On the summability of formal solutions for doubly singular nonlinear partial differential equations

Stéphane Malek
Université de Lille 1
UFR de mathématiques
59655 Villeneuve d'Ascq Cedex France
Stephane.Malek@math.univ-lille1.fr
Tel 03 20 43 42 40

May, 24 2011

Abstract

We study Gevrey asymptotic properties of solutions to singularly perturbed singular nonlinear partial differential equations of irregular type in the complex domain. We construct actual holomorphic solutions of these problems with the help of the Borel-Laplace transforms. Using the Malgrange-Sibuya theorem, we show that these holomorphic solutions have a common formal power series asymptotic expansion of Gevrey order 1 in the perturbation parameter.

Key words: asymptotic expansion, Borel-Laplace transform, Cauchy problem, formal power series, nonlinear integro-differential equation, nonlinear partial differential equation, singular perturbation. 2000 MSC: 35C10, 35C20.

1 Introduction

We study a family of singularly perturbed nonlinear partial differential equations of the form

$$\epsilon t^2 \partial_t \partial_z^S X(t,z,\epsilon) + (\epsilon t + 1)\partial_z^S X(t,z,\epsilon) = \sum_{(s,k_0,k_1)\in\mathcal{S}} b_{k_0,k_1}(z,\epsilon) t^s \partial_t^{k_0} \partial_z^{k_1} X(t,z,\epsilon) + P(t,z,\epsilon,X(t,z,\epsilon)) \quad (1)$$

for given initial conditions

$$(\partial_z^j X)(t,0,\epsilon) = \varphi_j(t,\epsilon) \quad , \quad 0 \leq j \leq S-1, \quad (2)$$

where ϵ is a complex perturbation parameter near the origin in $\mathbb{C}$, S is some positive integer, $\mathcal{S}$ is a finite subset of $\mathbb{N}^3$, the coefficients $b_{k_0,k_1}(z,\epsilon)$ of the linear part belong to $\mathcal{O}\{z,\epsilon\}$ and $P(t,z,\epsilon,X) \in \mathcal{O}\{z,\epsilon\}[t,X]$, where $\mathcal{O}\{z,\epsilon\}$ denotes the space of holomorphic functions in (z,ϵ) near the origin in $\mathbb{C}^2$. The initial data $\varphi_j(t,\epsilon)$ are assumed to be holomorphic functions on a product of two sectors with finite radius centered at the origin in $\mathbb{C}^2$.

For all $\epsilon \neq 0$, this family belongs to a class of partial differential equations which have a so-called *irregular singularity* at $t = 0$ (in the sense of [24]). Only a few results about the existence of solutions and their asymptotic properties are known for partial differential equations with irregular singularities, see for instance the papers [9], [10], [24], [26], [27], while partial differential equations with *fuchsian singularities* have been studied to a large extent, see for instance [1], [4], [14], [18], [23], [28], [32].

In a previous work, we have considered the situation when the coefficients of the linear part do not depend on the time variable t and when the nonlinear part $P(t,z,\epsilon,X)$ is replaced by an integro-differential convolution operator acting on X, see [21]. Here, we assume that the following condition holds for the shape of the equation (1) : there exists a real number $b > 1$ such that

$$S \geq b(s - k_0 + 2) + k_1 \quad , \quad s \geq 2k_0,$$

for all $(s, k_0, k_1) \in \mathcal{S}$.

Our result comes within the framework of the asymptotic analysis of singular perturbations of initial value problems of the form

$$\epsilon L_2(t,x,\partial_t,\partial_x)[u(t,x,\epsilon)] + L_1[u(t,x,\epsilon)] = 0 \tag{3}$$

where L_2 is a linear differential operator and L_1 is a nonlinear differential operator, for given initial data $(\partial_x^j u)(t,0,\epsilon) = h_j(t,\epsilon)$, $0 \leq j \leq \nu$, belonging to some space of functions. Most of the statements in the literature concern the situation when ϵ is a real parameter and when L_2 is an elliptic or hyperbolic second order operator acting on real functions spaces (for instance infinitely smooth functions spaces $\mathrm{C}^{\infty}(\mathbb{R}^d)$ or Sobolev spaces $\mathrm{H}^s(\mathbb{R}^d)$). These results concern sufficient conditions for a solution $u(t,x,\epsilon)$ of (3) to have an asymptotic expansion of the form

$$u(t,x,\epsilon) = \sum_{i=0}^{n-1} w_i(t,x)\epsilon^i + R_n(t,x,\epsilon)$$

where bounds of the remainders R_n are obtained, for all $n \geq 1$. The proofs use semi-group operators methods, see [17], the maximum principle and energy integrals estimates, see [15], [25], or fixed point theorems for the nonlinear equations, see [13], [15]. For a general survey on singular perturbations for both asymptotic and numerical aspects, we refer to [19]. But, there are very few informations on singularly perturbed partial differential equations with complex parameter ϵ and with solutions in spaces of analytic functions.

In this paper, we make the assumption that the coefficients in the equation (1) and the initial data (2) are holomorphic functions. Our goal is the construction of actual holomorphic solutions $X(t,z,\epsilon)$ of (1) and the statement of sufficient conditions for the existence and unicity of an asymptotic expansion

$$X(t,z,\epsilon) = \sum_{k=0}^{n-1} H_k(t,z)\epsilon^k + R_n(t,z,\epsilon)$$

with precise bounds for the remainder R_n of the form

$$|R_n(t,z,\epsilon)| \leq CM^n n! |\epsilon|^n$$

for some constants $C, M > 0$, for all ϵ on a sector, uniformly in (t,z) on a product of a sector and a small disc centered at 0, for all $n \geq 1$.

In a paper of C. Durand, J. Mozo and R. Schäfke, see [7], an analogous study has been performed for nonlinear doubly singular differential equations of the form

$$\varepsilon x^2 y'(x,\varepsilon) = f(x,\varepsilon,y(x,\varepsilon))$$

where $f(x,\varepsilon,y)$ is a holomorphic map from $\mathbb{C}^{n+2}$ into $\mathbb{C}^n$, for $n \geq 1$, extending some earlier work for linear differential equations of W. Balser and J. Mozo, c.f. [3].

Following the same strategy as in [7], using the linear map $t \mapsto t/\epsilon$, we transform the problem (1) into an auxiliary regularly perturbed nonlinear partial differential equation which has an irregular singularity at $t = 0$, see (60). The effect of this transformation is that the coefficients of this new equation now have poles with respect to ϵ at the origin. Notice that these kind of singularities in the perturbation parameter did not appear in our previous study [21].

The approach we follow is based on a resummation procedure of formal power series

$$\hat{Y}(t,z,\epsilon) = \sum_{m\geq 0} Y_m(z,\epsilon)t^m/m!$$

with respect to the variable t, which are solutions of the constructed auxiliary problem (60). This resummation method, called in the literature $\kappa-$summability, knows a great success in the study of Gevrey asymptotic expansions of analytic solutions to linear and nonlinear differential equations with irregular singularities, see [2], [5], [11], [16], [20], [29], [30].

We show that the Borel transform of order 1 of $\hat{Y}(t,z,\epsilon)$ with respect to t,

$$V(\tau,z,\epsilon) = \sum_{l\geq 0} Y_l(z,\epsilon)\frac{\tau^l}{(l!)^2}$$

satisfies a nonlinear convolution integro-differential Cauchy problem, with rational coefficients in τ, holomorphic in (τ,z) near the origin and meromorphic in ϵ with a pole at zero, see (73), (74). Under well chosen initial data, we show that $V(\tau,z,\epsilon)$ defines a holomorphic function near the origin with respect to (τ,z) and on a punctured disc at zero with respect to ϵ and can be analytically continued to functions $V_i(\tau,z,\epsilon)$ defined on products $U_i \times D(0,\delta) \times \mathcal{E}_i$, where $\{\mathcal{E}_i\}_{i\in I}$ is a finite set of open sectors centered at the origin whose union form a good covering (Definition 4) and U_i, $i \in I$, are suitable open sectors with small opening and infinite radius. Moreover, the functions V_i, $i \in I$, have exponential growth rate with respect to (τ,ϵ), namely that there exist constants $C, K > 0$ such that

$$\sup_{z\in D(0,\delta)} |V_i(\tau,z,\epsilon)| \leq Ce^{K|\tau|/|\epsilon|} \tag{4}$$

for all (τ,z,ϵ) in their domain of definition (Theorem 2). To obtain these estimates, we introduce some Banach spaces (depending on the parameter ϵ) of functions $v(\tau,z) = \sum_{l\geq 0} v_l(\tau)z^l/l! \in \mathcal{O}(U_i \cup D(0,r))\{z\}$, where $v_l(\tau)$ are bounded by $\exp(K_l|\tau|/|\epsilon|)$, for some constant K_l depending on $l \geq 0$, on $U_i \cup D(0,r)$, $i \in I$, and we solve the nonlinear convolution differential Cauchy problems (73), (74) within these spaces using a fixed point argument (Theorem 1).

We construct actual solutions $Y_i(t,z,\epsilon)$, $i \in I$, of the auxiliary equation (60) as Laplace transforms of the functions $V_i(\tau,z,\epsilon)$ with respect to τ, along a halfline $L_i = \mathbb{R}_+e^{\sqrt{-1}\gamma} \subset U_i \cup \{0\}$. For each $\epsilon \in \mathcal{E}_i$, the function $(t,z) \mapsto Y_i(t,z,\epsilon)$ is holomorphic on a domain $\mathcal{U}_{i,\epsilon} \times D(0,\delta)$ where $\mathcal{U}_{i,\epsilon}$ is a sector of opening larger than π, in direction γ, with radius $h|\epsilon|$ for some constant $h > 0$ (Theorem 2). The crucial observation is that the functions defined by $X_i(t,z,\epsilon) := Y_i(\epsilon t,z,\epsilon)$, $i \in I$, are holomorphic solutions of the initial singularly perturbed equation (1) on domains of the form $\mathcal{T} \times D(0,\delta) \times \mathcal{E}_i$, where $\mathcal{T}$ is a well chosen open sector centered at 0. Moreover, we show that the functions $G_i(\epsilon) := X_{i+1}(t,z,\epsilon) - X_i(t,z,\epsilon)$, $i \in I$, have exponentially small bounds as ϵ tends to 0 on $\mathcal{E}_{i+1} \cap \mathcal{E}_i$, seen as $\mathbb{E}$-valued functions, where $\mathbb{E}$ denotes the Banach space of holomorphic and bounded functions on $\mathcal{T} \times D(0,\delta)$ equipped with the supremum norm. In the proof, we use a deformation of the integration path in the integral representation of X_i and the estimates (4).

Using a cohomological criterion obtained by B. Malgrange and Y. Sibuya, we finally deduce the main result of this paper, namely the existence of a formal series

$$\hat{X}(\epsilon) = \sum_{k \geq 0} H_k \frac{\epsilon^k}{k!} \in \mathbb{E}[[\epsilon]]$$

solution of equation (1), which is the 1-Gevrey asymptotic expansion of the functions X_i on $\mathcal{E}_i$, for all $i \in I$.

The layout of this paper is as follows.
In Section 2, we consider parameter depending nonlinear convolution differential Cauchy problems with singular coefficients. We construct solutions of these equations in parameter depending Banach spaces of holomorphic functions on sectors with exponential growth.
In Section 3.1, we recall the definition of Borel-Laplace transforms and we give commutation formulas with multiplication and integro-differential operators.
In Section 3.2, we study a nonlinear Cauchy problem with irregular singularity having coefficients with poles singularities. We solve this problem using Laplace transforms of the solutions to the Cauchy problems introduced in Section 2.
In Section 4.1, we construct actual solutions X_i, $i \in I$, of our initial equation (1) and we show that the cocycle $G_i = X_{i+1} - X_i$ is exponentially small with respect to ϵ as $\mathbb{E}$-valued function.
In Section 4.2, we state the main result of this paper, that is the existence of a formal series $\hat{X}(\epsilon) \in \mathbb{E}[[\epsilon]]$, solution of (1) which is the 1-Gevrey asymptotic expansion of the functions X_i, $i \in I$.

2 A global Cauchy problem with singular complex parameter

2.1 Weighted Banach spaces of holomorphic functions on sectors

We denote by $D(0,r)$ the open disc centered at 0 with radius $r > 0$ in $\mathbb{C}$. Let S_d be an open sector of infinite or finite radius in direction $d \in \mathbb{R}$ and $\mathcal{E}$ be an open sector with finite radius $r_{\mathcal{E}}$, both centered at 0 in $\mathbb{C}$. By convention, these sectors do not contain the origin in $\mathbb{C}$. For any open set $\mathcal{D} \subset \mathbb{C}$, we denote by $\mathcal{O}(\mathcal{D})$ the vector space of holomorphic functions on $\mathcal{D}$. In this section 2, we denote $\Omega = (S_d \cup D(0,r)) \times \mathcal{E}$.

Definition 1 *Let $b > 1$ a real number and let $r_b(\beta) = \sum_{n=0}^{\beta} 1/(n+1)^b$ for all integers $\beta \geq 0$. Let $\epsilon \in \mathcal{E}$ and $\sigma > 0$ be a real number. We denote by $E_{\beta,\epsilon,\sigma,\Omega}$ the vector space of all functions $v \in \mathcal{O}(S_d \cup D(0,r))$ such that*

$$||v(\tau)||_{\beta,\epsilon,\sigma,\Omega} := \sup_{\tau \in S_d \cup D(0,r)} |v(\tau)|(1 + \frac{|\tau|^2}{|\epsilon|^2}) \exp\left(-\frac{\sigma}{|\epsilon|} r_b(\beta)|\tau|\right)$$

is finite. Let $\delta > 0$ be a real number. We denote by $G(\epsilon,\sigma,\delta,\Omega)$ the vector space of all functions $v(\tau,z) = \sum_{\beta \geq 0} v_\beta(\tau) z^\beta/\beta!$ that belong to $\mathcal{O}(S_d \cup D(0,r))\{z\}$ such that

$$||v(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)} := \sum_{\beta \geq 0} ||v_\beta(\tau)||_{\beta,\epsilon,\sigma,\Omega} \frac{\delta^\beta}{\beta!}$$

is finite. One can check that the normed space $(G(\epsilon,\sigma,\delta,\Omega), ||.||_{(\epsilon,\sigma,\delta,\Omega)})$ is a Banach space.

Remark: These norms are appropriate modifications of the norms defined by O. Costin in [12] and those of C. Stenger and the author introduced in the work [22].

In the next proposition, we study the rate of growth of the functions belonging to the latter Banach spaces.

Proposition 1 *Let* $v(\tau,z) \in G(\epsilon,\sigma,\delta,\Omega)$*. Let* $0 < \delta_1 < 1$*. There exists a constant* $C > 0$ *depending on* $||v||_{(\epsilon,\sigma,\delta,\Omega)}$ *and* δ_1 *such that*

$$|v(\tau,z)| \leq C(1 + \frac{|\tau|^2}{|\epsilon|^2})^{-1} \exp(\frac{\sigma\zeta(b)}{|\epsilon|}|\tau|) \tag{5}$$

for all $\tau \in S_d \cup D(0,r)$*, all* $z \in \mathbb{C}$ *such that* $\frac{|z|}{\delta} < \delta_1$*, where* $\zeta(b) = \sum_{n=0}^{\infty} 1/(n+1)^b$*.*

Proof Let $v(\tau,z) = \sum_{\beta \geq 0} v_\beta(\tau) z^\beta/\beta!$ be in $G(\epsilon,\sigma,\delta,\Omega)$. By definition, there exists a constant $c_1 > 0$ (depending on $||v||_{(\epsilon,\sigma,\delta,\Omega)}$) such that

$$|v_\beta(\tau)| \leq c_1(1 + \frac{|\tau|^2}{|\epsilon|^2})^{-1} \exp\left(\frac{\sigma}{|\epsilon|} r_b(\beta)|\tau|\right) \beta!(\frac{1}{\delta})^\beta$$

for all $\beta \geq 0$, all $\tau \in S_d \cup D(0,r)$. Let $0 < \delta_1 < 1$. From the definition of $\zeta(b)$, we deduce that

$$|v(\tau,z)| \leq c_1(1 + \frac{|\tau|^2}{|\epsilon|^2})^{-1} \sum_{\beta \geq 0} \exp(\frac{\sigma}{|\epsilon|} r_b(\beta)|\tau|)(\delta_1)^\beta \leq c_1(1 + \frac{|\tau|^2}{|\epsilon|^2})^{-1} \exp(\frac{\sigma\zeta(b)}{|\epsilon|}|\tau|)\frac{1}{1-\delta_1}, \tag{6}$$

for all $z \in \mathbb{C}$ such that $\frac{|z|}{\delta} < \delta_1 < 1$, all $\tau \in S_d \cup D(0,r)$. Finally, from (6), we deduce the estimates (5). □

In the next proposition, we study some parameter depending linear operators acting on the space $G(\epsilon,\sigma,\delta,\Omega)$.

Proposition 2 *Let* $s_1, s_2, k_1, k_2 \geq 0$ *be positive integers. Assume that the condition*

$$k_2 \geq b(s_1 + k_1 + 2) \tag{7}$$

hold. Then, for all $\epsilon \in \mathcal{E}$*, the operator* $(\tau^{s_1}/\epsilon^{s_2})\partial_\tau^{-k_1}\partial_z^{-k_2}$ *is a bounded linear operator from* $(G(\epsilon,\sigma,\delta,\Omega), ||.||_{(\epsilon,\sigma,\delta,\Omega)})$ *into itself. Moreover, there exists a constant* $C_1 > 0$ *(depending on* b, σ, s_1, k_1, k_2*), which does not depend on* $\epsilon \in \mathcal{E}$*, such that*

$$||\frac{\tau^{s_1}}{\epsilon^{s_2}}\partial_\tau^{-k_1}\partial_z^{-k_2} v(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)} \leq C_1 |\epsilon|^{s_1+k_1-s_2} \delta^{k_2} ||v(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)} \tag{8}$$

for all $v \in G(\epsilon,\sigma,\delta,\Omega)$*, all* $\epsilon \in \mathcal{E}$*.*

Proof Let $v(\tau,z) \in G(\epsilon,\sigma,\delta,\Omega)$. By definition, we have

$$||\frac{\tau^{s_1}}{\epsilon^{s_2}}\partial_\tau^{-k_1}\partial_z^{-k_2} v(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)} = \sum_{\beta \geq k_2} ||\frac{\tau^{s_1}}{\epsilon^{s_2}}\partial_\tau^{-k_1} v_{\beta-k_2}(\tau)||_{\beta,\epsilon,\sigma,\Omega} \frac{\delta^\beta}{\beta!}. \tag{9}$$

Lemma 1 *The following inequality holds*

$$(10)\quad ||\frac{\tau^{s_1}}{\epsilon^{s_2}}\partial_{\tau}^{-k_1}v_{\beta-k_2}(\tau)||_{\beta,\epsilon,\sigma,\Omega} \leq |\epsilon|^{s_1+k_1-s_2}\left((\frac{(s_1+k_1)e^{-1}}{\sigma k_2})^{s_1+k_1}(\beta+1)^{b(s_1+k_1)} + (\frac{(s_1+k_1+2)e^{-1}}{\sigma k_2})^{s_1+k_1+2}(\beta+1)^{b(s_1+k_1+2)}\right)||v_{\beta-k_2}(\tau)||_{\beta-k_2,\epsilon,\sigma,\Omega}$$

for all $\beta \geq k_2$.

Proof By definition, we have that $\partial_{\tau}^{-1}v_{\beta-k_2}(\tau) = \int_0^{\tau} v_{\beta-k_2}(\tau_1)d\tau_1$, for all $\tau \in S_d \cup D(0,r)$. Using the parametrization $\tau_1 = h_1\tau$ with $0 \leq h_1 \leq 1$, we get that

$$\partial_{\tau}^{-1}v_{\beta-k_2}(\tau) = \tau\int_0^1 v_{\beta-k_2}(h_1\tau)M_1(h_1)dh_1$$

where $M_1(h_1) = 1$. More generally, for all $k_1 \geq 2$, we have by definition

$$\partial_{\tau}^{-k_1}v_{\beta-k_2}(\tau) = \int_0^{\tau}\int_0^{\tau_1}\cdots\int_0^{\tau_{k_1-1}} v_{\beta-k_2}(\tau_{k_1})d\tau_{k_1}d\tau_{k_1-1}\cdots d\tau_1$$

for all $\tau \in S_d \cup D(0,r)$. Using the parametrization $\tau_j = h_j\tau_{j-1}$, $\tau_1 = h_1\tau$, with $0 \leq h_j \leq 1$, for $2 \leq j \leq k_1$, we can write

$$\partial_{\tau}^{-k_1}v_{\beta-k_2}(\tau) = \tau^{k_1}\int_0^1\cdots\int_0^1 v_{\beta-k_2}(h_{k_1}\cdots h_1\tau)M_{k_1}(h_1,\ldots,h_{k_1})dh_{k_1}dh_{k_1-1}\cdots dh_1$$

where $M_{k_1}(h_1,\ldots,h_{k_1})$ is a monomial in $h_1,\ldots,h_{k_1}$ whose coefficient is equal to 1. Using these latter expressions, we now write

$$(11)\quad |\frac{\tau^{s_1}}{\epsilon^{s_2}}\partial_{\tau}^{-k_1}v_{\beta-k_2}(\tau)| = |\frac{\tau^{s_1+k_1}}{\epsilon^{s_2}}\int_0^1\cdots\int_0^1 v_{\beta-k_2}(h_{k_1}\cdots h_1\tau)(1+\frac{|h_{k_1}\cdots h_1\tau|^2}{|\epsilon|^2})\exp\left(-\frac{\sigma}{|\epsilon|}r_b(\beta-k_2)|h_{k_1}\cdots h_1\tau|\right) \times \frac{\exp\left(\frac{\sigma}{|\epsilon|}r_b(\beta-k_2)|h_{k_1}\cdots h_1\tau|\right)}{1+|h_{k_1}\cdots h_1\tau|^2/|\epsilon|^2}M_{k_1}(h_1,\ldots,h_{k_1})dh_{k_1}\cdots dh_1|.$$

So that

$$(12)\quad |\frac{\tau^{s_1}}{\epsilon^{s_2}}\partial_{\tau}^{-k_1}v_{\beta-k_2}(\tau)|(1+\frac{|\tau|^2}{|\epsilon|^2})\exp\left(-\frac{\sigma}{|\epsilon|}r_b(\beta)|\tau|\right) \leq ||v_{\beta-k_2}(\tau)||_{\beta-k_2,\epsilon,\sigma,\Omega}\frac{|\tau|^{s_1+k_1}}{|\epsilon|^{s_2}}(1+\frac{|\tau|^2}{|\epsilon|^2})\exp\left(-\frac{\sigma}{|\epsilon|}(r_b(\beta)-r_b(\beta-k_2))|\tau|\right).$$

By construction of $r_b(\beta)$ we have that

$$(13)\quad r_b(\beta) - r_b(\beta-k_2) \geq \frac{k_2}{(\beta+1)^b}$$

for all $\beta \geq k_2$. From (12) and (13), we get that

$$(14)\quad |\frac{\tau^{s_1}}{\epsilon^{s_2}}\partial_{\tau}^{-k_1}v_{\beta-k_2}(\tau)|(1+\frac{|\tau|^2}{|\epsilon|^2})\exp\left(-\frac{\sigma}{|\epsilon|}r_b(\beta)|\tau|\right) \leq ||v_{\beta-k_2}(\tau)||_{\beta-k_2,\epsilon,\sigma,\Omega}\frac{|\tau|^{s_1+k_1}}{|\epsilon|^{s_2}}(1+\frac{|\tau|^2}{|\epsilon|^2})\exp\left(-\frac{\sigma}{|\epsilon|}\frac{k_2}{(\beta+1)^b}|\tau|\right)$$

for all $\beta \geq k_2$. Now, we recall the following classical estimates. Let $m_1, m_2 > 0$ two real numbers. Then, we have

$$\sup_{x \geq 0} x^{m_1} \exp(-m_2 x) = (\frac{m_1}{m_2})^{m_1} e^{-m_1} \tag{15}$$

We deduce that

$$\frac{|\tau|^{s_1+k_1}}{|\epsilon|^{s_2}}(1 + \frac{|\tau|^2}{|\epsilon|^2}) \exp\left(-\frac{\sigma}{|\epsilon|}\frac{k_2}{(\beta+1)^b}|\tau|\right) \tag{16}$$
$$\leq |\epsilon|^{s_1+k_1-s_2}\left((\frac{(s_1+k_1)e^{-1}}{\sigma k_2})^{s_1+k_1}(\beta+1)^{b(s_1+k_1)} + (\frac{(s_1+k_1+2)e^{-1}}{\sigma k_2})^{s_1+k_1+2}(\beta+1)^{b(s_1+k_1+2)}\right)$$

for all $\tau \in S_d \cup D(0,r)$. From the estimates (14) and (16), we deduce the inequality (10) $\Box$

From the equality (9) and Lemma 1, we get that

$$||\frac{\tau^{s_1}}{\epsilon^{s_2}}\partial_\tau^{-k_1}\partial_z^{-k_2}v(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)} \tag{17}$$
$$\leq \sum_{\beta \geq k_2} |\epsilon|^{s_1+k_1-s_2}\left((\frac{(s_1+k_1)e^{-1}}{\sigma k_2})^{s_1+k_1}(\beta+1)^{b(s_1+k_1)}\right.$$
$$\left.+(\frac{(s_1+k_1+2)e^{-1}}{\sigma k_2})^{s_1+k_1+2}(\beta+1)^{b(s_1+k_1+2)}\right)\frac{(\beta-k_2)!}{\beta!}$$
$$\times ||v_{\beta-k_2}(\tau)||_{\beta-k_2,\epsilon,\sigma,\Omega}\delta^{k_2}\frac{\delta^{\beta-k_2}}{(\beta-k_2)!}$$

From the assumptions (7), we get a constant $C_2 > 0$ (depending on b, s_1, k_1, k_2) such that

$$(\beta+1)^{b(s_1+k_1)}\frac{(\beta-k_2)!}{\beta!} \leq C_2 \quad , \quad (\beta+1)^{b(s_1+k_1+2)}\frac{(\beta-k_2)!}{\beta!} \leq C_2, \tag{18}$$

for all $\beta \geq k_2$. Finally, from the estimates (17) and (18), we get the inequality (8). $\Box$

Proposition 3 *Let $s_1, s_2, k_1 \geq 0$ be positive integers and $\sigma, \tilde{\sigma} > 0$ be real numbers such that $\sigma > \tilde{\sigma}$. Then, there exists a constant $\tilde{C}_1 > 0$ (depending on $\sigma, \tilde{\sigma}, s_1, k_1$), which does not depend on $\epsilon \in \mathcal{E}$, such that*

$$||\frac{\tau^{s_1}}{\epsilon^{s_2}}\partial_\tau^{-k_1}v(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)} \leq \tilde{C}_1|\epsilon|^{s_1+k_1-s_2}||v(\tau,z)||_{(\epsilon,\tilde{\sigma},\delta,\Omega)} \tag{19}$$

for all $v \in G(\epsilon,\tilde{\sigma},\delta,\Omega)$, all $\epsilon \in \mathcal{E}$.

Proof Let $v(\tau,z) \in G(\epsilon,\tilde{\sigma},\delta,\Omega)$. By definition, we have

$$||\frac{\tau^{s_1}}{\epsilon^{s_2}}\partial_\tau^{-k_1}v(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)} = \sum_{\beta \geq 0}||\frac{\tau^{s_1}}{\epsilon^{s_2}}\partial_\tau^{-k_1}v_\beta(\tau)||_{\beta,\epsilon,\sigma,\Omega}\frac{\delta^\beta}{\beta!}. \tag{20}$$

Lemma 2 *The following inequality holds*

$$||\frac{\tau^{s_1}}{\epsilon^{s_2}}\partial_\tau^{-k_1}v_\beta(\tau)||_{\beta,\epsilon,\sigma,\Omega} \leq |\epsilon|^{s_1+k_1-s_2}\left((\frac{(s_1+k_1)e^{-1}}{\sigma-\tilde{\sigma}})^{s_1+k_1}\right. \tag{21}$$
$$\left.+(\frac{(s_1+k_1+2)e^{-1}}{\sigma-\tilde{\sigma}})^{s_1+k_1+2}\right)||v_\beta(\tau)||_{\beta,\epsilon,\tilde{\sigma},\Omega}$$

for all $\beta \geq 0$.

Proof Using the same notations as in the proof of Lemma 1, we can write

$$
\begin{aligned}
(22)\quad & |\frac{\tau^{s_1}}{\epsilon^{s_2}}\partial_{\tau}^{-k_1}v_{\beta}(\tau)| \\
&= |\frac{\tau^{s_1+k_1}}{\epsilon^{s_2}}\int_0^1 \cdots \int_0^1 v_{\beta}(h_{k_1}\cdots h_1\tau)(1+\frac{|h_{k_1}\cdots h_1\tau|^2}{|\epsilon|^2})\exp\left(-\frac{\tilde{\sigma}}{|\epsilon|}r_b(\beta)|h_{k_1}\cdots h_1\tau|\right) \\
&\qquad \times \frac{\exp\left(\frac{\tilde{\sigma}}{|\epsilon|}r_b(\beta)|h_{k_1}\cdots h_1\tau|\right)}{1+|h_{k_1}\cdots h_1\tau|^2/|\epsilon|^2}M_{k_1}(h_1,\ldots,h_{k_1})dh_{k_1}\cdots dh_1|.
\end{aligned}
$$

Using the fact that $r_b(\beta) \geq 1$, for all $\beta \geq 0$, we deduce that

$$
\begin{aligned}
(23)\quad & |\frac{\tau^{s_1}}{\epsilon^{s_2}}\partial_{\tau}^{-k_1}v_{\beta}(\tau)|(1+\frac{|\tau|^2}{|\epsilon|^2})\exp(-\frac{\sigma}{|\epsilon|}r_b(\beta)|\tau|) \\
&\leq ||v_{\beta}(\tau)||_{\beta,\epsilon,\tilde{\sigma},\Omega}\frac{|\tau|^{s_1+k_1}}{|\epsilon|^{s_2}}(1+\frac{|\tau|^2}{|\epsilon|^2})\exp(-(\sigma-\tilde{\sigma})\frac{r_b(\beta)}{|\epsilon|}|\tau|) \\
&\leq |v_{\beta}(\tau)||_{\beta,\epsilon,\tilde{\sigma},\Omega}\frac{|\tau|^{s_1+k_1}}{|\epsilon|^{s_2}}(1+\frac{|\tau|^2}{|\epsilon|^2})\exp(-\frac{\sigma-\tilde{\sigma}}{|\epsilon|}|\tau|)
\end{aligned}
$$

for all $\beta \geq 0$. From the inequality (15), we deduce that

$$
\begin{aligned}
(24)\quad & \frac{|\tau|^{s_1+k_1}}{|\epsilon|^{s_2}}(1+\frac{|\tau|^2}{|\epsilon|^2})\exp(-\frac{\sigma-\tilde{\sigma}}{|\epsilon|}|\tau|) \\
&\leq |\epsilon|^{s_1+k_1-s_2}\left((\frac{(s_1+k_1)e^{-1}}{\sigma-\tilde{\sigma}})^{s_1+k_1}+(\frac{(s_1+k_1+2)e^{-1}}{\sigma-\tilde{\sigma}})^{s_1+k_1+2}\right)
\end{aligned}
$$

for all $\tau \in S_d \cup D(0,r)$. From (23) and (24) we deduce the inequality (21). □

Finally, from the equality (20) and Lemma 2, we deduce the estimates (19). □

In the next proposition, we study linear operators of multiplication by bounded holomorphic functions.

Proposition 4 *Let $h(\tau,z,\epsilon)$ be a holomorphic function on $(S_d \cup D(0,r)) \times D(0,\rho) \times \mathcal{E}$, for some $\rho > 0$, bounded by some constant $M > 0$. Let $0 < \delta < \rho$. Then, the linear operator of multiplication by $h(\tau,z,\epsilon)$ is continuous from $(G(\epsilon,\sigma,\delta,\Omega),||.||_{(\epsilon,\sigma,\delta,\Omega)})$ into itself, for all $\epsilon \in \mathcal{E}$. Moreover, there exists a constant C_2 (depending on M,δ,ρ), independent of ϵ, such that*

$$
(25)\qquad ||h(\tau,z,\epsilon)v(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)} \leq C_2||v(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)}
$$

for all $v(\tau,z) \in G(\epsilon,\sigma,\delta,\Omega)$, for all $\epsilon \in \mathcal{E}$.

Proof Let $h(\tau,z,\epsilon) = \sum_{\beta\geq 0} h_{\beta}(\tau,\epsilon)z^{\beta}/\beta!$ be holomorphic on $(S_d \cup D(0,r)) \times D(0,\rho) \times \mathcal{E}$ such that there exists $M > 0$ with

$$
\sup_{\tau\in S_d\cup D(0,r),z\in D(0,\rho),\epsilon\in\mathcal{E}} |h(\tau,z,\epsilon)| \leq M.
$$

Let $v(\tau,z) = \sum_{\beta\geq 0} v_{\beta}(\tau)z^{\beta}/\beta! \in G(\epsilon,\sigma,\delta,\Omega)$. By construction, we have that

$$
(26)\qquad ||h(\tau,z,\epsilon)v(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)} \leq \sum_{\beta\geq 0}(\sum_{\beta_1+\beta_2=\beta}||h_{\beta_1}(\tau,\epsilon)v_{\beta_2}(\tau)||_{\beta,\epsilon,\sigma,\Omega}\frac{\beta!}{\beta_1!\beta_2!})\frac{\delta^{\beta}}{\beta!}.
$$

From the Cauchy formula, we have

$$\sup_{\tau \in S_d \cup D(0,r), \epsilon \in \mathcal{E}} |h_\beta(\tau,\epsilon)| \leq M(\frac{1}{\delta'})^{\beta} \beta!$$

for any $\delta < \delta' < \rho$, for all $\beta \geq 0$. By definition, we deduce that

$$||h_{\beta_1}(\tau,\epsilon) v_{\beta_2}(\tau)||_{\beta,\epsilon,\sigma,\Omega} \leq M\beta_1!(\frac{1}{\delta'})^{\beta_1} ||v_{\beta_2}(\tau)||_{\beta,\epsilon,\sigma,\Omega} \leq M\beta_1!(\frac{1}{\delta'})^{\beta_1} ||v_{\beta_2}(\tau)||_{\beta_2,\epsilon,\sigma,\Omega} \tag{27}$$

for all $\beta_1, \beta_2 \geq 0$ such that $\beta_1 + \beta_2 = \beta$. From (26) and (27), we deduce that

$$||h(\tau,z,\epsilon) v(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)} \leq M(\sum_{\beta \geq 0} (\frac{\delta}{\delta'})^{\beta}) ||v(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)}$$

which yields (25). □

In the next proposition, we give norm estimates for the convolution product.

Proposition 5 *Let* f*,*g *be in* $G(\epsilon,\sigma,\delta,\Omega)$*. Then, the function*

$$(f * g)(\tau,z) = \int_0^{\tau} f(\tau - s, z) g(s,z) ds$$

belongs to $G(\epsilon,\sigma,\delta,\Omega)$*. Moreover, there exists a (universal) constant* $C_3 > 0$ *such that*

$$||(f * g)(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)} \leq C_3 |\epsilon| ||f(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)} ||g(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)} \tag{28}$$

for all $f, g \in G(\epsilon,\sigma,\delta,\Omega)$*.*

Proof Let

$$f(\tau,z) = \sum_{\beta \geq 0} f_\beta(\tau) z^\beta / \beta! \quad , \quad g(\tau,z) = \sum_{\beta \geq 0} g_\beta(\tau) z^\beta / \beta!$$

in $G(\epsilon,\sigma,\delta,\Omega)$. By construction of $f * g$, we have that

$$||\int_0^{\tau} f(\tau - s, z) g(s,z) ds||_{(\epsilon,\sigma,\delta,\Omega)} \leq \sum_{\beta \geq 0} (\sum_{\beta_1 + \beta_2 = \beta} \frac{\beta!}{\beta_1! \beta_2!} ||\int_0^{\tau} f_{\beta_1}(\tau - s) g_{\beta_2}(s) ds||_{\beta,\epsilon,\sigma,\Omega}) \frac{\delta^\beta}{\beta!}. \tag{29}$$

Lemma 3 *There exists a (universal) constant* $C_3 > 0$ *such that*

$$||\int_0^{\tau} f_{\beta_1}(\tau - s) g_{\beta_2}(s) ds||_{\beta,\epsilon,\sigma,\Omega} \leq C_3 |\epsilon| ||f_{\beta_1}(\tau)||_{\beta_1,\epsilon,\sigma,\Omega} ||g_{\beta_2}(\tau)||_{\beta_2,\epsilon,\sigma,\Omega} \tag{30}$$

for all $\beta \geq 0$ *and all* $\beta_1, \beta_2 \geq 0$ *with* $\beta_1 + \beta_2 = \beta$*.*

Proof We write

$$|\int_0^{\tau} f_{\beta_1}(\tau - s) g_{\beta_2}(s) ds| = |\int_0^{\tau} f_{\beta_1}(\tau - s)(1 + \frac{|\tau - s|^2}{|\epsilon|^2}) \exp\left(-\frac{\sigma}{|\epsilon|} r_b(\beta_1)|\tau - s|\right)$$
$$\times g_{\beta_2}(s)(1 + \frac{|s|^2}{|\epsilon|^2}) \exp\left(-\frac{\sigma}{|\epsilon|} r_b(\beta_2)|s|\right)$$
$$\times \frac{\exp\left(\frac{\sigma}{|\epsilon|}(r_b(\beta_1)|\tau - s| + r_b(\beta_2)|s|)\right)}{(1 + \frac{|s|^2}{|\epsilon|^2})(1 + \frac{|\tau - s|^2}{|\epsilon|^2})} ds|$$

for all $\tau \in S_d \cup D(0,r)$. We deduce that

$$(31)\quad |\int_0^\tau f_{\beta_1}(\tau-s)g_{\beta_2}(s)ds| \leq ||f_{\beta_1}(\tau)||_{\beta_1,\epsilon,\sigma,\Omega}||g_{\beta_2}(\tau)||_{\beta_2,\epsilon,\sigma,\Omega} \times \int_0^1 \frac{|\tau|\exp\left(\frac{\sigma|\tau|}{|\epsilon|}(r_b(\beta_1)(1-h)+r_b(\beta_2)h)\right)}{(1+\frac{|\tau|^2}{|\epsilon|^2}(1-h)^2)(1+\frac{|\tau|^2}{|\epsilon|^2}h^2)}dh.$$

In the next step we will show that there exists a constant $C_3 > 0$ such that

$$(32)\quad I(|\tau|,|\epsilon|,\beta,\beta_1,\beta_2) = (1+\frac{|\tau|^2}{|\epsilon|^2})\exp(-\frac{\sigma}{|\epsilon|}r_b(\beta)|\tau|) \times \int_0^1 \frac{|\tau|\exp\left(\frac{\sigma|\tau|}{|\epsilon|}(r_b(\beta_1)(1-h)+r_b(\beta_2)h)\right)}{(1+\frac{|\tau|^2}{|\epsilon|^2}(1-h)^2)(1+\frac{|\tau|^2}{|\epsilon|^2}h^2)}dh \leq |\epsilon|C_3$$

for all $\tau \in S_d \cup D(0,r)$, all $\epsilon \in \mathcal{E}$, for all $\beta \geq 0$, all $\beta_1,\beta_2 \geq 0$ with $\beta_1+\beta_2=\beta$. Indeed, from the fact that r_b is increasing, we first have that

$$(33)\qquad r_b(\beta_1)(1-h)+r_b(\beta_2)h \leq r_b(\beta)$$

for all $0 \leq h \leq 1$, all $\beta_1,\beta_2 \geq 0$ with $\beta_1+\beta_2=\beta$. Then, from (33), we get that

$$(34)\qquad I(|\tau|,|\epsilon|,\beta,\beta_1,\beta_2) \leq J(|\tau|,|\epsilon|) = \int_0^1 \frac{(1+\frac{|\tau|^2}{|\epsilon|^2})|\tau|}{(1+\frac{|\tau|^2}{|\epsilon|^2}(1-h)^2)(1+\frac{|\tau|^2}{|\epsilon|^2}h^2)}dh$$

for all $\tau \in S_d \cup D(0,r)$, all $\epsilon \in \mathcal{E}$. On the other hand, we have that

$$(35)\qquad \frac{J(|\epsilon||\tau|,|\epsilon|)}{|\epsilon|} = \int_0^1 \frac{(1+|\tau|^2)|\tau|}{(1+|\tau|^2(1-h)^2)(1+|\tau|^2h^2)}dh.$$

From Corollary 4.9 of [12], we know that the right hand side of (35) is a bounded function of $|\tau|$ on $\mathbb{R}_+$. We deduce that there exists a (universal) constant $C_3 > 0$ such that

$$(36)\qquad \sup_{|\tau|\geq 0}\frac{J(|\tau|,|\epsilon|)}{|\epsilon|} = \sup_{|\tau|\geq 0}\frac{J(|\epsilon||\tau|,|\epsilon|)}{|\epsilon|} \leq C_3$$

for all $\epsilon \in \mathcal{E}$. We get from (34) and (36) that the inequality (32) holds. Finally, the inequality (30) follows from (31) and (32). □

From (29) and (30), we get that (28) holds with the constant C_3 from Lemma 3. □

Corollary 1 *Let $k_1, s_2 \geq 0$ be positive integers. Then, for all $\epsilon \in \mathcal{E}$, the operator $(1/\epsilon^{s_2})\partial_\tau^{-k_1}$ is a bounded linear operator from $(G(\epsilon,\sigma,\delta,\Omega),||.||_{(\epsilon,\sigma,\delta,\Omega)})$ into itself. Moreover, there exists a constant $C_4 > 0$ (depending on σ, k_1) such that*

$$(37)\qquad ||\frac{1}{\epsilon^{s_2}}\partial_\tau^{-k_1}v(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)} \leq C_4|\epsilon|^{k_1-s_2}||v(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)}$$

for all $v(\tau,z) \in G(\epsilon,\sigma,\delta,\Omega)$, all $\epsilon \in \mathcal{E}$.

Proof Let $v(\tau,z) \in G(\epsilon,\sigma,\delta,\Omega)$. We denote by $\chi_{\mathbb{C}}$ the function equal to 1 on $\mathbb{C}$. By definition, we put $\chi_{\mathbb{C}}^{*1} = \chi_{\mathbb{C}}$ and $\chi_{\mathbb{C}}^{*l}$ means the convolution product of $\chi_{\mathbb{C}}$, $l-1$ times for $l \geq 2$. By definition, we can write $\partial_{\tau}^{-k_1} v(\tau,z) = (\chi_{\mathbb{C}}(\tau))^{*k_1} * v(\tau,z)$. From Proposition 5, there exists a (universal) constant $C_3 > 0$ such that

$$||\partial_{\tau}^{-k_1} v(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)} \leq C_3^{k_1}|\epsilon|^{k_1}||\chi_{\mathbb{C}}(\tau)||_{(\epsilon,\sigma,\delta,\Omega)}^{k_1}||v(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)} \tag{38}$$

By Definition 1 and using the formula (15), we have that

$$||\chi_{\mathbb{C}}(\tau)||_{(\epsilon,\sigma,\delta,\Omega)} = \sup_{\tau \in S_d \cup D(0,r)} (1 + \frac{|\tau|^2}{|\epsilon|^2}) \exp\left(-\frac{\sigma}{|\epsilon|}|\tau|\right) \leq 1 + (\frac{2e^{-1}}{\sigma})^2 \tag{39}$$

From the estimates (38) and (39), we get the inequality (37). □

2.2 A global Cauchy problem

We keep the same notations as in the previous section. In the following, we introduce some definitions. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be finite subsets of $\mathbb{N}^2$.
For all $(k_0,k_1) \in \mathcal{A}_1$, we denote by $I_{(k_0,k_1)}$ a finite subset of $\mathbb{N}^2$. For all $(s_1,s_2) \in I_{(k_0,k_1)}$, we denote by $a_{s_1,s_2,k_0,k_1}(\tau,z,\epsilon)$ some bounded holomorphic function on $(S_d \cup D(0,r)) \times D(0,\rho) \times \mathcal{E}$, for some $\rho > 0$. For all $(k_0,k_1) \in \mathcal{A}_1$, we consider

$$a_{(k_0,k_1)}(\tau,z,\epsilon) = \sum_{(s_1,s_2) \in I_{(k_0,k_1)}} a_{s_1,s_2,k_0,k_1}(\tau,z,\epsilon)\tau^{s_1}\epsilon^{-s_2}$$

which are holomorphic functions on $(S_d \cup D(0,r)) \times D(0,\rho) \times \mathcal{E}$.
For all $(l_0,l_1) \in \mathcal{A}_2$, we denote by $J_{(l_0,l_1)}$ a finite subset of $\mathbb{N}$. For all $m_1 \in J_{(l_0,l_1)}$, we denote by $\alpha_{m_1,l_0,l_1}(\tau,z,\epsilon)$ some bounded holomorphic function on $(S_d \cup D(0,r)) \times D(0,\rho) \times \mathcal{E}$. For all $(l_0,l_1) \in \mathcal{A}_2$, we consider

$$\alpha_{(l_0,l_1)}(\tau,z,\epsilon) = \sum_{m_1 \in J_{(l_0,l_1)}} \alpha_{m_1,l_0,l_1}(\tau,z,\epsilon)\epsilon^{-m_1}$$

which are holomorphic functions on $(S_d \cup D(0,r)) \times D(0,\rho) \times \mathcal{E}$. Let $S \geq 1$ be an integer. For all $0 \leq j \leq S-1$, we consider a function $\tau \mapsto V_j(\tau,\epsilon)$ that belongs to $E_{0,\epsilon,\tilde{\sigma},\Omega}$, for some $\tilde{\sigma} > 0$ and all $\epsilon \in \mathcal{E}$.

We consider the following Cauchy problem

$$\partial_z^S V(\tau,z,\epsilon) = \sum_{(k_0,k_1) \in \mathcal{A}_1} a_{(k_0,k_1)}(\tau,z,\epsilon)\partial_{\tau}^{-k_0}\partial_z^{k_1} V(\tau,z,\epsilon) + \sum_{(l_0,l_1) \in \mathcal{A}_2, l_1 \geq 2} \alpha_{(l_0,l_1)}(\tau,z,\epsilon)\partial_{\tau}^{-l_0}((V(\tau,z,\epsilon))^{*l_1}) \tag{40}$$

where $V^{*1} = V$ and V^{*l_1}, $l_1 \geq 2$, stands for the convolution product of V ($l_1 - 1$ times with respect to τ), for given initial conditions

$$(\partial_z^j V)(\tau,0,\epsilon) = V_j(\tau,\epsilon) \quad , \quad 0 \leq j \leq S-1. \tag{41}$$

We state the main result of this section.

Theorem 1 *We make the following assumptions.*

A1) *For all* $(k_0, k_1) \in \mathcal{A}_1$*, all* $(s_1, s_2) \in I_{(k_0,k_1)}$*, we have*

$$S \geq b(s_1 + k_0 + 2) + k_1 \quad , \quad s_1 + k_0 \geq s_2,$$

A2) *For all* $(l_0, l_1) \in \mathcal{A}_2$*, all* $m_1 \in J_{(l_0,l_1)}$*, we have*

$$l_0 + l_1 \geq m_1 + 1 \quad , \quad l_1 \geq 2.$$

We put

$$w(\tau, z, \epsilon) = \sum_{j=0}^{S-1} V_j(\tau, \epsilon) \frac{z^j}{j!}.$$

Then, there exist constants $I > 0$*,* $R > 0$ *and* $\delta > 0$ *(independent of* ϵ*) such that if we assume that*

$$\sum_{j=0}^{S-1-h} ||V_{j+h}(\tau, \epsilon)||_{0,\epsilon,\tilde{\sigma},\Omega} \frac{\delta^j}{j!} \leq I,$$

for all $0 \leq h \leq S-1$*, for all* $\epsilon \in \mathcal{E}$*, the problem (40) and (41) has a unique solution* $V(\tau, z, \epsilon)$ *in the space* $G(\epsilon, \sigma, \delta, \Omega)$*, for some* $\sigma > \tilde{\sigma}$*, for all* $\epsilon \in \mathcal{E}$*, which satisfies moreover the estimates*

$$||V(\tau, z, \epsilon)||_{(\epsilon,\sigma,\delta,\Omega)} \leq \delta^S R + I,$$

for all $\epsilon \in \mathcal{E}$*.*

Proof For all $\epsilon \in \mathcal{E}$, we define a map A_ϵ from $\mathcal{O}(S_d \cup D(0, r))\{z\}$ into itself by

$$\begin{aligned} A_\epsilon(U(\tau, z)) = &\sum_{(k_0,k_1)\in\mathcal{A}_1} a_{(k_0,k_1)}(\tau, z, \epsilon)\partial_\tau^{-k_0}\partial_z^{k_1-S}U(\tau, z) \\ &+ \sum_{(k_0,k_1)\in\mathcal{A}_1} a_{(k_0,k_1)}(\tau, z, \epsilon)\partial_\tau^{-k_0}\partial_z^{k_1} w(\tau, z, \epsilon) \\ &+ \sum_{(l_0,l_1)\in\mathcal{A}_2} \alpha_{(l_0,l_1)}(\tau, z, \epsilon)\partial_\tau^{-l_0}((\partial_z^{-S}U(\tau, z) + w(\tau, z, \epsilon))^{*l_1}) \end{aligned}$$

In the next lemma, we show that A_ϵ is a Lipschitz shrinking map from and into a small ball in a neighborhood of the origin of $G(\epsilon, \sigma, \delta, \Omega)$, for some $\sigma > \tilde{\sigma}$.

Lemma 4 *Under the conditions* **A1)**, **A2)***, let a real number* I *be such that*

$$\sum_{j=0}^{S-1-h} ||V_{j+h}(\tau, \epsilon)||_{0,\epsilon,\tilde{\sigma},\Omega} \frac{\delta^j}{j!} \leq I,$$

for all $0 \leq h \leq S-1$*, for all* $\epsilon \in \mathcal{E}$*. Then, for a good choice of* $I > 0$*,*
a) there exist real numbers $0 < \delta < \rho$*,* $\sigma > \tilde{\sigma}$ *and* $R > 0$ *(not depending on* ϵ*) such that*

$$||A_\epsilon(U(\tau, z))||_{(\epsilon,\sigma,\delta,\Omega)} \leq R \tag{42}$$

for all $U(\tau, z) \in B(0, R)$*, for all* $\epsilon \in \mathcal{E}$*, where* $B(0, R)$ *is the closed ball centered at 0 with radius* R *in* $G(\epsilon, \sigma, \delta, \Omega)$*,*
b) we have

$$||A_\epsilon(U_1(\tau, z)) - A_\epsilon(U_2(\tau, z))||_{(\epsilon,\sigma,\delta,\Omega)} \leq \frac{1}{2}||U_1(\tau, z) - U_2(\tau, z)||_{(\epsilon,\sigma,\delta,\Omega)} \tag{43}$$

for all $U_1, U_2 \in B(0, R)$*, for all* $\epsilon \in \mathcal{E}$*.*

Proof First of all, for all $0 \leq h \leq S-1$, $0 \leq j \leq S-1-h$, we have that

$$||V_{j+h}(\tau,\epsilon)||_{j,\epsilon,\tilde{\sigma},\Omega} \leq ||V_{j+h}(\tau,\epsilon)||_{0,\epsilon,\tilde{\sigma},\Omega}.$$

We deduce that $\partial_z^h w(\tau,z,\epsilon) \in G(\epsilon,\tilde{\sigma},\delta,\Omega)$ and that

$$||\partial_z^h w(\tau,z,\epsilon)||_{(\epsilon,\tilde{\sigma},\delta,\Omega)} \leq \sum_{j=0}^{S-1-h} ||V_{j+h}(\tau,\epsilon)||_{0,\epsilon,\tilde{\sigma},\Omega}\frac{\delta^j}{j!} \leq I \tag{44}$$

for all $0 \leq h \leq S-1$.

We first show the estimates (42).

Let $\sigma > \tilde{\sigma}$, $R > 0$ and $U(\tau,z) \in G(\epsilon,\sigma,\delta,\Omega)$ with $||U||_{(\epsilon,\sigma,\delta,\Omega)} \leq R$. From Propositions 2,4 we get that there exists a constant $d_1 > 0$ (independent of ϵ) such that

$$\begin{aligned} &||a_{s_1,s_2,k_0,k_1}(\tau,z,\epsilon)\tau^{s_1}\epsilon^{-s_2}\partial_\tau^{-k_0}\partial_z^{k_1-S}U(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)} \\ &\leq d_1|\epsilon|^{s_1+k_0-s_2}\delta^{S-k_1}||U(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)} \leq d_1|\epsilon|^{s_1+k_0-s_2}\delta^{S-k_1}R, \end{aligned} \tag{45}$$

for all $(k_0,k_1) \in \mathcal{A}_1$, all $(s_1,s_2) \in I_{(k_0,k_1)}$. From Propositions 3,4 and the inequality (44) we get that there exists a constant $\tilde{d}_1 > 0$ (independent of ϵ) such that

$$\begin{aligned} &||a_{s_1,s_2,k_0,k_1}(\tau,z,\epsilon)\tau^{s_1}\epsilon^{-s_2}\partial_\tau^{-k_0}\partial_z^{k_1}w(\tau,z,\epsilon)||_{(\epsilon,\sigma,\delta,\Omega)} \\ &\leq \tilde{d}_1|\epsilon|^{s_1+k_0-s_2}||\partial_z^{k_1}w(\tau,z,\epsilon)||_{(\epsilon,\tilde{\sigma},\delta,\Omega)} \leq \tilde{d}_1|\epsilon|^{s_1+k_0-s_2}I, \end{aligned} \tag{46}$$

for all $(k_0,k_1) \in \mathcal{A}_1$, all $(s_1,s_2) \in I_{(k_0,k_1)}$. On the other hand, since the convolution product is commutative, from the binomal formula, we can write

$$\begin{aligned} (\partial_z^{-S}U(\tau,z) + w(\tau,z,\epsilon))^{*l_1} &= (\partial_z^{-S}U(\tau,z))^{*l_1} + (w(\tau,z,\epsilon))^{*l_1} \\ &+ \sum_{l_1^1+l_1^2=l_1, l_1^1\geq 1, l_1^2 \geq 1} \frac{l_1!}{l_1^1!l_1^2!}(\partial_z^{-S}U(\tau,z))^{*l_1^1} * (w(\tau,z,\epsilon))^{*l_1^2} \end{aligned}$$

for all $l_1 \geq 2$. From Propositions 2,5 we get a constant $d_2 > 0$ (independent of ϵ) such that

$$\begin{aligned} &||(\partial_z^{-S}U(\tau,z) + w(\tau,z,\epsilon))^{*l_1}||_{(\epsilon,\sigma,\delta,\Omega)} \\ &\leq d_2|\epsilon|^{l_1-1}(\delta^{Sl_1}R^{l_1} + I^{l_1} + \sum_{l_1^1+l_1^2=l_1, l_1^1\geq 1, l_1^2\geq 1} \frac{l_1!}{l_1^1!l_1^2!}\delta^{Sl_1^1}R^{l_1^1}I^{l_1^2}) = d_2|\epsilon|^{l_1-1}(\delta^S R + I)^{l_1} \end{aligned} \tag{47}$$

for all $2 \leq l_1 \leq \max\{l \in \mathbb{N}/(l_0,l) \in \mathcal{A}_2\}$. From Proposition 4 and Corollary 1, we get a constant $d_3 > 0$ (independent of ϵ) such that

$$\begin{aligned} &||\alpha_{m_1,l_0,l_1}(\tau,z,\epsilon)\epsilon^{-m_1}\partial_\tau^{-l_0}((\partial_z^{-S}U(\tau,z) + w(\tau,z,\epsilon))^{*l_1})||_{(\epsilon,\sigma,\delta,\Omega)} \\ &\leq d_3|\epsilon|^{l_0-m_1}||(\partial_z^{-S}U(\tau,z) + w(\tau,z,\epsilon))^{*l_1}||_{(\epsilon,\sigma,\delta,\Omega)} \end{aligned} \tag{48}$$

for all $(l_0,l_1) \in \mathcal{A}_2$, all $m_1 \in J_{(l_0,l_1)}$. From (47) and (48), we get that

$$\begin{aligned} &||\alpha_{m_1,l_0,l_1}(\tau,z,\epsilon)\epsilon^{-m_1}\partial_\tau^{-l_0}((\partial_z^{-S}U(\tau,z) + w(\tau,z,\epsilon))^{*l_1})||_{(\epsilon,\sigma,\delta,\Omega)} \\ &\leq d_2 d_3|\epsilon|^{l_0+l_1-m_1-1}(\delta^S R + I)^{l_1} \end{aligned} \tag{49}$$

for all $(l_0,l_1) \in \mathcal{A}_2$, all $m_1 \in J_{(l_0,l_1)}$. Now, we choose $\delta,R,I>0$ such that

$$(50)\quad \sum_{(k_0,k_1)\in\mathcal{A}_1}\sum_{(s_1,s_2)\in I_{(k_0,k_1)}} |\epsilon|^{s_1+k_0-s_2}(d_1\delta^{S-k_1}R+\tilde{d}_1 I) + \sum_{(l_0,l_1)\in\mathcal{A}_2}\sum_{m_1\in J_{(l_0,l_1)}} d_2d_3|\epsilon|^{l_0+l_1-m_1-1}(\delta^S R+I)^{l_1} \leq R$$

for all $\epsilon \in \mathcal{E}$. From the inequalities (45), (46), (49), we deduce that $||\mathcal{A}_\epsilon(U(\tau,z))||_{(\epsilon,\sigma,\delta,\Omega)} \leq R$, for all $\epsilon \in \mathcal{E}$.

We prove now the estimates (43).

Let $R>0$ and let $U_1,U_2 \in B(0,R)$. From Propositions 2,4 we get that there exists a constant $d_4>0$ (independent of ϵ) such that

$$(51)\quad ||a_{s_1,s_2,k_0,k_1}(\tau,z,\epsilon)\tau^{s_1}\epsilon^{-s_2}\partial_\tau^{-k_0}\partial_z^{k_1-S}(U_1(\tau,z)-U_2(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)} \leq d_4|\epsilon|^{s_1+k_0-s_2}\delta^{S-k_1}||U_1(\tau,z)-U_2(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)}$$

for all $(k_0,k_1)\in\mathcal{A}_1$, all $(s_1,s_2)\in I_{(k_0,k_1)}$. As in the part a), we can write from the binomial formula

$$(52)\quad (\partial_z^{-S}U_1(\tau,z)+w(\tau,z,\epsilon))^{*l_1}-(\partial_z^{-S}U_2(\tau,z)+w(\tau,z,\epsilon))^{*l_1} = (\partial_z^{-S}U_1(\tau,z))^{*l_1}-(\partial_z^{-S}U_2(\tau,z))^{*l_1} + \sum_{l_1^1+l_1^2=l_1,l_1^1\geq 1,l_1^2\geq 1}\frac{l_1!}{l_1^1!l_1^2!}((\partial_z^{-S}U_1(\tau,z))^{*l_1^1}-(\partial_z^{-S}U_2(\tau,z))^{*l_1^1})*(w(\tau,z,\epsilon))^{*l_1^2}$$

for all $l_1 \geq 2$. On the other hand, we have that

$$(53)\quad (\partial_z^{-S}U_1(\tau,z))^{*2}-(\partial_z^{-S}U_2(\tau,z))^{*2} = (\partial_z^{-S}U_1(\tau,z)-\partial_z^{-S}U_2(\tau,z))*(\partial_z^{-S}U_1(\tau,z)+\partial_z^{-S}U_2(\tau,z))$$

and, for all $l\geq 3$, we can write

$$(54)\quad (\partial_z^{-S}U_1(\tau,z))^{*l}-(\partial_z^{-S}U_2(\tau,z))^{*l} = \big(\partial_z^{-S}U_1(\tau,z)-\partial_z^{-S}U_2(\tau,z)\big) * \Big((\partial_z^{-S}U_1(\tau,z))^{*l-1}+(\partial_z^{-S}U_2(\tau,z))^{*l-1} + \sum_{k=1}^{l-2}(\partial_z^{-S}U_2(\tau,z))^{*k}*(\partial_z^{-S}U_1(\tau,z))^{*l-k-1}\Big).$$

Using (53) and (54), from Propositions 2,5, we get a constant $d_5>0$ (independent of ϵ) such that

$$(55)\quad ||(\partial_z^{-S}U_1(\tau,z))^{*l}-(\partial_z^{-S}U_2(\tau,z))^{*l}||_{(\epsilon,\sigma,\delta,\Omega)} \leq (d_5|\epsilon|^{l-1}\delta^{Sl}R^{l-1})||U_1(\tau,z)-U_2(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)}$$

for all $1\leq l\leq \max\{l_1\in\mathbb{N}/(l_0,l_1)\in\mathcal{A}_2\}$. From (52), (55), we get a constant $d_6>0$ (independent of ϵ) such that

$$(56)\quad ||(\partial_z^{-S}U_1(\tau,z)+w(\tau,z,\epsilon))^{*l_1}-(\partial_z^{-S}U_2(\tau,z)+w(\tau,z,\epsilon))^{*l_1}||_{(\epsilon,\sigma,\delta,\Omega)}$$
$$\leq d_6|\epsilon|^{l_1-1}(\delta^{Sl_1}R^{l_1-1}+\sum_{l_1^1+l_1^2=l_1,l_1^1\geq 1,l_1^2\geq 1}\frac{l_1!}{l_1^1!l_1^2!}\delta^{Sl_1^1}R^{l_1^1-1}I^{l_1^2})||U_1(\tau,z)-U_2(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)}$$
$$= d_6|\epsilon|^{l_1-1}R^{-1}((\delta^S R+I)^{l_1}-I^{l_1})||U_1(\tau,z)-U_2(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)}$$

for all $2 \leq l_1 \leq \max\{l \in \mathbb{N}/(l_0,l) \in \mathcal{A}_2\}$. From Proposition 4 and Corollary 1 we get a constant $d_7 > 0$ (independent of ϵ) such that

(57)
$$||\alpha_{m_1,l_0,l_1}(\tau,z,\epsilon)\epsilon^{-m_1}\partial_\tau^{-l_0}((\partial_z^{-S}U_1(\tau,z)+w(\tau,z,\epsilon))^{*l_1} - (\partial_z^{-S}U_2(\tau,z)+w(\tau,z,\epsilon))^{*l_1})||_{(\epsilon,\sigma,\delta,\Omega)}$$
$$\leq d_7|\epsilon|^{l_0-m_1}||(\partial_z^{-S}U_1(\tau,z)+w(\tau,z,\epsilon))^{*l_1} - (\partial_z^{-S}U_2(\tau,z)+w(\tau,z,\epsilon))^{*l_1}||_{(\epsilon,\sigma,\delta,\Omega)}$$

for all $(l_0,l_1) \in \mathcal{A}_2$, all $m_1 \in J_{(l_0,l_1)}$. From (56), (57), we get that

(58)
$$||\alpha_{m_1,l_0,l_1}(\tau,z,\epsilon)\epsilon^{-m_1}\partial_\tau^{-l_0}((\partial_z^{-S}U_1(\tau,z)+w(\tau,z,\epsilon))^{*l_1} - (\partial_z^{-S}U_2(\tau,z)+w(\tau,z,\epsilon))^{*l_1})||_{(\epsilon,\sigma,\delta,\Omega)}$$
$$\leq d_6d_7|\epsilon|^{l_0+l_1-m_1-1}R^{-1}((\delta^S R+I)^{l_1} - I^{l_1})||U_1(\tau,z)-U_2(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)}$$

for all $(l_0,l_1) \in \mathcal{A}_2$, all $m_1 \in J_{(l_0,l_1)}$. Now, we choose $\delta, R, I > 0$ such that

(59)
$$\sum_{(k_0,k_1)\in\mathcal{A}_1}\sum_{(s_1,s_2)\in I_{(k_0,k_1)}} d_4|\epsilon|^{s_1+k_0-s_2}\delta^{S-k_1} + \sum_{(l_0,l_1)\in\mathcal{A}_2}\sum_{m_1\in J_{(l_0,l_1)}} d_6d_7|\epsilon|^{l_0+l_1-m_1-1}R^{-1}((\delta^S R+I)^{l_1} - I^{l_1}) \leq 1/2$$

for all $\epsilon \in \mathcal{E}$. From the inequalities (51), (58), we deduce that

$$||A_\epsilon(U_1(\tau,z)) - A_\epsilon(U_2(\tau,z))||_{(\epsilon,\sigma,\delta,\Omega)} \leq \frac{1}{2}||U_1(\tau,z)-U_2(\tau,z)||_{(\epsilon,\sigma,\delta,\Omega)},$$

for all $\epsilon \in \mathcal{E}$. Finally, we choose $\delta, R, I > 0$ is such a way that the conditions (50) and (59) hold simultaneously. This yields Lemma 4. □

Now, let the assumptions **A1)**, **A2)** hold. We choose the constants I, R, δ as in the lemma 4. Assume that

$$\sum_{j=0}^{S-1-h} ||V_{j+h}(\tau,\epsilon)||_{0,\epsilon,\tilde{\sigma},\Omega}\frac{\delta^j}{j!} \leq I,$$

for all $0 \leq h \leq S-1$, for all $\epsilon \in \mathcal{E}$. From Lemma 4 and the classical shrinking map theorem on complete metric spaces, we deduce that the map A_ϵ has a unique fixed point (called $U(\tau,z,\epsilon)$) in the closed ball $B(0,R) \subset G(\epsilon,\sigma,\delta,\Omega)$, for all ϵ in $\mathcal{E}$, which means that $A_\epsilon(U(\tau,z,\epsilon)) = U(\tau,z,\epsilon)$ with $||U||_{(\epsilon,\sigma,\delta,\Omega)} \leq R$. Finally, we get that the function

$$V(\tau,z,\epsilon) = \partial_z^{-S}U(\tau,z,\epsilon) + w(\tau,z,\epsilon)$$

satisfies the Cauchy problem (40), (41), for all $\tau \in S_d \cup D(0,r)$, all $z \in D(0,\delta)$, all $\epsilon \in \mathcal{E}$. Moreover, from Proposition 2, we deduce that

$$||V(\tau,z,\epsilon)||_{(\epsilon,\sigma,\delta,\Omega)} \leq \delta^S R + I,$$

for all $\epsilon \in \mathcal{E}$. □

3 Analytic solutions in a complex parameter of a singular Cauchy problem

3.1 Laplace transform and asymptotic expansions

We recall the definition of Borel summability of formal series with coefficients in a Banach space, see [2].

Definition 2 *A formal series*

$$\hat{X}(t) = \sum_{j=0}^{\infty} \frac{a_j}{j!} t^j \in \mathbb{E}[[t]]$$

with coefficients in a Banach space $(\mathbb{E}, ||.||_{\mathbb{E}})$ *is said to be* $1-$*summable with respect to* t *in the direction* $d \in [0, 2\pi)$ *if*

i) *there exists* $\rho \in \mathbb{R}_+$ *such that the following formal series, called formal Borel transform of* $\hat{X}$ *of order 1*

$$\mathcal{B}(\hat{X})(\tau) = \sum_{j=0}^{\infty} \frac{a_j \tau^j}{(j!)^2} \in \mathbb{E}[[\tau]],$$

is absolutely convergent for $|\tau| < \rho$,

ii) *there exists* $\delta > 0$ *such that the series* $\mathcal{B}(\hat{X})(\tau)$ *can be analytically continued with respect to* τ *in a sector* $S_{d,\delta} = \{\tau \in \mathbb{C}^* : |d - \arg(\tau)| < \delta\}$. *Moreover, there exist* $C > 0$, *and* $K > 0$ *such that*

$$||\mathcal{B}(\hat{X})(\tau)||_{\mathbb{E}} \leq C e^{K|\tau|}$$

for all $\tau \in S_{d,\delta}$.

If this is so, the vector valued Laplace transform of order 1 of $\mathcal{B}(\hat{X})(\tau)$ in the direction d is defined by

$$\mathcal{L}^d(\mathcal{B}(\hat{X}))(t) = t^{-1} \int_{L_\gamma} \mathcal{B}(\hat{X})(\tau) e^{-(\tau/t)} d\tau,$$

along a half-line $L_\gamma = \mathbb{R}_+ e^{i\gamma} \subset S_{d,\delta} \cup \{0\}$, where γ depends on t and is chosen in such a way that $\cos(\gamma - \arg(t)) \geq \delta_1 > 0$, for some fixed δ_1, for all t in a sector

$$S_{d,\theta,R} = \{t \in \mathbb{C}^* : |t| < R \ \ , \ \ |d - \arg(t)| < \theta/2\},$$

where $\pi < \theta < \pi + 2\delta$ and $0 < R < \delta_1/K$. The function $\mathcal{L}^d(\mathcal{B}(\hat{X}))(t)$ is called the $1-$sum of the formal series $\hat{X}(t)$ in the direction d. The function $\mathcal{L}^d(W)(t)$ is a holomorphic and a bounded function on the sector $S_{d,\theta,R}$. Moreover, the function $\mathcal{L}^d(W)(t)$ has the formal series $\hat{X}(t)$ as Gevrey asymptotic expansion of order 1 with respect to t on $S_{d,\theta,R}$. This means that for all $\theta_1 < \theta$, there exist $C, M > 0$ such that

$$||\mathcal{L}^d(W)(t) - \sum_{p=0}^{n-1} \frac{a_p}{p!} t^p||_{\mathbb{E}} \leq C M^n n! |t|^n$$

for all $n \geq 1$, all $t \in S_{d,\theta_1,R}$.

In the next proposition, we give some well known identities for the Borel transform that will be useful in the sequel.

Proposition 6 *Let $\hat{X}(t) = \sum_{n\geq 0} a_n t^n/n!$ and $\hat{G}(t) = \sum_{n\geq 0} b_n t^n/n!$ be formal series in $\mathbb{E}[[t]]$. We have the following equalities as formal series in $\mathbb{E}[[\tau]]$:*

$$(\tau\partial_\tau^2 + \partial_\tau)(\mathcal{B}(\hat{X})(\tau)) = \mathcal{B}(\partial_t\hat{X}(t))(\tau), \partial_\tau^{-1}(\mathcal{B}(\hat{X}))(\tau) = \mathcal{B}(t\hat{X}(t))(\tau),$$
$$\tau\mathcal{B}(\hat{X})(\tau) = \mathcal{B}((t^2\partial_t + t)\hat{X}(t))(\tau), \int_0^\tau (\mathcal{B}\hat{X})(\tau - s)(\mathcal{B}\hat{G})(s)ds = \mathcal{B}(t\hat{X}(t)\hat{G}(t))(\tau).$$

Proof By a direct computation, we have the following expansions from which the proposition 6 follows.

$$\partial_t\hat{X}(t) = \sum_{n\geq 0} a_{n+1}\frac{t^n}{n!}, (\tau\partial_\tau^2 + \partial_\tau)(\mathcal{B}(\hat{X})(\tau)) = \sum_{n\geq 0} a_{n+1}\frac{\tau^n}{(n!)^2}, t\hat{X}(t) = \sum_{n\geq 1} na_{n-1}\frac{t^n}{n!},$$
$$\partial_\tau^{-1}(\mathcal{B}(\hat{X}))(\tau) = \sum_{n\geq 1} na_{n-1}\frac{\tau^n}{(n!)^2}, (t^2\partial_t + t)\hat{X}(t) = \sum_{n\geq 1} n^2 a_{n-1}\frac{t^n}{n!}, \tau\mathcal{B}(\hat{X})(\tau) = \sum_{n\geq 1} n^2 a_{n-1}\frac{\tau^n}{(n!)^2},$$
$$t\hat{X}(t)\hat{G}(t) = \sum_{n\geq 1}(\sum_{l+m=n-1}\frac{n!}{l!m!}a_l b_m)\frac{t^n}{n!}, \int_0^\tau (\mathcal{B}\hat{X})(\tau - s)(\mathcal{B}\hat{G})(s)ds = \sum_{n\geq 1}(\sum_{l+m=n-1}\frac{n!}{l!m!}a_l b_m)\frac{\tau^n}{(n!)^2}$$

□

3.2 Analytic solutions of a singular Cauchy problem with irregular singularities

Let $S \geq 1$ be an integer. Let $\mathcal{S}$ be a finite subset of $\mathbb{N}^3$, $\mathcal{N}$ be a finite subset of $\mathbb{N}^2$ and let $b_{s,k_0,k_1}(z,\epsilon)$, $c_{l_0,l_1}(z,\epsilon)$ be holomorphic bounded functions on a polydisc $D(0,\rho) \times D(0,\epsilon_0)$, for some $\rho, \epsilon_0 > 0$, for all $(s,k_0,k_1) \in \mathcal{S}$, all $(l_0,l_1) \in \mathcal{N}$.

Definition 3 *Let $V(\tau,\epsilon)$ be a holomorphic function on some punctured polydisc*

$$\Omega_{\tau_0,\epsilon_0} = D(0,\tau_0) \times (D(0,\epsilon_0) \setminus \{0\})$$

where $0 < \tau_0 < 1$ and $\epsilon_0 > 0$. We make the assumption that the function $\tau \mapsto V(\tau,\epsilon)$ belongs to $E_{0,\epsilon,\tilde{\sigma},\Omega_{\tau_0,\epsilon_0}}$, for some $\tilde{\sigma} > 0$, all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$. Let $d \in [0,2\pi)$ with $d \neq \pi$. Let U_d be a closed sector centered at 0, with bisecting direction d, with infinite radius and with small opening such that $-1 \notin U_d$. Let $\mathcal{E}$ be an open sector centered at 0 such that $\mathcal{E} \subset D(0,\epsilon_0)$. We denote by

$$\Omega(d,\mathcal{E}) = (U_d \cup D(0,\tau_0)) \times \mathcal{E}.$$

We assume that the function $(\tau,\epsilon) \mapsto V(\tau,\epsilon)$ can be extended to an analytic function $(\tau,\epsilon) \mapsto V_{U_d,\mathcal{E}}(\tau,\epsilon)$ on $(U_d \cup D(0,\tau_0)) \times \mathcal{E}$ and that the function $\tau \mapsto V_{U_d,\mathcal{E}}(\tau,\epsilon)$ belongs to $E_{0,\epsilon,\tilde{\sigma},\Omega(d,\mathcal{E})}$, for all $\epsilon \in \mathcal{E}$. We say that the set $\{V, V_{U_d,\mathcal{E}}, \tilde{\sigma}\}$ is admissible.

We consider the following singular Cauchy problem

(60)
$$t^2\partial_t\partial_z^S Y_{U_d,\mathcal{E}}(t,z,\epsilon) + (t+1)\partial_z^S Y_{U_d,\mathcal{E}}(t,z,\epsilon) = \sum_{(s,k_0,k_1)\in\mathcal{S}} b_{s,k_0,k_1}(z,\epsilon)\epsilon^{k_0-s}t^s(\partial_t^{k_0}\partial_z^{k_1}Y_{U_d,\mathcal{E}})(t,z,\epsilon)$$
$$+ \sum_{(l_0,l_1)\in\mathcal{N}} c_{l_0,l_1}(z,\epsilon)\epsilon^{-(l_0+l_1-1)}t^{l_0+l_1-1}Y_{U_d,\mathcal{E}}^{l_1}(t,z,\epsilon)$$

for given initial conditions

$$\partial_z^j Y_{U_d,\mathcal{E}}(t,0,\epsilon) = Y_{U_d,\mathcal{E},j}(t,\epsilon) \quad , \quad 0 \leq j \leq S-1. \tag{61}$$

The initial conditions $Y_{U_d,\mathcal{E},j}(t,\epsilon)$, $0 \leq j \leq S-1$ are defined as follows: for all $0 \leq j \leq S-1$, let $\{V_j, V_{U_d,\mathcal{E},j}, \tilde{\sigma}\}$ be an admissible set. Let

$$V_j(\tau,\epsilon) = \sum_{m \geq 0} \frac{\chi_{m,j}(\epsilon)}{(m!)^2} \tau^m$$

be its Taylor expansion with respect to τ on $D(0,\tau_0)$, for all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$. We consider the formal series

$$\hat{Y}_j(t,\epsilon) = \sum_{m \geq 0} \frac{\chi_{m,j}(\epsilon)}{m!} t^m$$

for all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$. We define $Y_{U_d,\mathcal{E},j}(t,\epsilon)$ as the $1-$sum (in the sense of Definition 2) of $\hat{Y}_j(t,\epsilon)$ in direction d. From the fact that the function $\tau \mapsto V_{U_d,\mathcal{E},j}(\tau,\epsilon)$ belongs to $E_{0,\epsilon,\tilde{\sigma},\Omega(d,\mathcal{E})}$, we get that $t \mapsto Y_{U_d,\mathcal{E},j}(t,\epsilon)$ defines a holomorphic function for all $t \in U_{d,\theta,h|\epsilon|}$, all $\epsilon \in \mathcal{E}$, where

$$U_{d,\theta,h|\epsilon|} = \{t \in \mathbb{C} : |t| < h|\epsilon| \quad , \quad |d - \arg(t)| < \theta/2\},$$

for some $\theta > \pi$ and some constant $h > 0$ (independent of ϵ), for all $0 \leq j \leq S-1$.

We have the following result.

Theorem 2 *Let the initial data (61) be constructed as above. We make the following assumption.*

B) *For all $(s,k_0,k_1) \in \mathcal{S}$, we have that*

$$S \geq b(s-k_0+2) + k_1 \quad , \quad s \geq 2k_0$$

and for all $(l_0,l_1) \in \mathcal{N}$, we have that $l_1 \geq 2$.

Then, there exist two constants $I, \delta > 0$ (independent of ϵ) such that if we assume that

$$\sum_{j=0}^{S-1-m} ||V_{j+m}(\tau,\epsilon)||_{0,\epsilon,\tilde{\sigma},\Omega_{\tau_0,\epsilon_0}} \frac{\delta^j}{j!} \leq I \tag{62}$$

for all $0 \leq m \leq S-1$, for all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$, and

$$\sum_{j=0}^{S-1-m} ||V_{U_d,\mathcal{E},j+m}(\tau,\epsilon)||_{0,\epsilon,\tilde{\sigma},\Omega(d,\mathcal{E})} \frac{\delta^j}{j!} \leq I, \tag{63}$$

for all $0 \leq m \leq S-1$, for all $\epsilon \in \mathcal{E}$, the problem (60), (61) has a solution $(t,z) \mapsto Y_{U_d,\mathcal{E}}(t,z,\epsilon)$ which is holomorphic and bounded on the set $U_{d,\theta,h'|\epsilon|} \times D(0,\delta/2)$, for some $h' > 0$ (independent of ϵ), for all $\epsilon \in \mathcal{E}$. The function $Y_{U_d,\mathcal{E}}(t,z,\epsilon)$ can be written as the Laplace transform of order 1 in the direction d of a function $\tau \mapsto V_{U_d,\mathcal{E}}(\tau,z,\epsilon)$ which is holomorphic on the domain $(U_d \cup D(0,\tau_0)) \times D(0,\delta/2) \times \mathcal{E}$ and satisfies the estimates : there exists a constant $C_{\Omega(d,\mathcal{E})}$ (independent of ϵ) such that

$$|V_{U_d,\mathcal{E}}(\tau,z,\epsilon)| \leq C_{\Omega(d,\mathcal{E})}(1 + \frac{|\tau|^2}{|\epsilon|^2})^{-1} \exp(\frac{\sigma\zeta(b)}{|\epsilon|}|\tau|) \tag{64}$$

for all $(\tau, z, \epsilon) \in (U_d \cup D(0, \tau_0)) \times D(0, \delta/2) \times \mathcal{E}$*. Moreover, the function* $V_{U_d,\mathcal{E}}(\tau, z, \epsilon)$ *is the analytic continuation of a function* $V(\tau, z, \epsilon)$ *which is holomorphic on a punctured polydisc* $D(0, \tau_0) \times D(0, \delta/2) \times (D(0, \epsilon_0) \setminus \{0\})$ *and verifies the following estimates : there exists a constant* $C_{\Omega_{\tau_0,\epsilon_0}} > 0$ *(independent of* ϵ*) such that*

$$|V(\tau, z, \epsilon)| \leq C_{\Omega_{\tau_0,\epsilon_0}} (1 + \frac{|\tau|^2}{|\epsilon|^2})^{-1} \exp(\frac{\sigma\zeta(b)}{|\epsilon|}|\tau|) \tag{65}$$

for all $\tau \in D(0, \tau_0)$*, all* $z \in D(0, \delta/2)$*, all* $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$*.*

Proof We consider a formal series

$$\hat{Y}(t, z, \epsilon) = \sum_{m \geq 0} Y_m(z, \epsilon) \frac{t^m}{m!} \in \mathcal{O}(D(0, \rho))[[t]]$$

solution of the problem (60), with initial conditions

$$\partial_z^j \hat{Y}(t, 0, \epsilon) = \hat{Y}_j(t, \epsilon) \quad , \quad 0 \leq j \leq S - 1,$$

for all $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$. Let $V(\tau, z, \epsilon)$ be the Borel transform of $\hat{Y}$ with respect to τ,

$$V(\tau, z, \epsilon) = \sum_{m \geq 0} Y_m(z, \epsilon) \frac{\tau^m}{(m!)^2}.$$

In the first step of the proof, we show that $V(\tau, z, \epsilon)$ defines a holomorphic function on a punctured polydisc $D(0, \tau_0) \times D(0, \delta/2) \times (D(0, \epsilon_0) \setminus \{0\})$, which satisfies the estimates (65).

From the identities of Proposition 6, we get that $V(\tau, z, \epsilon)$ satisfies the following singular Cauchy problem

$$(\tau + 1)\partial_z^S V(\tau, z, \epsilon) = \sum_{(s,k_0,k_1) \in \mathcal{S}} b_{s,k_0,k_1}(z, \epsilon) \epsilon^{k_0 - s} \partial_\tau^{-s} (\tau \partial_\tau^2 + \partial_\tau)^{k_0} \partial_z^{k_1} V(\tau, z, \epsilon) + \sum_{(l_0,l_1) \in \mathcal{N}} c_{l_0,l_1}(z, \epsilon) \epsilon^{-(l_0 + l_1 - 1)} \partial_\tau^{-l_0} V^{*l_1}(\tau, z, \epsilon) \tag{66}$$

with initial conditions

$$(\partial_z^j V)(\tau, 0, \epsilon) = V_j(\tau, \epsilon) \quad , \quad 0 \leq j \leq S - 1. \tag{67}$$

for all $\tau \in D(0, \tau_0)$, all $z \in D(0, \rho)$, all $\epsilon \in D(0, \epsilon_0) \setminus \{0\}$. In the next step, we rewrite the equation (66), using the following two lemma.

Lemma 5 *For all* $k_0 \geq 1$*, there exist constants* $a_{k,k_0} \in \mathbb{N}$*,* $k_0 \leq k \leq 2k_0$*, such that*

$$(\tau \partial_\tau^2 + \partial_\tau)^{k_0} u(\tau) = \sum_{k=k_0}^{2k_0} a_{k,k_0} \tau^{k - k_0} \partial_\tau^k u(\tau) \tag{68}$$

for all holomorphic functions $u : \Omega \to \mathbb{C}$ *on an open set* $\Omega \subset \mathbb{C}$*.*

Proof Let the polynomials $p_{k,k_0}(\tau)$, for all $k_0 \geq 1$, all $0 \leq k \leq 2k_0$ defined by the following recursion : $p_{0,1}(\tau) \equiv 0$, $p_{1,1}(\tau) \equiv 1$, $p_{2,1}(\tau) = \tau$,

$$(69) \quad p_{0,k_0+1}(\tau) := (\tau\partial_\tau^2 + \partial_\tau)p_{0,k_0}(\tau) \quad , \quad p_{1,k_0+1}(\tau) := (\tau\partial_\tau^2 + \partial_\tau)p_{1,k_0}(\tau) + (2\tau\partial_\tau + 1)p_{0,k_0},$$
$$p_{k,k_0+1} := (\tau\partial_\tau^2 + \partial_\tau)p_{k,k_0}(\tau) + (2\tau\partial_\tau + 1)p_{k-1,k_0}(\tau) + \tau p_{k-2,k_0}(\tau),$$
$$p_{2k_0+1,k_0+1}(\tau) := (2\tau\partial_\tau + 1)p_{2k_0,k_0}(\tau) + \tau p_{2k_0-1,k_0}(\tau) \quad , \quad p_{2k_0+2,k_0+1}(\tau) := \tau p_{2k_0,k_0}(\tau)$$

for all $2 \leq k \leq 2k_0$. By a direct computation, we get that

$$(\tau\partial_\tau^2 + \partial_\tau)^{k_0} u(\tau) = \sum_{k=0}^{2k_0} p_{k,k_0}(\tau)\partial_\tau^k u(\tau),$$

for all holomorphic functions $u : \Omega \to \mathbb{C}$ on an open set $\Omega \subset \mathbb{C}$. By induction on k_0, we can show that the polynomials $p_{k,k_0}(\tau)$ have the following simple shape. We have that $p_{k,k_0}(\tau) \equiv 0$, for $0 \leq k \leq k_0 - 1$, and $p_{k,k_0}(\tau) = a_{k,k_0}\tau^{k-k_0}$, where a_{k,k_0}, $k_0 \leq k \leq 2k_0$, are integers defined by the recusion

$$(70) \quad a_{1,1} := 1, a_{2,1} := 1, a_{k_0+1,k_0+1} := a_{k_0+1,k_0} + a_{k_0,k_0}$$
$$a_{k,k_0+1} := a_{k,k_0}(k-k_0)^2 + a_{k-1,k_0}(1 + 2(k-1-k_0)) + a_{k-2,k_0}$$
$$a_{2k_0+1,k_0+1} := (2k_0+1)a_{2k_0,k_0} + a_{2k_0-1,k_0} \quad , \quad a_{2k_0+2,k_0+1} := a_{2k_0,k_0}$$

for all k such that $k_0 + 2 \leq k \leq 2k_0$, for any $k_0 \geq 1$. □

Lemma 6 *Let $a, b, c \geq 0$ be positive integers such that $a \geq b$ and $a \geq c$. We put $\delta = a + b - c$. Then, for all holomorphic functions $u : \Omega \to \mathbb{C}$, the function $\partial_\tau^{-a}(\tau^b\partial_\tau^c u(\tau))$ can be written in the form*

$$\partial_\tau^{-a}(\tau^b\partial_\tau^c u(\tau)) = \sum_{(b',c') \in \mathcal{O}_\delta} \alpha_{b',c'}\tau^{b'}\partial_\tau^{c'}u(\tau)$$

where $\mathcal{O}_\delta$ is a finite subset of $\mathbb{Z}^2$ such that for all $(b',c') \in \mathcal{O}_\delta$, $b' - c' = \delta$, $b' \geq 0$, $c' \leq 0$, and $\alpha_{b',c'} \in \mathbb{Z}$.

Proof For all integers $b \geq 0$ and $c \in \mathbb{Z}$, using b integrations by parts, we get $b+1$ integers $\alpha_b, \ldots, \alpha_0 \in \mathbb{Z}$ (depending on b) such that

$$(71) \qquad \int_0^\tau s^b\partial_s^c u(s)ds = \sum_{k=0}^{b-1} \alpha_{b-k}\tau^{b-k}\partial_\tau^{(c-(k+1))}u(\tau) + \alpha_0\int_0^\tau \partial_s^{c-b}u(s)ds$$

for all $\tau \in \Omega$. For all $a, b \geq 0$, $c \in \mathbb{Z}$ integers, we define

$$I_{a,b,c}(\tau) = \partial_\tau^{-a}(\tau^b\partial_\tau^c u(\tau))$$

for all $\tau \in \Omega$. Using the expression (71), we get that $I_{a,b,c}(\tau)$ satisfies the following recursion formula:

$$(72) \qquad I_{a,b,c}(\tau) = \sum_{k=0}^{b-1} \alpha_{b-k}I_{a-1,b-k,c-k-1}(\tau) + \alpha_0 I_{a,0,c-b}(\tau)$$

for all $a \geq 1$, $b \geq 0$, $c \in \mathbb{Z}$ integers and $\tau \in \Omega$.

First of all, we observe that if the condition $a+b-c=\delta$ is satisfied for the term $I_{a,b,c}$, then we have that $(a-1)+(b-k)-(c-k-1)=\delta$, for the terms $I_{a-1,b-k,c-k-1}(\tau)$ on the right handside of the equality (72), for all $0 \leq k \leq b-1$ and also for the term $I_{a,0,c-b}(\tau)$. Using the recursion (72), one can check that if $a \geq b$ and $a \geq c$, then $I_{a,b,c}(\tau)$ can be expressed as a linear combination with coefficients in $\mathbb{Z}$ of terms of the form $I_{0,b',c'}(\tau)$ where $b' \geq 0$ and $c' \leq 0$. Moreover, from the observation above, if one assumes that $a+b-c=\delta$, then we also have that $b'-c'=\delta$ for all the terms $I_{0,b',c'}(\tau)$ appearing in the linear combination. □

Using the lemma 5,6 and the assumption **B)** in Theorem 2, we can rewrite the Cauchy problem (66), (67) in the form

$$
(73)\quad \partial_z^S V(\tau,z,\epsilon) = \sum_{(s,k_0,k_1)\in\mathcal{S}} \frac{b_{s,k_0,k_1}(z,\epsilon)}{\tau+1}\epsilon^{k_0-s}\Big(\sum_{(r,p)\in\mathcal{O}_{s-k_0}} \alpha_{r,p}\tau^r \partial_\tau^{-p}\partial_z^{k_1}V(\tau,z,\epsilon)\Big) + \sum_{(l_0,l_1)\in\mathcal{N}} \frac{c_{l_0,l_1}(z,\epsilon)}{\tau+1}\epsilon^{-(l_0+l_1-1)}\partial_\tau^{-l_0}V^{*l_1}(\tau,z,\epsilon)
$$

where $\mathcal{O}_{s-k_0}$ is a finite subset of $\mathbb{N}^2$ such that for all $(r,p) \in \mathcal{O}_{s-k_0}$, we have $r+p=s-k_0$ and $\alpha_{r,p} \in \mathbb{Z}$, with the given initial conditions

$$
(74)\qquad (\partial_z^j V)(\tau,0,\epsilon) = V_j(\tau,\epsilon) \quad , \quad 0 \leq j \leq S-1.
$$

From the assumption **B)** in Theorem 2, we get that the assumptions **A1)**, **A2)** of Theorem 1 are fulfilled for the equation (73). From Theorem 1, we deduce that there exist two constants $I, \delta > 0$ such that if the inequality (62) holds, then the Cauchy problem (73), (74) has a solution $V(\tau,z,\epsilon) \in G(\epsilon,\sigma,\delta,\Omega_{\tau_0,\epsilon_0})$ for some $\sigma > \tilde{\sigma}$ with a constant $R > 0$ (independent of ϵ) such that $||V||_{(\epsilon,\sigma,\delta,\Omega_{\tau_0,\epsilon_0})} \leq \delta^S R + I$. From the proposition 1, we get the estimates (65).

In the second step of the proof, we show that the function $V(\tau,z,\epsilon)$ can be analytically continued to a function $V_{U_d,\mathcal{E}}(\tau,z,\epsilon)$ on $(U_d \cup D(0,\tau_0)) \times D(0,\delta/2) \times \mathcal{E}$ satisfying the estimates (64).

Indeed, by construction, the function $V(\tau,z,\epsilon)$ solves also the problem

$$
(75)\quad \partial_z^S V(\tau,z,\epsilon) = \sum_{(s,k_0,k_1)\in\mathcal{S}} \frac{b_{s,k_0,k_1}(z,\epsilon)}{\tau+1}\epsilon^{k_0-s}\Big(\sum_{(r,p)\in\mathcal{O}_{s-k_0}} \alpha_{r,p}\tau^r \partial_\tau^{-p}\partial_z^{k_1}V(\tau,z,\epsilon)\Big) + \sum_{(l_0,l_1)\in\mathcal{N}} \frac{c_{l_0,l_1}(z,\epsilon)}{\tau+1}\epsilon^{-(l_0+l_1-1)}\partial_\tau^{-l_0}V^{*l_1}(\tau,z,\epsilon)
$$

with the given initial conditions

$$
(76)\qquad (\partial_z^j V)(\tau,0,\epsilon) = V_{U_d,\mathcal{E},j}(\tau,\epsilon) \quad , \quad 0 \leq j \leq S-1.
$$

for all $\tau \in D(0,\tau_0)$, all $z \in D(0,\delta/2)$, all $\epsilon \in \mathcal{E}$. Due to the assumption **B)** in Theorem 2, the assumptions **A1)**, **A2)** of Theorem 1 are fulfilled for the equation (75). From Theorem 1, we get two constants $I, \delta > 0$ such that if the inequality (63) holds, then the Cauchy problem (75), (76) has a unique solution $V_{U_d,\mathcal{E}}(\tau,z,\epsilon) \in G(\epsilon,\sigma,\delta,\Omega(d,\mathcal{E}))$ for some $\sigma > \tilde{\sigma}$ with a constant $R > 0$ (independent of ϵ) such that $||V_{U_d,\mathcal{E}}||_{(\epsilon,\sigma,\delta,\Omega(d,\mathcal{E}))} \leq \delta^S R + I$. From the unicity, $V_{U_d,\mathcal{E}}$ coincides with V on the domain $D(0,\tau_0) \times D(0,\delta/2) \times \mathcal{E}$. Moreover, the estimates (64) follow from Proposition 1.

From the first and the second step of the proof, we deduce that the formal series $\hat{Y}(t,z,\epsilon)$ constructed before is 1-summable with respect to t in the direction d as series in the Banach space $\mathcal{O}(D(0,\delta/2))$, for all $\epsilon \in \mathcal{E}$. We denote by $Y_{U_d,\mathcal{E}}(t,z,\epsilon)$ it's 1-sum, which is a holomorphic function with respect to t on a domain $U_{d,\theta,h'|\epsilon|}$, due to Definition 2 and the estimates (64). Moreover, from the algebraic properties of the $\kappa-$summability procedure, see [2] section 6.3, we deduce that $Y_{U_d,\mathcal{E}}(t,z,\epsilon)$ is a solution of the Cauchy problem (60), (61). □

4 Formal series solutions and Gevrey asymptotic expansions in a complex parameter for a doubly singular Cauchy problem

4.1 Analytic solutions in a complex parameter for a singularly perturbed Cauchy problem

We recall the definition of a good covering.

Definition 4 *For all $0 \leq i \leq \nu - 1$, we consider open sectors $\mathcal{E}_i$ centered at 0, with radius ϵ_0 and opening $\pi + \delta_i$, with $\delta_i > 0$, such that $\mathcal{E}_i \cap \mathcal{E}_{i+1} \neq \emptyset$, for all $0 \leq i \leq \nu - 1$ (with the convention that $\mathcal{E}_\nu = \mathcal{E}_0$) and such that $\cup_{i=0}^{\nu-1} \mathcal{E}_i = \mathcal{U} \setminus \{0\}$, where $\mathcal{U}$ is some neighborhood of 0 in $\mathbb{C}$. Such a set of sectors $\{\mathcal{E}_i\}_{0 \leq i \leq \nu-1}$ is called a good covering in $\mathbb{C}^*$.*

Definition 5 *Let $\{\mathcal{E}_i\}_{0 \leq i \leq \nu-1}$ be a good covering in $\mathbb{C}^*$. Let $\mathcal{T}$ be an open sector centered at 0 with radius $r_{\mathcal{T}}$ and consider a family of open sectors*

$$U_{d_i,\theta,\epsilon_0 r_{\mathcal{T}}} := \{t \in \mathbb{C} : |t| < \epsilon_0 r_{\mathcal{T}} \quad , \quad |d_i - \arg(t)| < \theta/2\},$$

where $d_i \in [0, 2\pi)$, for $0 \leq i \leq \nu - 1$, where $\theta > \pi$, which satisfy the following properties:

1) For all $0 \leq i \leq \nu - 1$, $\arg(d_i) \neq \pi$.
2) For all $0 \leq i \leq \nu - 1$, for all $t \in \mathcal{T}$, all $\epsilon \in \mathcal{E}_i$, we have that $\epsilon t \in U_{d_i,\theta,\epsilon_0 r_{\mathcal{T}}}$.

We say that the family $\{\{U_{d_i,\theta,\epsilon_0 r_{\mathcal{T}}}\}_{0 \leq i \leq \nu-1}, \mathcal{T}\}$ is associated to the good covering $\{\mathcal{E}_i\}_{0 \leq i \leq \nu-1}$.

Let $S \geq 1$ be an integer. Let $\mathcal{S}$ be a finite subset of $\mathbb{N}^3$, $\mathcal{N}$ be a finite subset of $\mathbb{N}^2$ and let $b_{s,k_0,k_1}(z,\epsilon)$, $c_{l_0,l_1}(z,\epsilon)$ be holomorphic bounded functions on a polydisc $D(0,\rho) \times D(0,\epsilon_0)$, for some $\rho > 0$, for all $(s,k_0,k_1) \in \mathcal{S}$, all $(l_0,l_1) \in \mathcal{N}$. Let $\{\mathcal{E}_i\}_{0 \leq i \leq \nu-1}$ be a good covering in $\mathbb{C}^*$.

For all $0 \leq i \leq \nu - 1$, we consider the following singularly perturbed Cauchy problem

$$\epsilon t^2 \partial_t \partial_z^S X_i(t,z,\epsilon) + (\epsilon t + 1)\partial_z^S X_i(t,z,\epsilon) = \sum_{(s,k_0,k_1) \in \mathcal{S}} b_{s,k_0,k_1}(z,\epsilon) t^s (\partial_t^{k_0} \partial_z^{k_1} X_i)(t,z,\epsilon) + \sum_{(l_0,l_1) \in \mathcal{N}} c_{l_0,l_1}(z,\epsilon) t^{l_0+l_1-1} X_i^{l_1}(t,z,\epsilon) \tag{77}$$

for given initial conditions

$$\partial_z^j X_i(t,0,\epsilon) = \varphi_{i,j}(t,\epsilon) \quad , \quad 0 \leq j \leq S-1, \tag{78}$$

where the functions $\varphi_{i,j}(t,\epsilon)$ are constructed as follows. We consider a family of sectors $\{\{U_{d_i,\theta,\epsilon_0 r_{\mathcal{T}}}\}_{0 \leq i \leq \nu-1}, \mathcal{T}\}$ associated to the good covering $\{\mathcal{E}_i\}_{0 \leq i \leq \nu-1}$. For all $0 \leq i \leq \nu - 1$, let U_{d_i} be an open sector of infinite radius centered at 0, with bisecting direction d_i and with opening $n_i > \theta - \pi$. We choose θ and n_i in such a way that $-1 \notin U_{d_i}$, for all $0 \leq i \leq \nu - 1$. For all $0 \leq i \leq \nu - 1$, all $0 \leq j \leq S - 1$, we define

$$\varphi_{i,j}(t,\epsilon) = Y_{U_{d_i},\mathcal{E}_i,j}(\epsilon t,\epsilon)$$

where $Y_{U_{d_i},\mathcal{E}_i,j}(t,\epsilon)$ is constructed as the initial condition of the problem (60), (61) in section 3.2, with the help of admissible sets $\{V_j, V_{U_{d_i},\mathcal{E}_i,j}, \tilde{\sigma}\}$, for all $0 \leq i \leq \nu - 1$, all $0 \leq j \leq S - 1$. By construction, the function $\varphi_{i,j}(t,\epsilon)$ is a holomorphic and bounded function on $\mathcal{T} \times \mathcal{E}_i$, for all $0 \leq i \leq \nu - 1$, all $0 \leq j \leq S - 1$, for well chosen radius $r_{\mathcal{T}}$ and opening θ.

Proposition 7 *Let the initial data (78) constructed as above. We make the following assumption.*

B) *For all $(s, k_0, k_1) \in \mathcal{S}$, we have that*

$$S \geq b(s - k_0 + 2) + k_1 \quad , \quad s \geq 2k_0$$

and for all $(l_0, l_1) \in \mathcal{N}$, we have that $l_1 \geq 2$.

Then, there exist two constants $I, \delta > 0$ (independent of ϵ) such that if we assume that

$$\sum_{j=0}^{S-1-m} ||V_{j+m}(\tau,\epsilon)||_{0,\epsilon,\tilde{\sigma},\Omega_{\tau_0,\epsilon_0}} \frac{\delta^j}{j!} \leq I \tag{79}$$

for all $0 \leq m \leq S-1$, for all $\epsilon \in D(0,\epsilon_0) \setminus \{0\}$, and

$$\sum_{j=0}^{S-1-m} ||V_{U_{d_i},\mathcal{E}_i,j+m}(\tau,\epsilon)||_{0,\epsilon,\tilde{\sigma},\Omega(d_i,\mathcal{E}_i)} \frac{\delta^j}{j!} \leq I, \tag{80}$$

for all $0 \leq m \leq S-1$, for all $\epsilon \in \mathcal{E}_i$, for all $0 \leq i \leq \nu - 1$, the problem (77), (78) has a solution $X_i(t,z,\epsilon)$ which is holomorphic and bounded on a set $(\mathcal{T} \cap D(0,h')) \times D(0,\delta/2) \times \mathcal{E}_i$, for all $0 \leq i \leq \nu - 1$, for some $h' > 0$. Moreover, there exist constants $0 < h'' \leq h'$, $K_i, M_i > 0$ (independent of ϵ) such that

$$\sup_{t \in \mathcal{T} \cap D(0,h''), z \in D(0,\delta/2)} |X_{i+1}(t,z,\epsilon) - X_i(t,z,\epsilon)| \leq K_i e^{-\frac{M_i}{|\epsilon|}} \tag{81}$$

for all $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$, for all $0 \leq i \leq \nu - 1$ (where by convention $X_\nu = X_0$).

Proof For $0 \leq i \leq \nu - 1$, we consider the Cauchy problem (60), (61) for the initial conditions

$$(\partial_z^j Y_{U_{d_i},\mathcal{E}_i})(t,0,\epsilon) = Y_{U_{d_i},\mathcal{E}_i,j}(t,\epsilon),$$

for $0 \leq j \leq S - 1$. From our hypotheses, the assumptions of Theorem 2 are satisfied for this problem, which then has a solution $(t,z) \mapsto Y_{U_{d_i},\mathcal{E}_i}(t,z,\epsilon)$ which is holomorphic and bounded on the set $U_{d_i,\theta,h'|\epsilon|} \times D(0,\delta/2)$, for some $h' > 0$ (independent of ϵ), for all $\epsilon \in \mathcal{E}_i$. Now, we put

$$X_i(t,z,\epsilon) = Y_{U_{d_i},\mathcal{E}_i}(\epsilon t, z, \epsilon)$$

which defines a holomorphic and bounded function on $(\mathcal{T} \cap D(0,h')) \times D(0,\delta/2) \times \mathcal{E}_i$, for $0 \leq i \leq \nu - 1$. By construction, one can check that $X_i(t,z,\epsilon)$ satisfies the problem (77), (78) on $(\mathcal{T} \cap D(0,h')) \times D(0,\delta/2) \times \mathcal{E}_i$, for all $0 \leq i \leq \nu - 1$.

In the second part of the proof, we will show the estimates (81). Let i an integer such that $0 \leq i \leq \nu - 1$. From Theorem 2, we can write

$$X_i(t,z,\epsilon) = (\epsilon t)^{-1} \int_{L_{\gamma_i}} V_{U_{d_i},\mathcal{E}_i}(\tau,z,\epsilon) e^{-\frac{\tau}{\epsilon t}} d\tau,$$
$$X_{i+1}(t,z,\epsilon) = (\epsilon t)^{-1} \int_{L_{\gamma_{i+1}}} V_{U_{d_{i+1}},\mathcal{E}_{i+1}}(\tau,z,\epsilon) e^{-\frac{\tau}{\epsilon t}} d\tau$$

where $L_{\gamma_i} = \mathbb{R}_+ e^{\sqrt{-1}\gamma_i} \subset U_{d_i} \cup \{0\}$, $L_{\gamma_{i+1}} = \mathbb{R}_+ e^{\sqrt{-1}\gamma_{i+1}} \subset U_{d_{i+1}} \cup \{0\}$, and $V_{U_{d_i},\mathcal{E}_i}$ (resp. $V_{U_{d_{i+1}},\mathcal{E}_{i+1}}$) is a holomorphic function on $(U_{d_i} \cup D(0,\tau_0)) \times D(0,\delta/2) \times \mathcal{E}_i$ (resp. on $(U_{d_{i+1}} \cup D(0,\tau_0)) \times D(0,\delta/2) \times \mathcal{E}_{i+1}$) for which the estimates (64) hold and which is moreover an analytic continuation of a function $V(\tau,z,\epsilon)$ which satisfy the estimates (65).

From the fact that $\tau \mapsto V(\tau,z,\epsilon)$ is holomorphic on $D(0,\tau_0)$ for all $(z,\epsilon) \in D(0,\delta/2) \times (D(0,\epsilon_0) \setminus \{0\})$, the integral of $\tau \mapsto V(\tau,z,\epsilon)$ along the union of a segment starting from 0 to $(\tau_0/2)e^{\sqrt{-1}\gamma_{i+1}}$, an arc of circle with radius $\tau_0/2$ connecting $(\tau_0/2)e^{\sqrt{-1}\gamma_{i+1}}$ and $(\tau_0/2)e^{\sqrt{-1}\gamma_i}$ and a segment starting from $(\tau_0/2)e^{\sqrt{-1}\gamma_i}$ to 0, is equal to zero. So that, we can rewrite the difference $X_{i+1} - X_i$ as a sum of three integrals,

$$X_{i+1}(t,z,\epsilon) - X_i(t,z,\epsilon) = (\epsilon t)^{-1} \left(\int_{L_{\tau_0/2,\gamma_{i+1}}} V_{U_{d_{i+1}},\mathcal{E}_{i+1}}(\tau,z,\epsilon) e^{-\frac{\tau}{\epsilon t}} d\tau \right.$$
$$\left. - \int_{L_{\tau_0/2,\gamma_i}} V_{U_{d_i},\mathcal{E}_i}(\tau,z,\epsilon) e^{-\frac{\tau}{\epsilon t}} d\tau + \int_{C(\tau_0/2,\gamma_i,\gamma_{i+1})} V(\tau,z,\epsilon) e^{-\frac{\tau}{\epsilon t}} d\tau \right) \quad (82)$$

where $L_{\tau_0/2,\gamma_i} = [\tau_0/2, +\infty)e^{\sqrt{-1}\gamma_i}$, $L_{\tau_0/2,\gamma_{i+1}} = [\tau_0/2,+\infty)e^{\sqrt{-1}\gamma_{i+1}}$ and $C(\tau_0/2,\gamma_i,\gamma_{i+1})$ is an arc of circle with radius $\tau_0/2$ connecting $(\tau_0/2)e^{\sqrt{-1}\gamma_i}$ with $(\tau_0/2)e^{\sqrt{-1}\gamma_{i+1}}$ with a well chosen orientation.

We give estimates for $I_1 = |(\epsilon t)^{-1} \int_{L_{\tau_0/2,\gamma_{i+1}}} V_{U_{d_{i+1}},\mathcal{E}_{i+1}}(\tau,z,\epsilon) e^{-\frac{\tau}{\epsilon t}} d\tau|$. By construction, the direction γ_{i+1} (which depends on ϵt) is chosen in such a way that $\cos(\gamma_{i+1} - \arg(\epsilon t)) \geq \delta_1$, for all $\epsilon \in \mathcal{E}_{i+1} \cap \mathcal{E}_i$, all $t \in \mathcal{T} \cap D(0,h')$, for some fixed $\delta_1 > 0$. From the estimates (64), we get

$$I_1 \leq |\epsilon t|^{-1} \int_{\tau_0/2}^{+\infty} C_{\Omega(d_{i+1},\mathcal{E}_{i+1})} (1 + \frac{r^2}{|\epsilon|^2})^{-1} e^{\frac{\sigma\zeta(b)r}{|\epsilon|}} e^{-\frac{r}{|\epsilon||t|}\cos(\gamma_{i+1} - \arg(\epsilon t))} dr \quad (83)$$
$$\leq |\epsilon t|^{-1} \int_{\tau_0/2}^{+\infty} C_{\Omega(d_{i+1},\mathcal{E}_{i+1})} e^{(\sigma\zeta(b) - \frac{\delta_1}{|t|})\frac{r}{|\epsilon|}} dr$$
$$= \frac{C_{\Omega(d_{i+1},\mathcal{E}_{i+1})}}{\delta_1 - \sigma\zeta(b)|t|} e^{-((\frac{\delta_1}{|t|} - \sigma\zeta(b))\frac{\tau_0}{2})\frac{1}{|\epsilon|}} \leq \frac{C_{\Omega(d_{i+1},\mathcal{E}_{i+1})}}{\delta_2} e^{-\frac{\delta_2 \tau_0/2}{|\epsilon| h'}}$$

for all $t \in \mathcal{T} \cap D(0,h')$, with $|t| < (\delta_1 - \delta_2)/(\sigma\zeta(b))$, for some $0 < \delta_2 < \delta_1$, and for all $\epsilon \in \mathcal{E}_{i+1} \cap \mathcal{E}_i$.

We give estimates for $I_2 = |(\epsilon t)^{-1} \int_{L_{\tau_0/2,\gamma_i}} V_{U_{d_i},\mathcal{E}_i}(\tau,z,\epsilon) e^{-\frac{\tau}{\epsilon t}} d\tau|$. By construction, the direction γ_i (which depends on ϵt) is chosen in such a way that there exists a fixed $\delta_1 > 0$ with $\cos(\gamma_i - \arg(\epsilon t)) \geq \delta_1$, for all $\epsilon \in \mathcal{E}_{i+1} \cap \mathcal{E}_i$, all $t \in \mathcal{T} \cap D(0,h')$. From the estimates (64), we deduce as before that

$$I_2 \leq \frac{C_{\Omega(d_i,\mathcal{E}_i)}}{\delta_2} e^{-\frac{\delta_2 \tau_0/2}{|\epsilon| h'}} \quad (84)$$

for all $t \in \mathcal{T} \cap D(0,h')$, with $|t| < (\delta_1 - \delta_2)/(\sigma\zeta(b))$, for some $0 < \delta_2 < \delta_1$, and for all $\epsilon \in \mathcal{E}_{i+1} \cap \mathcal{E}_i$.

Finally, we get estimates for $I_3 = |\epsilon t|^{-1} |\int_{C(\tau_0/2,\gamma_i,\gamma_{i+1})} V(\tau,z,\epsilon) e^{-\frac{\tau}{\epsilon t}} d\tau|$. From the estimates (65), we have

$$I_3 \leq |\epsilon t|^{-1} |\int_{\gamma_i}^{\gamma_{i+1}} C_{\Omega_{\tau_0,\epsilon_0}} (1 + \frac{(\tau_0/2)^2}{|\epsilon|^2})^{-1} e^{\frac{\sigma\zeta(b)\tau_0}{2|\epsilon|}} e^{-\frac{\tau_0}{2|\epsilon||t|}\cos(\theta - \arg(\epsilon t))} \frac{\tau_0}{2} d\theta| \tag{85}$$

By construction, the arc of cicle $C(\tau_0/2,\gamma_i,\gamma_{i+1})$ is chosen in such a way that that $\cos(\theta - \arg(\epsilon t)) \geq \delta_1$, for all $\theta \in [\gamma_i,\gamma_{i+1}]$ (if $\gamma_i < \gamma_{i+1}$)), $\theta \in [\gamma_{i+1},\gamma_i]$ (if $\gamma_{i+1} < \gamma_i$)), for all $t \in \mathcal{T}$, all $\epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$. From (85), we deduce that

$$I_3 \leq |\gamma_{i+1} - \gamma_i| C_{\Omega_{\tau_0,\epsilon_0}} \frac{\tau_0}{2} \frac{1}{|\epsilon t|} e^{-((\frac{\delta_1}{|t|} - \sigma\zeta(b))\frac{\tau_0}{2})\frac{1}{|\epsilon|}} \leq |\gamma_{i+1} - \gamma_i| C_{\Omega_{\tau_0,\epsilon_0}} \frac{\tau_0}{2} \frac{1}{|\epsilon t|} e^{-\frac{\delta_2\tau_0/4}{|\epsilon t|}} e^{-\frac{\delta_2\tau_0/4}{|\epsilon|h'}} \tag{86}$$

for all $t \in \mathcal{T} \cap D(0,h')$, with $|t| < (\delta_1 - \delta_2)/(\sigma\zeta(b))$, for some $0 < \delta_2 < \delta_1$, and for all $\epsilon \in \mathcal{E}_{i+1} \cap \mathcal{E}_i$. Using the inequality (86) and the estimates (15), we deduce that

$$I_3 \leq |\gamma_{i+1} - \gamma_i| C_{\Omega_{\tau_0,\epsilon_0}} \frac{2e^{-1}}{\delta_2} e^{-\frac{\delta_2\tau_0/4}{|\epsilon|h'}} \tag{87}$$

for all $t \in \mathcal{T} \cap D(0,h')$, with $|t| < (\delta_1 - \delta_2)/(\sigma\zeta(b))$ and for all $\epsilon \in \mathcal{E}_{i+1} \cap \mathcal{E}_i$.

Finally, collecting the inequalities (83), (84), (87), we deduce from (82), that

$$|X_{i+1}(t,z,\epsilon) - X_i(t,z,\epsilon)| \leq \frac{C_{\Omega(d_{i+1},\mathcal{E}_i)} + C_{\Omega(d_i,\mathcal{E}_i)}}{\delta_2} e^{-\frac{\delta_2\tau_0/2}{|\epsilon|h'}} + |\gamma_{i+1} - \gamma_i| C_{\Omega_{\tau_0,\epsilon_0}} \frac{2e^{-1}}{\delta_2} e^{-\frac{\delta_2\tau_0/4}{|\epsilon|h'}}$$

for all $t \in \mathcal{T} \cap D(0,h')$, with $|t| < (\delta_1 - \delta_2)/(\sigma\zeta(b))$, for some $0 < \delta_2 < \delta_1$, for all $\epsilon \in \mathcal{E}_{i+1} \cap \mathcal{E}_i$, for all $0 \leq i \leq \nu - 1$. So that the estimates (81) hold. □

4.2 Existence of formal series solutions in the complex parameter for the singularly perturbed problem

We keep the same notations as in the previous subsection. In this subsection, we establish the existence of formal power series

$$\hat{X}(t,z,\epsilon) \in \mathcal{O}((\mathcal{T} \cap D(0,h'')) \times D(0,\delta/2))[[\epsilon]]$$

which are solutions of (77) and satisfy the property that the solutions $X_i(t,z,\epsilon)$ of the problem (77), (78) have $\hat{X}$ as Gevrey asymptotic expansion of order 1 on $\mathcal{E}_i$, for $0 \leq i \leq \nu - 1$.

The proof is based on a cohomological criterion for summability of formal series with coefficients in a Banach space, see [2], page 121, which is known as the Malgrange-Sibuya theorem in the literature.

Theorem (MS) Let $(\mathbb{E},||.||_{\mathbb{E}})$ be a Banach space over $\mathbb{C}$ and $\{\mathcal{E}_i\}_{0\leq i\leq\nu-1}$ be a good covering in $\mathbb{C}^*$. For all $0 \leq i \leq \nu - 1$, let G_i be a holomorphic function from $\mathcal{E}_i$ into the Banach space $(\mathbb{E},||.||_{\mathbb{E}})$ and let the cocycle $\Delta_i(\epsilon) = G_{i+1}(\epsilon) - G_i(\epsilon)$ be a holomorphic function from the sector $Z_i = \mathcal{E}_{i+1} \cap \mathcal{E}_i$ into $\mathbb{E}$ (with the convention that $\mathcal{E}_\nu = \mathcal{E}_0$ and $G_\nu = G_0$). We make the following assumptions.

1) The functions $G_i(\epsilon)$ are bounded as $\epsilon \in \mathcal{E}_i$ tends to the origin in $\mathbb{C}$, for all $0 \leq i \leq \nu - 1$.

2) The functions $\Delta_i(\epsilon)$ are exponentially flat on Z_i, for all $0 \leq i \leq \nu - 1$. This means that there exist constants $C_i, A_i > 0$ such that

$$||\Delta_i(\epsilon)||_{\mathbb{E}} \leq C_i e^{-A_i/|\epsilon|}$$

for all $\epsilon \in Z_i$, all $0 \leq i \leq \nu - 1$.

Then, for all $0 \leq i \leq \nu - 1$, the functions $G_i(\epsilon)$ are the $1-$sums on $\mathcal{E}_i$ of a $1-$summable formal series $\hat{G}(\epsilon)$ in ϵ with coefficients in the Banach space $\mathbb{E}$.

We are now ready to state the main result of this paper.

Theorem 3 *Let us assume that the hypotheses of Proposition 7 hold and that the functions $\varphi_{i,j}(t,\epsilon)$, $0 \leq i \leq \nu - 1$, $0 \leq j \leq S - 1$, constructed in (78) satisfy the estimates (79) and (80) for some constants $I, \delta > 0$. Then, there exists a formal series*

$$\hat{X}(t,z,\epsilon) = \sum_{k \geq 0} H_k(t,z) \frac{\epsilon^k}{k!} \in \mathbb{E}[[\epsilon]]$$

where the functions H_k belong to the Banach space $\mathbb{E} = \mathcal{O}((\mathcal{T} \cap D(0,h'')) \times D(0,\delta/2))$ of holomorphic and bounded functions on the set $(\mathcal{T} \cap D(0,h'')) \times D(0,\delta/2)$ equipped with the supremum norm, for some $h'' > 0$, which solves the doubly singular equation

$$(88) \quad \epsilon t^2 \partial_t \partial_z^S \hat{X}(t,z,\epsilon) + (\epsilon t + 1)\partial_z^S \hat{X}(t,z,\epsilon) = \sum_{(s,k_0,k_1) \in \mathcal{S}} b_{s,k_0,k_1}(z,\epsilon) t^s (\partial_t^{k_0} \partial_z^{k_1} \hat{X})(t,z,\epsilon) + \sum_{(l_0,l_1) \in \mathcal{N}} c_{l_0,l_1}(z,\epsilon) t^{l_0+l_1-1} \hat{X}^{l_1}(t,z,\epsilon)$$

and has the $\mathbb{E}-$valued functions $\epsilon \mapsto X_i(t,z,\epsilon)$ constructed in Proposition 7 as $1-$sums on $\mathcal{E}_i$, for all $0 \leq i \leq \nu - 1$.

Proof Let us consider the tuple of functions $(X_i(t,z,\epsilon))_{0 \leq i \leq \nu-1}$ constructed in the proposition 7. For all $0 \leq i \leq \nu - 1$, we define $G_i(\epsilon) := (t,z) \mapsto X_i(t,z,\epsilon)$, which is, by construction, a holomorphic and bounded function from $\mathcal{E}_i$ into the Banach space $\mathbb{E} = \mathcal{O}((\mathcal{T} \cap D(0,h'')) \times D(0,\delta/2))$, where $\mathcal{T}, h'', \delta$ are defined in Proposition 7. From the estimates (81), we get that the cocycle $\Delta_i(\epsilon) = G_{i+1}(\epsilon) - G_i(\epsilon)$ is exponentially flat on $Z_i = \mathcal{E}_{i+1} \cap \mathcal{E}_i$, for all $0 \leq i \leq \nu - 1$.

From the theorem (MS) stated above, there exists a formal series $\hat{G}(\epsilon) \in \mathbb{E}[[\epsilon]]$ such that the functions $G_i(\epsilon)$ are the $1-$sums on $\mathcal{E}_i$ of $\hat{G}(\epsilon)$ as $\mathbb{E}-$valued functions. We put

$$\hat{G}(\epsilon) =: \hat{X}(t,z,\epsilon) = \sum_{k \geq 0} H_k(t,z) \frac{\epsilon^k}{k!}.$$

It remains to show that the formal series $\hat{X}(t,z,\epsilon)$ satisfies the equation (88). Since the $G_i(\epsilon)$ are the $1-$sums of $\hat{G}(\epsilon)$ we have that

$$(89) \qquad \lim_{\epsilon \to 0, \epsilon \in \mathcal{E}_i} \sup_{(t,z) \in (\mathcal{T} \cap D(0,h'')) \times D(0,\delta/2)} |\partial_\epsilon^l X_i(t,z,\epsilon) - H_l(t,z)| = 0$$

for all $0 \leq i \leq \nu - 1$, all $l \geq 0$. Let an integer i such that $0 \leq i \leq \nu - 1$. By construction, the function $X_i(t,z,\epsilon)$ satisfies the equation (77). Taking the derivative of order l with respect to

ϵ of the left and the right hand side of the equation (77) and using the Leibniz rule, we deduce that $\partial_\epsilon^l X_i(t,z,\epsilon)$ satisfies the following equation

$$
\begin{aligned}
(90)\quad & \epsilon t^2 \partial_t \partial_z^S \partial_\epsilon^l X_i(t,z,\epsilon) + (\epsilon t + 1)\partial_z^S \partial_\epsilon^l X_i(t,z,\epsilon) + l t^2 \partial_t \partial_z^S \partial_\epsilon^{l-1} X_i(t,z,\epsilon) + l t \partial_z^S \partial_\epsilon^{l-1} X_i(t,z,\epsilon) \\
&= \sum_{(s,k_0,k_1)\in\mathcal{S}} \Big(\sum_{h_1+h_2=l} \frac{l!}{h_1! h_2!} \partial_\epsilon^{h_1} b_{s,k_0,k_1}(z,\epsilon) t^s \partial_t^{k_0} \partial_z^{k_1} \partial_\epsilon^{h_2} X_i(t,z,\epsilon)\Big) \\
&\quad + \sum_{(l_0,l_1)\in\mathcal{N}} \Big(\sum_{h_0+\ldots+h_{l_1}=l} \frac{l!}{h_0!\cdots h_{l_1}!} \partial_\epsilon^{h_0}(c_{l_0 l_1}(z,\epsilon)) t^{l_0+l_1-1} \Pi_{j=1}^{l_1} \partial_\epsilon^{h_j} X_i(t,z,\epsilon)\Big)
\end{aligned}
$$

for all $l \geq 1$, all $(t,z,\epsilon) \in (\mathcal{T} \cap D(0,h'')) \times D(0,\delta/2) \times \mathcal{E}_i$. Letting ϵ tends to zero in (90) and using (89) yields the recursion

$$
\begin{aligned}
(91)\quad & t^2 \partial_t \partial_z^S \Big(\frac{H_{l-1}(t,z)}{(l-1)!}\Big) + t\partial_z^S\Big(\frac{H_{l-1}(t,z)}{(l-1)!}\Big) + \partial_z^S\Big(\frac{H_l(t,z)}{l!}\Big) \\
&= \sum_{(s,k_0,k_1)\in\mathcal{S}} \Big(\sum_{h_1+h_2=l} \frac{(\partial_\epsilon^{h_1} b_{s,k_0,k_1})(z,0)}{h_1!} t^s \partial_t^{k_0} \partial_z^{k_1} \Big(\frac{H_{h_2}(t,z)}{h_2!}\Big)\Big) \\
&\quad + \sum_{(l_0,l_1)\in\mathcal{N}} \Big(\sum_{h_0+\ldots+h_{l_1}=l} \frac{(\partial_\epsilon^{h_0} c_{l_0 l_1})(z,0)}{h_0!} t^{l_0+l_1-1} \Pi_{j=1}^{l_1} \frac{H_{h_j}(t,z)}{h_j!}\Big)
\end{aligned}
$$

for all $l \geq 1$, all $(t,z) \in (\mathcal{T} \cap D(0,h'')) \times D(0,\delta/2)$. Since $b_{s,k_0,k_1}(z,\epsilon)$ and $c_{l_0,l_1}(z,\epsilon)$ are analytic with respect to ϵ at 0, we have that

$$
(92)\qquad b_{s,k_0,k_1}(z,\epsilon) = \sum_{h\geq 0} \frac{(\partial_\epsilon^h b_{s,k_0,k_1})(z,0)}{h!}\epsilon^h \quad , \quad c_{l_0,l_1}(z,\epsilon) = \sum_{h\geq 0} \frac{(\partial_\epsilon^h c_{l_0,l_1})(z,0)}{h!}\epsilon^h
$$

for all (z,ϵ) near the origin in $\mathbb{C}^2$. Finally, one can check from the recursion (91) and (92) that the formal series $\hat{X}(t,z,\epsilon) = \sum_{k\geq 0} H_k(t,z)\epsilon^k/k!$ solves the equation (88). □

References

[1] S. Alinhac, *Problèmes de Cauchy pour des opérateurs singuliers.* (French) Bull. Soc. Math. France 102 (1974), 289–315.

[2] W. Balser, *Formal power series and linear systems of meromorphic ordinary differential equations*, Springer-Verlag, New-York, 2000.

[3] W. Balser, J. Mozo-Fernandez, *Multisummability of formal solutions of singular perturbation problems.* J. Differential Equations 183 (2002), no. 2, 526–545.

[4] A. Bove, J. Lewis, C. Parenti, *Propagation of singularities for Fuchsian operators.* Lecture Notes in Mathematics, 984. Springer-Verlag, Berlin, 1983. ii+161 pp.

[5] B. Braaksma, *Multisummability of formal power series solutions of nonlinear meromorphic differential equations.* Ann. Inst. Fourier (Grenoble) 42 (1992), no. 3, 517–540.

[6] E. Benoît, A. El Hamidi, A. Fruchard, *On combined asymptotic expansions in singular perturbations.* Electron. J. Differential Equations 2002, No. 51, 27 pp.

[7] M. Canalis-Durand, J. Mozo-Fernandez, R. Schäfke, *Monomial summability and doubly singular differential equations.* J. Differential Equations 233 (2007), no. 2, 485–511.

[8] M. Canalis-Durand, J-P. Ramis, R. Schäfke, Y. Sibuya, *Gevrey solutions of singularly perturbed differential equations.* J. Reine Angew. Math. 518 (2000), 95–129.

[9] H. Chen, Z. Luo, H. Tahara, *Formal solutions of nonlinear first order totally characteristic type PDE with irregular singularity.* Ann. Inst. Fourier (Grenoble) 51 (2001), no. 6, 1599–1620.

[10] H. Chen, Z. Luo, C. Zhang, *On the summability of formal solutions for a class of nonlinear singular PDEs with irregular singularity.* Recent progress on some problems in several complex variables and partial differential equations, 53–64, Contemp. Math., 400, Amer. Math. Soc., Providence, RI, 2006.

[11] O. Costin, *Asymptotics and Borel summability.* Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 141. CRC Press, Boca Raton, FL, 2009. xiv+250 pp.

[12] O. Costin, S. Tanveer, *Existence and uniqueness for a class of nonlinear higher-order partial differential equations in the complex plane.* Comm. Pure Appl. Math. 53 (2000), no. 9, 1092–1117.

[13] R. Geel, E. de Jager, *Hyperbolic singular perturbations of nonlinear first order differential equations.* Differential equations and applications (Proc. Third Scheveningen Conf., Scheveningen, 1977), pp. 75–83, North-Holland Math. Stud., 31, North-Holland, Amsterdam-New York, 1978.

[14] R. Gérard, H. Tahara, *Singular nonlinear partial differential equations.* Aspects of Mathematics. Friedr. Vieweg and Sohn, Braunschweig, 1996. viii+269 pp.

[15] E. de Jager, F. Jiang, *The theory of singular perturbations.* North-Holland Series in Applied Mathematics and Mechanics, 42. North-Holland Publishing Co., Amsterdam, 1996. xii+340 pp.

[16] J. Écalle, *Les fonctions résurgentes.* Publications Mathématiques d'Orsay, (1981).

[17] H. Fattorini, *The hyperbolic singular perturbation problem: an operator theoretic approach.* J. Differential Equations 70 (1987), no. 1, 1–41.

[18] K. Igari, *Removable singularities of holomorphic solutions of linear partial differential equations.* J. Math. Soc. Japan 56 (2004), no. 1, 87–113.

[19] M. K. Kadalbajoo, K. C. Patidar, *Singularly perturbed problems in partial differential equations: a survey.* Appl. Math. Comput. 134 (2003), no. 2-3, 371–429.

[20] B. Malgrange, *Sommation des séries divergentes.* Exposition. Math. **13** (1995), no. 2-3, 163–222.

[21] S. Malek, *On singularly perturbed partial integro-differential equations with irregular singularity.* J. Dyn. Control Syst. 13 (2007), no. 3, 419–449.

[22] S. Malek, C. Stenger, *Complex singularity analysis of holomorphic solutions of linear partial differential equations*, to appear in Adv. Dyn. Syst. Appl. (2011).

[23] T. Mandai, *The method of Frobenius to Fuchsian partial differential equations.* J. Math. Soc. Japan 52 (2000), no. 3, 645–672.

[24] T. Mandai, *Existence and nonexistence of null-solutions for some non-Fuchsian partial differential operators with T-dependent coefficients.* Nagoya Math. J. 122 (1991), 115–137.

[25] J. Melenk, C. Schwab, *Analytic regularity for a singularly perturbed problem.* SIAM J. Math. Anal. 30 (1999), no. 2, 379–400.

[26] S. Ouchi, *Multisummability of formal solutions of some linear partial differential equations.* J. Differential Equations 185 (2002), no. 2, 513–549.

[27] S. Ouchi, *Multisummability of formal power series solutions of nonlinear partial differential equations in complex domains.* Asymptot. Anal. 47 (2006), no. 3-4, 187–225.

[28] P. Pongérard, C. Wagschal, *Opérateurs de Fuchs non linéaires.* (French) [Nonlinear Fuchs operators] Ann. Fac. Sci. Toulouse Math. (6) 16 (2007), no. 2, 303–329.

[29] J-P. Ramis, *Dévissage Gevrey.* Journées Singulières de Dijon (Univ. Dijon, Dijon, 1978), pp. 4, 173–204, Astérisque, 59-60, Soc. Math. France, Paris, 1978.

[30] J-P. Ramis, Y. Sibuya, *A new proof of multisummability of formal solutions of nonlinear meromorphic differential equations.* Ann. Inst. Fourier (Grenoble) 44 (1994), no. 3, 811–848.

[31] C. Stenger, *On a conjecture of Wolfgang Wasow in linear turning point theory.* C. R. Acad. Sci. Paris Sér. I Math. 325 (1997), no. 1, 27–32.

[32] H. Yamane, *Global Fuchsian Cauchy problem.* J. Math. Sci. Univ. Tokyo 7 (2000), no. 1, 147–162.