On the Borel-Cantelli Lemma

Alexei Stepanov *, *Izmir University of Economics, Turkey*

In the present note, we propose a new form of the Borel-Cantelli lemma.

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1 Introduction

Suppose $A_1, A_2, \cdots$ is a sequence of events on a common probability space and that A_i^c denotes the complement of event A_i. The Borel-Cantelli lemma (presented below as Lemma 1.1) is used extensively for producing strong limit theorems.

Lemma 1.1. *1. If, for any sequence $A_1, A_2, \cdots$ of events,*

$$\sum_{n=1}^{\infty} P(A_n) < \infty, \tag{1.1}$$

then $P(A_n \text{ i.o.}) = 0$, where i.o. is an abbreviation for "infinitively often";

2. If $A_1, A_2, \cdots$ is a sequence of independent events and if

$$\sum_{n=1}^{\infty} P(A_n) = \infty, \tag{1.2}$$

then $P(A_n \text{ i.o.}) = 1$.

The first part of the Borel-Cantelli lemma is generalized in Barndorff-Nielsen (1961), and Balakrishnan and Stepanov (2010).

*Department of Mathematics, Izmir University of Economics, 35330, Balcova, Izmir, Turkey;

The independence condition in the second part of the Borel-Cantelli lemma is weakened by a number of authors, including Chung and Erdos (1952), Erdos and Renyi (1959), Lamperti (1963), Kochen and Stone (1964), Spitzer (1964), Ortega and Wschebor (1983), and Petrov (2002), (2004). One can also refer to Martikainen and Petrov (1990), and Petrov (1995) for related topics. It should be noted that in all existing publications the sufficient condition in the second part of the Borel-Cantelli lemma is based on equality (1.2) and some additional assumption.

In our work, we prove the second part of the Borel-Cantelli lemma without any additional assumption. This allows us to derive a new and nice form of the Borel-Cantelli lemma.

The rest of this paper is organized as follows. In Section 2 we present our main result. All technical results and their proofs are gathered in Appendix (Section 3).

2 Results

Our main result is the following.

Lemma 2.1. *Let $A_1, A_2, \cdots$ be a sequence of events. Then*

1. $P(A_n \text{ } i.o.) = 0$ *iff (1.1) holds true, and*
2. $P(A_n \text{ } i.o.) = 1$ *iff (1.2) holds true.*

Proof The proof of this lemma consists of three parts.

1. In the first part, we state that

$$(1.2) \quad \Rightarrow P(A_n \text{ } i.o.) = 1.$$

This statement follows from Lemma 2.2.

Lemma 2.2. *Let $A_1, A_2, \ldots$ be a sequence of events for which (1.2) holds true. Then*

$$P(A_n \text{ } i.o.) = 1.$$

2. In the second part, we state that

$$P(A_n \text{ } i.o.) = 1 \quad \Rightarrow \quad (1.2).$$

This statement follows from Proposition 2.1

Proposition 2.1. *Let $A_1, A_2, \ldots$ be a sequence of events such that $P(A_n \text{ } i.o.) = 1$. Then (1.2) holds true.*

3. To conclude the proof of Lemma 2.1, we analyze the above results. By parts **1.**, **2.** of this proof and part 1. of the Borel-Cantelli lemma, we have

$$\begin{cases} P(A_n \ i.o.) = 1 \qquad \Leftrightarrow \quad (1.2), \\ \\ \qquad (1.1) \qquad \Rightarrow \quad P(A_n \ i.o.) = 0. \end{cases}$$

It follows that

$$P(A_n \ i.o.) = 0 \quad \Rightarrow \quad (1.1).$$

Lemma 2.1 is proved. □

3 Appendix

Proof of Lemma 2.2 Observe that

$$P\{A_n \ i.o.\} = \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} A_k\right) \tag{3.1}$$

and

$$1 - P\{A_n \ i.o.\} = \lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty} A_k^c\right). \tag{3.2}$$

To estimate the limit in (3.2) we need the following proposition.

Proposition 3.1. *Let $B_1, B_2, \ldots$ be a sequence of events and $p_i > 1, q_i > 1$ $(i \geq 1)$ two number sequences such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$ $(i \geq 1)$. Then for $n \geq 2$*

$$P(\bigcap_{i=1}^{n} B_i) \leq$$

$$[P(B_1)]^{\frac{1}{p_1}} [P(B_2)]^{\frac{1}{q_1 p_2}} \ldots [P(B_{n-1})]^{\frac{1}{q_1 \cdots q_{n-2} p_{n-1}}} [P(B_n)]^{\frac{1}{q_1 \cdots q_{n-1}}} \tag{3.3}$$

and

$$P(\bigcap_{i=1}^{\infty} B_i) \leq \prod_{i=1}^{\infty} [P(B_i)]^{\frac{1}{q_1 \cdots q_{i-1} p_i}}. \tag{3.4}$$

The proof of Proposition 3.1 will be given after the proof of Lemma 2.2.

By (3.4), we have

$$P\left(\bigcap_{k=n}^{\infty} A_k^c\right) \leq \prod_{k=n}^{\infty} [1 - P(A_k)]^{\frac{1}{q_n \cdots q_{k-1} p_k}} = T_n.$$

Then

$$\log(T_n) = \sum_{k=n}^{\infty} \frac{\log(1 - P(A_k))}{q_n \dots q_{k-1} p_k} \le -\sum_{k=n}^{\infty} \frac{P(A_k)}{q_n \dots q_{k-1} p_k} = -K_n.$$

Our goal now is to find the conditions on the sequences p_i and q_i for $K_n \to \infty$ to be valid. The following auxiliary proposition is well-known and given without proof.

Proposition 3.2. *Let $A_1, A_2, \dots$ be a sequence of events for which (1.2) holds true. Then*

$$\sum_{n=1}^{\infty} \left[\frac{P(A_n)}{\sum_{i=1}^{n} P(A_i)} \right] = \infty. \tag{3.5}$$

Let $S_n = \sum_{i=1}^{n} P(A_i)$ and L_n be the 'tail' of the series in (3.5), i.e. $L_n = \sum_{k=n}^{\infty} \left[\frac{P(A_k)}{S_k} \right]$. Suppose now that $K_n = L_n$, and all the terms in K_n and L_n are equal. Then we get the system of equations

$$\begin{cases} p_n = S_n \\ q_n p_{n+1} = S_{n+1} \\ q_n q_{n+1} p_{n+2} = S_{n+2} \\ \dots, \end{cases} \tag{3.6}$$

where p_i, q_i are unknown variables and S_i are known values. Choose n such that $S_n > 1$. The solution of (3.6) is given by

$$\begin{cases} p_n = S_n \\ q_n = \frac{S_n}{S_n - 1} \\ \\ p_{n+j} = \frac{S_{n+j}\left[S_n \dots S_{n+j-1} - \sum_{k=0}^{j-1} \prod_{i=0, i\ne k}^{j-1} S_{n+i}\right]}{S_n \cdots S_{n+j-1}} \quad (j \ge 1) \\ q_{n+j} = \frac{S_{n+j}\left[S_n \dots S_{n+j-1} - \sum_{k=0}^{j-1} \prod_{i=0, i\ne k}^{j-1} S_{n+i}\right]}{S_n \cdots S_{n+j} - \sum_{k=0}^{j} \prod_{i=0, i\ne k}^{j} S_{n+i}} \quad (j \ge 1), \end{cases} \tag{3.7}$$

where $\sum_{k=0}^{0} \prod_{i=0, i\ne k}^{0} S_{n+i} = 1$. Observe that

$$p_{n+j} \sim S_{n+j} \to \infty \quad \text{and} \quad q_{n+j} \to 1 \quad (j \ge 0, n \to \infty).$$

The series in (3.5) is divergent, and $K_n = L_n$. Then $K_n \to \infty$ provided that the sequences p_i and q_i $(i \ge n)$ in K_n are determined by system (3.7). It follows from (3.2) that

$$1 - P\{A_n \ i.o.\} = 0.$$

The last observation concludes the proof of Lemma 2.2. □

Proof of Proposition 3.1 By Holder's inequality, for B_1 and B_2, we have

$$P(B_1 B_2) \le [P(B_1)]^{\frac{1}{p_1}} [P(B_2)]^{\frac{1}{q_1}}.$$

Replacing B_1 and B_2 by C_1 and C_2C_3, respectively, and applying again the Holder inequality, we obtain

$$P(C_1C_2C_3) \leq [P(C_1)]^{\frac{1}{p_1}} [P(C_2)]^{\frac{1}{q_1 p_2}} [P(C_3)]^{\frac{1}{q_1 q_2}}.$$

By this argument one can come to (3.3). Since

$$P(\lim_{n\to\infty} \bigcap_{i=1}^{n} B_i) = \lim_{n\to\infty} P(\bigcap_{i=1}^{n} B_i),$$

inequality (3.4) can be derived from (3.3). □

Proof of Proposition 2.1 It follows from (3.1) that

$$P(A_n \ i.o.) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} P(A_k). \tag{3.8}$$

The series in (3.8) can not be convergent under the condition $P(A_n \ i.o.) = 1$. Otherwise, we would obtain a contradiction, because it would give us $\lim_{n\to\infty} \sum_{k=n}^{\infty} P(A_k) = 0$. □

We expect this work will be published soon in a statistical journal.

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