounded in X_0. Up to a subsequence, we get $v_n \rightharpoonup v^*$. v^* is a solution of $(P)_\Lambda$ and $\mathscr{F}_\Lambda(v^*) \leqslant 0$. Since $\{v_n\}$ is a non-negative sequence, and non-decreasing, v^* is a non-trivial solution.

3. This follows from the definition of Λ. □

Before finishing this section, we will prove the strongest result for the case in which there are symmetrical conditions imposed on the nonlinearity. I.e. if we assume that $g(x, u) = g(x, -u)$, using the Lusternik-Schnirelman theory one can also obtain the existence of infinitely many pairs of solutions.

Theorem 6.4. Assume g satisfies (H1)-(H4) (or (H1)-(H3), (H4)*). $h \in L^\infty(\Omega)$ with $h \geqslant h_0$, where h_0 is a positive constant. Then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem $(P)_\lambda$ has infinitely many pairs of solutions $\{u_n\}$, $\{v_n\}$ such that $\mathscr{F}_\lambda(u_n) < 0 < \mathscr{F}_\lambda(v_n)$.

Proof. We give a brief sketch here because the arguments are similar to those of [1]. Set

$$\Sigma := \{A \in X_0, 0 \notin A, \text{ if } u \in A \text{ then } -u \in A\}.$$

For $A \in \Sigma$ the genus $\gamma(A)$ is defined as the least integer n for which there exists $\phi \in C(X_0, \mathbb{R}^n)$ such that ϕ is odd and $\phi(x) \neq 0$ for all $x \in A$. We define $\gamma(A) = +\infty$ if there are no integers with the above property and $\gamma(\emptyset) = 0$.

From Lemma 4.3 we know that for $\lambda > 0$ small enough, there exist $\rho, R > 0$ such that $\mathscr{F}^{\pm}_\lambda(u) \geqslant R$, if $\|u\|_{X_0} = \rho$.

We set also

$$\mathscr{A}_{n,\rho} := \{A \in \Sigma : A \text{ compact}, A \subset \overline{B}(\rho), \gamma(A) \geqslant n\}.$$

Obviously, $\mathscr{A}_{n,\rho} \neq \emptyset$ for all $n = 1, 2, \cdots$, since

$$S_{n,\rho} := \partial(X_n \cap \overline{B}(\rho)) \in \mathscr{A}_{n,\rho},$$

where X_n denotes an $n-$dimensional subspace of X_0.

Let

$$c_{n,\rho} = \inf_{A \in \mathscr{A}_{n,\rho}} \max_{u \in A} \mathscr{F}_\lambda(u).$$

Each $c_{n,\rho}$ is finite since the functional is bounded on $\overline{B}(\rho)$. We claim

$$c_{n,\rho} < 0, \quad n \in \mathbb{N}.$$

Indeed, let $w \in X_n$ be such that $\|w\|_{X_0} = \rho$. From (4.2), we know for $\rho > 0$ small enough, $\mathscr{F}_\lambda(w) < 0$. We note that for all $u \in \overline{B}(\rho) \cap \{\mathscr{F}_\lambda \leqslant 0\}$ the steepest descent flow η_t is well-defined for $t \in [0, \infty)$ and

$$\eta_t(u) \in \overline{B}(\rho) \cap \{\mathscr{F}_\lambda \leqslant 0\}, \quad \text{for any } t \geqslant 0,$$

since $\mathscr{F}^{\pm}_\lambda(u) > 0$, if $\|u\|_{X_0} = \rho$. Moreover, we know $c_{n,\rho} < 0$ from above and Palais-Smale condition holds from Lemma 4.4. Then, we can make use of Lusternik-Schnirelman theory to find infinitely many critical points of $\mathscr{F}_\lambda$ in $\overline{B}(\rho)$, denotes by $\{u_n\}$, such that $\mathscr{F}_\lambda(u_n) < 0$. Since Palais-Smale condition is satisfied, and Lemma 5.8 holds, from [27] we also obtain infinitely many critical points $\{v_n\}$, satisfying $\mathscr{F}_\lambda(v_n) > 0$. □

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Academy of Mathematics and Systems Science,Chinese Academy of Sciences, No.55 Zhongguancun East Road, Beijing, 100190, CHINA

Academy of Mathematics and Systems Science,Chinese Academy of Sciences, No.55 Zhongguancun East Road, Beijing, 100190, CHINA
E-mail address: billy3492@gmail.com, xfsu@amss.ac.cn
E-mail address: yhwei@amss.ac.cn