ZEROES OF THE SPECTRAL DENSITY OF THE PERIODIC SCHRÖDINGER OPERATOR WITH WIGNER-VON NEUMANN POTENTIAL

SERGEY NABOKO AND SERGEY SIMONOV

Abstract. We consider the Schrödinger operator $\mathcal{L}_\alpha$ on the half-line with a periodic background potential and the Wigner-von Neumann potential of Coulomb type: $\frac{c\sin(2\omega x+\delta)}{x+1}$. It is known that the continuous spectrum of the operator $\mathcal{L}_\alpha$ has the same band-gap structure as the free periodic operator, whereas in each band of the absolutely continuous spectrum there exist two points (so-called critical or resonance) where the operator $\mathcal{L}_\alpha$ has a subordinate solution, which can be either an eigenvalue or a "half-bound" state. The phenomenon of an embedded eigenvalue is unstable under the change of the boundary condition as well as under the local change of the potential, in other words, it is not generic. We prove that in the general case the spectral density of the operator $\mathcal{L}_\alpha$ has power-like zeroes at critical points (i.e., the absolutely continuous spectrum has pseudogaps). This phenomenon is stable in the above-mentioned sense.

1. Introduction

It is well-known that the Schrödinger operator on the half-line with a summable potential has purely absolutely continuous spectrum on $\mathbb{R}_+$ [28]. It is also known [12] that if the potential is square summable, then the absolutely continuous spectrum still covers the positive half-line, see also the work [17]. In this case however there is no guarantee that the positive singular spectrum is empty. Moreover, there are explicit examples of potentials [25, 27], which produce dense point spectrum on $\mathbb{R}_+$. In a sense, the simplest example of the potential producing positive eigenvalues is the so-called Wigner-von Neumann potential $\frac{c\sin(2\omega x)}{x}$. The self-adjoint operator $\mathcal{L}_{0,\alpha}$ given by the differential expression $l_0 := -\frac{d^2}{dx^2} + \frac{c\sin(2\omega x)}{x}$ and the boundary condition $\psi(0)\cos\alpha - \psi'(0)\sin\alpha = 0$, acting in $L_2(\mathbb{R}_+)$ on the domain

$$\operatorname{Dom}\mathcal{L}_{0,\alpha} = \left\{\psi \in L_2(\mathbb{R}_+) \cap H^2_{loc}(\mathbb{R}_+) : l_0\psi \in L_2(\mathbb{R}_+), \psi(0)\cos\alpha - \psi'(0)\sin\alpha = 0\right\}$$

may have an eigenvalue at the point ω^2 [30], while the rest of $\mathbb{R}_+$ is covered by the purely absolutely continuous spectrum [2]. Due to the subordinacy theory [13], this can be seen from the behavior of generalized eigenvectors (i.e., the solutions of the spectral equation $l_0\psi = \lambda\psi$): for $\lambda \in \mathbb{R}_+\backslash\{\omega^2\}$ there is a base of them with asymptotics $e^{\pm i\sqrt{\lambda}x} + o(1)$ as $x \to +\infty$. For $\lambda = \omega^2$ the base is different: $x^{\frac{c}{4\omega}}(\sin(\omega x) + o(1))$ and $x^{-\frac{c}{4\omega}}(\cos(\omega x) + o(1))$. This point is called the resonance (or critical) point due to the change of the type of asymptotics of generalized eigenvectors. One of the solutions (up to a constant) of the equation $l_0\psi = \omega^2\psi$ is subordinate. If on top of that it belongs to $L_2(\mathbb{R}_+)$ and satisfies the boundary condition at the origin, then it represents an eigenfunction of $\mathcal{L}_{0,\alpha}$. It is clear that the effect of appearance of positive eigenvalue is highly unstable. It happens with probability one in a suitable sense: the eigenvalue disappears if one changes slightly

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the boundary condition or adds a summable (or even compactly supported) perturbation to the potential. If the notion of resonances makes sense for the operator (i.e., if the kernel of the resolvent admits the analytic continuation through the continuous spectrum), this eigenvalue may become a resonance under such perturbation. The set of summable functions which one could utilize as perturbations of the potential preserving the eigenvalue at the point ω^2 was studied in [10]. On the other hand, the phenomenon of the change of the type of asymptotics of generalized eigenvectors at a point of the absolutely continuous spectrum is stable under summable perturbations of the potential and does not depend on the boundary condition. It is therefore meaningful to consider objects which are a more or less stable in this sense: one can study the Weyl-Titchmarsh function m or the spectral density ρ' (the derivative of the spectral function of the operator), which are related by the equality

$$\rho'(\lambda) = \frac{1}{\pi} \operatorname{Im} m(\lambda + i0) \text{ for a.a. } \lambda \in \mathbb{R}.$$

The behavior of the Weyl function near resonance points has been studied by Hinton-Klaus-Shaw [15, 18] and Behncke [2, 3, 4] for Schrödinger and Dirac operators with potentials of, in particular, Wigner-von Neumann type.

In the present paper we consider the differential Schrödinger operator $\mathcal{L}_\alpha$ with the potential that is the sum of the following three parts: a periodic background q, a potential of Wigner-von Neumann type $\frac{c\sin(2\omega x+\delta)}{x+1}$ and a summable part q_1. The operator is defined by the differential expression

$$l := -\frac{d^2}{dx^2} + q(x) + \frac{c\sin(2\omega x + \delta)}{x+1} + q_1(x) \tag{1}$$

and the boundary condition $\psi(0)\cos\alpha - \psi'(0)\sin\alpha = 0$, $\alpha \in [0;\pi)$. It acts in $L_2(\mathbb{R}_+)$ on a suitable domain. We assume that the function q has the period a and is summable over this period: $q \in L_1(0;a)$. The parameters c and δ are real constants. The operator $\mathcal{L}_\alpha$ is then self-adjoint in $L_2(\mathbb{R}_+)$.

The geometry of the spectrum in the case of periodic background differs from the case $q(x) \equiv 0$. It is well-known (see, e.g., [22]) that the absolutely continuous spectrum of the operator given by the expression l on the whole real line coincides with the spectrum of the corresponding periodic operator on the whole real line,

$$\mathcal{L}_{per} = -\frac{d^2}{dx^2} + q(x), \tag{2}$$

i.e., has a band-gap structure:

$$\sigma(\mathcal{L}_{per}) =: \bigcup_{j=0}^{\infty} ([\lambda_{2j};\mu_{2j}] \cup [\mu_{2j+1};\lambda_{2j+1}]),$$

where $\lambda_0 < \mu_0 \le \mu_1 < \lambda_1 \le \lambda_2 < \mu_2 \le \mu_3 < \lambda_3 \le \lambda_4 < ...$ It is clear that the absolutely continuous spectrum of $\mathcal{L}_\alpha$ coincides set-wise with $\sigma(\mathcal{L}_{per})$, although $\sigma(\mathcal{L}_{per})$ has multiplicity two, whereas $\sigma(\mathcal{L}_\alpha)$ is simple; note that $\mathcal{L}_\alpha$ is considered on the half-line. In every band there exist two resonance points $\nu_{j,+}$ and $\nu_{j,-}$. The type of asymptotics of generalized eigenvectors at these points is different from that in other points of the absolutely continuous spectrum. Subordinate solution appears and thus each of the resonance points can be an eigenvalue of the operator $\mathcal{L}_\alpha$ as long as this solution satisfies the boundary condition and belongs to $L_2(\mathbb{R}_+)$. The exact locations of the points $\nu_{j,\pm}$ are determined by the "quantization conditions" [22]

$$k(\nu_{j,+}) = \pi\left(j + 1 - \left\{\frac{a\omega}{\pi}\right\}\right), \; k(\nu_{j,-}) = \pi\left(j + \left\{\frac{a\omega}{\pi}\right\}\right), \; j \ge 0,$$

where $k(\lambda)$ is the quasi-momentum of the periodic operator $\mathcal{L}_{per}$ and $\{\cdot\}$ is the standard fractional part function. The above-mentioned condition $\omega \notin \frac{\pi\mathbb{Z}}{2a}$ guarantees that resonance points firstly do not coincide with the band boundaries and secondly that they do not glue up together.

The main goal of the present paper is to analyze the interplay between the periodic structure and the "singular" Wigner-von Neumann perturbation. The main result is Theorem 2 of Section 5, which essentially tells us that the spectral density of the operator $\mathcal{L}_\alpha$ has power-like zeroes at each of the resonance points. Let $\psi_+(x,\lambda)$ and $\psi_-(x,\lambda)$ be the Bloch solutions of the periodic equation $-\psi''(x)+q(x)\psi(x)=\lambda\psi(x)$. Let $\varphi_\alpha(x,\lambda)$ be the solution of the Cauchy problem

$$l\varphi_\alpha = \lambda\varphi_\alpha,\ \varphi_\alpha(0)=\sin\alpha,\ \varphi'_\alpha(0)=\cos\alpha. \tag{3}$$

Denote by $W\{\psi_+(\lambda),\psi_-(\lambda)\} = W\{\psi_+(\cdot,\lambda),\psi_-(\cdot,\lambda)\}$ the Wronskian of two Bloch solutions.

Theorem. *Let $q_1 \in L_1(\mathbb{R}_+)$, $\omega \notin \frac{\pi\mathbb{Z}}{2a}$, and $\rho'_\alpha(\lambda)$ be the spectral density of the operator $\mathcal{L}_\alpha$. Let the index $j \geq 0$. If α is such that the solution $\varphi_\alpha(x,\nu_{j,+})$ of (3) is not a subordinate one, then there exist two non-zero limits*

$$\lim_{\lambda\to\nu_{j,+}\pm 0} \frac{\rho'_\alpha(\lambda)}{|\lambda-\nu_{j,+}|^{\frac{2|c|}{a|W\{\psi_+(\nu_{j,+}),\psi_-(\nu_{j,+})\}|}\left|\int_0^a \psi_+^2(t,\nu_{j,+})e^{2i\omega t}dt\right|}}.$$

Analogously, if α is such that $\varphi_\alpha(x,\nu_{j,-})$ is not subordinate, then there exist two non-zero limits

$$\lim_{\lambda\to\nu_{j,-}\pm 0} \frac{\rho'_\alpha(\lambda)}{|\lambda-\nu_{j,-}|^{\frac{2|c|}{a|W\{\psi_+(\nu_{j,+}),\psi_-(\nu_{j,+})\}|}\left|\int_0^a \psi_-^2(t,\nu_{j,-})e^{2i\omega t}dt\right|}}.$$

Note that the condition of non-subordinacy of φ_α is indeed satisfied in the generic case. Consider one of the critical points ν_{cr}. There is a unique $\alpha \in [0;\pi)$ for which φ_α is subordinate, denote it by α_{cr}. Denote also $\beta := \frac{|c|}{a|W\{\psi_+(\nu_{cr}),\psi_-(\nu_{cr})\}|}\left|\int_0^a \psi_\pm^2(t,\nu_{cr})e^{2i\omega t}dt\right|$ where the choice of sign coincides with the one in $\nu_{cr} = \nu_{j,\pm}$. It follows from [22] and the analysis below that $\varphi_{\alpha_{cr}}(x,\nu_{cr}) = x^{-\beta}(c_+\psi_+(x,\nu_{cr}) + c_-\psi_-(x,\nu_{cr}) + o(1))$ as $x\to+\infty$ where ψ_+,ψ_- are the Bloch solutions of the periodic equation $-\psi''(x)+q(x)\psi(x)=\lambda\psi(x)$ and c_+, c_- are some constants. Therefore ν_{cr} is an eigenvalue of $\mathcal{L}_{\alpha_{cr}}$ iff $\beta > \frac{1}{2}$. At the same time, for $\beta > \frac{1}{2}$ and $\alpha \neq \alpha_{cr}$ the order of the zero of $\rho'_\alpha(\lambda)$ at the point ν_{cr} is greater than one. This fact is in exact correspondence with the result of Aronszajn-Donoghue [1], by which if the point ν is an eigenvalue of the operator $\mathcal{L}_{\alpha_{cr}}$, then the integral $\int_{\mathbb{R}} \frac{d\rho_\alpha(\lambda)}{(\lambda-\nu)^2}$ is finite for every $\alpha \neq \alpha_{cr}$.

It should be mentioned, that this result in the particular case of zero background periodic potential q follows from the work of Hinton-Klaus-Shaw [15]. They considered the differential Schrödinger operator on the half-line with the potential, which is an infinite sum of terms of Wigner-von Neumann type plus a rapidly decreasing term analogous to our q_1 but decaying faster. They studied the behavior of the Weyl-Titchmarsh function near the critical points in the case $\alpha = \alpha_{cr}$ and $\beta < \frac{1}{2}$, i.e., when the solution φ_α is subordinate but is not an eigenfunction (the so-called half-bound state). The corresponding result for the case $\alpha \neq \alpha_{cr}$ and any $\beta > 0$ was a by-product of their analysis. Nevertheless, the most essential part of the problem in their work is similar to the one in our case. However, our approach is different from that of papers [15, 18], see the detailed discussion in Section 6.

Our goal was to elaborate a more general approach (reduction to a model problem), which would enable us to apply it without any significant changes to different operators.

Zeroes of the spectral density of the Schrödinger operator divide the absolutely continuous spectrum of the operator into independent parts. This phenomenon is called a pseudogap and has a clear physical meaning. Eigenvalues embedded into the continuous spectrum have been observed in experiment [9]. Operators with Wigner-von Neumann potentials attracted attention of many other authors, e.g., [7, 8, 24, 15, 18, 4, 3, 2, 21, 20, 22, 26].

The paper is organized as follows. In Section 2, we state the Weyl-Titchmarsh type formula for the operator $\mathcal{L}_\alpha$ which was proved in [23] and serves as our main tool in the work with the spectral density. In Section 3, we write the spectral equation for the operator $\mathcal{L}_\alpha$ in an equivalent form of a discrete linear system and restate the Weyl-Titchmarsh type formula in terms of the asymptotic behavior of solutions of that system. In Section 4, we study this discrete system (which can be regarded as a model problem). This can be viewed as an independent task. In Section 5, we use the results of the preceding analysis to find the asymptotics of the spectral density of the operator $\mathcal{L}_\alpha$ near the critical points. In Section 6, we make final comments characterizing our method in comparison to the one utilized in the work [15] and give a discussion of a few examples of its applicability.

2. Preliminaries

Spectral properties of second-order differential Schrödinger operators are related to the asymptotic behavior of their generalized eigenvectors [13]. One of illustrations of this relation is the Weyl-Titchmarsh type formula for the spectral density in terms of asymptotic coefficients of the solution to the spectral equation that satisfies the boundary condition. The spectral equation $l\psi = \lambda\psi$ can be considered as a perturbation of the periodic equation $-\psi''(x) + q(x)\psi(x) = \lambda\psi(x)$.

Let us denote by $\partial := \{\lambda_j, \mu_j,\ j \geq 0\}$ the "generalized boundary" of the spectrum of the periodic operator $\mathcal{L}_{per}$ (it can be not exactly the boundary of the set $\sigma(\mathcal{L}_{per})$, because the endpoints of the neighboring bands can coincide). We will use the following result (Weyl-Titchmarsh type formula).

Proposition 1 ([23]). *Let $\omega \notin \frac{\pi\mathbb{Z}}{2a}$ and $q_1 \in L_1(\mathbb{R}_+)$. Then for every fixed $\lambda \in \sigma(\mathcal{L}_{per})\backslash(\partial \cup \{\nu_{j,+}, \nu_{j,-}, j \geq 0\})$ there exists a non-zero constant $A_\alpha(\lambda)$ depending on λ such that*

$$\begin{aligned}\varphi_\alpha(x,\lambda) &= A_\alpha(\lambda)\psi_-(x,\lambda) + \overline{A_\alpha(\lambda)}\psi_+(x,\lambda) + o(1) \text{ as } x \to +\infty,\\ \varphi'_\alpha(x,\lambda) &= A_\alpha(\lambda)\psi'_-(x,\lambda) + \overline{A_\alpha(\lambda)}\psi'_+(x,\lambda) + o(1) \text{ as } x \to +\infty,\end{aligned} \tag{4}$$

and the following equality holds:

$$\rho'_\alpha(\lambda) = \frac{1}{2\pi|W\{\psi_+(\lambda), \psi_-(\lambda)\}|\,|A_\alpha(\lambda)|^2}, \tag{5}$$

where $\rho'_\alpha(\lambda)$ is the spectral density of the operator $\mathcal{L}_\alpha$.

This result is a generalization of the classical Weyl-Titchmarsh (or Kodaira) formula [29, 19]. A variant of this formula for the Schrödinger operator with Wigner-von Neumann potential without the periodic background ($q(x) \equiv 0$) follows from the results of [24, 6]. In the case of discrete Schrödinger operator with Wigner-von Neumann potential an analogous formula is also known, see [11, 16].

3. Discretization

In this section we pass over from the spectral equation $l\psi = \lambda\psi$ to a specially constructed discrete linear system essentially equivalent to $l\psi = \lambda\psi$ and rewrite the Weyl-Titchmarsh type formula expressing the spectral density in terms of solutions of this system.

Step 1. Variation of parameters and discretization: perturbation of the monodromy matrix. First we perform a transformation of the spectral equation to the differential system in order to "get rid" of the periodic background potential (in fact, this simply is a variation of parameters). Consider $\lambda \in \sigma(\mathcal{L}_{per})\backslash\partial$ and equation $l\psi = \lambda\psi$. Define new vector-valued function $\eta(x)$ by the following equality:

$$\begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix} = \begin{pmatrix} \psi_-(x,\lambda) & \psi_+(x,\lambda) \\ \psi'_-(x,\lambda) & \psi'_+(x,\lambda) \end{pmatrix} \eta(x).$$

By a straightforward substitution one has:

$$\eta'(x) = L(x,\lambda)\eta(x), \tag{6}$$

where

$$L(x,\lambda) := \frac{\frac{c\sin(2\omega x+\delta)}{x+1} + q_1(x)}{W\{\psi_+(\lambda),\psi_-(\lambda)\}} \begin{pmatrix} -\psi_+(x,\lambda)\psi_-(x,\lambda) & -\psi_+^2(x,\lambda) \\ \psi_-^2(x,\lambda) & \psi_+(x,\lambda)\psi_-(x,\lambda) \end{pmatrix}.$$

Denote by $\Phi(x_0,x,\lambda)$ the fundamental matrix for the system (6) (i.e., the matrix solution of (6) satisfying $\Psi(x_0,x_0,\lambda) \equiv I$). Denote also by $M_n(\lambda)$ the monodromy matrix corresponding to the shift by the period a of the background potential, $M_n(\lambda) := \Phi(a(n-1), an, \lambda)$. Instead of the solution $\eta(x)$ of the differential system (6) consider the following sequence $\{w_n\}_{n=1}^{\infty}$ of vectors from $\mathbb{C}^2$:

$$w_n := \eta(a(n-1)),$$

which obviously solves the discrete linear system $w_{n+1} = M_n(\lambda)w_n$. In what follows, we deal with this system transforming it to a simpler form. Using the standard perturbation methods (see for example [22]) we can determine the matrix $M_n(\lambda)$ up to a summable (in n) term.

We have passed from the continuous variable x to the discrete variable n. Later, in Section 4 we will return to continuous variables considering a new one, y. There is no strict necessity in such a trick. One could think that the discretization determined by the period a of the background potential q helps to get rid of this potential. However, this is not completely true: this role is taken by the above variation of parameters transformation. One could do simpler: the coefficient matrix of the differential system (6) is the sum of two terms. The first term has a factor $\frac{1}{x+1}$ multiplied by an infinite sum of the exponential terms coming from the Fourier decomposition of the periodic parts of Bloch solutions. Each of these terms can "resonate" (become constant in x) for certain values of λ. This happens exactly at the resonance points $\nu_{j,\pm}$. The second term is a summable matrix-valued function. Non-resonating exponential terms can be eliminated using the uniform Harris-Lutz transformation in a fashion of [23]. This approach would lead to a simple differential model system equivalent to the equation $l\psi = \lambda\psi$. However, we use the approach of the discretization, which brings some extra technical difficulties and complicates the notation. The reason is that we obtain as a result a discrete model system, which can serve wider needs. In particular, it is possible to consider the discrete Schrödinger operator with the discrete Wigner-von Neumann potential which has two critical points on the interval $[-2;2]$ (covered by the absolutely coninuous spectrum of this operator) and to prove that its spectral density has zeroes of the power type at the critical points. See Discussion for the exact formulation of this result. We plan to prove it in a forthcoming paper.

Let us introduce the following notation for uniformly summable sequences. Let $R_n(\lambda)$ be 2×2 matrices for $n\in\mathbb{N}$ and $\lambda\in S$ (where S is an arbitrary set). We write $\{R_n(\lambda)\}_{n=1}^{\infty}\in l^1(S)$, if there exists a sequence of positive numbers $\{r_n\}_{n=1}^{\infty}\in l^1$ such that for every $\lambda\in S$ and $n\in\mathbb{N}$, $\|R_n(\lambda)\|<r_n$. We start with a simple asymptotic formula for the monodromy matrix:

Lemma 1. *For every* $\lambda\in\sigma(\mathcal{L}_{per})\backslash\partial$,

$$M_n(\lambda)=I+\frac{1}{an}\int_{a(n-1)}^{an}\frac{c\sin(2\omega t+\delta)}{W\{\psi_+(\lambda),\psi_-(\lambda)\}}\times\begin{pmatrix}-\psi_+(t,\lambda)\psi_-(t,\lambda) & -\psi_+^2(t,\lambda)\\ \psi_-^2(t,\lambda) & \psi_+(t,\lambda)\psi_-(t,\lambda)\end{pmatrix}dt+R_n^{(1)}(\lambda),$$

where $\{R_n^{(1)}(\lambda)\}_{n=1}^{\infty}\in l^1(K)$ *for every compact set* $K\subset\sigma(\mathcal{L}_{per})\backslash\partial$.

Note that it follows that for every $n\in\mathbb{N}$ the matrix $R_n^{(1)}(\lambda)$ depends continuously on $\lambda\in\sigma(\mathcal{L}_{per})\backslash\partial$.

Proof. We give only a sketch of the proof, because it is rather standard. Fundamental matrix for (6) satisfies the integral equation

$$\Phi(x_0,x,\lambda)=I+\int_{x_0}^{x}L(t,\lambda)\Phi(x_0,t,\lambda)dt. \tag{7}$$

To study $M_n(\lambda)$ put $x_0=a(n-1)$ and consider Volterra integral operator $\mathcal{V}_n(\lambda)$ in Banach space $C([0;a],M^{2,2})$ (of 2×2 matrix functions, with any matrix norm) defined by the rule

$$\mathcal{V}_n(\lambda):u(x)\mapsto\int_0^x L(a(n-1)+t,\lambda)u(t)dt.$$

Using Neumann series and direct estimate of the norm we can show the existence of $(I-\mathcal{V}_n)^{-1}$ and the following estimate for its norm:

$$\|(I-\mathcal{V}_n(\lambda))^{-1}\|\le\exp\left(\int_{a(n-1)}^{an}\|L(t,\lambda)\|\,dt\right). \tag{8}$$

Returning to (7) we can write the matrix equality

$$\Phi(a(n-1),a(n-1)+x,\lambda)=((I-\mathcal{V}_n(\lambda))^{-1}I)(x)=$$
$$=I+(\mathcal{V}_n(\lambda)I)(x)+(\mathcal{V}_n^2(\lambda)(I-\mathcal{V}_n(\lambda))^{-1}I)(x),$$

and so putting $x=a$ we have:

$$M_n(\lambda)=I+\int_{a(n-1)}^{an}L(t,\lambda)dt+\underbrace{(\mathcal{V}_n^2(\lambda)(I-\mathcal{V}_n(\lambda))^{-1}I)(a)}_{=:R_n^{(2)}(\lambda)}.$$

The remainder $R_n^{(2)}(\cdot)$ is continuous in $\sigma(\mathcal{L}_{per})\backslash\partial$ for every n and satisfies the uniform estimate

$$R_n^{(2)}(\lambda)=O\left(\left(\int_{a(n-1)}^{an}\|L(t,\lambda)\|\,dt\right)^2\right)\text{ as }n\to\infty.$$

Fix the compact set $K\subset\sigma(\mathcal{L}_{per})\backslash\partial$. One can always choose Bloch solutions so that their Wronskian is analytic functions and does not vanish on $\sigma(\mathcal{L}_{per})$. From the properties of Bloch solutions it follows that there exists $c_1(K)$ such that for every $\lambda\in K$ and x,

$$\left\|\frac{1}{W\{\psi_+(\lambda),\psi_-(\lambda)\}}\begin{pmatrix}-\psi_+(x,\lambda)\psi_-(x,\lambda) & -\psi_+^2(x,\lambda)\\ \psi_-^2(x,\lambda) & \psi_+(x,\lambda)\psi_-(x,\lambda)\end{pmatrix}\right\|\le c_1(K).$$

A straightforward estimate yields:

$$\int_{a(n-1)}^{an} \|L(t,\lambda)\|\, dt \le c_1(K)\left(|c| \ln\left(\frac{an+1}{an+1-a}\right) + \int_{a(n-1)}^{an} |q_1(t)| dt\right),$$

and thus $\{R_n^{(2)}(\lambda)\}_{n=1}^{\infty} \in l^1(K)$. Now consider the expression for $\int_{a(n-1)}^{an} L(t,\lambda)dt$. Fix the parameter t in the denominator, putting it equal to an (this gives precisely the expression for the integral in the assertion of the lemma). The difference is:

$$(9)\quad \int_{a(n-1)}^{an} \frac{c\sin(2\omega t+\delta)}{W\{\psi_+(\lambda),\psi_-(\lambda)\}}\left(\frac{1}{t+1}-\frac{1}{an}\right)\begin{pmatrix} -\psi_+(t,\lambda)\psi_-(t,\lambda) & -\psi_+^2(t,\lambda) \\ \psi_-^2(t,\lambda) & \psi_+(t,\lambda)\psi_-(t,\lambda)\end{pmatrix} dt,$$

which admits an estimate by $\frac{c_1(K)|c|(a+1)}{n(an+1-a)}$. The total error term $R_n^{(1)}$ is the sum of $R_n^{(2)}$ and the expression (9), and therefore it possesses the required properties. This completes the proof. □

Step 2. Fourier decomposition of Bloch solutions. Rewrite the expression from the second term of the formula for $M_n(\lambda)$ given by Lemma 1,

$$(10)\quad \frac{1}{a}\int_{a(n-1)}^{an} \frac{c\sin(2\omega t+\delta)}{W\{\psi_+(\lambda),\psi_-(\lambda)\}}\begin{pmatrix} -\psi_+(t,\lambda)\psi_-(t,\lambda) & -\psi_+^2(t,\lambda) \\ \psi_-^2(t,\lambda) & \psi_+(t,\lambda)\psi_-(t,\lambda)\end{pmatrix} dt,$$

using Fourier decompositions for periodic parts of Bloch solutions. Let $\{b_l^+(\lambda)\}_{l=-\infty}^{\infty}$, $\{b_l^-(\lambda)\}_{l=-\infty}^{\infty}$ and $\{b_l(\lambda)\}_{l=-\infty}^{\infty}$ be Fourier coefficients defined by identities

$$(11)\quad \psi_+(x,\lambda)\psi_-(x,\lambda) \equiv \sum_{l=-\infty}^{+\infty} b_l(\lambda)e^{2i\pi l\frac{x}{a}},\ \psi_\pm^2(x,\lambda) \equiv \sum_{l=-\infty}^{+\infty} b_l^\pm(\lambda)e^{2i(\pi l\pm k(\lambda))\frac{x}{a}}.$$

These Fourier coefficients and their derivatives with respect to λ are locally uniformly bounded in $\sigma(\mathcal{L}_{per})\backslash\partial$ and obey the locally uniform (on the same set) estimate $O(1/l^2)$ as $l\to\infty$. This fact is rather standard, see, e.g., [23] for the details. Since ψ_+ and ψ_- are complex conjugate on $\sigma(\mathcal{L}_{per})\backslash\partial$, one has:

$$b_l^+(\lambda) = \overline{b_{-l}^-(\lambda)},\ b_l(\lambda) = \overline{b_{-l}(\lambda)} \text{ for every } l\in\mathbb{Z} \text{ and } \lambda\in\sigma(\mathcal{L}_{per})\backslash\partial.$$

Substituting identities (11) into the expression (10) and changing the order of summation and integration (which is possible due to the properties of Fourier coefficients mentioned above), one obtains the following result. Expression (10) equals to $\begin{pmatrix} \beta_n^{(d)}(\lambda) & \beta_n^{(ad)}(\lambda) \\ \overline{\beta_n^{(ad)}}(\lambda) & -\beta_n^{(d)}(\lambda)\end{pmatrix}$, and the monodromy matrix can be written in the form

$$(12)\quad M_n(\lambda) = I + \frac{1}{n}\begin{pmatrix} \beta_n^{(d)}(\lambda) & \beta_n^{(ad)}(\lambda) \\ \overline{\beta_n^{(ad)}}(\lambda) & -\beta_n^{(d)}(\lambda)\end{pmatrix} + R_n^{(1)}(\lambda).$$

Here we have introduced the following notations:

$$(13)\quad \begin{aligned} \beta_0(\lambda) &:= -\tfrac{ce^{i(\delta-a\omega)}}{2iW\{\psi_+(\lambda),\psi_-(\lambda)\}}\sum_{l=-\infty}^{+\infty} b_l(\lambda)\tfrac{\sin(a\omega)}{\pi l+a\omega}, \\ \beta_\pm(\lambda) &:= \mp\tfrac{ce^{i(\delta-(k(\lambda)\pm a\omega))}}{2iW\{\psi_+(\lambda),\psi_-(\lambda)\}}\sum_{l=-\infty}^{+\infty} b_l^+(\lambda)\tfrac{\sin(k(\lambda)\pm a\omega)}{\pi l+k(\lambda)\pm a\omega}, \end{aligned}$$

and

$$(14)\quad \begin{aligned} \beta_n^{(d)}(\lambda) &:= \beta_0(\lambda)e^{2ia\omega n} - \overline{\beta_0(\lambda)}e^{-2ia\omega n}, \\ \beta_n^{(ad)}(\lambda) &:= \beta_+(\lambda)e^{2i(k(\lambda)+a\omega)n} + \beta_-(\lambda)e^{2i(k(\lambda)-a\omega)n}. \end{aligned}$$

Step 3. Elimination of non-resonant terms by Harris-Lutz method. For every fixed $\lambda \in \sigma(\mathcal{L}_{per})\backslash\partial$ the entries of the matrix $\begin{pmatrix} \beta_n^{(d)}(\lambda) & \beta_n^{(ad)}(\lambda) \\ \overline{\beta_n^{(ad)}(\lambda)} & -\beta_n^{(d)}(\lambda) \end{pmatrix}$ are linear combinations of four exponential terms: $e^{2ia\omega n}$, $e^{-2ia\omega n}$, $e^{2i(k(\lambda)+a\omega)n}$ and $e^{2i(k(\lambda)-a\omega)n}$. The first two of them do not depend on λ and since $\omega \notin \frac{\pi\mathbb{Z}}{2a}$ they do oscillate. The third and fourth terms do depend on λ. They oscillate, if $k(\lambda)+a\omega \notin \pi\mathbb{Z}$ and $k(\lambda)-a\omega \notin \pi\mathbb{Z}$, respectively, otherwise they are constant (we call resonance this matching of the quasi-momentum $k(\lambda)$ and the frequency ω). Oscillating exponential terms can be dropped with the help of Harris-Lutz transformation and do not affect the asymptotical type of solutions of the system $w_{n+1} = M_n(\lambda)w_n$. Therefore (see the rigorous proof of this fact below) if λ is not one of the resonance points $\nu_{j,\pm}$, which are defined by conditions

$$k(\nu_{j,+}) := \pi\left(j+1-\left\{\frac{a\omega}{\pi}\right\}\right),\ k(\nu_{j,-}) := \pi\left(j+\left\{\frac{a\omega}{\pi}\right\}\right),\ j \geq 0,$$

then every solution of the system $w_{n+1} = M_n(\lambda)w_n$ has a limit as $n \to \infty$. Note that due to the condition $\omega \notin \frac{\pi\mathbb{Z}}{2a}$, for every $j \geq 0$ resonance points $\nu_{j,+} \neq \nu_{j,-}$ do not coincide with the endpoints of the j-th spectral band.

Pick a spectral band and one of the two critical points in this band, e.g., $\nu_{j,+}$ for some index j. To simplify the notation, we write $\nu_{cr} := \nu_{j,+}$. Consider an open neighborhood U_{cr} of the point ν_{cr} such that it contains neither the critical point $\nu_{j,-}$ nor the endpoints of the band μ_j and λ_j. In what follows we assume that ν_{cr} and U_{cr} are fixed and we drop the indices $j,+$ in most cases. To remove the oscillating terms from the system we need the following elementary lemma belonging to the domain of the mathematical folklore.

Lemma 2. *For every real $\xi \notin 2\pi\mathbb{Z}$ and $n \in \mathbb{N}$,*

$$\left|\sum_{m=n}^{\infty} \frac{e^{im\xi}}{m}\right| \leq \frac{1}{n\left|\sin\frac{\xi}{2}\right|}.$$

Proof.

$$\left|(e^{i\xi}-1)\sum_{m=n}^{\infty}\frac{e^{im\xi}}{m}\right| = \left|\sum_{m=n}^{\infty} e^{i(m+1)\xi}\left(\frac{1}{m}-\frac{1}{m+1}\right)-\frac{e^{in\xi}}{n}\right|$$
$$\leq \sum_{m=n}^{\infty}\left(\frac{1}{m}-\frac{1}{m+1}\right)+\frac{1}{n} = \frac{2}{n}.$$

Since $|e^{i\xi}-1| = 2\left|\sin\frac{\xi}{2}\right|$, this argument completes the proof. □

By a Harris-Lutz transformation uniform in U_{cr} it is possible to eliminate non-resonating exponential terms and to stabilize coefficients at resonating terms (i.e., make them independent of λ). Doing this we can provide for the uniform summability of the remainder. We formulate this argument in the following lemma.

Lemma 3. *There exists a sequence of matrices $\{Q_n^{(1)}(\lambda)\}_{n=1}^{\infty}$ defined in U_{cr} such that $Q_n^{(1)}(\lambda) = O\left(\frac{1}{n}\right)$ as $n \to \infty$ uniformly in U_{cr} and*

$$\exp\Big(-Q_{n+1}^{(1)}(\lambda)\Big)M_n(\lambda)\exp\Big(Q_n^{(1)}(\lambda)\Big)$$
$$= I + \frac{1}{n}\begin{pmatrix} 0 & \beta_+(\nu_{cr})e^{2i(k(\lambda)+a\omega)n} \\ \overline{\beta_+(\nu_{cr})}e^{-2i(k(\lambda)+a\omega)n} & 0 \end{pmatrix} + R_n^{(3)}(\lambda) \qquad (15)$$

with some $\{R_n^{(3)}(\lambda)\}_{n=1}^{\infty} \in l^1(U_{cr})$ such that $R_n^{(3)}(\cdot)$ is continuous in U_{cr} for every n.

Proof. We follow the scheme of [5] and need to take care of the uniformity only. The explicit formula for $Q_n^{(1)}$ is

$$Q_n^{(1)}(\lambda) := -\sum_{m=n}^{\infty} \frac{1}{m} \begin{pmatrix} \beta_m^{(d)}(\lambda) & \beta_m^{(ad)}(\lambda) - \beta_+(\nu_{cr}) e^{2i(k(\lambda)+a\omega)m} \\ \overline{\beta_m^{(ad)}(\lambda) - \beta_+(\nu_{cr}) e^{-2i(k(\lambda)+a\omega)m}} & -\beta_m^{(d)}(\lambda) \end{pmatrix}. \tag{16}$$

Lemma 2 yields immediately the estimates

$$\left| \sum_{m=n}^{\infty} \frac{\beta_m^{(d)}(\lambda)}{m} \right| \le \frac{2|\beta_0(\lambda)|}{n|\sin a\omega|}$$

and

$$\left| \sum_{m=n}^{\infty} \frac{\beta_m^{(ad)}(\lambda) - \beta_+(\nu_{cr}) e^{2i(k(\lambda)+a\omega)}}{m} \right| \le \frac{|\beta_+(\lambda) - \beta_+(\nu_{cr})|}{n|\sin(k(\lambda)+a\omega)|} + \frac{|\beta_-(\lambda)|}{n|\sin(k(\lambda)-a\omega)|}.$$

Functions β_0 and β_- are continuous and the denominators $\sin a\omega$ and $\sin(k(\lambda) - a\omega)$ are separated from zero in U_{cr}. On the other hand, the denominator $\sin(k(\lambda) + a\omega)$ has the only zero at the point ν_{cr}, which is simple and thus is compensated by the zero of the numerator. Therefore the estimate $Q_n^{(1)}(\lambda) = O\left(\frac{1}{n}\right)$ holds and is uniform in U_{cr}. Using the obvious property

$$Q_{n+1}^{(1)}(\lambda) - Q_n^{(1)}(\lambda) = \frac{1}{n} \begin{pmatrix} \beta_n^{(d)}(\lambda) & \beta_n^{(ad)}(\lambda) - \beta_+(\nu_{cr}) e^{2i(k(\lambda)+a\omega)n} \\ \overline{\beta_n^{(ad)}(\lambda) - \beta_+(\nu_{cr}) e^{-2i(k(\lambda)+a\omega)n}} & -\beta_n^{(d)}(\lambda) \end{pmatrix},$$

one obtains:

$$M_n(\lambda) = I + \frac{1}{n} \begin{pmatrix} 0 & \beta_+(\nu_{cr}) e^{2i(k(\lambda)+a\omega)n} \\ \overline{\beta_+(\nu_{cr}) e^{-2i(k(\lambda)+a\omega)n}} & 0 \end{pmatrix} + Q_{n+1}^{(1)}(\lambda) - Q_n^{(1)}(\lambda) + R_n^{(1)}(\lambda).$$

Multiplying by $\exp(Q_n^{(1)}(\lambda))$ from the right; by $\exp(-Q_{n+1}^{(1)}(\lambda))$ form the left, expanding exponents and absorbing the terms of the order $1/n^2$ in the remainder, we have:

$$\begin{aligned} &\exp\left(-Q_{n+1}^{(1)}(\lambda)\right) M_n(\lambda) \exp\left(Q_n^{(1)}(\lambda)\right) \\ &\quad = I + \frac{1}{n} \begin{pmatrix} 0 & \beta_+(\nu_{cr}) e^{2i(k(\lambda)+a\omega)n} \\ \overline{\beta_+(\nu_{cr}) e^{-2i(k(\lambda)+a\omega)n}} & 0 \end{pmatrix} + \underbrace{R_n^{(1)}(\lambda) + O\left(\frac{1}{n^2}\right)}_{=R_n^{(3)}(\lambda)}, \end{aligned}$$

where the estimate $O\left(\frac{1}{n^2}\right)$ is uniform in U_{cr}. It is clear that $R_n^{(3)}(\lambda)$ is continuous in U_{cr} for every n. This completes the proof. $\square$

Step 4. Reduction to the model problem. After the Harris-Lutz transformation $\{w_n\}_{n=1}^{\infty} \mapsto \{\widetilde{w}_n\}_{n=1}^{\infty}$ with $\widetilde{w}_n := \exp(-Q_n^{(1)}(\lambda)) w_n$ the system $w_{n+1} = M_n(\lambda) w_n$ is reduced to the system

$$\widetilde{w}_{n+1} = \left[I + \frac{1}{n} \begin{pmatrix} 0 & \beta_+(\nu_{cr}) e^{2i(k(\lambda)+a\omega)n} \\ \overline{\beta_+(\nu_{cr}) e^{-2i(k(\lambda)+a\omega)n}} & 0 \end{pmatrix} + R_n^{(3)}(\lambda) \right] \widetilde{w}_n.$$

At the critical point it takes the following form:

$$\widetilde{w}_{n+1} = \left[I + \frac{1}{n} \begin{pmatrix} 0 & \beta_+(\nu_{cr}) \\ \overline{\beta_+(\nu_{cr})} & 0 \end{pmatrix} + R_n^{(3)}(\nu_{cr}) \right] \widetilde{w}_n.$$

It is convenient to diagonalize the constant matrix in the second term. One has:

$$\begin{pmatrix} e^{\frac{i}{2}\arg\beta_+(\nu_{cr})} & ie^{\frac{i}{2}\arg\beta_+(\nu_{cr})} \\ e^{-\frac{i}{2}\arg\beta_+(\nu_{cr})} & -ie^{-\frac{i}{2}\arg\beta_+(\nu_{cr})} \end{pmatrix}^{-1} \begin{pmatrix} 0 & \beta_+(\nu_{cr}) \\ \overline{\beta_+(\nu_{cr})} & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} e^{\frac{i}{2}\arg\beta_+(\nu_{cr})} & ie^{\frac{i}{2}\arg\beta_+(\nu_{cr})} \\ e^{-\frac{i}{2}\arg\beta_+(\nu_{cr})} & -ie^{-\frac{i}{2}\arg\beta_+(\nu_{cr})} \end{pmatrix} = |\beta_+(\nu_{cr})| \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Consider the following sequence $\{v_n\}_{n=1}^\infty$ instead of $\{\widetilde{w}_n\}_{n=1}^\infty$:

$$v_n := \begin{pmatrix} e^{\frac{i}{2}\arg\beta_+(\nu_{cr})} & ie^{\frac{i}{2}\arg\beta_+(\nu_{cr})} \\ e^{-\frac{i}{2}\arg\beta_+(\nu_{cr})} & -ie^{-\frac{i}{2}\arg\beta_+(\nu_{cr})} \end{pmatrix}^{-1} \widetilde{w}_n.$$

Due to (15) the system $w_{n+1} = M_n(\lambda)w_n$ is equivalent to:

$$v_{n+1} = \left[I + \frac{\beta}{n} \begin{pmatrix} \cos(2(k(\lambda)+a\omega)n) & \sin(2(k(\lambda)+a\omega)n) \\ \sin(2(k(\lambda)+a\omega)n) & -\cos(2(k(\lambda)+a\omega)n) \end{pmatrix} + R_n^{(4)}(\lambda) \right] v_n, \tag{17}$$

where the sequence $\{R_n^{(4)}(\lambda)\}_{n=1}^\infty \in l^1(U_{cr})$, the functions $R_n^{(4)}(\cdot)$ are continuous in U_{cr} and

$$\beta := |\beta_+(\nu_{cr})| = \frac{|c|}{2a|W\{\psi_+(\nu_{cr}), \psi_-(\nu_{cr})\}|} \left| \int_0^a \psi_+^2(t,\nu_{cr}) e^{2i\omega t} dt \right|.$$

Replace the parameter λ on the set U_{cr} by the new small parameter

$$\varepsilon := 2(k(\lambda) - k(\nu_{cr})) = 2(k(\lambda) + a\omega) - 2\pi \left(j + 1 + \left\lfloor \frac{a\omega}{\pi} \right\rfloor \right), \tag{18}$$

where $\lfloor \cdot \rfloor$ is the standard floor function. The set of values taken by ε is $U := \{2(k(\lambda) - k(\nu_{cr})), \lambda \in U_{cr}\}$. By the property of the quasi-momentum (that $k(\lambda) \in [\pi j; \pi(j+1)]$ in j-th spectral band) and the condition $\omega \notin \frac{\pi\mathbb{Z}}{2a}$, which guarantees that the critical point is in the interior of the spectral band, we have $\overline{U} \subset (-2\pi; 2\pi)$. Denote $R_n(\varepsilon) = R_n^{(4)}(\lambda)$ for λ corresponding to ε according to (18). System (17) then reads:

$$v_{n+1} = \left[I + \frac{\beta}{n} \begin{pmatrix} \cos(\varepsilon n) & \sin(\varepsilon n) \\ \sin(\varepsilon n) & -\cos(\varepsilon n) \end{pmatrix} + R_n(\varepsilon) \right] v_n, \tag{19}$$

The aim of this section is to rewrite the Weyl-Titchmarsh type formula in terms of the solutions of the system (19). Proposition 1 deals with the solution φ_α of the spectral equation for the operator $\mathcal{L}_\alpha$. Combining all the transformations described above and denoting the result by $\Xi : \psi(x) \mapsto \{v_n\}_{n=1}^\infty$, one ends up with the following model image of the initial solution φ_α:

$$v_{\alpha,n}(\varepsilon) = v_{\alpha,n}(2(k(\lambda) - k(\nu_{cr}))) := \begin{pmatrix} e^{\frac{i}{2}\arg\beta_+(\nu_{cr})} & ie^{\frac{i}{2}\arg\beta_+(\nu_{cr})} \\ e^{-\frac{i}{2}\arg\beta_+(\nu_{cr})} & -ie^{-\frac{i}{2}\arg\beta_+(\nu_{cr})} \end{pmatrix}^{-1} \tag{20}$$
$$\times \exp\left(-Q_n^{(1)}(\lambda)\right) \begin{pmatrix} \psi_-(a(n-1),\lambda) & \psi_+(a(n-1),\lambda) \\ \psi'_-(a(n-1),\lambda) & \psi'_+(a(n-1),\lambda) \end{pmatrix}^{-1} \begin{pmatrix} \varphi_\alpha(a(n-1),\lambda) \\ \varphi'_\alpha(a(n-1),\lambda) \end{pmatrix}.$$

This is obviously a solution of the system (19) and it is continuous in U for every n.

Lemma 4. *For every $\varepsilon \in U\backslash\{0\}$ there exists a limit $\lim_{n\to\infty} v_{\alpha,n}(\varepsilon) \neq 0$, which is continuous in $\varepsilon \in U\backslash\{0\}$ as a vector-valued function of ε. The spectral density of $\mathcal{L}_\alpha$ equals*

$$\rho'_\alpha(\lambda) = \frac{2}{\pi|W\{\psi_+(\lambda), \psi_-(\lambda)\}| \left\| \lim_{n\to\infty} v_{\alpha,n}(2(k(\lambda) - k(\nu_{cr}))) \right\|^2} \quad \text{a.e. in } U_{cr}. \tag{21}$$

Proof. A straightforward substitution of the transformation Ξ yields this result. Indeed, asymptotics of the solution φ and of its derivative (4) together with the boundedness of $\begin{pmatrix} \psi_-(x,\lambda) & \psi_+(x,\lambda) \\ \psi_-'(x,\lambda) & \psi_+'(x,\lambda) \end{pmatrix}^{-1}$ imply that there exists the limit

$$\lim_{x\to+\infty} \begin{pmatrix} \psi_-(x,\lambda) & \psi_+(x,\lambda) \\ \psi_-'(x,\lambda) & \psi_+'(x,\lambda) \end{pmatrix}^{-1} \begin{pmatrix} \varphi_\alpha(x,\lambda) \\ \varphi_\alpha'(x,\lambda) \end{pmatrix} = \begin{pmatrix} A_\alpha(\lambda) \\ \overline{A_\alpha}(\lambda) \end{pmatrix}.$$

Furthermore, since $\exp\bigl(-Q_n^{(1)}(\lambda)\bigr) \to I$ as $n\to\infty$, there exists the limit

$$\lim_{n\to\infty} v_{\alpha,n}(\varepsilon) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^{-\frac{i}{2}\arg\beta_+(\nu_{cr})}A_\alpha(\lambda) \\ e^{\frac{i}{2}\arg\beta_+(\nu_{cr})}\overline{A_\alpha}(\lambda) \end{pmatrix}.$$

It follows that $|A_\alpha(\lambda)|^2 = \left\|\lim\limits_{n\to\infty} v_{\alpha,n}(\varepsilon)\right\|^2/4$, and substitution of this to the Weyl-Titchmarsh type formula (5) completes the proof. □

The summary of this section is that the study of the spectral density of $\mathcal{L}_\alpha$ is reduceable to the study of the system (19) and, more precisely, to the study of the behavior of $\lim\limits_{n\to\infty} v_{\alpha,n}(\varepsilon)$ for small ε.

In the general case ($\nu_{cr} = \nu_{j,\pm}$) denote the coefficients β, which may be different at different resonance points, as $\beta_{j,\pm}$. Explicit calculations give: $\beta_{j,\pm} = \left|\frac{c\int_0^a \psi_\pm^2(t,\nu_{j,\pm})e^{2i\omega t}dt}{2aW\{\psi_+(\nu_{j,\pm}),\psi_-(\nu_{j,\pm})\}}\right|$.

Remark 1. *Coefficients $\beta_{j,\pm}$ are not necessarily non-zero, because they are proportional to the Fourier coefficients of $p_+(\cdot,\nu_{j,\pm})$, which might be zero. E.g., consider the case of zero periodic potential, $q(x)\equiv 0$. In this case one can choose the period a arbitrarily and the result is independent of the choice (except for the case $\omega\in\frac{\pi\mathbb{Z}}{a}$). For any fixed a, the half-line is divided into spectral bands with coinciding endpoints, $\left[\left(\frac{\pi j}{a}\right)^2; \left(\frac{\pi(j+1)}{a}\right)^2\right]$, $j\ge 0$. The quasi-momentum is $k(\lambda) = a\sqrt{\lambda}$, the critical points are $\nu_{j,+} = \left(\frac{\pi}{a}\left(j+1-\left\{\frac{a\omega}{\pi}\right\}\right)\right)^2$ and $\nu_{j,-} = \left(\frac{\pi}{a}\left(j+\left\{\frac{a\omega}{\pi}\right\}\right)\right)^2$, $j\ge 0$. Bloch solutions are $\psi_\pm(x,\lambda) = e^{\pm ik(\lambda)\frac{x}{a}} = e^{\pm\sqrt{\lambda}x}$ and their periodic parts $p_+(x,\lambda)\equiv p_-(x,\lambda)\equiv 1$ have only one non-zero Fourier coefficient. Explicit calculation shows that $\beta_{j,+}\equiv 0$ and $\beta_{j,-}=0$ for every $j\ne\lfloor\frac{a\omega}{\pi}\rfloor$. For the single existing resonance point one has: $\nu_{\lfloor\frac{a\omega}{\pi}\rfloor,-} = \omega^2$ and $\beta_{\lfloor\frac{a\omega}{\pi}\rfloor,-} = \frac{|c|}{4\omega}$. This coincides with the classical results on the Wigner-von Neumann potential* [30]*. Our result concerning zeroes of the spectral density in this case is in accordance with the result of Hinton-Klaus-Shaw* [15].

4. Model problem

In this section our aim is to study the dependence on ε, which can be arbitrarily small, of the limits of solutions to the system (19). As we have shown in the previous section, this is equivalent to the study of the behavior of the spectral density of $\mathcal{L}_\alpha$ near critical points.

Let us make few comments on the structure of the coefficient matrix of the model system. One can write it as follows: $I + \frac{\beta}{n}D_{-\varepsilon n}\sigma_3 + R_n(\varepsilon)$, where $D_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$ is the matrix of rotation by the angle φ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (notation for the Pauli matrix) is the reflection matrix. The presence of the latter plays a very important role. With σ_3, the system is elliptic for $\varepsilon\in U\backslash\{0\}$ and hyperbolic for $\varepsilon=0$ (we call the system elliptic, if it has a base of solutions of the same order of magnitude and hyperbolic in the opposite case). Without σ_3, there is no change of type of the system at the point $\varepsilon = 0$, the system is always elliptic. If σ_3 is absent, one can factor out the diagonal term of the first two summands $1 + \beta\frac{\cos(\varepsilon n)}{n}$ and the coefficient matrix reduces to $I + \frac{\beta\sin(\varepsilon n)}{n}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + R_n(\varepsilon)$ with some other uniformly summable sequence $\{R_n(\varepsilon)\}_{n=1}^{\infty}$. The matrix in the second term here is constant and diagonalizable. Its spectrum is purely imaginary, and the Levinson theorem immediately gives the answer (=asymptotics of solutions), which is uniform in $\varepsilon \in U$ including the point $\varepsilon = 0$. We could say that the problem is "scalarized" in this case. The situation is different in our case, since the difference between the two eigenvalues is not pure imaginary: $\sigma\left(I + \frac{\beta}{n}D_{-\varepsilon n}\sigma_3\right) = \left\{1 \pm \frac{\beta}{n}\right\}$. This leads to serious troubles in the analysis and exhibits a new phenomenon.

One may expect that for sufficiently small values of ε the magnitude of solutions is determined mostly by diagonal elements $1 \pm \beta\frac{\cos(\varepsilon n)}{n}$. We want to transform the system in a way such that for every $\varepsilon \in U$ (including $\varepsilon = 0$) the limit of every solution as $n \to \infty$ exists (for the system (19) this is not true: if $\varepsilon = 0$, then one of the solutions grows as n^{β}). To this end, we make the following substitution: $v_n = \exp\left(\beta\int_1^n \frac{\cos(\varepsilon r)}{r}dr\right)u_n$. This leads to the system $u_{n+1} = B_n(\varepsilon)u_n$ with

$$B_n(\varepsilon) := \exp\left(-\beta\int_n^{n+1}\frac{\cos(\varepsilon r)}{r}dr\right)\left[I + \frac{\beta}{n}\begin{pmatrix} \cos(\varepsilon n) & \sin(\varepsilon n) \\ \sin(\varepsilon n) & -\cos(\varepsilon n)\end{pmatrix} + R_n(\varepsilon)\right]. \tag{22}$$

The existence of the limit of any solution of the system $u_{n+1} = B_n(\varepsilon)u_n$ is equivalent to the convergence of the infinite product $\prod_{n=1}^{\infty} B_n(\varepsilon)$ (in fact for $\varepsilon \neq 0$ this follows from Lemma 4). Moreover, most of the statements that we make about the asymptotic behavior of solutions of the system $u_{n+1} = B_n(\varepsilon)u_n$ can be formulated in terms of products of matrices $B_n(\varepsilon)$. In what follows we will choose the way of formulation depending on the convenience of its use. We are going to work with particular solutions determined by fixing their initial values, this is a discrete analogue of the Cauchy problem. To this end, we introduce the following notation: for given $\varepsilon \in U$ and the vector of initial data $f \in \mathbb{C}^2$ define the vector sequence $\{u_n(\varepsilon, f)\}_{n=1}^{\infty}$ by the recurrence relation

$$\begin{aligned} u_1(\varepsilon, f) &:= f, \\ u_{n+1}(\varepsilon, f) &:= B_n(\varepsilon)u_n(\varepsilon, f),\ n \geq 1. \end{aligned} \tag{23}$$

Note that due to the decomposition of the exponential term the matrix $B_n(\varepsilon)$ can be written as

$$B_n(\varepsilon) = I + \frac{\beta}{n}\begin{pmatrix} \cos(\varepsilon n) - \int_n^{n+1}\cos(\varepsilon r)dr & \sin(\varepsilon n) \\ \sin(\varepsilon n) & -\cos(\varepsilon n) - \int_n^{n+1}\cos(\varepsilon r)dr \end{pmatrix} + \widetilde{R}_n(\varepsilon)$$

with a sequence $\{\widetilde{R}_n(\varepsilon)\}_{n=1}^{\infty} \in l^1(U)$. One can rewrite the system $u_{n+1} = B_n(\varepsilon)u_n$ as

$$u_{n+1} - u_n = \left[\frac{\beta}{n}\begin{pmatrix} \cos(\varepsilon n) - \int_n^{n+1}\cos(\varepsilon r)dr & \sin(\varepsilon n) \\ \sin(\varepsilon n) & -\cos(\varepsilon n) - \int_n^{n+1}\cos(\varepsilon r)dr \end{pmatrix} + \widetilde{R}_n(\varepsilon)\right]u_n. \tag{24}$$

The behavior of the solutions can be observed in a scale of the variable $y = n|\varepsilon|$ ("slow variable"). If one puts $z(y, \varepsilon, f) := u_{\lfloor \frac{y}{|\varepsilon|}\rfloor}(\varepsilon, f)$ and divides by $|\varepsilon|$, then (24) becomes approximately

$$\frac{z(y + |\varepsilon|) - z(y)}{|\varepsilon|} \approx \left[\frac{\beta\,\mathrm{sign}\,\varepsilon}{y}\begin{pmatrix} 0 & \sin y \\ \sin y & -2\cos y \end{pmatrix} + \frac{1}{|\varepsilon|}\widetilde{R}_{\lfloor \frac{y}{|\varepsilon|}\rfloor}(\varepsilon)\right]z(y) \tag{25}$$

The remainder $\frac{1}{|\varepsilon|}\widetilde{R}_{\lfloor \frac{y}{|\varepsilon|}\rfloor}(\varepsilon)$ is a step-wise constant matrix-valued function which is compressed in $\frac{1}{|\varepsilon|}$ times in the horizontal scale and stretched in $\frac{1}{|\varepsilon|}$ times in the vertical scale, therefore it concentrates near the origin of the variable y and its L_1 norm is preserved. So we may expect that the remainder term will be absorbed into a new boundary condition of the limit problem. Expression on the left hand side of (25) becomes the derivative as $\varepsilon \to 0$, and one "obtains" the following equation for the limits $h_\pm(y,f) := \lim_{\varepsilon\to\pm0} z(y,\varepsilon,f)$:

$$h'_\pm(y) = \pm\frac{\beta}{y}\begin{pmatrix} 0 & \sin y \\ \sin y & -2\cos y \end{pmatrix} h_\pm(y), \tag{26}$$

cf. (54). The remainder $\frac{1}{|\varepsilon|}\widetilde{R}_{\lfloor \frac{y}{|\varepsilon|}\rfloor}(\varepsilon)$ plays a role only in determining the initial values $h_\pm(0)$ for the solutions of (26). It even suffices to know only $\widetilde{R}_n(0)$ to determine this initial value, under the condition of continuity of $\widetilde{R}_n(\varepsilon)$ for every n. Thus the picture of the whole phenomenon can be described as follows: there exist two scales: "fast", discrete, $n \in \mathbb{N}$, and "slow", continuous, $y \in \mathbb{R}_+$. The system first moves along the first ("fast") scale with $\varepsilon = 0$. The limit of the solution as $n \to \infty$ for $\varepsilon = 0$ serves as initial value for the differential equation in the second ("slow") scale. Our aim in this section is to prove the following result which gives the exact formulation of the above considerations. We prefer to write an integral equation in the slow variable instead of the differential one, because the first has a unique solution while with the second one can have troubles due to the different behavior of the solutions near the origin in different cases depending on the value of β.

Theorem 1. *Assume that functions $R_n(\cdot)$ are continuous in U for every $n \in \mathbb{N}$, the matrices $B_n(\varepsilon)$ are invertible for every $n \in \mathbb{N}$, $\varepsilon \in U$ and the sequence $\{R_n(\varepsilon)\}_{n=1}^\infty \in l^1(U)$. Then for every $y > 0$ and $f \in \mathbb{C}^2$ there exist two limits*

$$h_\pm(y,f) := \lim_{\varepsilon\to\pm0} u_{\lfloor \frac{y}{|\varepsilon|}\rfloor}(\varepsilon,f),$$

which satisfy the following integral equations:

$$h_\pm(y,f) = \lim_{n\to\infty} u_n(0,f) \pm \int_0^y \begin{pmatrix} 0 & 1 \\ \exp\left(-2\beta\int_t^y \frac{\cos s}{s}ds\right) & 0 \end{pmatrix} \frac{\beta\sin t}{t} h_\pm(t,f)dt. \tag{27}$$

Moreover, the following four limits exist and are equal:

$$\lim_{\varepsilon\to\pm0}\lim_{n\to\infty} u_n(\varepsilon,f) = \lim_{y\to+\infty} h_\pm(y,f). \tag{28}$$

Additionally, the linear map $\Theta : f \mapsto \lim_{n\to\infty} u_n(0,f)$ has rank one.

Remark 2. *1. It follows that the linear map $f \mapsto \lim_{\varepsilon\to\pm0}\lim_{n\to\infty} u_n(\varepsilon,f)$ also has rank one.*
2. Note that $\lim_{n\to\infty} u_n(0,f) = \lim_{n\to\infty}\lim_{\varepsilon\to\pm0} u_n(\varepsilon,f)$ and $\lim_{\varepsilon\to\pm0}\lim_{n\to\infty} u_n(\varepsilon,f)$ are the limits of the same expression taken in the different order and that they do not coincide. Our aim is to prove that the second limit of these two exists. It will follow then that the spectral density of $\mathcal{L}_\alpha$ can have zeros at critical points. In fact we can rewrite the second expression as $\lim_{\varepsilon\to\pm0}\lim_{y\to+\infty} u_{\lfloor \frac{y}{|\varepsilon|}\rfloor}(\varepsilon,f)$. Here the convergence of the first limit is uniform in ε (unlike the convergence of $\lim_{n\to\infty} u_n(\varepsilon,f)$), and this makes it possible to change the order of limits as in (28).

Proof. We divide the proof of Theorem 1 into four steps.

Step I. A priori estimate and uniform convergence of the tail for the matrix product. We start with few technical results concerning the system $u_{n+1} = B_n(\varepsilon)u_n$. These include uniform boundedness of its solutions (in both variables, n and ε) and the uniform with respect to ε convergence in the slow variable y.

Lemma 5. *Let $B_n(\varepsilon)$ be given by (22) and $\{R_n(\varepsilon)\}_{n=1}^{\infty} \in l^1(U)$. Then for every $\varepsilon \in U$ the product*

$$\prod_{n=1}^{\infty} B_n(\varepsilon)$$

converges. If matrices $B_n(\varepsilon)$ are invertible for every $n \in \mathbb{N}$ and $\varepsilon \in U$, then for every non-zero ε the product is invertible, while for zero ε it is of rank one.

Proof. This follows from the discrete Levinson theorem [5]. Using the expansion

$$\exp\left(-\beta \int_n^{n+1} \frac{\cos r}{r} dr\right) = 1 - \frac{\beta}{n} \int_n^{n+1} \cos(\varepsilon r) dr + O\left(\frac{1}{n^2}\right) \text{ as } n \to \infty, \tag{29}$$

we rewrite each matrix of the sequence $B_n(\varepsilon)$ in the following form (at places, we drop the argument ε in order to simplify the notation, hoping that this will not lead to any confusion):

$$B_n = I + V_n^{(1)} + R_n^{(5)}, \tag{30}$$

where $\{R_n^{(5)}(\varepsilon)\}_{n=1}^{\infty} \in l^1(U)$ and

$$V_n^{(1)}(\varepsilon) := \frac{\beta}{n} \begin{pmatrix} \cos(\varepsilon n) - \int_n^{n+1} \cos(\varepsilon r) dr & \sin(\varepsilon n) \\ \sin(\varepsilon n) & -\cos(\varepsilon n) - \int_n^{n+1} \cos(\varepsilon r) dr \end{pmatrix}. \tag{31}$$

One has:

$$\int_n^{n+1} \cos(\varepsilon r) dr = \frac{\cos \varepsilon - 1}{\varepsilon} \sin(\varepsilon n) + \frac{\sin \varepsilon}{\varepsilon} \cos(\varepsilon n). \tag{32}$$

For $\varepsilon \neq 0$ the conditions of Theorem 3.1 from [5] are satisfied. The named theorem yields the existence of a base of solutions of the system $u_{n+1} = B_n(\varepsilon) u_n$, which have the asymptotics

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(1) \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1) \text{ as } n \to \infty.$$

This is equivalent to the convergence of the product $\prod_{n=1}^{\infty} B_n(\varepsilon)$ and its invertibility.

For $\varepsilon = 0$ the matrix is reduced to

$$B_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 - \frac{2\beta}{n} \end{pmatrix} + R_n^{(5)},$$

and the base of solutions changes accordingly:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(1) \text{ and } \frac{1}{n^{2\beta}} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1)\right] \text{ as } n \to \infty,$$

which follows from the discrete Levinson theorem [5, Theorem 2.2]. The existence of such a base of solutions is in its turn equivalent to the convergence of the product $\prod_{n=1}^{\infty} B_n(0)$ to a rank one matrix (since the second solution goes to zero as $n \to \infty$). □

Remark 3. *The rank one matrix $\Theta = \prod_{n=1}^{\infty} B_n(0)$ defines the linear map $f \mapsto \lim_{n \to \infty} u_n(0, f)$.*

Asymptotics of solutions of the equation $u_{n+1} = B_n(\varepsilon) u_n$ given above demonstrate the change of the system type. For non-zero ε the system is elliptic (i.e., its solutions have the same rate of growth), while for $\varepsilon = 0$ the system is hyperbolic (i.e., there exists a base of solutions, which have uncomparable magnitudes; this yields the existence of a subordinate solution).

The following lemma states the uniform convergence of the product of matrices $B_n(\varepsilon)$ in the slow scale.

Lemma 6. *Let* $\{R_n(\varepsilon)\}_{n=1}^{\infty} \in l^1(U)$. *Then*

$$\prod_{n>\frac{y}{|\varepsilon|}} B_n(\varepsilon) \to I \text{ as } y \to +\infty \text{ uniformly in } \varepsilon \in U\setminus\{0\}.$$

Proof. The sequence $V_n^{(1)}$ given by (31) has the following property:

$$\left\| \sum_{k\geq n} V_k^{(1)}(\varepsilon) \right\| \leq \frac{4\beta}{n\left|\sin\frac{\varepsilon}{2}\right|} \text{ for every } n \in \mathbb{N} \text{ and } \varepsilon \in U\setminus\{0\}. \tag{33}$$

This easily seen using the equality (32), elementary estimates $\left|\frac{\sin\varepsilon}{\varepsilon}\right| \leq 1$, $\left|\frac{\cos\varepsilon-1}{\varepsilon}\right| \leq 1$ and Lemma 2. It enables to define the sequence

$$Q_n^{(2)} := \sum_{k=n}^{\infty} V_k^{(1)}.$$

Then

$$B_n = I + Q_n^{(2)} - Q_{n+1}^{(2)} + R_n^{(5)}.$$

Following the ideas of [14, 5], we want to consider the Harris-Lutz transformation: $B_n^{(1)} := \left(I - Q_{n+1}^{(2)}\right)^{-1} B_n \left(I - Q_n^{(2)}\right)$. If $n > \frac{c_2}{|\varepsilon|}$ with, say,

$$c_2 := 8\beta \sup_{\varepsilon\in U\setminus\{0\}} \frac{|\varepsilon|}{\left|\sin\frac{\varepsilon}{2}\right|}, \tag{34}$$

then it is easy to see that the estimate $\|Q_n^{(2)}(\varepsilon)\| < \frac{1}{2}$ holds yielding the invertibility of $\left(I - Q_{n+1}^{(2)}\right)$. For such values of n by a straightforward calculation one has:

$$\begin{aligned} B_n^{(1)} &= \left(I - Q_{n+1}^{(2)}\right)^{-1}\left(I + Q_n^{(2)} - Q_{n+1}^{(2)}\right)\left(I - Q_n^{(2)}\right) + \left(I - Q_{n+1}^{(2)}\right)^{-1} R_n^{(5)}\left(I - Q_n^{(2)}\right) \\ &= I + \underbrace{\left(I - Q_{n+1}^{(2)}\right)^{-1}\left(Q_{n+1}^{(2)} - Q_n^{(2)}\right)Q_n^{(2)}}_{=:V_n^{(2)}} + \underbrace{\left(I - Q_{n+1}^{(2)}\right)^{-1} R_n^{(5)}\left(I - Q_n^{(2)}\right)}_{=:R_n^{(6)}}. \end{aligned} \tag{35}$$

Using the trivial bounds $\|V_n^{(2)}\| < 2\|V_n^{(1)}\|\|Q_n^{(2)}\|$, $\|R_n^{(6)}\| < 3\|R_n^{(5)}\|$, $\|V_n^{(1)}\| \leq \frac{2\beta}{n}$ and the bound $\|Q_n^{(2)}(\varepsilon)\| \leq \frac{4\beta}{n\left|\sin\frac{\varepsilon}{2}\right|}$, one has

$$\sum_{n>\frac{y}{|\varepsilon|}} \left\|V_n^{(2)}(\varepsilon) + R_n^{(6)}(\varepsilon)\right\| \leq \sum_{n>\frac{y}{|\varepsilon|}} \frac{16\beta^2|\varepsilon|}{y\left|\sin\frac{\varepsilon}{2}\right|} + 3\sum_{n>\frac{y}{|\varepsilon|}} \left\|R_n^{(5)}(\varepsilon)\right\| \to 0 \text{ as } y \to +\infty$$

uniformly in $\varepsilon \in U\setminus\{0\}$. A rather rough argument repeating the scalar estimates yields:

$$\left\| \left(\prod_{n>\frac{y}{|\varepsilon|}} B_n^{(1)}(\varepsilon) \right) - I \right\| \leq \exp\left(\sum_{n>\frac{y}{|\varepsilon|}} \|V_n^{(2)}(\varepsilon) + R_n^{(6)}(\varepsilon)\| \right),$$

hence the assertion of the lemma holds, if B_n is replaced by $B_n^{(1)}$. Then, coming back to the product of matrices $B_n(\varepsilon)$:

$$\prod_{k>\frac{y}{|\varepsilon|}} B_k(\varepsilon) = \left(I - \lim_{n\to\infty} Q_n^{(2)}(\varepsilon)\right) \left(\prod_{k>\frac{y}{|\varepsilon|}} B_k^{(1)}(\varepsilon) \right) \left(I - Q^{(2)}_{\left\lfloor\frac{y}{|\varepsilon|}\right\rfloor+1}(\varepsilon)\right)^{-1}. \tag{36}$$

Due to the estimate (33), $Q^{(2)}_{\left\lfloor\frac{y}{|\varepsilon|}\right\rfloor}(\varepsilon) \to 0$ as $y \to +\infty$ uniformly in $\varepsilon \in U\setminus\{0\}$. Therefore the convergence to the identity matrix in (36) is uniform. This completes the proof. $\square$

The next lemma completes Step I and proves the uniform boundedness of all partial products of matrices $B_n(\varepsilon)$. This lemma is rather non-trivial and plays an important role in the rest of the proof. It is only due to the choice of the scaling factor in $v_n = \exp\left(\beta\int_1^n \frac{\cos(\varepsilon r)}{r}dr\right) u_n$ that these products are uniformly bounded.

Lemma 7. *Let $\{R_n(\varepsilon)\}_{n=1}^\infty \in l^1(U)$. Then there exists a constant c_3 such that for every $\varepsilon \in U$ and every $n \le \infty$*

$$\left\|\prod_{k=1}^n B_k(\varepsilon)\right\| < c_3. \tag{37}$$

Proof. Using the decomposition of the exponent (29) again one can rewrite the sequence B_n in the following form:

$$B_n(\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-2\beta\int\limits_n^{n+1}\frac{\cos(\varepsilon r)}{r}dr\right)\end{pmatrix} + V_n^{(3)}(\varepsilon) + R_n^{(7)}(\varepsilon)$$

with some $\{R_n^{(7)}(\varepsilon)\}_{n=1}^\infty \in l^1(U)$ and

$$\begin{aligned} V_n^{(3)}(\varepsilon) &:= \frac{\beta}{n}\int_n^{n+1}\cos(\varepsilon r)dr\begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix} + \frac{\beta}{n}\begin{pmatrix}\cos(\varepsilon n) & \sin(\varepsilon n) \\ \sin(\varepsilon n) & -\cos(\varepsilon n)\end{pmatrix} \\ &= \frac{\beta}{n}\left(\cos(\varepsilon n) - \int_n^{n+1}\cos(\varepsilon r)dr\right)\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} + \frac{\beta}{n}\sin(\varepsilon n)\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}. \end{aligned} \tag{38}$$

We are going to perform the variation of parameters in the discrete equation

$$u_{n+1} = \left[\begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-2\beta\int\limits_n^{n+1}\frac{\cos(\varepsilon r)}{r}dr\right)\end{pmatrix} + V_n^{(3)}(\varepsilon) + R_n^{(7)}(\varepsilon)\right] u_n,$$

considering it as a perturbation of the equation

$$u_{n+1} = \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-2\beta\int\limits_n^{n+1}\frac{\cos(\varepsilon r)}{r}dr\right)\end{pmatrix} u_n.$$

This leads to the following:

$$\begin{aligned} u_n(\varepsilon, f) = &\begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-2\beta\int_1^n\frac{\cos(\varepsilon r)}{r}dr\right)\end{pmatrix} f \\ &+ \sum_{k=1}^{n-1}\begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-2\beta\int_{k+1}^n\frac{\cos(\varepsilon r)}{r}dr\right)\end{pmatrix}\left(V_k^{(3)}(\varepsilon) + R_k^{(7)}(\varepsilon)\right) u_k(\varepsilon, f). \end{aligned} \tag{39}$$

Using the Gronwall's lemma and a simple estimate

$$\exp\left(-2\beta\int_t^y\frac{\cos s}{s}ds\right) \le \exp\left(-2\beta\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{\cos s}{s}ds\right) \le 3^{2\beta}, \text{ if } 0 \le t \le y \le \infty, \tag{40}$$

one gets the following bound for the solution.

$$\|u_n(\varepsilon, f)\| \le 3^{2\beta}\exp\left(3^{2\beta}\sum_{k=1}^{n-1}\left\|V_k^{(3)}(\varepsilon) + R_k^{(7)}(\varepsilon)\right\|\right)\|f\|.$$

In other terms,

$$\left\| \prod_{k=1}^{n} B_k(\varepsilon) \right\| \le 3^{3\beta} \exp\left(3^{3\beta} \sum_{k=1}^{n} \left\| V_k^{(3)}(\varepsilon) + R_k^{(7)}(\varepsilon) \right\| \right). \tag{41}$$

Furthermore, the identity (38) leads to the following estimate:

$$\left\| V_n^{(3)}(\varepsilon) \right\| \le \frac{5\beta}{2} |\varepsilon| \text{ for every } n \in \mathbb{N} \text{ and } \varepsilon \in U,$$

which can be easily obtained with the help of the equality (32) and explicit bounds $\left| \frac{\cos\varepsilon - 1}{\varepsilon} \right| \le 1$, $1 - \frac{\sin\varepsilon}{\varepsilon} \le \frac{|\varepsilon|}{2}$. Thus, if $n \le \frac{c_2}{|\varepsilon|}$ (where c_2 is defined by (34)), then

$$\sum_{k=1}^{n} \left\| V_k^{(3)}(\varepsilon) + R^{(7)}(\varepsilon) \right\| \le \frac{5\beta c_2}{2} + \sum_{k=1}^{\infty} \left\| R^{(7)}(\varepsilon) \right\| \tag{42}$$

including the case of $\varepsilon = 0, n = \infty$.

The estimate (41) grows to infinity with n, therefore the tail of the matrix product ought to be considered separately. If $\varepsilon \neq 0$ and $n > \frac{c_2}{|\varepsilon|}$, one obtains from the equality $B_n^{(1)} = I + V_n^{(2)} + R_n^{(6)}$, see (35):

$$\left\| \prod_{k=\left\lfloor \frac{c_2}{|\varepsilon|} \right\rfloor + 1}^{n} B_k^{(1)}(\varepsilon) \right\| < \exp\left(\sum_{k=\left\lfloor \frac{c_2}{|\varepsilon|} \right\rfloor + 1}^{n} \left\| V_k^{(2)}(\varepsilon) + R_k^{(6)}(\varepsilon) \right\| \right).$$

Estimating the exponent in the same way as in the previous lemma one gets:

$$\sum_{k=\left\lfloor \frac{c_2}{|\varepsilon|} \right\rfloor + 1}^{n} \left\| V_k^{(2)}(\varepsilon) + R_k^{(6)}(\varepsilon) \right\| < \frac{16\beta^2 |\varepsilon|}{c_2 \left| \sin\frac{\varepsilon}{2} \right|} + 3 \sum_{k=1}^{\infty} \left\| R_k^{(5)}(\varepsilon) \right\|. \tag{43}$$

Since both expressions (42) and (43) are bounded uniformly with respect to ε, $B_n = \left(I - Q_{n+1}^{(2)}\right) B_n^{(1)} \left(I - Q_n^{(2)}\right)^{-1}$ and $\|Q_n^{(2)}\| < 1/2$, the product $\prod_{k=\left\lfloor \frac{c_2}{|\varepsilon|} \right\rfloor + 1}^{n} B_k(\varepsilon)$ is bounded for $n > \frac{c_2}{|\varepsilon|}$ uniformly with respect to ε. Assertion of the lemma follows. ☐

Step II. Rewriting the system $u_{n+1} = B_n(\varepsilon) u_n$ in the form of a Volterra integral equation in the slow scale. Consider the equation (39),

$$\begin{aligned} u_n(\varepsilon, f) = &\begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-2\beta \int_1^n \frac{\cos(\varepsilon r)}{r} dr \right) \end{pmatrix} f \\ &+ \sum_{k=1}^{n-1} \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-2\beta \int_{k+1}^n \frac{\cos(\varepsilon r)}{r} dr \right) \end{pmatrix} \left(V_k^{(3)}(\varepsilon) + R_k^{(7)}(\varepsilon) \right) u_k(\varepsilon, f). \end{aligned}$$

which is equivalent to the system $u_{n+1} = B_n(\varepsilon) u_n$. On this step we rewrite it in an integral operator form. Fix an $\varepsilon \in U \backslash \{0\}$. Put again $n = \left\lfloor \frac{y}{|\varepsilon|} \right\rfloor$ and divide the sum in (39) into two sums, which contain terms $R^{(7)}$ and $V^{(3)}$, respectively. Then write the second sum of the two as an integral in a new variable τ putting $k = \lfloor \tau \rfloor$. Doing this we get for $y \ge |\varepsilon|$, since a piece-wise constant function appears in the second sum:

$$(44)\quad u_{\left\lfloor \frac{y}{|\varepsilon|}\right\rfloor}(\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-2\beta \int_1^{\left\lfloor \frac{y}{|\varepsilon|}\right\rfloor} \frac{\cos(\varepsilon r)}{r} dr\right) \end{pmatrix} f$$
$$+ \sum_{k=1}^{\left\lfloor \frac{y}{|\varepsilon|}\right\rfloor - 1} \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-2\beta \int_{k+1}^{\left\lfloor \frac{y}{|\varepsilon|}\right\rfloor} \frac{\cos(\varepsilon r)}{r} dr\right) \end{pmatrix} R_k^{(7)}(\varepsilon) u_k(\varepsilon)$$
$$+ \int_1^{\left\lfloor \frac{y}{|\varepsilon|}\right\rfloor} \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-2\beta \int_{\lfloor \tau \rfloor + 1}^{\left\lfloor \frac{y}{|\varepsilon|}\right\rfloor} \frac{\cos(\varepsilon r)}{r} dr\right) \end{pmatrix} V_{\lfloor \tau \rfloor}^{(3)}(\varepsilon) u_{\lfloor \tau \rfloor}(\varepsilon) d\tau.$$

Scaling the variable of the integration $\tau = \frac{t}{|\varepsilon|}$ we write this as a Volterra integral equation:

$$(45)\qquad z(y) = g(y) + \int_0^y K(y,t) z(t) dt.$$

Here we denote by z, g and K the following piecewise-constant (on intervals of length ε) functions:

$$z(y, \varepsilon, f) := u_{\left\lfloor \frac{y}{|\varepsilon|}\right\rfloor}(\varepsilon, f), \text{ if } y \geq |\varepsilon|,$$

$$(46)\quad g(y, \varepsilon, f) := \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-2\beta \int_{|\varepsilon|}^{|\varepsilon|\left\lfloor \frac{y}{|\varepsilon|}\right\rfloor} \frac{\cos s}{s} ds\right) \end{pmatrix} f$$
$$+ \sum_{k=1}^{\left\lfloor \frac{y}{|\varepsilon|}\right\rfloor - 1} \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-2\beta \int_{k+1}^{\left\lfloor \frac{y}{|\varepsilon|}\right\rfloor} \frac{\cos(\varepsilon r)}{r} dr\right) \end{pmatrix} R_k^{(7)}(\varepsilon) u_k(\varepsilon, f), \text{ if } y \geq |\varepsilon|,$$

$$K(y, t, \varepsilon) := \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-2\beta \int_{|\varepsilon|\left(\left\lfloor \frac{t}{|\varepsilon|}\right\rfloor + 1\right)}^{|\varepsilon|\left\lfloor \frac{y}{|\varepsilon|}\right\rfloor} \frac{\cos s}{s} ds\right) \end{pmatrix} \frac{V_{\left\lfloor \frac{t}{|\varepsilon|}\right\rfloor}^{(3)}(\varepsilon)}{|\varepsilon|}, \text{ if } |\varepsilon| \leq t < |\varepsilon| \left\lfloor \frac{y}{|\varepsilon|} \right\rfloor.$$

For all the other values of y (and t, if it is present) define these functions to be equal zero. After we have done this, we can successfully use standard operator methods. Before doing that let us observe the point-wise convergence of the kernel K and the "free term" g of the integral equation (45). It is easy to see from the definition

$$V_n^{(3)}(\varepsilon) = \frac{\beta}{n} \int_n^{n+1} \cos(\varepsilon r) dr \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\beta}{n} \begin{pmatrix} \cos(\varepsilon n) & \sin(\varepsilon n) \\ \sin(\varepsilon n) & -\cos(\varepsilon n) \end{pmatrix}$$

that

$$\frac{V_{\left\lfloor \frac{y}{|\varepsilon|}\right\rfloor}^{(3)}(\varepsilon)}{\varepsilon} \to \frac{\beta \sin y}{y} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ as } \varepsilon \to 0$$

and hence for $y > t > 0$

$$(47)\qquad K(y, t, \varepsilon) \to \pm \begin{pmatrix} 0 & 1 \\ \exp\left(-2\beta \int_t^y \frac{\cos s}{s} ds\right) & 0 \end{pmatrix} \frac{\beta \sin t}{t} \text{ as } \varepsilon \to \pm 0.$$

Additionally, for every $y > 0$

$$(48)\qquad g(y, \varepsilon) \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left(f + \sum_{k=1}^{\infty} R_k^{(7)}(0) u_k(0, f) \right) \text{ as } \varepsilon \to 0.$$

This follows from the uniform bound given by Lemma 7 and from the property $\{R_n^{(7)}(\varepsilon)\}_{n=1}^{\infty} \in l^1(U)$ by Lebesgue dominated convergence theorem.

Remark 4. *It is very important that the limit in* (48) *coincides with the* $\lim_{n\to\infty} u_n(0,f)$. *The latter can be obtained from the following variant of discrete Levinson theorem* [16, Lemma 4.4, case (b)].

Proposition 2 ([16]). *Suppose that* $\sum_{k=1}^{\infty} \frac{\|\hat{R}_k\|}{|\hat{\lambda}_k|} < \infty$ *and that there exist an* M *such that for every* $m \geq n$ *the estimate* $\prod_{l=n+1}^{m} |\hat{\lambda}_l| \geq \frac{1}{M}$ *holds. Let* $\hat{u}$ *be a solution of the system*

$$\hat{u}_{n+1} = \left[\begin{pmatrix} \hat{\lambda}_n & 0 \\ 0 & 1/\hat{\lambda}_n \end{pmatrix} + \hat{R}_n \right] \hat{u}_n.$$

If $\prod_{l=1}^{\infty} |\hat{\lambda}_l| = \infty$, *then both sides of the following equality exist and the equality holds:*

$$\lim_{n\to\infty} \frac{\hat{u}_n}{\prod_{l=1}^{n-1} \hat{\lambda}_l} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left[\hat{u}_1 + \sum_{k=1}^{\infty} \frac{\hat{R}_k \hat{u}_k}{\prod_{l=1}^{k} \hat{\lambda}_l} \right].$$

In the case of our system $u_{n+1} = B_n(\varepsilon) u_n$, one can successfully apply this argument, putting $\hat{u}_n = n^{\beta} u_n(0,f)$, $\hat{\lambda}_n = \left(\frac{n+1}{n}\right)^{\beta}$ and $\hat{R}_n = \left(\frac{n+1}{n}\right)^{\beta} R_n^{(7)}(0)$.

Step III. Convergence in the slow scale. Now we establish the convergence as $\varepsilon \to \pm 0$ of the solution $u(\varepsilon)$ in the slow scale. We remind that in our notation $u_{\lfloor \frac{y}{|\varepsilon|} \rfloor}(\varepsilon) = z(y,\varepsilon)$.

Lemma 8. *For every* $y > 0$ *and* $f \in \mathbb{C}^2$ *there exist two limits*

$$h_{\pm}(y,f) := \lim_{\varepsilon\to\pm 0} z(y,\varepsilon,f),$$

which satisfy the following integral equations:

$$h_{\pm}(y,f) = \lim_{n\to\infty} u_n(0,f) \pm \int_0^y \begin{pmatrix} 0 & 1 \\ \exp\left(-2\beta \int_t^y \frac{\cos s}{s} ds\right) & 0 \end{pmatrix} \frac{\beta \sin t}{t} h_{\pm}(t,f) dt. \tag{49}$$

Proof. For every $y_0 > 0$ and $\varepsilon \neq 0$ define the operator $\mathcal{K}_{y_0}(\varepsilon)$ in the Banach space $L_{\infty}((0;y_0),\mathbb{C}^2)$ by the rule

$$\mathcal{K}_{y_0}(\varepsilon) : u(y) \mapsto \int_0^y K(y,t,\varepsilon) u(t) dt, \ y \in (0;y_0). \tag{50}$$

Denote $K(y,t,\pm 0) := \lim_{\varepsilon\to\pm 0} K(y,t,\varepsilon)$ and analogously define two operators $\mathcal{K}_{y_0}(\pm 0)$. We consider only the case $\varepsilon \to +0$ here. The second case can be treated in the same way.

First let us see that the operator $\mathcal{K}_{y_0}(\varepsilon)$ converges to $\mathcal{K}_{y_0}(+0)$ as $\varepsilon \to +0$ in the space of bounded linear operators in $L_{\infty}((0;y_0),\mathbb{C}^2)$. It is enough to show that

$$\max_{y\in[0;y_0]} \int_0^y \|K(y,t,\varepsilon) - K(y,t,+0)\| dt \to 0 \text{ as } \varepsilon \to +0. \tag{51}$$

Fix a $\Delta > 0$. Since both kernels are uniformly bounded in all variables, there exists a $y_1(\Delta) < y_0$ such that for every positive $\varepsilon \in U$

$$\max_{y\in[0;y_1(\Delta)]} \int_0^y \|K(y,t,\varepsilon) - K(y,t,+0)\| dt < \Delta.$$

For the same reason

$$\max_{y\in[y_1(\Delta);y_0]}\int_0^y \|K(y,t,\varepsilon)-K(y,t,+0)\|dt \tag{52}$$
$$=\max_{y\in[y_1(\Delta);y_0]}\int_\varepsilon^{y-\varepsilon}\|K(y,t,\varepsilon)-K(y,t,+0)\|dt+O(\varepsilon) \text{ as } \varepsilon\to+0.$$

Using the estimates $\frac{1-\cos\varepsilon}{\varepsilon}=O(\varepsilon)$ and $1-\frac{\sin\varepsilon}{\varepsilon}=O(\varepsilon^2)$ as $\varepsilon\to 0$, one can write

$$K(y,t,\varepsilon)=\begin{pmatrix}0 & 1\\ \exp\left(-2\beta\int_{\varepsilon\left(\left\lfloor\frac{t}{\varepsilon}\right\rfloor+1\right)}^{\varepsilon\left\lfloor\frac{y}{\varepsilon}\right\rfloor}\frac{\cos s}{s}ds\right) & 0\end{pmatrix}\frac{\beta\sin\left(\varepsilon\left\lfloor\frac{t}{\varepsilon}\right\rfloor\right)}{\varepsilon\left\lfloor\frac{t}{\varepsilon}\right\rfloor}+O(\varepsilon) \text{ as } \varepsilon\to+0$$

for y and t such that $2\varepsilon<y<y_0$ and $\varepsilon<t<y-\varepsilon$, where $O(\varepsilon)$ is uniform with respect to these y and t. We remind that

$$K(y,t,+0)=\begin{pmatrix}0 & 1\\ \exp\left(-2\beta\int_t^y\frac{\cos s}{s}ds\right) & 0\end{pmatrix}\frac{\beta\sin t}{t}.$$

The mapping $(y;t)\mapsto\exp\left(-2\beta\int_t^y\frac{\cos s ds}{s}\right)$ is uniformly continuous on the compact set $y_1(\Delta)\le y\le y_0$, $0\le t\le y$ (notice that it has a discontinuity at the point $y=t=0$), the function $t\mapsto\frac{\sin t}{t}$ is uniformly continuous in the interval $[0;y_0]$. Therefore $\max_{t\in[\varepsilon;y-\varepsilon]}\|K(y,t,\varepsilon)-K(y,t,+0)\|=o(1)$ as $\varepsilon\to+0$ uniformly in $y\in[y_1(\Delta);y_0]$. Together with (52) this means that for sufficiently small positive values of ε the maximum

$$\max_{y\in[y_1(\Delta);y_0]}\int_0^y\|K(y,t,\varepsilon)-K(y,t,+0)\|dt<\Delta,$$

and hence the maximum in (51) has the same property. Since Δ is an arbitraty positive number, the convergence of operators $\mathcal{K}_{y_0}(\varepsilon)$ follows.

Secondly, as mentioned in Remark 4, for every $y>0$ one has:

$$g(y,\varepsilon)\to\lim_{n\to\infty}u_n(0) \text{ as } \varepsilon\to 0.$$

We face an obstacle here: this convergence is only point-wise and is not in the norm of $L_\infty((0;y_0),\mathbb{C}^2)$. The same is true for the convergence of $z(y,\varepsilon)$, which we are going to prove. There is no sense in directly applying the inverse of the operator $I-\mathcal{K}_{y_0}(\varepsilon)$. The proper object to consider is the difference $z(y,\varepsilon)-g(y,\varepsilon)$, which converges in $L_\infty((0;y_0),\mathbb{C}^2)$. To see this let us rewrite the equation (45) in the following form:

$$z(\varepsilon)-g(\varepsilon)=(I-\mathcal{K}_{y_0}(\varepsilon))^{-1}\mathcal{K}_{y_0}(\varepsilon)g(\varepsilon) \tag{53}$$

(since $\mathcal{K}_{y_0}(\varepsilon)$ is a Volterra integral operator, $(I-\mathcal{K}_{y_0}(\varepsilon))^{-1}$ exists). It suffices to prove that $K_{y_0}(+0)g(\varepsilon)\to K_{y_0}(+0)\lim_{n\to\infty}u_n(0)$ in $L_\infty((0;y_0),\mathbb{C}^2)$. By a direct estimate we have:

$$\max_{y\in[0;y_0]}\int_0^y\left\|K(y,t,+0)\left(g(t,\varepsilon)-\lim_{n\to\infty}u_n(0)\right)\right\|dt\le\beta 3^{2\beta}\int_0^{y_0}\|g(t,\varepsilon)-\lim_{n\to\infty}u_n(0)\|dt.$$

The right-hand side tends to zero as $\varepsilon\to+0$ due to the point-wise convergence and the uniform boundedness of $g(t,\varepsilon)$ by Lebesgue dominated convergence theorem. From (53) we obtain:

$$z(\varepsilon)-g(\varepsilon)\to(I-\mathcal{K}_{y_0}(+0))^{-1}\mathcal{K}_{y_0}(+0)\lim_{n\to\infty}u_n(0) \text{ as } \varepsilon\to+0 \text{ in } L_\infty((0;y_0),\mathbb{C}^2)$$

From the point-wise convergence of $g(y,\varepsilon)$ it follows that for every $y>0$

$$z(y,\varepsilon)\to\left(\left(I+(I-\mathcal{K}_{y_0}(+0))^{-1}\mathcal{K}_{y_0}(+0)\right)\lim_{n\to\infty}u_n(0)\right)(y)=:h_+(y).$$

Consider the integral equation (45) again. Taking the limit as $\varepsilon \to +0$ turns it into the equation for h_+, (49). This is possible due to the uniform boundedness of the solution z and of the kernel K by Lebesgue dominated convergence theorem. This completes the proof. □

Remark 5. *The assertion of the last lemma includes the case when* $\lim_{n\to\infty} u_n(0,f) = 0$. *In this case* $h_\pm(y,f) \equiv 0$. *This happens only for one particular direction of the vector* $f \in \mathbb{C}^2$, *because the rank of* Θ *is one.*

Step IV. Convergence as $y \to +\infty$ and changing the order of limits. It remains to show that every solution of integral equations (49) has a limit at infinity and to see that it is possible to change the order of limits in $\lim\limits_{y\to+\infty} \lim\limits_{\varepsilon\to\pm 0} u_{\lfloor \frac{y}{|\varepsilon|} \rfloor}(\varepsilon, f)$.

Lemma 9. *The following two limits exist:* $\lim\limits_{y\to+\infty} h_\pm(y,f)$. *These limits are non-zero, iff* $\lim\limits_{n\to\infty} u_n(0,f) \neq 0$.

Proof. Consider the integral equation for h_+:

$$h_+(y) = \lim_{n\to\infty} u_n(0,f) + \int_0^y \begin{pmatrix} 0 & 1 \\ \exp\left(-2\beta \int_t^y \frac{\cos s}{s} ds\right) & 0 \end{pmatrix} \frac{\beta \sin t}{t} h_+(t) dt.$$

Let as write it as a differential equation using the following factorization property of the matrix in the integral:

$$\begin{pmatrix} 0 & 1 \\ \exp\left(-2\beta \int_t^y \frac{\cos s}{s} ds\right) & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(2\beta \int_y^\infty \frac{\cos s}{s} ds\right) \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ \exp\left(-2\beta \int_t^\infty \frac{\cos s}{s} ds\right) & 0 \end{pmatrix}.$$

Multiplying from the left by $\begin{pmatrix} 1 & 0 \\ 0 & \exp\left(2\beta \int_y^\infty \frac{\cos s}{s} ds\right) \end{pmatrix}^{-1}$ and taking the derivative in y one has:

$$h_+'(y) - \frac{\beta}{y} \begin{pmatrix} 0 & \sin y \\ \sin y & -2\cos y \end{pmatrix} h_+(y) = \frac{2\beta \cos y}{y} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \lim_{n\to\infty} u_n(0,f).$$

By Lemma 5 the vector $\lim\limits_{n\to\infty} u_n(0,f) \in \mathbb{C}^2$ is either proportional to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or is equal to zero and therefore

$$h_+'(y) = \frac{\beta}{y} \begin{pmatrix} 0 & \sin y \\ \sin y & -2\cos y \end{pmatrix} h_+(y). \tag{54}$$

If $\lim\limits_{n\to\infty} u_n(0,f) \neq 0$, then $h_+(y)$ is not identically zero. Every non-zero solution of (54) has a non-zero limit as $y \to +\infty$, which follows for example from Harris-Lutz results [14, Theorem 3.1, p.85] (one should take there $\Lambda(y) = -\frac{2\beta\cos y}{y} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $V(y) = \frac{\beta \sin y}{y} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $R(y) = 0$). For the second limit (of $h_-(y,f)$) the proof is absolutely analogous. □

The following lemma is a combination of definitions and previous results: Lemmas 6, 8, 9.

Lemma 10. *The following two limits exist:* $\lim\limits_{\varepsilon\to\pm 0} \lim\limits_{n\to\infty} u_n(\varepsilon, f)$ *and equal to* $\lim\limits_{y\to+\infty} h_\pm(y,f)$, *respectively.*

Proof. By definition of $h_\pm$, $\lim\limits_{y\to+\infty} h_\pm(y,f) = \lim\limits_{y\to+\infty}\lim\limits_{\varepsilon\to\pm0} u_{\lfloor\frac{y}{|\varepsilon|}\rfloor}(\varepsilon,f)$. By Lemma 6, $u_{\lfloor\frac{y}{|\varepsilon|}\rfloor}(\varepsilon,f)$ converges uniformly with respect to $\varepsilon\in U\backslash\{0\}$. Therefore there exist two limits, $\lim\limits_{\varepsilon\to\pm0}\lim\limits_{y\to+\infty} u_{\lfloor\frac{y}{|\varepsilon|}\rfloor}(\varepsilon,f)$ which coincide with the limits $\lim\limits_{y\to+\infty}\lim\limits_{\varepsilon\to\pm0} u_{\lfloor\frac{y}{|\varepsilon|}\rfloor}(\varepsilon,f)$, respectively. Clearly $\lim\limits_{y\to+\infty} u_{\lfloor\frac{y}{|\varepsilon|}\rfloor}(\varepsilon,f) = \lim\limits_{n\to\infty} u_n(\varepsilon,f)$ for $\varepsilon\neq0$, which implies that $\lim\limits_{\varepsilon\to\pm0}\lim\limits_{n\to\infty} u_n(\varepsilon,f) = \lim\limits_{y\to+\infty}\lim\limits_{\varepsilon\to\pm0} u_{\lfloor\frac{y}{|\varepsilon|}\rfloor}(\varepsilon,f) = \lim\limits_{y\to+\infty} h_\pm(y,f)$. This completes the proof. □

We are ready now to finish the proof of Theorem 1. Limits $\lim\limits_{\varepsilon\to\pm0} u_{\lfloor\frac{y}{|\varepsilon|}\rfloor}(\varepsilon,f)$ exist by Lemma 8, the equality $\lim\limits_{\varepsilon\to\pm0}\lim\limits_{n\to\infty} u_n(\varepsilon,f) = \lim\limits_{y\to+\infty} h_\pm(y,f)$ has sense and holds true by Lemma 10. The map Θ has the rank one by Lemma 5 (see also Remark 3). □

5. Zeroes of the spectral density

In this section, we prove the main result of the paper by applying Theorem 1.

Theorem 2. *Let $q_1\in L_1(\mathbb{R}_+)$, $\omega\notin\frac{\pi\mathbb{Z}}{2a}$, and $\rho'_\alpha(\lambda)$ be the spectral density of the operator $\mathcal{L}_\alpha$. Let the index j be greater or equal to zero. If the solution $\varphi_\alpha(y,\nu_{j,+})$ of (3) is not a subordinate one, then there exist two non-zero limits*

$$\lim_{\lambda\to\nu_{j,+}\pm0}\frac{\rho'_\alpha(\lambda)}{|\lambda-\nu_{j,+}|^{\frac{2|c|}{a|W\{\psi_+(\nu_{j,+}),\psi_-(\nu_{j,+})\}|}\left|\int_0^a\psi_+^2(t,\nu_{j,+})e^{2i\omega t}dt\right|}}.$$

Analogously, if $\varphi_\alpha(y,\nu_{j,-})$ is not subordinate, then there exist two non-zero limits

$$\lim_{\lambda\to\nu_{j,-}\pm0}\frac{\rho'_\alpha(\lambda)}{|\lambda-\nu_{j,-}|^{\frac{2|c|}{a|W\{\psi_+(\nu_{j,+}),\psi_-(\nu_{j,+})\}|}\left|\int_0^a\psi_-^2(t,\nu_{j,-})e^{2i\omega t}dt\right|}}.$$

Proof. We use the notation of Section 3: ν_{cr}, β instead of $\nu_{j,\pm}$, $\beta_{j,\pm}$. This will yield both claims of the theorem. By (20), the solution $\varphi_\alpha(y,\lambda)$ is related to the solution $\{v_{\alpha,n}(\varepsilon)\}_{n=1}^\infty$ of (19). In turn, the latter is related by the scaling $v_n = \exp\left(\beta\int_1^n\frac{\cos(\varepsilon r)}{r}dr\right)u_n$ to the solution $\{u_n(\varepsilon,v_{\alpha,1}(\varepsilon))\}_{n=1}^\infty$ of the system $u_{n+1} = B_n(\varepsilon)u_n$. We now apply Theorem to the sequence 1 to $\{u_n(\varepsilon,v_{\alpha,1}(\varepsilon))\}_{n=1}^\infty$. Non-subordinacy of $\varphi_\alpha(\nu_{cr})$ means that $\lim\limits_{n\to\infty} u_n(0,v_{\alpha,1}(0))\neq0$. Due to continuity of $v_{1,\alpha}(\cdot)$ in U, Theorem 1 implies the existence of the following two limits

$$\lim_{\varepsilon\to\pm0}\lim_{n\to\infty} u_n(\varepsilon,v_{\alpha,1}(\varepsilon))\neq0.$$

Taking into account the scaling $v_n = \exp\left(\beta\int_1^n\frac{\cos(\varepsilon r)}{r}dr\right)u_n$ this means that there exist two non-zero limits $\lim\limits_{\varepsilon\to\pm0}\lim\limits_{n\to\infty}|\varepsilon|^\beta v_{\alpha,n}(\varepsilon)$. This in turn yields the assertion of the theorem due to the Weyl-Titchmarsh type formula (21) and the property of the quasi-momentum that its derivative is positive inside spectral bands. □

6. Discussion

As mentioned in Introduction, our result is to be compared to the one of Hinton-Klaus-Shaw [15]. The difference in the classes of operators considered may seem significant: in [15] there is no periodic background and the non-summable part of the potential is an infinite sum of Wigner-von Neumann type terms. Nevertheless, both problems can be written after a few suitable transformations in virtually the same form, see system (6) in our case and the formula prior to (2.2) in [15]. We are also able to consider finite or even infinite sum of Wigner-von Neumann type terms and to reduce the problem to the same form by analogous transformations. The methods that we use also have common traits like usage of the slow variable $y = |\varepsilon|n = |\varepsilon|x/a$ or work with the limit integral Volterra equation like (27) which actually represents an equation of the same type as (2.75) in [15] but is written in a different, more explicit form. However, the approach that we develop in the present paper to our mind has several essential distinctions compared to the approach of [15].

1. We use a discretization of the differential system and work with the discrete model system of a "simple" form (19). In addition to bringing certain technical (inessential) difficulties it makes our method more universal and enables one to consider a wider class of problems. As an example, we can analyze by a direct application of Theorem 1 the structure of zeroes of the spectral density of the discrete Schrödinger operator with Wigner-von Neumann potential. This is by far not the only possible example. Using the solution of the model problem one can also consider differential Schrödinger operators with point interactions at integer points with strengths of interaction forming a sequence of the Wigner-von Neumann form.

2. We do not reduce the problem to a scalar one and use the analysis of vector equations instead. This makes the analysis simpler and more transparent. As a consequence of this, the form of the limit equation (27) is explicit and can be guessed from heuristic considerations (see (24)-(26)).

3. We avoid fine technically involved estimates of oscillatory integrals and in fact our proof is divided into two parts. First (Step I), we prove rather rough a priori estimates by dividing the positive half-line into subintervals $\mathbb{R}_+ = \left(0; \frac{c_2}{|\varepsilon|}\right] \cup \left(\frac{c_2}{|\varepsilon|}; +\infty\right)$ for sufficiently large c_2. Second (Steps II-IV), we deal with the slow scale and prove the convergence using operator techniques. This is the main distinction of our method which makes it sufficiently simpler than the method of [15].

We expect that using the technique developed in the present paper one can treat a more complicated case of the Wigner-von Neumann perturbation of non-Coulomb type with slowly decaying power part. We plan to address this problem in the future.

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Department of Mathematical Physics, Institute of Physics, St. Petersburg University, Ulianovskaia 1, St. Petergoff, St. Petersburg, Russia, 198904

E-mail address: sergey.naboko@gmail.com

Chebyshev Laboratory, Department of Mathematics and Mechanics, Saint-Petersburg State University 14th Line, 29b, Saint-Petersburg, 199178 Russia

E-mail address: sergey_simonov@mail.ru