A Fourier Series approach to solving the Navier-Stokes equations with spatially periodic data

William McLeod Rivera
College Park, MD 20740

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Email address: WILLIAM_RIVERA35@comcast.net

Abstract: The symbol $\hat{\vec{U}}$ denotes the velocity or momentum (the mass multiplied by the velocity). Transform the Navier Stokes momentum and density equations into infinite systems of ordinary differential and linear equations for the classical Fourier coefficients. Prove theorems on existence, uniqueness and smoothness of solutions. Interpret the results using the Fourier series representation $\vec{U} = \hat{\hat{\vec{U}}}, P = \hat{\hat{P}}$.
Key Words: Fluid Mechanics, incompressible Navier-Stokes equations, Fourier series

1. Introduction

The main result in this paper can be stated as follows. If the data are jointly smooth, spatially periodic, and, furthermore, the body force and its higher order time derivatives satisfy the generalized sector conditions of theorem 2-2, a unique physical solution $(\vec{U}, P)$ exists which is separately smooth in $(t, \vec{x})$ on $t \geq 0, \vec{x} \in R^3$. This solution is the extension of the unique regular (smooth) short time solution determined by the data. The solution $(\vec{U}, P)$ is also bounded for all forward time.

In 1934 Leray (in [9]) formulated the regularity problem and related it to the smoothness problem. In the year 2000, Fefferman formulated the problem (in [5]). In that same year, Bardos wrote a monograph on the problem ([3]) which summarized the then literature. The author interprets the remarks in [3] Bardos to indicate that the problem of regularity/smoothness can be solved as formulated in (A) of [5] Fefferman.

What is new?

As far as this author knows the formulas for the Navier Stokes ordinary differential equations in this paper are new. However for the problem defined on the aperiodic space domain Cannone- in the formula immediately prior to (27) on page 15 of *Harmonic Analysis Tools for Solving the Incompressible Navier-Stokes Equations*. (2003)- uses the Fourier transform to rewrite the variation of constants formula solution of the Navier-Stokes evolution equations.

The vector form of the Navier-Stokes equations

The vector form of the Navier-Stokes equations for an incompressible fluid on the unit cube is

$$\begin{aligned}
&\vec{U}_t + \vec{U}\cdot\nabla\vec{U} = \eta\Delta\vec{U} - \nabla P + \vec{F}, \eta > 0, t \ge 0, \vec{x} \in [0,1]^3 \\
&\vec{U}\cdot\nabla = 0, t \ge 0, \vec{x} \in [0,1]^3 \\
&\vec{U}(0,x,y,z) = \vec{U}_0(x,y,z), \vec{x} \in [0,1]^3 \\
&\vec{U}(t,x,y,0) = \vec{U}(t,x,y,1) = \vec{h}^1(x,y), (x,y) \in [0,1]^2, t \ge 0 \\
&\vec{U}(t,x,0,z) = \vec{U}(t,x,1,z) = \vec{h}^2(x,z), (x,z) \in [0,1]^2, t \ge 0 \\
&\vec{U}(t,0,y,z) = \vec{U}(t,1,y,z)) = \vec{h}^3(y,z), (y,z) \in [0,1]^2, t \ge 0.
\end{aligned} \tag{1-1}$$

One seeks a unique solution of (1-1) on the unit cube which is jointly smooth in time and space and bounded for all forward time given that the data $\vec{U}_0, \vec{F}, \vec{h}^i, i = 1,2,3$ is smooth. Such solutions extend to periodic solutions of period 1 in each space variable holding the others fixed.

In (1-1) $\vec{U}$ is the velocity vector field, ∇P is the pressure gradient to be determined, $\nabla\cdot\vec{U}$ is the divergence of $\vec{U}$, and $\vec{U}\cdot\nabla$ is the matrix tensor $D_{\vec{x}}\vec{U} = \begin{pmatrix} \nabla U \\ \nabla V \\ \nabla W \end{pmatrix}, \vec{U} = (U,V,W)$.

The first equation of (1-1) determines the momentum. The body force $\vec{F}$ is smooth on $[0,\infty)\times[0,1]^3$. The initial function $\vec{U}_0$ is smooth on $[0,1]^3$. The boundary functions $\vec{h}^i, i = 1,2,3$ are smooth on $[0,1]\times[0,1]$.

The momentum equation can be interpreted as Newton's second law of motion for fluids combined with a dynamic version of Archimedes' law of hydrostatics. If $\vec{U}(t,\vec{x}) = \vec{0}$, the equation reduces to Archimedes' law $\nabla P(t,\vec{x}) = \vec{F}(t,\vec{x})$.

The second equation of (1-1) is the equation of continuity. It is the Navier-Stokes equation for the density for an incompressible fluid.

The following equations for the classical Fourier series coefficients are equivalent to (1-1). The conditions on the data functions are equivalent to the conditions that their Fourier coefficients belong to discrete Schwartz frequency spaces

$$\frac{d\hat{\vec{U}}(t,\vec{r})}{dt} = -4\eta\,|\vec{r}|^2\pi^2\hat{\vec{U}}(t,\vec{r}) - 2\pi\hat{\vec{U}}(t,\vec{r})\vec{r}^{\,t} * \hat{\vec{U}}(t,\vec{r}) - 2\pi\vec{r}\hat{P}(t,\vec{r}) + \hat{\vec{F}}(t,\vec{r}),$$
$$t \ge 0, \vec{r} \in N^3, \eta > 0 \qquad (1\text{-}2a)$$
$$\hat{\vec{U}}(0,\vec{r}) = \hat{\vec{U}}_0(\vec{r}), \vec{r} \in N^3$$
$$\vec{r} \cdot \hat{\vec{U}}(t,\vec{r}) = 0, \vec{r} \in N^3, t \ge 0.$$

The boundary conditions for the Fourier coefficents are

$$\hat{\vec{U}}(t,0,r_2,r_3) = \hat{\vec{h}}_1(r_2,r_3), t \ge 0, \neg(r_2 \wedge r_3) = 0$$
$$\hat{\vec{U}}(t,r_1,0,r_3) = \hat{\vec{h}}_2(r_1,r_3), t \ge 0, \neg(r_1 \wedge r_3) = 0 \qquad (1\text{-}2b)$$
$$\hat{\vec{U}}(t,r_1,r_2,0) = \hat{\vec{h}}_3(r_1,r_2), t \ge 0, \neg(r_1 \wedge r_2) = 0$$
$$t \ge 0, \vec{r} \in N^3.$$

In (1-2), $\hat{\vec{U}}, \hat{P}, \hat{\vec{F}}, \hat{\vec{h}}^i, i = 1,2,3$ denote the classical Fourier sine series coefficients of $\vec{U}, P, \vec{F}, \vec{h}^i, i = 1,2,3$ and N^3, W^3 denote the set of natural (respectively whole) number triples.

The law governing the average mechanical energy of an incompressible fluid

Theorem 2-4 establishes the existence of a unique solution defined (bounded) for all forward time smooth in *t* uniformly in $\vec{x} \in [0,1]^3$ and smooth in $\vec{x}$ uniformly in $t \ge 0$. The following formulas provide a smooth generalization of Leray's mechanical energy law ([12] section 17 formula 3.4) for the Navier-Stokes momentum equation and its time derivatives of order *k*

$$\frac{1}{2}\int_{[0,1]^3} |\vec{U}|^2 d\vec{y} - \frac{1}{2}\int_{[0,1]^3} |\vec{U}_0|^2 d\vec{y} = -\eta\int_0^t \int_{[0,1]^3} |\nabla\vec{U}|^2 d\vec{y}ds + \int_0^t \int_{[0,1]^3} \vec{U} \cdot \vec{F} d\vec{y}ds$$
$$\frac{1}{2}\int_{[0,1]^3} |\vec{U}^{(k)}|^2 d\vec{y} = -\eta\int_0^t \int_{[0,1]^3} |\nabla\vec{U}^{(k)}|^2 d\vec{y}ds + \int_0^t \int_{[0,1]^3} \vec{U}^{(k)} \cdot \vec{F}^{(k)} d\vec{y}ds, \eta > 0, t \ge 0, \qquad (1\text{-}3)$$
$$k = 1,2,3,...$$

Equations (1-3) state that the difference of the space average of the kinetic energy (at time $t > 0$) minus that at time $t = 0^+$ is equal to the viscosity times the potential energy minus the average work done by the body force acting on the incompressible fluid where $|\nabla\vec{U}|^2 = \nabla u \cdot \nabla u + \nabla v \cdot \nabla v + \nabla w \cdot \nabla w, \vec{U}^t = (u,v,w)$. The energy formulas (1-3) are equivalent to the formulas

$$\sum_{\vec r>\vec 0}|\hat{\vec U}(t,\vec r)|^2-\sum_{\vec r>\vec 0}|\hat{\vec U}(0,\vec r)|^2=$$
$$-4\pi^2\eta\int_0^t\sum_{\vec r>\vec 0}|\vec r|^2|\hat{\vec U}|^2\;ds+\int_0^t\sum_{\vec r>\vec 0}\hat{\vec U}(s,\vec r)\cdot\hat{\vec F}(s,\vec r)ds,$$
$$\sum_{\vec r>\vec 0}|\hat{\vec U}^{(k)}(t,\vec r)|^2= \tag{1-4}$$
$$-4\pi^2\eta\int_0^t\sum_{\vec r>\vec 0}|\vec r|^2|\hat{\vec U}^{(k)}|^2\;ds+\int_0^t\sum_{\vec r>\vec 0}\hat{\vec U}^{(k)}(s,\vec r)\cdot\hat{\vec F}^{(k)}(s,\vec r)ds,t\ge 0,\eta>0,k=1,2,3,...$$

established in theorem 2-2.

By Parseval's theorem, the quantities on the left side of (1-3) and (1-4) are both equal to the average kinetic energy.

The problem of finite time blow up

In theorem 2-3 the author shows that finite time blow up of solutions of (1-1) is impossible given smooth data, the equation of continuity, and the following conditions

$$-\eta\int_0^t\int_{[0,1]^3}|\vec U\cdot\nabla|^2d\vec x ds>-\infty,\int_0^t\int_{[0,1]^3}\vec U^{(k)}\cdot\vec F^{(k)}d\vec x ds=M_k(t)<\infty,t\ge 0,\eta>0,k=0,1,2,... \tag{1-5}$$

In the frequency domain, inequalities (1-5) become

$$-4\pi^2\eta\int_0^t\sum_{\vec r>\vec 0}|\vec r|^2|\hat{\vec U}^{(k)}(s,\vec r)|^2ds>-\infty,\int_0^t\sum_{\vec r>\vec 0}\hat{\vec U}^{(k)}(s,\vec r)\cdot\hat{\vec F}^{(k)}ds=\hat M_k(t), \tag{1-6}$$
$$t\ge 0,\eta>0,k=0,1,2,...$$

as proved in lemma 2-3.

Under the conditions (1-5) on $\vec F$ the solutions of (1-1) are absolutely bounded. Absolute stability/boundedness extends the concept of Lyapunov stability/boundedness from homogeneous nonlinear systems to nonlinear systems with a forcing function.

2. Existence of a unique smooth short time solution and its forward time extension

The Navier-Stokes ordinary differential equations for the momentum coefficients in the discrete frequency domain comprise an infinite system of ordinary differential equations for the time dependent Fourier coefficients $\hat{\vec U}(t,\vec r)$ of the velocity $\vec U(t,\vec x)$. It is possible to use the classical Fourier sine series coefficients because the inhomogeneous Dirichlet boundary conditions match on opposite faces of the unit cube.

The notation

$\hat{\vec{U}} \in \{\vec{C}^{\infty}([0,T]), \vec{r} \in N^3\} \cap \{\hat{\vec{S}}(N^3), t \geq 0\}$

indicates the space of Fourier coefficients of momentum vectors which are smooth on $t \in [0,T]$ and discrete Schwartz in the vector of frequency parameters. A Fourier coefficient is discrete Schwartz in a parameter if it decays more rapidly than any fixed power of the norm of the frequency $|\vec{r}|^p, p \in W$ grows. Thus $\lim_{|\vec{r}| \to \infty} |\vec{r}|^p |\hat{\vec{U}}(t,\vec{r}) = 0, t \geq 0, p \in W$. Equivalently the discrete Schwartz coefficients can be defined with upper bounds replacing limits (see definition 2-2 below).

Lemma 2-1. The Navier-Stokes ordinary differential equations for the Fourier coefficients of the solution of the spatially periodic problem (1-1) are

$$\begin{aligned}
&\frac{d\hat{\vec{U}}(t,\vec{r})}{dt} = -4\eta |\vec{r}|^2 \pi^2 \hat{\vec{U}}(t,\vec{r}) - 2\pi \hat{\vec{U}}(t,\vec{r})\vec{r}^{\,t} * \hat{\vec{U}}(t,\vec{r}) - 2\pi\vec{r}\hat{P}(t,\vec{r}) + \hat{\vec{F}}(t,\vec{r}), \\
&t \geq 0, \vec{r} \in N^3, \eta > 0 \\
&\vec{r} \cdot \hat{\vec{U}}(t,\vec{r}) = 0, \vec{r} \in N^3, t \geq 0 \\
&\hat{\vec{U}}(0,\vec{r}) = \hat{\vec{U}}_0(\vec{r}), \vec{r} \in N^3.
\end{aligned} \tag{2-1a}$$

where

$$\begin{aligned}
&\hat{\vec{U}}(t,r_1,r_2,0) = \hat{\vec{h}}_3(r_1,r_2), t \geq 0, \neg(r_1 \wedge r_2) = 0 \\
&\hat{\vec{U}}(t,r_1,0,r_3) = \hat{\vec{h}}_2(r_1,r_3), t \geq 0, \neg(r_1 \wedge r_3) = 0 \\
&\hat{\vec{U}}(t,0,r_2,r_3) = \hat{\vec{h}}_1(r_2,r_3), t \geq 0, \neg(r_2 \wedge r_3) = 0.
\end{aligned} \tag{2-1b}$$

In line 1 of (2-1a) $*$ denotes the discrete convolution

$$\hat{\vec{U}}(t,\vec{r})\vec{r}^{\,t} * \hat{\vec{U}}(t,\vec{r}) = \sum_{\vec{q} > \vec{0}} [\hat{\vec{U}}(t,\vec{r}-\vec{q})(\vec{r}-\vec{q})^t] \hat{\vec{U}}(t,\vec{q}). \tag{2-2}$$

The discrete Fourier transform (the Fourier coefficient) of the velocity is

$$\begin{aligned}
&\hat{\vec{U}}(t,\vec{r}) = \Im\{\vec{U}(t,\vec{x})\} = \int_{[0,1]^3} \vec{U}(t,\vec{x}) \sin 2\pi r_1 x \sin 2\pi r_2 y \sin 2\pi r_3 z dxdydz, \\
&\vec{r} \in N^3, t \geq 0.
\end{aligned} \tag{2-3}$$

and similarly for $\hat{\vec{F}}(t,\vec{r}), \hat{P}(t,\vec{r})$.

PROOF

Note that all triple integrals over $[0,1]^3$ in the definition of the discrete Fourier transform defined by (2-3) are well defined for all forward time since $\vec{F}, \vec{U}_0$ are smooth in $\vec{x} \in D$ and smooth and bounded in *t* with all time derivatives continuous and bounded on $t \in [0,\infty)$. The boundary functions $\vec{h}^i, i = 1,2,3$ are smooth on $[0,1]^2$.

The terms $\frac{d\hat{\vec{U}}}{dt}, \vec{r}\hat{P}, \hat{\vec{F}}$ are calculated from the discrete Fourier transform to the terms of (1-1) and the differentiation property of discrete Fourier transforms applied to the first partial derivatives in the case of the pressure gradient term.

The Fourier series coefficient of the diffusion term can be calculated using integration by parts twice. Since the three calculations are identical, it suffices to calculate the transform of the second partial derivative of $\vec{U}$ with respect to the first component *x,*

$$\begin{aligned}
&\int_{[0,1]^3} \vec{U}_{xx} \sin 2\pi r_1 x \sin 2\pi r_2 y \sin 2\pi r_3 z dxdydz = \\
&\int_{y=0}^{1}\int_{z=0}^{1} \vec{U}_x \sin 2\pi r_1 x \,|_{x=0}^{1} \sin 2\pi r_2 y \sin 2\pi r_3 z dydz = \\
&-2\pi r_1 \int_{[0,1]^3} \vec{U}_x \cos 2\pi r_1 x \sin 2\pi r_2 y \sin 2\pi r_3 z dxdydz = \\
&-4\pi^2 r_1^2 \int_{[0,1]^3} \vec{U} \sin 2\pi r_2 y \sin 2\pi r_3 z dxdydz = -4\pi^2 r_1^2 \hat{\vec{U}}(t,\vec{r}), t \geq 0, \vec{r} \in N^3.
\end{aligned} \tag{2-4}$$

Thus the diffusion term is

$$\begin{aligned}
&\Im\{\Delta\vec{U}\} = (r_1^2 + r_2^2 + r_3^2)\hat{\vec{U}}(t,\vec{r}) + r_1[\frac{\partial\vec{U}(t,1,x_2,x_3)}{\partial x_1} - \frac{\partial\vec{U}(t,0,x_2,x_3)}{\partial x_1}] + \\
&r_2[\frac{\partial\vec{U}(t,x_1,1,x_3)}{\partial x_2} - \frac{\partial\vec{U}(t,x_1,0,x_3)}{\partial x_2}] + r_3[\frac{\partial\vec{U}(t,x_1,x_2,1)}{\partial x_3} - \frac{\partial\vec{U}(t,x_1,x_2,0)}{\partial x_3}] + \\
&\vec{U}(t,1,x_2,x_3) - \vec{U}(t,0,x_2,x_3) + \vec{U}(t,x_1,1,x_3) - \vec{U}(t,x_1,0,x_3) + \vec{U}(t,x_1,x_2,1) - \vec{U}(t,x_1,x_2,0) \\
&= (r_1^2 + r_2^2 + r_3^2)\hat{\vec{U}}(t,\vec{r}) \neq 0, t \geq 0, \vec{r} \in N^3.
\end{aligned} \tag{2-5}$$

It follows that

$$\Im\{\eta\Delta\vec{U}\} = -4\eta\pi^2 |\vec{r}|^2 \hat{\vec{U}}, \vec{r} \in N^3, t \geq 0. \tag{2-6}$$

The transform of the Euler term is

$$\Im\{\vec{U}\cdot\nabla\vec{U}\} = \sum_{\vec{q}\in N^3} 2\pi[\hat{\hat{\vec{U}}}(t,\vec{r}-\vec{q})(\vec{r}-\vec{q})^t]\hat{\hat{\vec{U}}}(t,\vec{q}), t \ge 0, \vec{r} \in N^3. \tag{2-7}$$

The transform of the pressure gradient term is

$$\int_{[0,1]^3} \nabla P \sin 2\pi r_1 x \sin 2\pi r_2 y \sin 2\pi r_3 z d\vec{x} = 2\pi \vec{r}\hat{P}, t \ge 0, \vec{r} \in N^3. \tag{2-8}$$

The transform of the equation of continuity is

$$\int_{[0,1]^3} (U_x + V_y + W_z) \sin 2\pi r_1 x \sin 2\pi r_2 y \sin 2\pi r_3 z d\vec{x} = 2\pi[r_1 U + r_2 V + r_3 W] = 0, \\ t \ge 0, \vec{r} \in N^3. \tag{2-9}$$

The boundary conditions (lines 3-6 of formula (2-1)) are derived by solving the homogeneous Laplace's equation constrained by the face matching boundary conditions using transform (Fourier series) methods.

END PROOF

Remark 2-1. Equations (2-1) specify an infinite system of ordinary differential equations with a discrete vector parameter whose solutions are the time dependent Fourier coefficients in the Fourier expansions of $\vec{U}(t,\vec{x}), \vec{F}(t,\vec{x}), \vec{U}_0(\vec{x}), P(t,\vec{x})$. In fact formulas (2-4) through (2-9) hold on $\vec{r} \in W^3$. But, since the boundary conditions are Dirichlet and the Fourier sine series is used, there is no constant term corresponding to $\vec{r} = \vec{0}$ while cases with one or two indices r_i equal to 0 occur on the boundary of the frequency domain.

Definition 2-1. The family of linear operators defined by the transformed equation of continuity is

$$L_{\vec{r}}[\hat{\hat{\vec{F}}}(t,\vec{r})] = \vec{r}\cdot\hat{\hat{\vec{F}}}(t,\vec{r}), \vec{r} \in N^3, t \ge 0. \tag{2-10}$$

Remark 2-2. For any $\vec{r} \in N^3$, $L_{\vec{r}} : \vec{C}^\infty[0,\infty) \to C^\infty[0,\infty)$. The linear operator $L_{\vec{r}}$ is a map from vectors of smooth (infinitely continuous differentiable) functions on $t \in [0,\infty)$ to smooth scalar functions on $[0,\infty)$.

Proposition 2-1. Any derivative of finite order of the velocity $\frac{d^k\hat{\hat{\vec{U}}}}{dt^k}, k = 0,1,2,...$ is in $\eta(L_{\vec{r}})$ (the null space of $L_{\vec{r}}$).

PROOF

By (1-1) this relation holds for the discrete Fourier transform of the equation of continuity ($k=0$). Take derivatives of order $k=1,2,3,...$ with respect to t of both sides of

$$\hat{\vec{U}}(t,\vec{r})\cdot\vec{r}=0, t\geq 0, \vec{r}>\vec{0} \tag{2-11}$$

to complete the proof.

END PROOF

Since differentiation of the momentum function with respect to the space variables corresponds to multiplication of its transform by frequency variables, the following Banach space is the one needed to establish that solutions of the Navier-Stokes equations are smooth in $(t,\vec{x})\in[0,\infty)\times R^3$.

Definition 2-2. The discrete Schwartz space (uniform in $t\in[0,T]$) for Fourier coefficients $\hat{\vec{U}}$ defined on $[0,T]\times N^3$ is

$$\hat{\vec{S}}=\{\hat{\vec{U}}\in\vec{C}^{\infty}([0,T]):\| r_1^{s(1)}r_2^{s(2)}r_3^{s(3)}\hat{\vec{U}}\|=\sup_{\vec{r}\in N^3}|r_1^{s(1)}r_2^{s(2)}r_3^{s(3)}\hat{\vec{U}}|\leq M(\vec{s}), \vec{r}\in N^3, \vec{s}\in W^3\}, \tag{2-12}$$
$0\leq t\leq T$.

The following lemma establishes that $\hat{P}$ is an auxiliary function for (2-1)- i.e. a function which can be removed in an equivalent form of the equation.

Lemma 2-2. Equations (2-1) can be placed into the following equivalent form

$$\hat{\vec{U}}_t=-4\pi^2\eta|\vec{r}|^2\hat{\vec{U}}+\left[\frac{\vec{r}\vec{r}^{\,t}}{|\vec{r}|^2}-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right][2\pi(\hat{\vec{U}}\vec{r}^{\,t})*\hat{\vec{U}}-\hat{\vec{F}}], t\geq 0, \vec{r}\in N^3$$

$$\hat{\vec{U}}(t,\vec{r})\cdot\vec{r}=0, t\geq 0, \vec{r}\in N^3 \tag{2-13a}$$

$$\hat{\vec{U}}(0,\vec{r})=\hat{\vec{U}}_0(\vec{r}), \vec{r}\in N^3$$

$$\hat{\vec{U}}\in\{\hat{\vec{U}}:\hat{\vec{U}}\in(\hat{\vec{C}}^{\infty}\cap\hat{\vec{L}}^{\infty})[0,\infty), \vec{r}\in N^3, \hat{\vec{U}}\in\hat{\vec{S}}(N^3), t\geq 0).$$

The boundary conditions for the velocity coefficients are

$$\hat{\vec{U}}(t,r_1,r_2,0)=\hat{\vec{h}}_3(r_1,r_2), t\geq 0, \neg(r_1\wedge r_2)=0$$

$$\hat{\vec{U}}(t,r_1,0,r_3)=\hat{\vec{h}}_2(r_1,r_3), t\geq 0, \neg(r_1\wedge r_3)=0 \tag{2-13b}$$

$$\hat{\vec{U}}(t,0,r_2,r_3)=\hat{\vec{h}}_1(r_2,r_3), t\geq 0, \neg(r_2\wedge r_3)=0.$$

The pressure coefficients satisfy the following equations

$$\hat{P}(t,\vec{r}) = \frac{1}{2\pi|\vec{r}|^2}\{-\vec{r}\cdot 2\pi(\hat{\vec{U}}\vec{r}^{\,t})*\hat{\vec{U}} + \vec{r}\cdot\hat{\vec{F}}(t,\vec{r})\}, t \ge 0, \vec{r} \in N^3$$

$$\hat{P}(0^+,\vec{r}) = \frac{1}{2\pi|\vec{r}|^2}\{-\vec{r}\cdot 2\pi(\hat{\vec{U}}_0(\vec{r})\vec{r}^{\,t})*\hat{\vec{U}}_0(\vec{r}) + \vec{r}\cdot\hat{\vec{F}}(0^+,\vec{r}\}, \vec{r} \in N^3. \qquad (2\text{-}14a)$$

The pressure coefficients satisfy the following boundary conditions

$$\hat{P}(t,r_1,r_2,0) = \frac{1}{2\pi|\vec{r}|^2}\{-\sum_{\vec{q}\in N^2} 2\pi(r_1,r_2,0)\cdot[\hat{\vec{U}}(t,q_1,q_2,0)(q_1,q_2,0)^t][\hat{\vec{U}}(t,r_1-q_1,r_2-q_2,0)]$$
$$+(r_1,r_2,0)\cdot\hat{\vec{F}}(t,r_1,r_2,0)\}, (r_1,r_2) > (0,0), t \ge 0,$$
$$\hat{P}(t,r_1,0,r_3) = \frac{1}{2\pi|\vec{r}|^2}\{-\sum_{\vec{q}\in N^2} 2\pi(r_1,0,r_3)\cdot[\hat{\vec{U}}(t,q_1,0,q_3)(q_1,0,q_3)^t][\hat{\vec{U}}(t,r_1-q_1,0,r_3-q_3)]$$
$$+(r_1,0,r_3)\cdot\hat{\vec{F}}(t,r_1,0,r_3)\}, (r_1,r_3) > (0,0), t \ge 0, \qquad (2\text{-}14b)$$
$$\hat{P}(t,0,r_2,r_3) = \frac{1}{2\pi|\vec{r}|^2}\{-\sum_{\vec{q}\in N^2} 2\pi(0,r_2,r_3)\cdot[\hat{\vec{U}}(t,q_1,q_2,0)(0,q_2,q_3)^t][\hat{\vec{U}}(t,0,r_2-q_2,r_3-q_3)]$$
$$+(0,r_2,r_3)\cdot\hat{\vec{F}}(t,0,r_2,r_3)\}, (r_2,r_3) > (0,0), t \ge 0.$$

PROOF

The equations in the first line of (2-13a) form an infinite system of vector ordinary differential equations-one for each Fourier coefficient as a function of time. The equations in the second line of (2-13a) define an infinite set of homogeneous linear equations.

Apply the linear operator $L_{\vec{r}}$ to each side of equation (2-1) by forming the dot product of each term with the vector $\vec{r}$ to obtain

$$\vec{r}\cdot\frac{d\hat{\vec{U}}(t,\vec{r})}{dt} = -\eta\vec{r}\cdot|\vec{r}|^2\pi^2\hat{\vec{U}}(t,\vec{r})$$
$$-\vec{r}\cdot\sum_{\vec{q}\in N^3}[2\pi\hat{\vec{U}}(t,\vec{r}-\vec{q})(\vec{r}-\vec{q})^t]\hat{\vec{U}}(t,\vec{q}) - \vec{r}\cdot\vec{r}\,2\pi\hat{P}(t,\vec{r}) + \vec{r}\cdot\hat{\vec{F}}(t,\vec{r}), \vec{r} \in N^3, t \ge 0. \qquad (2\text{-}15)$$

By proposition 2-1, $\hat{\vec{U}}(t,\vec{r}), \hat{\vec{U}}_t(t,\vec{r}) \in \eta(L_{\vec{r}}), \vec{r} \in W^3$ hence (2-15) reduces to

$$0 = -\vec{r}\cdot\sum_{\vec{q}\in N^3}2\pi[\hat{\vec{U}}(t,\vec{r}-\vec{q})(\vec{r}-\vec{q})^t]\hat{\vec{U}}(t,\vec{q}) - \vec{r}\cdot\vec{r}\,2\pi\hat{P}(t,\vec{r}) + \vec{r}\cdot\hat{\vec{F}}(t,\vec{r}). \qquad (2\text{-}16)$$

Solve (2-16) for $\hat{P}(t,\vec{r})$ to obtain the first line of (2-14a). Insert the first line of the pressure formula into (2-1) to obtain the first line of (2-13a). The second line of (2-14a) is obtained from the transform of the initial function $\hat{\vec{U}}_0(\vec{r})$. The coefficients $\hat{\vec{h}}^i, i = 1,2,3$ on the boundary of the frequency domain can be inserted for $\hat{\vec{U}}$ to further reduce the formulas (2-14b). The pressure coefficients are independent of time on the boundary.

END PROOF

Remark 2-3. From higher order derivatives of (2-1) and the projection defined by the kth order equation of continuity, one can calculate formulas for $\hat{P}^{(k)}(t,\vec{r})$ similar to (2-14).

The equation that results from calculating any finite order time derivative of equations (2-13a) is

$$\hat{\vec{U}}^{(k+1)} = -4\pi^2\eta\,|\,\vec{r}\,|^2\,\hat{\vec{U}}^{(k)} + \left[\frac{\vec{r}\vec{r}^{\,t}}{|\,\vec{r}\,|^2} - \begin{pmatrix}1\,0\,0\\0\,1\,0\\0\,0\,1\end{pmatrix}\right][(2\pi(\hat{\vec{U}}\vec{r}^{\,t}) * \hat{\vec{U}})^{(k)} - \hat{\vec{F}}^{(k)}],$$

$$t \ge 0, \vec{r} \in N^3, k \in N \qquad (2\text{-}17)$$

$$\hat{\vec{U}}^{(k)}(0,\vec{r}) = \vec{0}, \vec{r} \in N^3, k \in N$$

$$\hat{\vec{U}}^{(k)}(t,\vec{r}) \cdot \vec{r} = 0, t \ge 0, \vec{r} \in N^3, k \in N.$$

The following theorem establishes the existence of smooth short time solutions of equation (2-13).

Theorem 2-1. Suppose $\hat{\vec{F}}, \hat{\vec{U}} \in \vec{C}^{\infty}([0,\infty) \cap \hat{\vec{S}}(N^3), \hat{\vec{U}}_0 \in \hat{\vec{S}}(N^3), \hat{\vec{h}}_i \in \hat{\vec{S}}(N^2), t \ge 0, i = 1,2,3$,then there exists $T > 0$ such that $\hat{\vec{U}} \in \vec{C}^{\infty}([0,T)) \cap \hat{\vec{S}}(N^3)$, satisfies (2-11) and $\hat{P} \in C^{\infty}([0,T)) \cap \hat{\vec{S}}(N^3)$.

PROOF

The goal is to establish

a. For any fixed $\vec{r} \in N^3$, and any non negative integer *k*, $\hat{\vec{U}}^{(k)}$ is continuous for sufficiently short time.

$$\exists T > 0 : |\hat{\vec{U}}^{(k)}(t_2) - \hat{\vec{U}}^{(k)}(t_1)| < M(k,\vec{r})\,|\,t_2 - t_1| < \infty, \forall t_1, t_2 \in [0,T], M = \sup_{0 \le t \le T} \hat{\vec{U}}^{(k+1)}(t) \qquad (2\text{-}18)$$

b. The continuity of $\hat{\vec{U}}^{(k)}$ is uniform in $k \in W$

$$\exists T > 0 : |\hat{\vec{U}}^{(k)}(t_2) - \hat{\vec{U}}^{(k)}(t_1)| < M(\vec{r})\,|\,t_2 - t_1| < \infty, \forall t_1, t_2 \in [0,T],$$

$$\widetilde{M} = \sup_{k \in W} \sup_{0 \le t \le T} \hat{\vec{U}}^{(k+1)}(t) \qquad (2\text{-}19)$$

c. The continuity of $\hat{\vec{U}}^{(k)}$ is uniform in $\vec{r} \in N^3$

$$\exists T>0:|\hat{\vec{U}}^{(k)}(t_2)-\hat{\vec{U}}^{(k)}(t_1)|<M\,|t_2-t_1|<\infty,\forall t_1,t_2\in[0,T],\tilde{\tilde{M}} = \sup_{\vec{r}\in N^3}\sup_{k\in W}\sup_{0\le t\le T}\hat{\vec{U}}^{(k+1)}(t) \tag{2-20}$$

PROOF

For derivatives of the momentum coefficients of any finite order $k=0,1,2,3,...$ the variation of constants formula yields

$$\hat{\vec{U}}^{(k)}(t,\vec{r})=(-4\eta)^k\pi^{2k}\,|\vec{r}\,|^{2k}\,e^{-\eta 4\pi^2|\vec{r}|^2 t}\hat{\vec{U}}_0 + \int_0^t e^{-\eta 4\pi^2|\vec{r}|^2(t-s)}\left[\frac{\vec{r}\vec{r}^{\,t}}{|\vec{r}\,|^2}-\begin{pmatrix}1\,0\,0\\0\,1\,0\\0\,0\,1\end{pmatrix}\right][(2\pi(\hat{\vec{U}}\vec{r}^{\,t})*\hat{\vec{U}})^{(k)}-\hat{\vec{F}}^{(k)}]ds,.t\ge 0,\vec{r}\in N^3, \tag{2-21}$$

$t\ge 0,\vec{r}\in N^3,k=0,1,2,3,...$

To prove continuity of the k[th] derivative of the momentum coefficient in time, calculate

$$\begin{aligned}\hat{\vec{U}}^{(k)}(t_2,\vec{r})-\hat{\vec{U}}^{(k)}(t_1,\vec{r}) &= (-4\eta)^k\pi^{2k}\,|\vec{r}\,|^{2k}\,e^{-\eta 4\pi^2|\vec{r}|^2 t(2)}\hat{\vec{U}}_0-(-4\eta)^k\pi^{2k}\,|\vec{r}\,|^{2k}\,e^{-\eta 4\pi^2|\vec{r}|^2 t(1)}\hat{\vec{U}}_0\\ &+\int_0^{t(2)} e^{-\eta 4\pi^2|\vec{r}|^2(t(2)-s)}\left[\frac{\vec{r}\vec{r}^{\,t}}{|\vec{r}\,|^2}-\begin{pmatrix}1\,0\,0\\0\,1\,0\\0\,0\,1\end{pmatrix}\right][(2\pi(\hat{\vec{U}}\vec{r}^{\,t})*\hat{\vec{U}})^{(k)}-\hat{\vec{F}}^{(k)}]ds-\\ &+\int_0^{t(1)} e^{-\eta 4\pi^2|\vec{r}|^2(t(1)-s)}\left[\frac{\vec{r}\vec{r}^{\,t}}{|\vec{r}\,|^2}-\begin{pmatrix}1\,0\,0\\0\,1\,0\\0\,0\,1\end{pmatrix}\right][(2\pi(\hat{\vec{U}}\vec{r}^{\,t})*\hat{\vec{U}})^{(k)}-\hat{\vec{F}}^{(k)}]ds\end{aligned} \tag{2-22}$$

$t_2>t_1\ge 0,\vec{r}\in N^3,k=0,1,2,3,..$

Simplify the difference of $\hat{\vec{U}}^{(k)}$ at two distinct times

$$\hat{\hat{U}}^{(k)}(t_2,\vec{r}) - \hat{\hat{U}}^{(k)}(t_1,\vec{r}) = (-4\eta)^k \pi^{2k} \, |\vec{r}|^{2k} \, [e^{-\eta 4\pi^2|\vec{r}|^2 t(2)} - e^{-\eta 4\pi^2|\vec{r}|^2 t(1)}]\hat{\hat{U}}_0$$
$$+ \int_{t(1)}^{t(2)} e^{-\eta 4\pi^2|\vec{r}|^2(t(2)-s)} \left[\frac{\vec{r}\vec{r}^{\,t}}{|\vec{r}|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] [(2\pi(\hat{\hat{U}}\vec{r}^{\,t}) * \hat{\hat{U}})^{(k)} - \hat{\hat{F}}^{(k)}]ds - \tag{2-23}$$
$$\int_0^{t(1)} [e^{-\eta 4\pi^2|\vec{r}|^2(t(2)-s)} - e^{-\eta 4\pi^2|\vec{r}|^2(t(1)-s)}] \left[\frac{\vec{r}\vec{r}^{\,t}}{|\vec{r}|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] [(2\pi(\hat{\hat{U}}\vec{r}^{\,t}) * \hat{\hat{U}})^{(k)} - \hat{\hat{F}}^{(k)}]ds$$
$$t_2 > t_1 \geq 0, \vec{r} \in N^3, k = 0,1,2,3,...$$

To investigate behavior near time 0, evaluate (2-22) at $t_1 = 0, t_2 = t$

$$\hat{\hat{U}}^{(k)}(t,\vec{r}) = (-4\eta)^k \pi^{2k} \, |\vec{r}|^{2k} \, [e^{-\eta 4\pi^2|\vec{r}|^2 t} - 1]\hat{\hat{U}}_0$$
$$+ \int_0^t e^{-\eta 4\pi^2|\vec{r}|^2(t-s)} \left[\frac{\vec{r}\vec{r}^{\,t}}{|\vec{r}|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] [(2\pi(\hat{\hat{U}}\vec{r}^{\,t}) * \hat{\hat{U}})^{(k)} - \hat{\hat{F}}^{(k)}]ds \tag{2-24}$$
$$t \geq 0, \vec{r} \in N^3, k = 0,1,2,3,...$$

The upper bound on the momentum coefficient is

$$|\hat{\hat{U}}^{(k)}(t,\vec{r})| \leq |4\eta|^k \, \pi^{2k} \, |\vec{r}|^{2k} \, [1 - e^{-\eta 4\pi^2|\vec{r}|^2 t}] \, |\hat{\hat{U}}_0|$$
$$+ \int_0^t e^{-\eta 4\pi^2|\vec{r}|^2(t-s)} \left| \left[\frac{\vec{r}\vec{r}^{\,t}}{|\vec{r}|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] \right| [|(2\pi(\hat{\hat{U}}\vec{r}^{\,t}) * \hat{\hat{U}})^{(k)}| + |\hat{\hat{F}}^{(k)}|]ds \tag{2-25}$$
$$t \geq 0, \vec{r} \in N^3, k = 0,1,2,3,...$$

The upper bound can be simplified as follows.

$$|\hat{\hat{U}}^{(k)}(t,\vec{r})| \leq |4\eta|^k \, \pi^{2k} \, |\vec{r}|^{2k} \, 4\eta\pi^2 \, |\vec{r}|^2 |\hat{\hat{U}}_0| \, t$$
$$+ \int_0^t e^{-\eta 4\pi^2|\vec{r}|^2(t-s)} \left| \left[\frac{1}{|\vec{r}|^2} \begin{pmatrix} r_1^2 & r_1r_2 & r_1r_3 \\ r_1r_2 & r_2^2 & r_2r_3 \\ r_1r_3 & r_2r_3 & r_3^2 \end{pmatrix} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] \right| [|(2\pi(\hat{\hat{U}}\vec{r}^{\,t}) * \hat{\hat{U}})^{(k)}| + |\hat{\hat{F}}^{(k)}|]ds \tag{2-26}$$
$$t \geq 0, \vec{r} \in N^3, k = 0,1,2,3,...$$

Use the trace norm to bound the matrix operator inside the integral above

$$\frac{1}{|\vec{r}|^2}\left|\begin{pmatrix} r_1^2 & r_1r_2 & r_1r_3 \\ r_1r_2 & r_2^2 & r_2r_3 \\ r_1r_3 & r_2r_3 & r_3^2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right| < \left|\begin{pmatrix} \frac{r_1^2 - |\vec{r}|^2}{|\vec{r}|^2} & 0 & 0 \\ 0 & \frac{r_2^2 - |\vec{r}|^2}{|\vec{r}|^2} & 0 \\ 0 & 0 & \frac{r_3^2 - |\vec{r}|^2}{|\vec{r}|^2} \end{pmatrix}\right| < \frac{3|\vec{r}|^2 - |\vec{r}|^2}{|\vec{r}|^2} = 2. \tag{2-27}$$

By the hypothesis on the data

$$\sup_{t \ge 0} |\hat{\hat{U}}_0(\vec{r})| < L(r) < L \tag{2-28a}$$

$$\sup_{s \ge 0} |4\eta|^k \ \pi^{2k} \ |\vec{r}|^{2k} |\hat{\hat{F}}^{(k)}(s,\vec{r})| < |\vec{r}|^{5k} |\hat{\hat{F}}^{(k)}(s,\vec{r})| < M_2(\vec{r}) < M_2, |\vec{r}| > \max\{4\pi, \eta\} \tag{2-28b}$$

Since the variation of constants operator is defined on coefficients which are smooth time/ discrete Schwartz frequency, $\hat{\hat{U}}(s,\vec{r})\vec{r}^{\,t}$ is discrete Schwartz in the frequency. The convolution of discrete Schwartz functions is discrete Schwartz in the frequency $\vec{r} \in N^3$ which is also smooth in the time variable,

$$\sup_{s \ge 0} |(2\pi[(\hat{\hat{U}}(s,\vec{r})\vec{r}^{\,t}) * \hat{\hat{U}}(s,\vec{r})]^{(k)}| < M_1(\vec{r}) < M_1. \tag{2-29}$$

It follows that

$$\begin{aligned} &|\hat{\hat{U}}^{(k)}(t,\vec{r})| \le Lt \\ &+ \int_0^t e^{-\eta 4\pi^2 |\vec{r}|^2 (t-s)} 2\{M_1 + M_2\} ds = \\ &Lt + \left(\frac{1 - e^{-\eta 4\pi^2 |\vec{r}|^2 t}}{\eta 4\pi^2 |\vec{r}|^2}\right) 2\{M_1 + M_2\} \\ &t \ge 0, \vec{r} \in N^3, k = 1,2,3,... \end{aligned} \tag{2-30}$$

Not only are the coefficients $|\hat{\hat{U}}^{(k)}(t,\vec{r})|$ bounded for sufficiently short time but for all finite forward time.

The formulas (2-13b) and (2-14b) for $\hat{\hat{U}}, \hat{P}$ on the integer lattice “boundaries” of the frequency domain and their discrete Schwartz property in N^3 for all forward time follow automatically from the given conditions on the boundary data.

END PROOF

Proposition 2-2. The inner product of $\hat{\vec{U}}^{(k)}(t,\vec{r})$ with the transformed Euler (convolution) term

$\hat{\vec{U}}^{(k)}(t,\vec{r})[\hat{\vec{U}}\vec{r}^{\,t} * \hat{\vec{U}}(t,r)]^{(k)} = 0, k = 0,1,2,3,..., t \geq 0, \vec{r} > \vec{0}.$

PROOF

By the Liebnitz rule, the k[th] order Euler term can be written

$$\hat{\vec{U}}_{\vec{r}}^{(k)} \cdot \frac{d^k}{dt^k}[\hat{\vec{U}}_{\vec{r}}(t)\vec{r}^{\,t} * \hat{\vec{U}}_{\vec{r}}(t)] = \hat{\vec{U}}_{\vec{r}}^{(k)} \cdot \sum_{l=0}^{k} \binom{k}{l} (\hat{\vec{U}}_{\vec{r}}(t)\vec{r}^{\,t})^{(l)} * (\hat{\vec{U}}_{\vec{r}}(t))^{(k-l)}$$
$$= \hat{\vec{U}}_{\vec{r}}^{(k)}(t) \cdot \sum_{\vec{r}>\vec{q}} \sum_{l=0}^{k} \binom{k}{l} (\hat{\vec{U}}_{\vec{r}-\vec{q}}(t)(\vec{r}-\vec{q})^t)^{(l)} (\hat{\vec{U}}_{\vec{q}}(t))^{(k-l)}. \tag{2-31}$$

The equality in the second line follows by the definition of convolution and the fact that the coefficients for the classical Fourier sign series are one sided $\vec{r} > \vec{0}$.

The next equality follows by matrix vector multiplication,

$$\hat{\vec{U}}_{\vec{r}}^{(k)} \cdot \frac{d^k}{dt^k}[\hat{\vec{U}}_{\vec{r}}(t)\vec{r}^{\,t} * \hat{\vec{U}}_{\vec{r}}(t)] = \sum_{\vec{r}>\vec{q}} \sum_{l=0}^{k} \binom{k}{l} (\vec{r}-\vec{q}) \cdot (\hat{\vec{U}}_{\vec{q}}(t))^{(k-l)} (\hat{\vec{U}}_{\vec{r}-\vec{q}}(t))^{(l)} \cdot (\hat{\vec{U}}_{\vec{q}}(t))^{(k-l)}. \tag{2-32}$$

The next inequality follows by the Schwartz inequality for vectors in R^3 applied to each term and both dot products,

$$\hat{\vec{U}}_{\vec{r}}^{(k)} \cdot \frac{d^k}{dt^k}[\hat{\vec{U}}_{\vec{r}}(t)\vec{r}^{\,t} * \hat{\vec{U}}_{\vec{r}}(t)] \leq$$
$$\sum_{\vec{r}>\vec{q}} \sum_{l=0}^{k} \binom{k}{l} |\vec{r}-\vec{q}|^2 |(\hat{\vec{U}}_{\vec{q}})^{(k-l)}(t)|^2 |(\hat{\vec{U}}_{\vec{r}-\vec{q}})^{(l)}(t)|^2 |(\hat{\vec{U}}_{\vec{q}})^{(k-l)}(t)|. \tag{2-33}$$

The next equality follows by the definition of the inner product of a vector with itself $\vec{x} \cdot \vec{x} = |\vec{x}|^2$.

$$\sum_{\vec{r}>\vec{q}} \sum_{l=0}^{k} \binom{k}{l} |\vec{r}-\vec{q}|^2 |(\hat{\vec{U}}_{\vec{q}})^{(k-l)}(t)|^2 |(\hat{\vec{U}}_{\vec{r}-\vec{q}})^{(l)}(t)|^2 |(\hat{\vec{U}}_{\vec{q}})^{(k-l)}(t) =$$
$$\sum_{\vec{r}>\vec{q}} \sum_{l=0}^{k} \binom{k}{l} |(\hat{\vec{U}}_{\vec{q}})^{(k-l)}(t)|^2 |(\hat{\vec{U}}_{\vec{q}})^{(k-l)}(t)|^2 |(\hat{\vec{U}}_{\vec{r}-\vec{q}})^{(l)}(t) \cdot (\vec{r}-\vec{q})|^2 = 0, k \in W. \tag{2-34}$$

The final inequality follows by applying the higher order equation of continuity to each term

$$\hat{\vec{U}}_{\vec{r}}^{(l)}(t) \cdot \vec{r} = 0, l = 0,1,2,..., k, t \geq 0, \vec{r} > \vec{0}. \tag{2-35}$$

Similarly

$$\hat{\hat{U}}_{\vec{r}}^{(k)} \cdot \frac{d^k}{dt^k}[\hat{\hat{U}}_{\vec{r}}(t)\vec{r}^{\,t} * \hat{\hat{U}}_{\vec{r}}(t)] \geq$$
$$-\sum_{\vec{r}>\vec{q}}\sum_{l=0}^{k}\binom{k}{l}|(\hat{\hat{U}}_{\vec{q}})^{(k-l)}(t)|^2|(\hat{\hat{U}}_{\vec{q}})^{(k-l)}(t)|^2|(\hat{\hat{U}}_{\vec{r}-\vec{q}})^{(l)}(t)\cdot(\vec{r}-\vec{q})|^2 = 0, k \in N. \tag{2-36}$$

Hence

$$\hat{\hat{U}}_{\vec{r}}^{(k)} \cdot \frac{d^k}{dt^k}[\hat{\hat{U}}_{\vec{r}}(t)\vec{r}^{\,t} * \hat{\hat{U}}_{\vec{r}}(t)] = 0, t \geq 0, \vec{r} > \vec{0}, k \in N\,. \tag{2-37}$$

END PROOF

The notation $\vec{r} > \vec{0}$ indicates that all discrete frequency indices are non negative integers and not all indices are 0.

The domain of definition of solutions of (2-13) can be extended by showing that they and all of their finite time derivatives are bounded and continuous for all forward time. A frequency domain formula for the total mechanical energy of the average velocity and any time derivative of it also appears. This is the frequency domain analog of the extension of Leray's energy law augmented by the body force.

In theorem 2-2 I assume that the velocity vector fields of interest are those whose time integral of the potential energy of order k is bounded below (this suffices because the potential energy is a decreasing function of time). I also assume that the time integral of the work done by the body force on the incompressible fluid converges.

Theorem 2-2. If

$$-4\pi^2\eta\int_0^t\sum_{\vec{r}>\vec{0}}|\vec{r}|^2|\hat{\hat{U}}^{(k)}(s,\vec{r})|^2 ds > -\infty, \int_0^t\sum_{\vec{r}>\vec{0}}\hat{\hat{U}}^{(k)}(s,\vec{r})\cdot\hat{\hat{F}}^{(k)} ds = \hat{M}_k(t),$$
$$t \geq 0, \eta > 0, k = 0,1,2,... \tag{2-38}$$

then

a. the following formulas for the total mechanical energy are well defined

$$\sum_{\vec r>\vec 0}|\hat{\vec U}(t,\vec r)|^2-\sum_{\vec r>\vec 0}|\hat{\vec U}(0,\vec r)|^2=-4\pi^2\eta\int_0^t\sum_{\vec r>\vec 0}|\vec r|^2|\hat{\vec U}(t,\vec r)|^2\ ds$$
$$+\int_0^t\sum_{\vec r>\vec 0}\hat{\vec U}(t,\vec r)\cdot\hat{\vec F}(t,\vec r)ds,$$
$$\sum_{\vec r>\vec 0}|\frac{\partial^k\hat{\vec U}(t,\vec r)}{\partial t^k}|^2=-4\pi^2\eta\int_0^t\sum_{\vec r>\vec 0}|\vec r|^2|\frac{\partial^k\hat{\vec U}(t,\vec r)}{\partial t^k}|^2\ ds \tag{2-39}$$
$$+\int_0^t\sum_{\vec r>\vec 0}\frac{\partial^k\hat{\vec U}^t(s,\vec r)}{\partial s^k}\cdot\frac{\partial^k\hat{\vec F}^t(s,\vec r)}{\partial s^k}ds,k=1,2,3,...,t\geq 0.$$

b. The solution of (2-1)/ (2-13a) and every finite time derivative $\hat{\vec U}^{(k)}(t,\vec r),k=0,1,2,...,t\geq 0$ of it is unique, continuous, and bounded in t for all forward time, square summable in $\vec r\in N^3$ for all forward time and jointly bounded in $(t,\vec r)\in[0,\infty)\times N^3$.

PROOF

a. First note that, by continuity, the hypothesis implies $\int_D|\vec U_0^{(k)}|^2(\vec x)d\vec x>0,k=0,1,2,...$ hence

$$\sum_{\vec r>\vec 0}|\hat{\vec U}_0^{(k)}(\vec r)|^2>0,k=0,1,2,... \quad . \tag{2-40}$$

By Parseval's theorem

$$\int_D|\vec U_0^{(k)}|^2(\vec x)d\vec x<\infty\Leftrightarrow\sum_{\vec r>\vec 0}|\hat{\vec U}_0^{(k)}(\vec r)|^2<\infty,k=0,1,2,... \tag{2-41}$$

Construct formulas for the average energy $\vec U^{(k)}$ in terms of $\hat{\vec U}^{(k)}(t,\vec r),k=0,1,2,...$ Then apply proposition 2-2 to eliminate the $\hat{\vec U}^{(k)}(t,\vec r)$ dotted time derivative of the Euler terms to simplify them. Form the dot product of each equation (2-13a) with $\hat{\vec U}^{(k)}(t,\vec r)$, sum over $\vec r\in W^3-\{\vec 0\}$ and integrate from 0 to t to obtain

$$\sum_{\vec{r}>\vec{0}}|\hat{\vec{U}}(t,\vec{r})|^2-\sum_{\vec{r}>\vec{0}}|\hat{\vec{U}}(0,\vec{r})|^2=-4\pi^2\eta\int_0^t\sum_{\vec{r}>\vec{0}}|\vec{r}\,|^2|\hat{\vec{U}}(t,\vec{r})\,|^2\ ds$$
$$+\int_0^t\sum_{\vec{r}>\vec{0}}\hat{\vec{U}}(t,\vec{r})\cdot\hat{\vec{F}}(t,\vec{r})ds,$$
$$\sum_{\vec{r}>\vec{0}}|\hat{\vec{U}}^{(k)}(t,\vec{r})|^2=-4\pi^2\eta\int_0^t\sum_{r>\vec{0}}|\vec{r}\,|^2|\hat{\vec{U}}^{(k)}\,|^2\ ds+ \tag{2-42}$$
$$2\pi\int_0^t\sum_{\vec{r}\geq\vec{q}>\vec{0}}\sum_{\vec{q}>\vec{0}}\frac{\partial^k}{\partial t^k}\{\hat{\vec{U}}^{(k)}(s,\vec{r}-\vec{q})\cdot(\vec{r}-\vec{q}))(\hat{\vec{U}}(s,\vec{r})\cdot\hat{\vec{U}}^{(k)}(s,\vec{q})\}ds+$$
$$\int_0^t\sum_{\vec{r}>\vec{0}}\hat{\vec{U}}^{(k)}(s,\vec{r})\cdot\hat{\vec{F}}^{(k)}(s,\vec{r})ds, k=1,2,3,...$$

By proposition 2-2 the $\hat{\vec{U}}$ dotted transformed Euler term summed over $\vec{r}>\vec{0}$ satisfies

$$2\pi\sum_{\vec{r}\geq\vec{q}>\vec{0}}\sum_{\vec{q}>\vec{0}}\hat{\vec{U}}(s,\vec{r})\cdot(\hat{\vec{U}}(s,\vec{r}-\vec{q})\cdot\vec{q})\hat{\vec{U}}(s,\vec{q})=0. \tag{2-43}$$

By proposition 2-2 the $\hat{\vec{U}}^{(k)}$ dotted transformed Euler term of the k$^{\text{th}}$ order Navier-Stokes ordinary differential equations, likewise summed over $\vec{r}\in W^3$ satisfies

$$2\pi\sum_{\vec{r}\geq\vec{q}>\vec{0}}\sum_{\vec{q}>\vec{0}}[\hat{\vec{U}}(s,\vec{r})]^{(k)}\cdot[(\hat{\vec{U}}(s,\vec{r}-\vec{q})\cdot\vec{r}-\vec{q})\hat{\vec{U}}(s,\vec{q})]^{(k)}=0. \tag{2-44}$$

It follows by (2-38) that the space average of the kinetic energy is bounded for all forward time. Thus all terms appearing in (2-39) are well defined for all forward time.

b. If a series converges absolutely (for all $t\geq 0$) then each term is finite. It converges uniformly with respect to *t* treated as a parameter in its full range.

$$\frac{1}{2}\sum_{\vec{r}>\vec{0}}|\hat{\vec{U}}(t,\vec{r})\,|^2<\infty, t\in[0,\infty)\Rightarrow$$
$$\sup_{t\geq 0}|\hat{\vec{U}}(t,\vec{r})\,|^2<\infty, t\in[0,\infty),\vec{r}\in W^3-\{\vec{0}\}\Rightarrow \tag{2-45}$$
$$\sup_{t\geq 0}|\hat{\vec{U}}(t,\vec{r})\,|<\infty, t\in[0,\infty),\vec{r}\in W^3-\{\vec{0}\}$$
$$\hat{\vec{U}}\in\vec{L}^\infty[0,\infty),\vec{r}\in W^3-\{\vec{0}\}$$

By the fundamental (extension) theory of ordinary differential equations it follows that for each $\vec{r} \in N^3$, $\hat{\vec{U}}^{(k)}(t,\vec{r})$ is unique and continuous in *t* for all forward time uniformly with respect to $\vec{r} \in N^3$.

END PROOF

The following lemma establishes a link between the inequalities (2-18) in the hypothesis of theorem 2-2 and inequality (5) and its higher order analogs.

Lemma 2-3. If $\vec{U}^{(k)}, k = 0,1,2,...$ satisfy the Navier-Stokes partial differential equations such that

$$\vec{U}^{(k)} \cdot \vec{F}^{(k)}, \nabla \vec{U}^{(k)} \in L_2[0,1]^3, k = 0,1,2,..., t \geq 0 \qquad (2\text{-}46)$$

such that the velocity, the body force and any finite order time derivatives of them satisfy

$$-\eta \int_0^t \int_{[0,1]^3} |\vec{U}^{(k)} \cdot \nabla|^2 d\vec{x} ds > -\infty, \int_0^t \int_{[0,1]^3} \vec{U}^{(k)} \cdot \vec{F}^{(k)} d\vec{x} ds = M_k(t) < \infty, t \geq 0, \eta > 0, k = 0,1,2,... \qquad (2\text{-}47)$$

for all forward time if and only if $\hat{\vec{U}}^{(k)} \cdot \hat{\vec{F}}^{(k)}, \nabla \hat{\vec{U}}^{(k)} \in l_2(N^3), k = 0,1,2,..., t \geq 0$ such that the momentum $\hat{\vec{U}}^{(k)}, k = 0,1,2,...$ satisfy the Navier-Stokes ordinary differential equations. Moreover the finite order time derivatives of the momentum and the body force satisfy

$$-4\pi^2 \eta \int_0^t \sum_{\vec{r} > \vec{0}} |\vec{r}|^2 |\hat{\vec{U}}^{(k)}(s,\vec{r})|^2 ds > -\infty, \int_0^t \sum_{\vec{r} > \vec{0}} \hat{\vec{U}}^{(k)}(s,\vec{r}) \cdot \hat{\vec{F}}^{(k)} ds = \hat{M}_k(t), \qquad (2\text{-}48)$$
$$t \geq 0, \eta > 0, k = 0,1,2,...$$

PROOF

By the strict inequality, the integral on the left of (2-47) over R^3 must be finite. In order for the Fourier transforms which appear in (2-48) to be well defined
$\vec{F}^{(k)} \cdot \vec{U}^{(k)}, \nabla \vec{U}^{(k)} \in \vec{L}^2([0,1]^3), k = 0,1,2,..., t \geq 0$ if and only if
$\hat{\vec{U}}^{(k)} \cdot \hat{\vec{F}}^{(k)}, |\vec{\omega}|^2 \hat{\vec{U}}^{(k)} \in \vec{l}^2(W^3), k = 0,1,2,..., t \geq 0.$
Note that, since $\vec{F}^{(k)}, k = 0,1,2,...$ is smooth in $\vec{x} \in [0,1]^3$, $\vec{U}^{(k)} \cdot \vec{F}^{(k)}$ is integrable by *Hölder's* inequality if only $\vec{U}^{(k)} \in \vec{L}^1([0,1]^3)$.

Now suppose

$$-\eta \int_0^t \int_{[0,1]^3} |\vec{U}^{(k)} \cdot \nabla|^2 d\vec{x} ds > -\infty, \int_0^t \int_{[0,1]^3} \vec{U}^{(k)} \cdot \vec{F}^{(k)} d\vec{x} ds = M_k(t) < \infty, t \geq 0, \eta > 0, k = 0,1,2,... \qquad (2\text{-}49)$$

By the definition of the discrete Fourier transform (i.e. the Fourier coefficient) the previous inequality is equivalent to

$$\int_0^t \sum_{\vec{r}\in N^3} \int_{[0,1]^3} \vec{U}^{(k)}(s,\vec{x})\cdot \vec{F}^{(k)}(s,\vec{x}) \,|\times|\, \sin 2\pi r_1 x \sin 2\pi r_2 y \sin 2\pi r_3 z|^2 d\vec{x}ds <$$
$$4\eta\pi^2 \int_0^t \sum_{\vec{r}\in N^3} \int_{[0,1]^3} |\vec{U}^{(k)}(s,\vec{x})\cdot\nabla|^2 |\sin 2\pi r_1 x \sin 2\pi r_2 y \sin 2\pi r_3 z|^2 d\vec{x}ds, t\geq 0, k=0,1,2,... \quad (2\text{-}50)$$

Since $0 \leq |\sin 2\pi r_1 x \sin 2\pi r_2 y \sin 2\pi r_3 z|^2 \leq 1, \vec{x}\in R^3, r\in N^3$ (2-50) holds if and only if

$$-4\pi^2\eta \int_0^t \sum_{\vec{r}>\vec{0}} |\vec{r}|^2 |\hat{\vec{U}}^{(k)}(s,\vec{r})|^2 ds > -\infty, \int_0^t \sum_{\vec{r}>\vec{0}} \hat{\vec{U}}^{(k)}(s,\vec{r})\cdot \hat{\vec{F}}^{(k)} ds = \hat{M}_k(t),$$
$$t\geq 0, \eta>0, k=0,1,2,... \quad (2\text{-}51)$$

One can argue that this follows immediately by Parseval's theorem.

Note that $M_k(t) = \hat{M}_k(t)$

END PROOF

Theorem 2-3. If $\hat{\vec{F}}(t,\vec{r}), \hat{\vec{h}}_i, i=1,2,3$, and all time derivatives are discrete Schwartz in $\vec{r}\in N^3$ for all forward time and $\hat{\vec{U}}_0(\vec{r})$ is discrete Schwartz in $\vec{r}\in N^3$ then any finite time derivative of $\hat{\vec{U}}(t,\vec{r})$ (thus $\hat{P}(t,\vec{r})$) (the solution of the equation of lemma 2-2) is discrete Schwartz in $\vec{r}\in N^3$.

PROOF

a. For any fixed t,k $\hat{\vec{U}}^{(k)}$ is discrete Schwartz in $\vec{r}\in N^3$

$$\sup_{\vec{r}\in N^3} |\vec{r}|^p |\hat{\vec{U}}^{(k)}| < \infty, \forall p\in W \quad (2\text{-}52)$$

b. The upper bound for $\hat{\vec{U}}^{(k)}$ is uniform in k

$$\sup_{k\in W} \sup_{\vec{r}\in N^3} |\vec{r}|^p |\hat{\vec{U}}^{(k)}| < \infty, \forall p\in W \quad (2\text{-}53)$$

c. The upper bound for $\hat{\vec{U}}^{(k)}$ is uniform for all $t\geq 0$.

$$\sup_{t\geq 0} \sup_{k\in W} \sup_{\vec{r}\in N^3} |\vec{r}|^p |\hat{\vec{U}}^{(k)}| < \infty, \forall p\in W \quad (2\text{-}54)$$

By the variation of constants formula, it suffices to show that the convolution integral is discrete Schwartz since $\hat{\vec{\Psi}}(\vec{r}), \hat{\vec{\Phi}}(t,\vec{r}) = e^{-\eta 4\pi^2|\vec{r}|^2 t}\hat{\vec{U}}_0$ are discrete Schwartz in $\vec{r} \in N^3$ by inspection given the hypotheses on the initial and boundary conditions.

Multiply the time convolution by any monomial formed by the product of any finite powers of the frequency components

$$
\begin{aligned}
&\sup_{\vec{r}\in N^3} | r_1^{p(1)} r_2^{p(2)} r_3^{p(3)} || \hat{\vec{U}} | \le \\
&\sup_{\vec{r}\in N^3} | r_1^{p(1)} r_2^{p(2)} r_3^{p(3)} || \hat{\vec{\Psi}}(\vec{r}) | + \\
&\sup_{\vec{r}\in N^3} e^{-\eta 4\pi^2|\vec{r}|^2 t} | \hat{\vec{U}}_0 || r_1^{p(1)} r_2^{p(2)} r_3^{p(3)} | + \\
&\sup_{\vec{r}\in N^3} | r_1^{p(1)} r_2^{p(2)} r_3^{p(3)} || \int_0^t e^{-\eta 4\pi^2|\vec{r}|^2 (t-s)} \left[\frac{\vec{r}\vec{r}^{\,t}}{|\vec{r}|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] [2\pi((\hat{\vec{U}}\vec{r}^{\,t}) * \hat{\vec{U}}) - \hat{\vec{F}}] ds | \\
&\eta > 0, \vec{p} \in W, t \ge 0, \vec{r} \in N^3, k = 1,2,3,
\end{aligned}
\tag{2-55}
$$

Simplify the upper bound on the Schwartz weighted momentum coefficient of (2-55)

$$
\begin{aligned}
&\sup_{\vec{r}\in N^3} | r_1^{p(1)} r_2^{p(2)} r_3^{p(3)} || \hat{\vec{U}}^{(k)}(t,\vec{r}) | \le \\
&\sup_{\vec{r}\in N^3} | r_1^{p(1)} r_2^{p(2)} r_3^{p(3)} || (-4\eta)^k | \pi^{2k} | \vec{r} |^{2k} e^{-\eta 4\pi^2|\vec{r}|^2 t} | \hat{\vec{U}}_0 | \\
&+ \sup_{\vec{r}\in N^3} | r_1^{p(1)} r_2^{p(2)} r_3^{p(3)} || \int_0^t e^{-\eta 4\pi^2|\vec{r}|^2 (t-s)} \left[\frac{\vec{r}\vec{r}^{\,t}}{|\vec{r}|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] [(2\pi(\hat{\vec{U}}\vec{r}^{\,t}) * \hat{\vec{U}})^{(k)} - \hat{\vec{F}}^{(k)}] ds | \\
&\eta > 0, \vec{p} \in W, t \ge 0, \vec{r} \in N^3, k = 1,2,3,
\end{aligned}
\tag{2-56}
$$

The Fourier series coefficient of the solution of Laplace's equation is discrete Schwartz

$$
\begin{aligned}
&\vec{h}(\vec{r}) \in \hat{\vec{S}}(N^2) \Rightarrow \hat{\vec{\Psi}}(\vec{r}) \in \hat{\vec{S}}(N^3) \Rightarrow \\
&\forall \vec{p} \in W^3, \sup_{\vec{r}\in N^3} | r_1^{p(1)} r_2^{p(2)} r_3^{p(3)} || \hat{\vec{\Psi}}(\vec{r}) | < M_1.
\end{aligned}
\tag{2-57}
$$

Since $\hat{\vec{U}}_0 \in \hat{\vec{S}}(N^3)$

$$\forall t \geq 0, \vec{p} \in W^3$$

$$\sup_{\vec{r} \in N^3} e^{-\eta 4\pi^2 |\vec{r}|^2 t} \, | \hat{\vec{U}}_0 | \, r_1^{p(1)} r_2^{p(2)} r_3^{p(3)} \leq \sup_{\vec{r} \in N^3} | \hat{\vec{U}}_0 | \, r_1^{p(1)} r_2^{p(2)} r_3^{p(3)} \leq M_2 \tag{2-58}$$

Since

$$\hat{\vec{U}} \in \hat{\vec{S}}(N^3) \Rightarrow (\hat{\vec{U}}\vec{r}^{\,t}) * \hat{\vec{U}} \in \hat{\vec{S}}(N^3), \hat{\vec{F}} \in \hat{\vec{S}}(N^3)$$

and

$$\left| \left[\frac{\vec{r}\vec{r}^{\,t}}{|\vec{r}|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] \right| < 2, r \in N^3.$$

Use the norm on the matrix operator immediately above to get the following upper bound

$$\begin{aligned}
&\sup_{\vec{r} \in N^3} | r_1^{p(1)} r_2^{p(2)} r_3^{p(3)} || \hat{\vec{U}}^{(k)} | \leq \\
&\sup_{\vec{r} \in N^3} | r_1^{p(1)} r_2^{p(2)} r_3^{p(3)} || \int_0^t e^{-\eta 4\pi^2 |\vec{r}|^2 (t-s)} \left[\frac{\vec{r}\vec{r}^{\,t}}{|\vec{r}|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] [2\pi((\hat{\vec{U}}\vec{r}^{\,t}) * \hat{\vec{U}}) - \hat{\vec{F}}] ds | \\
&\leq 2 \int_0^t e^{-\eta 4\pi^2 |\vec{r}|^2 (t-s)} 2\pi \sup_{\vec{r} \in N^3} \sup_{s \geq 0} | r_1^{p(1)} r_2^{p(2)} r_3^{p(3)} | ((\hat{\vec{U}}\vec{r}^{\,t}) * \hat{\vec{U}}) ds | \\
&+ 2 \int_0^t e^{-\eta 4\pi^2 |\vec{r}|^2 (t-s)} \sup_{s \geq 0} | r_1^{p(1)} r_2^{p(2)} r_3^{p(3)} || \hat{\vec{F}} | ds.
\end{aligned} \tag{2-59}$$

The convolution and body force terms in the integrand have upper bounds which are uniform over all time and any order k of the derivative.

$$\begin{aligned}
&\sup_{\vec{r} \in N^3} | r_1^{p(1)} r_2^{p(2)} r_3^{p(3)} || \hat{\vec{U}}^{(k)} | \leq \\
&2 \int_0^t e^{-\eta 4\pi^2 |\vec{r}|^2 (t-s)} [N_1 + N_2] ds \leq \\
&2[N_1 + N_2] \frac{1}{\eta 4\pi^2 |\vec{r}|^2} \leq 2[N_1 + N_2] \frac{1}{\eta 12\pi^2}, |\vec{r}| \geq 3
\end{aligned} \tag{2-60}$$

For the k[th] derivative with respect to time of the classical Fourier coefficient

$$\sup_{\vec r\in N^3} |r_1^{p(1)}r_2^{p(2)}r_3^{p(3)}|| \int_0^t e^{-\eta 4\pi^2|\vec r|^2(t-s)}\left[\frac{\vec r\vec r^{\,t}}{|\vec r|^2}-\begin{pmatrix}1\,0\,0\\0\,1\,0\\0\,0\,1\end{pmatrix}\right][(2\pi(\hat{\vec U}\vec r^{\,t})*\hat{\vec U})^{(k)}-\hat{\vec F}^{(k)}]ds$$

$$\le 2\int_0^t e^{-\eta 4\pi^2|\vec r|^2(t-s)}2\pi \sup_{\vec r\in N^3}\sup_{s\ge 0}|r_1^{p(1)}r_2^{p(2)}r_3^{p(3)}|(2\pi(\hat{\vec U}\vec r^{\,t})*\hat{\vec U})^{(k)}ds|$$

$$+2\int_0^t e^{-\eta 4\pi^2|\vec r|^2(t-s)}\sup_{s\ge 0}|r_1^{p(1)}r_2^{p(2)}r_3^{p(3)}||\hat{\vec F}^{(k)}|ds \le \qquad (2\text{-}61)$$

$$2\int_0^t e^{-\eta 4\pi^2|\vec r|^2(t-s)}[P_1+P_2]ds \le$$

$$2[P_1+P_2]\frac{1}{\eta 4\pi^2|\vec r|^2}\le 2[P_1+P_2]\frac{1}{\eta 12\pi^2},|\vec r|\ge 3.$$

For coefficients in discrete Schwartz spaces the integer weights are unrestricted. Hence any sum of squares can be exceeded by a product which is a single square. Thus the Schwartz norm has two equivalent formulations

$$\sup_{\vec p\in W^3}|\hat{\vec U}_0|r_1^{p(1)}r_2^{p(2)}r_3^{p(3)}=\sup_{p\in W}|\vec r|^p|\hat{\vec U}_0|. \qquad (2\text{-}62)$$

The homogeneous term has a uniform upper bound

$$\forall t\ge 0,\vec p\in W^3$$
$$\sup_{\vec r\in N^3} e^{-\eta 4\pi^2|\vec r|^2 t}|\hat{\vec U}_0|r_1^{p(1)}r_2^{p(2)}r_3^{p(3)}\le \sup_{\vec r\in N^3}|\hat{\vec U}_0|r_1^{p(1)}r_2^{p(2)}r_3^{p(3)}\le M_2 \qquad (2\text{-}63)$$

The pressure function

$$\hat P(t,\vec r)=\frac{1}{2\pi|\vec r|^2}\{\vec r\cdot[2\pi\hat{\vec U}\vec r^{\,t}*\hat{\vec U}-\hat{\vec F}(t,\vec r)]\},t\ge 0,\vec r\in N^3 \qquad (2\text{-}64)$$

is discrete Schwartz in $\vec r$ since $\hat{\vec U},\hat{\vec U}\vec r^{\,t}$ are discrete Schwartz by hypothesis, the convolution of discrete Schwartz coefficient is discrete Schwartz and the sum of discrete Schwartz functions $[\hat{\vec U}\vec r^{\,t}*\hat{\vec U}-\hat{\vec F}(t,\vec r)]$ is discrete Schwartz.

By the same reasoning any finite order time derivative of the pressure transform $\hat P(t,\vec r)$ is discrete Schwartz.

END PROOF

The following theorem interprets the results obtained for the Fourier coefficients $\hat{\vec U},\hat P$ which solve the Navier-Stokes ODE to the Fourier series representation of the functions $\hat{\hat{\vec{U}}},\hat{\hat{P}}$ which solve the Navier-Stokes PDE.

Theorem 2-4. Suppose $\vec{U}^{(k)}\cdot\vec{F}^{(k)},\nabla\vec{U}^{(k)}\in L^2[0,1]^3, k=0,1,2,...,t\geq 0$ where the $\vec{U}^{(k)}, k=0,1,2,...$ satisfy the Navier-Stokes partial differential equations and its finite order time derivatives such that

$$\int_0^t\int_D\vec{U}^{(k)}\cdot\vec{F}^{(k)}d\vec{x}ds<\eta\int_0^t\int_D\|\vec{U}^{(k)}\cdot\nabla\|^2d\vec{x}ds, t\geq 0,\eta>0,k=0,1,2,... \tag{2-65}$$

where $\vec{F}^{(k)}$ is smooth in $(t,\vec{x})$ and bounded in $t\in[0,\infty)$. Then every finite time derivative of the solution $(\hat{\hat{\vec{U}}},\hat{\hat{P}})$ of the Navier-Stokes momentum equation is bounded, continuous and uniquely determined in t –that is to say it is smooth and bounded in $t\in[0,\infty)$ for each fixed $\vec{x}\in[0,1]^3$ It is also smooth in $\vec{x}\in[0,1]^3$ for each fixed $t\in[0,\infty)$.

PROOF

The conclusion of the theorem follows directly by the properties of the Fourier series representation $\hat{\hat{\vec{U}}}$ of the velocity function and theorem 2-2. In particular,

$$\begin{aligned}&\hat{\vec{U}}\in\hat{\vec{S}}(N^3),t\geq 0\Rightarrow\hat{\hat{\vec{U}}}=\vec{U}\in\vec{C}^\infty([0,1]^3),t\geq 0\\&\hat{\vec{U}}\in(\vec{C}^\infty\cap\vec{L}^\infty)([0,\infty)),\vec{r}\in N^3\Rightarrow\hat{\hat{\vec{U}}}=\vec{U}\in(\vec{C}^\infty\cap\vec{L}^\infty)([0,\infty)),\vec{x}\in[0,1]^3\end{aligned}. \tag{2-66}$$

By the projection formula (or by solving the Navier-Stokes momentum equation for ∇P) and the first part of this theorem, the pressure gradient satisfies the same properties as each component of the momentum vector (marginal smoothness in $t,\vec{x}$, uniqueness, and boundedness for all forward time).

END PROOF

The variation of constants formulas for the incompressible Navier Stokes equation on the unit cube for all forward time are

$$\vec{U}(t,\vec{x}) = \vec{\Psi}(\vec{x}) + \vec{\Phi}(t,\vec{x}) +$$

$$\int_0^t \int_D K(t-s,\vec{x},\vec{x}')\{-\vec{U}\cdot\nabla\vec{U} - \nabla P + \vec{F}\}d\vec{x}'ds, \vec{x}\in D, t\geq 0, \eta > 0$$

$$K(t,\vec{x},\vec{x}') = \sum_{l,m,n}^{\infty} e^{-\pi^2\eta(l^2+m^2+n^2)t}\sin\pi lx \sin\pi lx' \sin\pi my \sin\pi my' \sin\pi nz \sin\pi nz'$$

$$= \sum_{l,m,n}^{\infty} e^{-\pi^2\eta(l^2+m^2+n^2)t}\sin\pi lx \sin\pi my \sin\pi nz \int_{[0,1]^3}\delta(\vec{x}'-\vec{\xi})\sin\pi l\xi_1 \sin\pi m\xi_2 \sin\pi n\xi_3 d\vec{\xi} \qquad (2\text{-}67)$$

$$\Phi(t,\vec{x}) = \int_D K(t,\vec{x},\vec{x}')\vec{U}_0(\vec{x}')d\vec{x}'$$

$$\vec{\Psi}(\vec{x}) = \iint_{\partial D}\vec{h}(\vec{y})\frac{\partial G(\vec{y},\vec{x})}{\partial\vec{n}}dS$$

$\vec{\Psi}(\vec{x})$ is the Poisson form with vector Green's function kernel solution of the solution of the Laplacian with opposite face matching boundary conditions and ∂D consists of the faces of the unit cube. The variation of constants formula (2-67) generalizes that in [18] Strauss.

To obtain a second variation of constants formula equivalent to the one above, it is possible to integrate over D to obtain a simplification of formula (2-67) in the domain of sequences of vector Fourier coefficients.

$$\vec{U}(t,\vec{x}) = \vec{\Psi}(\vec{x}) + \vec{\Phi}(t,\vec{x}) +$$

$$\int_0^t K(t-s,\vec{x})\{-\hat{\vec{U}}\vec{r} * \hat{\vec{U}}\}_{l,m.n}(s) - \hat{P}_{l,m,n}(s)\vec{r} + \hat{\vec{F}}_{l.m,n}(s)\}ds, \vec{x}\in D, t\geq 0, \eta > 0$$

$$K(t,\vec{x}) = \sum_{l,m,n}^{\infty} e^{-\pi^2\eta(l^2+m^2+n^2)t}\sin\pi lx \sin\pi my \sin\pi nz \qquad (2\text{-}68)$$

$$\Phi(t,\vec{x}) = \sum_{l,m,n=1}^{\infty}\hat{\vec{U}}_{l,m,n}(0)e^{-\pi^2\eta(l^2+m^2+n^2)t}\sin\pi lx \sin\pi my \sin\pi nz$$

with $\vec{\Psi}$ defined in series form as

$$\vec{\Psi} = \sum_{m,n} \hat{\vec{h}}^1_{m,n} [\frac{\sinh \pi x\sqrt{m^2+n^2} + \sinh \pi(1-x)\sqrt{m^2+n^2}}{\sinh \pi\sqrt{m^2+n^2}}] \sin \pi my \sin \pi nz +$$

$$\sum_{l,n} \hat{\vec{h}}^2_{l,n} [\frac{\sinh \pi y\sqrt{l^2+n^2} + \sinh \pi(1-y)\sqrt{l^2+n^2}}{\sinh \pi\sqrt{l^2+n^2}}] \sin \pi lx \sin \pi nz + \qquad (2\text{-}69)$$

$$\sum_{m,n} \hat{\vec{h}}^3_{l,m} [\frac{\sinh \pi z\sqrt{l^2+m^2} + \sinh \pi(1-z)\sqrt{l^2+m^2}}{\sinh \pi\sqrt{l^2+m^2}}] \sin \pi lx \sin \pi my.$$

In (2-69) a doubly indexed coefficient with an arrow overhead means $\vec{A}_{ij} = (A^1_{i,j}, A^2_{i,j}, A^3_{i,j})^t, i, j = 1,2,3,...$ while $\sum_{m,n}$ denotes $\sum_{m=}^{\infty}\sum_{n=1}^{\infty}$ and similarly for triple series.

Formulas (2-67) and (2-68)/(2-69) are equivalent except for the fact that, in (2-67), the initial value at $t = 0+$ must be established by continuous extension $\lim_{t\to 0+} \vec{U}(t,\vec{x}) = \vec{U}_0$ since the diffusion kernel contains a (removable) singularity at $t = 0+$.

The variation of constants formulas above are used to prove the main result of this paper. Since the space domain is bounded, smoothness at the boundary means smoothness on one side relative to the interior of the unit cube.

Theorem 2-5. If the data is smooth and bounded for all forward time, then the solution $(\vec{U}, P)$ of the incompressible Navier-Stokes equations is jointly smooth on $[0,\infty)\times[0,1]^3$ and bounded for all forward time.

PROOF

It suffices to show that the arbitrary tensor derivatives of any non negative integer order m of the Fourier series representation of the solution are jointly continuous on $[0,\infty)\times[0,1]^3$ and bounded for all forward time.

The first (steady state) term (which solves the Laplacian and satisfies the boundary conditions) is independent of time. Thus it suffices to show that the general tensor of order m is jointly continuous in $\vec{x} \in R^3$.

$$\begin{pmatrix} \sum_{m,n} \vec{h}^1_{m,n} \frac{(\pi\sqrt{m^2+n^2})^{k(1)}}{\sinh \pi\sqrt{m^2+n^2}} (\pi m)^{k(2)} (\pi n)^{k(3)} (\begin{cases} \pm\sin \pi my, k_2 \bmod 4 \in \{0,2\} \\ \pm\cos \pi my, k_2 \bmod 4 \in \{1,3\} \end{cases})(\begin{cases} \pm\sin \pi nz, k_2 \bmod 4 \in \{0,2\} \\ \pm\cos \pi nz, k_2 \bmod 4 \in \{1,3\} \end{cases}) \\ (\begin{cases} \sinh \pi\sqrt{m^2+n^2})x + \sinh \pi\sqrt{m^2+n^2})(1-x), k_1 \bmod 2 = 0 \\ \cosh \pi\sqrt{m^2+n^2})x - \cosh \pi\sqrt{m^2+n^2})(1-x), k_1 \bmod 2 = 1 \end{cases}), \\ \sum_{l,n} \vec{h}^2_{l,n} \frac{(\pi\sqrt{l^2+n^2})^{k(2)}}{\sinh \pi\sqrt{l^2+n^2}} (\pi l)^{k(1)} (\pi n)^{k(3)} (\begin{cases} \pm\sin \pi lx, k_1 \bmod 4 \in \{0,2\} \\ \pm\cos \pi lx, k_1 \bmod 4 \in \{1,3\} \end{cases})(\begin{cases} \pm\sin \pi nz, k_3 \bmod 4 \in \{0,2\} \\ \pm\cos \pi nz, k_3 \bmod 4 \in \{1,3\} \end{cases}) \\ (\begin{cases} \sinh \pi\sqrt{l^2+n^2})y + \sinh \pi\sqrt{l^2+n^2})(1-y), k_2 \bmod 2 = 0 \\ \cosh \pi\sqrt{l^2+n^2})y - \cosh \pi\sqrt{l^2+n^2})(1-y), k_2 \bmod 2 = 1 \end{cases}), \\ \sum_{l,m} \vec{h}^3_{l,m} \frac{(\pi\sqrt{l^2+m^2})^{k(3)}}{\sinh \pi\sqrt{l^2+m^2}} (\pi l)^{k(1)} (\pi m)^{k(2)} (\begin{cases} \pm\sin \pi lx, k_1 \bmod 4 \in \{0,2\} \\ \pm\cos \pi lx, k_1 \bmod 4 \in \{1,3\} \end{cases})(\begin{cases} \pm\sin \pi my, k_3 \bmod 4 \in \{0,2\} \\ \pm\cos \pi my, k_3 \bmod 4 \in \{1,3\} \end{cases}) \\ (\begin{cases} \sinh \pi\sqrt{l^2+m^2})z + \sinh \pi\sqrt{l^2+m^2})(1-z), k_3 \bmod 2 = 0 \\ \cosh \pi\sqrt{l^2+m^2})z - \cosh \pi\sqrt{l^2+m^2})(1-z), k_3 \bmod 2 = 1 \end{cases}) \end{pmatrix} \quad (2\text{-}70)$$

Since $\vec{h}^i_{-,-}$ are discrete Schwartz in two integer indices each the series tensor component converges and the (norm) of the tensor is finite on the boundaries $[0,1]^2$ of the unit cube.

By direct calculation (inspection), the tensor in (2-70) is smooth and bounded in $\vec{x} \in R^3$.

The generic term of the tensor of order m is

$$\frac{\partial^l \partial^{k(1)} \partial^{k(2)} \partial^{k(3)} \Phi(t,\vec{x})}{\partial t^l \partial x^{k(1)} \partial y^{k(2)} \partial z^{k(3)}} = \sum_{l,m,n=1}^{\infty} \hat{\vec{U}}_{l,m,n}(0)(-\eta\pi^2 |\vec{r}|^2)^l e^{-\pi^2\eta(l^2+m^2+n^2)t} (\pi l)^{k(1)} (\pi m)^{k(1)} (\pi m)^{k(3)}$$

$$(\begin{cases} \pm\sin \pi lx, k_1 \bmod 4 \in \{0,2\} \\ \pm\cos \pi lx, k_1 \bmod 4 \in \{1,3\} \end{cases})(\begin{cases} \pm\sin \pi my, k_2 \bmod 4 \in \{0,2\} \\ \pm\cos \pi my, k_2 \bmod 4 \in \{1,3\} \end{cases})(\begin{cases} \pm\sin \pi nz, k_3 \bmod 4 \in \{0,2\} \\ \pm\cos \pi nz, k_3 \bmod 4 \in \{1,3\} \end{cases}) \quad (2\text{-}71)$$

$$\vec{x} = (x,y,z), \vec{r} = (l,m,n)$$

Since $\vec{U}_0(\vec{r})$ is discrete Schwartz in $\vec{r} \in N^3$ the series converges. The homogeneous part of the solution is bounded for all forward time since the series converges and $\sup_{t\geq 0} e^{-\pi^2\eta(l^2+m^2+n^2)t} < 1$. The tensor $\nabla^{l,\vec{k}}_{t,\vec{x}} \Phi(t,\vec{x}), l + |\vec{k}| = m$ is jointly continuous (jointly smooth) in $(t,\vec{x}) \in [0,\infty) \times [0,1]^3$ because $e^{-\pi^2\eta(l^2+m^2+n^2)t} \sin \pi lx \sin \pi my \sin \pi nz$ is jointly smooth in $(t,\vec{x}) \in [0,\infty) \times [0,1]^3$.

To summarize progress to this point, for the steady state and homogeneous part, the proof that the momentum is jointly smooth and bounded on the domain of interest follows automatically by inspection (recognition of known smoothness of exponential, hyperbolic, and trigonometric functions) and the given conditions on the data.

The generic component of the tensor of order *m* is

$$l+|\vec{k}|=m$$

$$\frac{\partial \partial^{k(1)}\partial^{k(2)}\partial^{k(3)}\vec{U}}{\partial t\partial x_1\partial x_2\partial x_3}=$$

$$\sum_{l,m,n}\sum_{p=0}^{l}\binom{l}{p}[e^{-\eta\pi^2|\vec{r}|^2 t}]^{(p)}[\int_0^t e^{\eta\pi^2|\vec{r}|^2 s}\{-\hat{\vec{U}}\vec{r}*\hat{\vec{U}}\}_{l,m,n}(s)-\hat{P}_{l,m,n}(s)\vec{r}+\hat{\vec{F}}_{l,m,n}(s)\}ds]^{l-p}\sin\pi my\sin\pi nz\times$$

$$(\pi l)^{k(1)}\sum_{l,m,n}\{[\int_0^t e^{-\eta\pi^2|\vec{r}|^2(t-s)}\{-\hat{\vec{U}}\vec{r}*\hat{\vec{U}}\}_{l,m,n}(s)-\hat{P}_{l,m,n}(s)\vec{r}+\hat{\vec{F}}_{l,m,n}(s)\}ds(\begin{cases}\pm\sin\pi lx, k_1 \bmod 4\in\{0,2\}\\ \pm\cos\pi lx, k_1 \bmod 4\in\{1,3\}\end{cases})\sin\pi my\sin\pi nz\}\times$$

$$(\pi m)^{k(2)}\sum_{l,m,n}\{[\int_0^t e^{-\eta\pi^2|\vec{r}|^2(t-s)}\{-\hat{\vec{U}}\vec{r}*\hat{\vec{U}}\}_{l,m,n}(s)-\hat{P}_{l,m,n}(s)\vec{r}+\hat{\vec{F}}_{l,m,n}(s)\}ds\sin\pi lx(\begin{cases}\pm\sin\pi my, k_3 \bmod 4\in\{0,2\}\\ \pm\cos\pi my, k_3 \bmod 4\in\{1,3\}\end{cases})\sin\pi nz\}\times$$

$$(\pi n)^{k(3)}\sum_{l,m,n}\{[\int_0^t e^{-\eta\pi^2|\vec{r}|^2(t-s)}\{-\hat{\vec{U}}\vec{r}*\hat{\vec{U}}\}_{l,m,n}(s)-\hat{P}_{l,m,n}(s)\vec{r}+\hat{\vec{F}}_{l,m,n}(s)\}ds\sin\pi lx\sin\pi my(\begin{cases}\pm\sin\pi nz, k_3 \bmod 4\in\{0,2\}\\ \pm\cos\pi nz, k_3 \bmod 4\in\{1,3\}\end{cases})\}$$

(2-72)

Note that

$$[e^{-\eta|\vec{\omega}|^2 t}]^{(p)}=[(-\eta\,|\,\vec{\omega}\,|^2)^p e^{-\eta|\vec{\omega}|^2 t}]$$

$$[\int_0^t e^{\eta|\vec{\omega}|^2 s}\{-j\vec{\omega}\hat{\vec{U}}*\hat{\vec{U}}-j\vec{\omega}\hat{P}+\hat{\vec{F}}\}(s,\vec{\omega})]ds]^{(l-p)}=e^{\eta|\vec{\omega}|^2 t}\{-j\vec{\omega}\hat{\vec{U}}*\hat{\vec{U}}-j\vec{\omega}\hat{P}+\hat{\vec{F}}\}(t,\vec{\omega})]^{(l-p-1)}, \quad (2\text{-}73)$$

$$l\geq 1, 1\leq l-p-1\leq m-1$$

The previous formula shows that strong mathematical induction (theorem 2-3) that all derivatives up to order $0\leq l-p-1\leq m-1$ are continuous and bounded can be applied to prove that derivatives of order m are jointly continuous and bounded. By mathematical induction it follows that general component of the tensor of the inhomogeneous term is jointly smooth and bounded on $[0,\infty)\times[0,1]^3$.

The remaining portion of the solution $P(t,\vec{x})$ is automatically smooth and bounded on $(t,\vec{x}) \in [0,\infty)\times[0,1]^3$ since, in 2-14, its transform is defined in terms of $\vec{U},\vec{F},\vec{h}^i,\vec{U}_0, i=1,2,3$ which have all of those properties.

END PROOF

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