SZEGÖ LIMIT THEOREM ON THE LATTICE

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ABSTRACT. In this paper, we prove a Szegö type limit theorem on $\ell^2(\mathbb{Z}^d)$. We consider operators of the form $H = \rho\Delta + |\xi|^k, 0 \leq \rho,\ 0 < k < 2$ on $\ell^2(\mathbb{Z}^d)$ and π_λ the orthogonal projection of $\ell^2(\mathbb{Z}^d)$ on to the space of eigenfunctions of H with eigenvalues $\leq \lambda$. We take A be a 0th order self adjoint pseudo difference operator with symbol $a(\xi, x)$ satisfying $[A, H](H + 1)^{-\sigma}$ bounded for some $0 < \sigma < \frac{1}{2}$. Then for $f \in \mathcal{C}(\mathbb{R})$ and $(\xi, x) \in \mathbb{Z}^d \times \mathbb{T}^d$,

$$\lim_{\lambda\to\infty} \frac{\mathrm{tr}\ \mathrm{f}(\pi_\lambda \mathrm{A} \pi_\lambda)}{\mathrm{rank}\ \pi_\lambda} = \lim_{\lambda\to\infty} \frac{1}{(2\pi)^d} \frac{1}{vol(h(\xi, x) \leq \lambda)} \sum_{(\xi,x):h(\xi,x)\leq\lambda} \int f(a(\xi, x))dx$$

assuming one of the limits exists. The limits are invariant under compact perturbation of A.

1. Introduction

Let Δ be the discrete Laplacian on $\ell^2(\mathbb{Z}^d)$ defined by $\Delta = \sum_{j=1}^{d} \Delta_{\xi_j}^+ \Delta_{\xi_j}^-$, where Δ_ξ^+ and Δ_ξ^- acting on $u(\xi)$ as $u(\xi + e_j) - u(\xi)$ and $u(\xi) - u(\xi - e_j)$ respectively for $\xi \in \mathbb{Z}^d$ and e_j is a vector in $\mathbb{Z}^d$ with 1 in jth place zero elsewhere. Let $|\xi| = \sqrt{\sum_{j=1}^d \xi_j^2}$ and let V be a positive function on $\mathbb{Z}^d$ such that $V(\xi) = |\xi|^k,\ 0 < k < 2$ for large $|\xi|$. We denote by V the operator of multiplication by the function $V(\xi)$ on $\ell^2(\mathbb{Z}^d)$. Our choice of the normalization in the definition of Δ makes it a positive operator with purely absolutely continuous spectrum in $[0, 4d]$. For this reason we add a constant $4d$ and normalize and it will not affect the problem. We will stick to this normalization of discrete Laplacian which we define explicitly as

$$(\Delta u)(\xi) = \sum_{j=1}^{d} [u(\xi + e_j) + u(\xi - e_j)] + 2du(\xi).$$

We take $V(\xi)$ as above with some $0 < k < 2,\ 0 \leq \rho$ fixed and consider

$$H = \rho\Delta + V. \tag{1.1}$$

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It turns out that H is a pseudo difference operator with symbol $h(\xi, x) = 2\rho \sum_{k=1}^{d} \cos(x_k) + V(\xi) + 2d\rho$ where $(\xi, x) \in \mathbb{Z}^d \times \mathbb{T}^d$.

The H is self adjoint on the domain $\{u \in \ell^2(\mathbb{Z}^d) : Vu \in \ell^2(\mathbb{Z}^d)\}$, since Δ is bounded. It is easy to see that the resolvent $(H - z)^{-1}$ is compact for some (hence for all) $z \in \mathbb{C}^+$, so the spectrum of H is discrete and the multiplicity of each eigen value is finite.

Let $\{(\lambda_i, \eta_i)\}$ be the set of eigenvalues and eigenfunctions of H counted with multiplicity. Let π_λ denote the (finite rank) orthogonal projection of $\ell^2(\mathbb{Z})$ on to span$\{\eta_j : \lambda_j \leq \lambda\}$. In this paper, we consider a Szegö type theorem associated with H.

The classical theorem of Szegö is stated as follows: Let P_n be the orthogonal projection of $L^2[0, 2\pi]$ onto the linear subspace spanned by the functions $\{e^{im\theta} : 0 \leq m \leq n; 0 \leq \theta < 2\pi\}$. For a positive function $f \in \mathcal{C}^{1+\alpha}[0, 2\pi], \alpha > 0$ the operator T_f defined by the operator of multiplication by the function f on $L^2[0, 2\pi]$ the following result holds

$$\lim_{n\to\infty} \frac{1}{n+1} \log \det P_n T_f P_n = \frac{1}{2\pi} \int_0^{2\pi} \log f(\theta) d\theta.$$

The above result is well known as Szegö limit theorem. We refer to [7, 4] for details and related results. In fact, Szegö limit theorem is a special case of a more general result proved by Szegö (see [4]) in section 5.3 as follows. Let f be a bounded, real valued integrable function and $\{\lambda_i^n\}_{i=1}^n$ be the eigenvalues of $P_n T_f P_n$. Then for any continuous function F on $[\inf f, \sup f]$ it was proved in (see [4], sect. 5.3) that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\lambda_i^n) = \frac{1}{2\pi} \int_0^{2\pi} F(f(\theta)) d\theta.$$

Notice that $e^{im\theta}$ is an eigen function of $\Delta = -\frac{d^2}{dx^2}$, one can view the above results on $L^2[0, 2\pi]$ as a special cases of Szegö limit theorem for the Laplace-Beltrami operator or more generally one can consider such results for pseudo differential operators on compact manifolds.

We however consider such a result on $\ell^2(\mathbb{Z}^d)$ in this paper. Consider H, π_λ on $\ell^2(\mathbb{Z}^d)$ as stated earlier. Let A be a bounded, pseudo difference operator on

$\ell(\mathbb{Z}^d)$ with symbol $a(\xi, x)$. Then $\pi_\lambda A \pi_\lambda$ is a finite rank operator and hence its spectral measure μ_λ can be defined as the sum of atomic measures supported at its eigenvalues. (In [11], Zelditch considered a Schrödinger operator on $\mathbb{R}^n$ of the form $H = -\frac{1}{2}\Delta + V$, where V is a smooth positive function which grows like $V_0|x|^k,\ k > 0$.)

In the context of a pair of pseudo differential operators A, B with symbols $a(x, \xi), b(x, \xi)$ the Szegö type theorem is to compute a limit like

$$\lim_{\lambda \to \infty} \frac{Tr(\pi_\lambda f(B))}{Tr(\pi_\lambda)}$$

where π_λ is the orthogonal projection onto the spectral subspace $\{A \leq \lambda\}$ and show that it equals

$$\lim_{\lambda \to \infty} \frac{\int_{\{(x,\xi): b(x,\xi) \leq \lambda\}} f(a(x, \xi))\ dx d\xi}{\int_{\{(x,\xi): b(x,\xi) \leq \lambda\}}\ dx d\xi},$$

which is the main content of the theorems of Zelditch or Robert.

To see the ideas involved in proving such a theorem, suppose $(\Omega, \mathbb{B})$ is a measurable space and X is a non-negative measurable function. Let μ, ν be σ-finite measures on Ω. Define $\Omega_R = \{\omega : X(\omega) < R\}$ and suppose $0 < \mu(\Omega_R) < \infty$ for all large R. Define $\mu_R, R \in \mathbb{R}^+$ on $\mathbb{B}$ by

$$\mu_R(A) = \frac{\nu(A \cap \Omega_R)}{\mu(\Omega_R)},\ \ A \in \mathbb{B}.$$

The question is to find if the vague limits of μ_R exist. To answer this question one has to show that for each continuous function f of compact support, the limits

$$\lim_{R \to \infty} \int f(\omega) d\mu_R(\omega)$$

exist. Given the structure of μ_R this is written as the limit of ratios of two distribution functions on $\mathbb{R}^+$ namely

$$\lim_{R \to \infty} \int f(\omega) d\mu_R(\omega) = \lim_{R \to \infty} \frac{\nu^f \circ X^{-1}((0, R))}{\mu \circ X^{-1}((0, R))},$$

where $d\nu^f(\omega) = f(\omega) d\nu(\omega)$.

Such limits are computed using Tauberian theorems where some transforms of these measures are considered and limits taken for such transforms. While Zelditch [11] used the Laplace transform (via Karamata's Tauberian theorem

([10],p-192), Robert [6] suggested the use of Stieltjes transform (via Keldysh Tauberian theorem[1]). The application of Keldysh theorem requires one of the measures μ or ν to be absolutely continuous. We don't have this feature in our problem, so we use the Tauberian theorem of Grishin and Poedintseva (theorem 8,[5]).

Thus using the Stieltjes transform method and an appropriate Tauberian theorem, the limit on the right hand side of the above equation is the same as the limit

$$\lim_{\lambda\to\infty} \frac{\int \frac{1}{(w+\lambda)^k} d\nu^f \circ X^{-1}(w)}{\int \frac{1}{(w+\lambda)^k} d\mu \circ X^{-1}(\omega)},$$

for some $k > -1$, which is nothing but

$$\lim_{\lambda\to\infty} \frac{\int f(\omega) \frac{1}{(X(\omega)+\lambda)^k} d\nu(\omega)}{\int \frac{1}{(X(\omega)+\lambda)^k} d\mu(\omega)},$$

for the same k, showing the existence of which is relatively easier.

This type of procedure also works for pseudo differential operators but involves a bit more analysis involved in passing between operators and their symbols and one actually shows the existence of the limits

$$\lim_{\lambda\to\infty} \frac{\int f(b(x,\xi)) \frac{1}{(a(x,\xi)+\lambda)^k} dx d\xi}{\int \frac{1}{(a(x,\xi)+\lambda)^k} dx d\xi}.$$

We do such an analysis in this paper for pseudo difference operators and in addition, our proof also requires the use of an improved version of Laptev-Safarov [2], [3] type estimate which essentially says that as $\lambda \to \infty$, the quantities $\frac{Tr(\pi_\lambda f(B))}{Tr(\pi_\lambda)}$ and $\frac{Tr(f(\pi_\lambda B \pi_\lambda))}{Tr(\pi_\lambda)}$ are the same. . The Laptev-Safarov result requires that the commutator $[A, H]$ is bounded. Such an assumption implies, in our case, that $v(\xi)$ is a positive constant which makes the problem trivial.

The improved version of Laptev-Safarov theorem in the following:

Theorem 1.1. *Let S be a positive selfadjoint operator and T be a bounded selfadjoint operator in a Hilbert space. Let π_λ be the spectral projection of S corresponding to the interval $[0, \lambda]$ and $N(\lambda)$ be the counting eigenvalues function with*

$$N_\epsilon(\lambda) = \sup_{\beta\le\lambda}[N(\beta) - N(\beta - \epsilon)].$$

Assume that the commutator $[S,T]$ *satisfies* $\tilde{T} = [S,T](S+I)^{-\sigma}$ *is bounded for some* $0 < \sigma < 1/2$*. Then for any* $\epsilon > 0$ *and for any function* $f \in \mathcal{C}^2(K)$ *the following inequality holds.*

$$|\mathrm{tr}\ \pi_\lambda \mathrm{f}(\mathrm{T})\pi_\lambda - \mathrm{tr}\ \mathrm{f}(\pi_\lambda \mathrm{T} \pi_\lambda)| \leq (2\|\mathrm{T}\|^2 + \mathrm{C}_\epsilon \|\tilde{\mathrm{T}}\|^2)\mathrm{N}_\epsilon(\lambda)\lambda^{2\sigma} \max_{\mathrm{K}} |\mathrm{f}''|,$$

where $K = [-\|T\|, \|T\|]$ *and the constant* C_ϵ *depends on* ϵ *only.*

In this paper, in our main Theorem 1.2, we present a Szegö type theorem for H given in the equation (1.1) and A a zeroth order pseudo difference operator relative to the symbol of H (see the next section for the definitions involved in pseudo difference operator theory).

Consider π_λ, $\{\eta_j\}$ associated with H defined in the paragraphs following equation (1.1) and let $f \in \mathcal{C}(\mathbb{R})$. Then let

$$\begin{aligned} \mu_\lambda(f) &= \sum_{\lambda_j \leq \lambda} (f(A)\eta_j, \eta_j) = Tr(\pi_\lambda f(A) \pi_\lambda),\ \nu_\lambda(f) = \frac{\mu_\lambda(f)}{Tr(\pi_\lambda)} \\ \mu(f) &= \lim_{\lambda \to \infty} \frac{1}{vol(h(\xi,x) \leq \lambda)} \times \sum_{h(\xi,x) \leq \lambda} \int f(a(\xi,x)) \tilde{d}x, \end{aligned} \tag{1.2}$$

where $\tilde{d}x (= dx/(2\pi)^d)$ is the normalized invariant measure on $\mathbb{T}^d$.

We fix a ρ, k in the equation (1.1). The definitions for pseudo difference operators are collected in the next section.

Theorem 1.2. *Consider the positive self adjoint operator* H *given in equation (1.1) on* $\ell^2(\mathbb{Z}^d)$*. Let* A *be a zeroth order pseudo difference operator such that* $[A,H](H+I)^{-\sigma}$ *is bounded for some* $0 < \sigma < \frac{1}{2}$*. For* $f \in \mathcal{C}(\mathbb{R})$ *define* $\mu_\lambda(f), \nu_\lambda(f), \mu(f)$ *as in equation (1.2). Then we have*

$$\lim_{\lambda \to \infty} \nu_\lambda(f) = \mu(f). \tag{1.3}$$

Remark. We need the restriction on k to be in the interval $(0,2)$ so that we can find a $\sigma \in (0, \frac{1}{2})$ (depending on k) for which the relative boundedness condition in the theorem is satisfied.

2. Preliminaries

In this section we provide few definitions and notations which we will be using frequently. We set $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$.

Definition 2.1. *Let $m \in \mathbb{R}, 0 \leq \delta, \kappa \leq 1$. Then $S^m_{\kappa,\delta}(\mathbb{Z}^d, \mathbb{T}^d)$ consisting of those functions $a(\xi, x)$ which are smooth in x for all $\xi \in \mathbb{Z}^d$ satisfying*

$$|\partial_x^\alpha \Delta_\xi^\gamma a(\xi, x)| \leq C_{a\alpha\gamma m} \langle\xi\rangle^{m-\kappa|\gamma|+\delta|\alpha|},$$

where $\Delta_\xi^\gamma = \Delta_{\xi_1}^{\gamma_1}\Delta_{\xi_2}^{\gamma_2}\cdots, \Delta_{\xi_d}^{\gamma_d}$ and $\Delta_{\xi_j} u(\xi) = u(\xi + e_j) - u(\xi)$ for every $x \in \mathbb{T}^d, \alpha, \gamma \in \mathbb{N}_0^d$ and $\xi \in \mathbb{Z}^d$.

We denote $S^m_{1,0}(\mathbb{Z}^d, \mathbb{T}^d)$ as simply $S^m(\mathbb{Z}^d, \mathbb{T}^d)$ and call its elements as symbols of order m. If $a \in S^m(\mathbb{Z}^d, \mathbb{T}^d)$ then the corresponding pseudo difference operator A is given by

$$Au(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^n} \sum_{\eta\in\mathbb{Z}^d} e^{i(\xi-\eta)\cdot x} a(\xi, x) u(\eta) dx.$$

More generally one can define a symbol class relative to another symbol. Let $h(\xi, x)$ be a smooth function on $\mathbb{Z}^d \times \mathbb{T}^d$ positive for large $|\xi|$. Then

Definition 2.2. *Let $m \in \mathbb{R}$. Then $S^m(h, \mathbb{Z}^d, \mathbb{T}^d)$, the space of symbols of order m relative to h consisting of those functions $a(\xi, x)$ which are smooth in x for all $\xi \in \mathbb{Z}^d$ satisfying*

$$|\partial_x^\alpha \Delta_\xi^\gamma a(\xi, x)| \leq C_{a\alpha\gamma m} (h(\xi, x))^{m-|\gamma|},$$

where $\Delta_\xi^\gamma = \Delta_{\xi_1}^{\gamma_1}\Delta_{\xi_2}^{\gamma_2}\cdots, \Delta_{\xi_d}^{\gamma_d}$ and $\Delta_{\xi_j} u(\xi) = u(\xi + e_j) - u(\xi)$ for every $x \in \mathbb{T}^d, \alpha, \gamma \in \mathbb{N}_0^d$ and $\xi \in \mathbb{Z}^d$.

We will use the notation Δ_{ξ_j} for forward difference instead of $\Delta^+_{\xi_j}$ for convenience. Now we discuss some of the properties of pseudo difference calculus which is developed by Ruzhansky and Turunen in [9]. The pseudo difference operators are closed under composition.

Example 2.3. (i) The symbol $a(x, \xi) = \frac{c}{(1+|\xi|^2)^\ell}$ is in $S^0(\mathbb{Z}^d, \mathbb{T}^d)$ while the symbol $a_1(x, \xi) = \cos(\xi)$ is not in this symbol class, since the finite differences of a_1 do not decay with $|\xi|$. Therefore also $\frac{\cos(\xi)}{(1+|\xi|^2)}$ is not in this symbol class. However $\cos(\frac{1}{1+|\xi|^2})$ is in the symbol class.

(ii) Let p be a smooth function on $\mathbb{T}^d$ and define $a_2 = a + p$, with a given above, then $a_2 \in S^0(\mathbb{Z}^d, \mathbb{T}^d)$.

(iii) Let $q(\xi) = |\xi|^{\frac{m}{2}}$. Then the symbol $q \in S^m(\mathbb{Z}^d, \mathbb{T}^d)$.

If a and b are two symbols then we denote by $a\tilde{b}$ to mean that the difference $a-b \in S^{-\infty}(\mathbb{Z}^d, \mathbb{T}^d)$, which means that as a function of ξ it is finitely supported in $\mathbb{Z}^d$. With this notation we recall a theorem on composition of two symbols from [9].

Theorem 2.4. *(Theorem 4.3,* [9]*) Let P and Q be pseudo difference operators with symbols $p(\xi, x) \in S^{m_1}(\mathbb{Z}^d, \mathbb{T}^d)$ and $q(\xi, x) \in S^{m_2}(\mathbb{Z}^d, \mathbb{T}^d)$. Then PQ is a pseudo difference operator with symbol $r \in S^{m_1+m_2}(\mathbb{Z}^d, \mathbb{T}^d)$ and*

$$r(\xi, x) \sim \sum_{\alpha} \frac{1}{\alpha!} \Delta_\xi^\alpha p(\xi, x) \frac{\partial^\alpha}{\partial x^\alpha} q(\xi, x)$$

where $\Delta_\xi^\gamma = \Delta_{\xi_1}^{\gamma_1} \Delta_{\xi_2}^{\gamma_2} \cdots, \Delta_{\xi_d}^{\gamma_d}$.

Fix a $\rho \in [0, \infty)$, the operator H defined in equation (1.1) is a pseudo difference operator with symbol $h(\xi, x) = 2\rho \sum_{j=1}^d \cos x_j + V(\xi) + 2d\rho$. We take a positive number λ and consider the pseudo difference operator $H + \lambda$ with symbol

$$(h_{\lambda,1})(\xi, x) = h(\xi, x) + \lambda.$$

Let $h_{\lambda,m}(\xi, x)$ denote the symbol of the operator $(H + \lambda)^m$, the symbol obtained using Theorem 2.4 . Then we find that for $m = 2$, the symbol $h_{\lambda,2}$ of the operator $(H + \lambda)^2$ is given by

$$\begin{aligned} h_{\lambda,2}(\xi, x) &= \sum_{\alpha} \frac{1}{\alpha!} \Delta_\xi^\alpha h_{\lambda,1}(\xi, x) \frac{\partial^\alpha}{\partial x^\alpha} h_{\lambda,1}(\xi, x) \\ &= (h_{\lambda,1})^2(\xi, x) + \sum_{|\alpha|=1}^{\infty} \frac{1}{\alpha!} \Delta_\xi^\alpha h_{\lambda,1}(\xi, x) \frac{\partial^\alpha}{\partial x^\alpha} h_{\lambda,1}(\xi, x) \end{aligned}$$

But

$$\begin{aligned}
\frac{h_{\lambda,2}(\xi,x)}{(h_{\lambda,1})^2} &\leq 1+2d\sum_{|\alpha|=1}^{\infty}\frac{1}{\alpha!}\frac{|\Delta_\xi^\alpha h_{\lambda,1}(\xi,x)|}{h_{\lambda,1}^2(\xi,x)}\\
&= 1+2d\sum_{|\alpha|=1}^{\infty}\frac{1}{\alpha!}|\sum_{\gamma\leq\alpha}(-1)^{|\alpha-\gamma|}\binom{\alpha}{\gamma}\frac{|h_{0,1}(\xi+\gamma,x)|}{h_{\lambda,1}^2(\xi,x)}\\
&\leq 1+2d\frac{(e^2-1)^d}{\lambda+|\xi|^k}
\end{aligned}$$

By induction we extend the above argument to get the symbols associated with higher powers of the operator $H+\lambda$.

We need to find symbols of inverses of pseudo difference operators. For large λ the operator $(H+\lambda)$ is invertible so the inverse operator $(H+\lambda)^{-m}$ is a bounded pseudo difference operator with symbol $h_{\lambda,-m}$ and we describe now the procedure to find an approximate expression for this symbol for large λ.

We note that the symbol $h_{\lambda,-m+1}$ of the operator $(H+\lambda)^{-m+1}$ is given by a composition of the symbols $h_{\lambda,-m}$ and $h_{\lambda,1}$ by using Theorem 2.4 and we can write this as

$$h_{\lambda,-m+1}(\xi,x)=h_{\lambda,-m}(\xi,x)h_{\lambda,1}(\xi,x)+s_1(\xi,x), \tag{2.1}$$

where $s_1(\xi,x)=\sum_{|\alpha|=1}^{\infty}\dfrac{1}{\alpha!}\Delta_\xi^\alpha h_{\lambda,-m}(\xi,x)\dfrac{\partial^\alpha}{\partial x^\alpha}h_{\lambda,1}(\xi,x)$. Since $|\dfrac{\partial^\alpha}{\partial x^\alpha}h_{\lambda,1}(\xi,x)|\leq 2d$ and $|\Delta_\xi^\alpha h_{\lambda,1}(\xi,x)|\leq\sum_{\gamma\leq\gamma}\binom{\alpha}{\gamma}|[h_{0,1}(\xi+\gamma,x)]^{-m}|$ (by Proposition 3.1 of [9]), we have $|s_1(\xi,x)|\leq 2d\frac{(e^2-1)^d}{(\lambda+|\xi|^k)^m}$. Proceeding as above, the composition of the pseudo difference operators corresponding to $h_{\lambda,-m}(\xi,x)h_{\lambda,1}(\xi,x)$ (which is the first term in the equation (2.1)) and $h_{\lambda,1}(\xi,x)$ can be calculated with reminder term $s_2(\xi,x)$. The remainder term $s_2(\xi,x)$ has a bound similar to that of $s_1(\xi,x)$ for large λ. We repeat this process $m-1$ times to obtain an approximation for $h_{\lambda,-1}$ which is the symbol of the operator $(H+\lambda)^{-1}$.

Notice that the symbol obtained composing $h_{\lambda,-m}$ and $h_{\lambda,m}$ is

$$1+\sum_{j=1}^{m}s_j(\xi,x)h_{\lambda,1}^{m-j}(\xi,x),\ \text{with}\ |s_j(\xi,x)|\leq 2d\frac{(e^2-1)^d}{(\lambda+|\xi|^k)^m}\quad j=1,2,\cdots,m.$$

For large λ, $|\sum_{j=1}^{m}s_j(\xi,x)h_{\lambda,1}^{m-j}(\xi,x)|\leq C\sum_{j=1}^{m}\frac{1}{\lambda^j}=\frac{\lambda^m-1}{\lambda^m(\lambda-1)}$.

Thus for large λ we have the approximations for the symbols as stated in the following remark.

Remark 2.5. For large λ and $m \in \mathbb{Z}^+$ we have

(i) $h_{\lambda,m}(\xi,x) \sim h^m_{\lambda,1}[1+O(\lambda^{-1})]$
(ii) $h_{\lambda,-m}(\xi,x) \sim h^{-m}_{\lambda,1}[1+O(\lambda^{-1})]$

It is clear from the pseudo difference calculus that if $p(\xi,x)$ is the symbol of a pseudo difference operator P, then the symbol of P^2 is NOT $p(\xi,x)^2$. However there is a pseudo difference operator with symbol $p(\xi,x)^2$. So to take care of such possibilities we denote henceforth the operator associated with a symbol $p(\xi,x)$ as p^W when there is no operator already associated with the symbol.

We have the following proposition.

Proposition 2.6. *Let* $H = \rho\Delta + |\xi|^k, 0 \le \rho \le 1,\ k > 0$ *on* $\ell(\mathbb{Z}^d)$ *and let* $h_{\lambda,1}(\xi,x)$ *be the symbol of* $H+\lambda$. *Let* A *be a* 0*th order pseudo difference operator. Let* m *be a positive integer. Then*

(i) $\dfrac{|\mathrm{tr}\,((\mathrm{H}+\lambda)^{-\mathrm{m}}) - \mathrm{tr}\,(\mathrm{h}^{-\mathrm{m}}_{\lambda,1})^{\mathrm{W}}|}{|\mathrm{tr}\,((\mathrm{h}(\xi,\mathrm{x})+\lambda)^{-\mathrm{m}})^{\mathrm{w}}|} \to 0$ *as* $\lambda \to \infty$

(ii) $\dfrac{|\mathrm{tr}\,(\mathrm{A}(\mathrm{H}+\lambda)^{-\mathrm{m}}) - \mathrm{tr}\,(\mathrm{A}(\mathrm{h}^{-\mathrm{m}}_{\lambda,1})^{\mathrm{W}}|}{|\mathrm{tr}\,(\mathrm{A}(\mathrm{h}^{-\mathrm{m}}_{\lambda,1})^{\mathrm{W}}|} \to 0$ *as* $\lambda \to \infty$,

Proof.

Using remark 2.5(ii) we have for large λ,

$$(h^m_{\lambda,1})^W (h^{-m}_{\lambda,1})^W = I + B^W_\lambda(\xi,x), \tag{2.2}$$

where $B_\lambda(\xi,x)$
$= \sum_{j=1}^m s_j(\xi,x) h^{m-j}_{\lambda,1}(\xi,x)$.

Thus we have $\|B^W_\lambda f\|_2 \le \dfrac{C}{\lambda}\|f\|_2$ for $f \in \mathcal{S}(\mathbb{Z}^d)$. So $\|B^W_\lambda f\|_2 \to 0$ as $\lambda \to \infty$ and hence $\|B^W_\lambda\| \to 0$ as $\lambda \to \infty$.

We have the identity,

$$I + B^W_\lambda \;=\; (H+\lambda)^m (H+\lambda)^{-m} + B^W_\lambda$$

which can be written as, using equation (2.2) and the fact that for large λ, $(h^m_{\lambda,1})^W \approx (H+\lambda)^m$.

$$(h^{-m}_{\lambda,1})^W - (H+\lambda)^{-m} = (H+\lambda)^{-m} B^W_\lambda \tag{2.3}$$

Now composing the operator A from the left in equation 2.3, applying trace and dividing by $\mathrm{tr\,A(H+\lambda)^{-m})}$ we have

$$\left|\frac{\mathrm{tr\,A(h^{-m}_{\lambda,1})^W) - tr\,A(H+\lambda)^{-m})}}{\mathrm{tr\,A(H+\lambda)^{-m})}}\right| = \left|\frac{\mathrm{tr\,[A(H+\lambda)^{-m})B^w_\lambda]}}{\mathrm{tr\,A(H+\lambda)^{-m})}}\right| \le \|B^w_\lambda\| \to 0$$

as $\lambda \to \infty$. This proves (ii). Now (i) is a particular case of (ii) by replacing A with the identity operator on $\ell^2(\mathbb{Z}^d)$. $\square$

In the next lemma we give a technical result, whose proof is easy from the equivalence of the norms $|\xi|_\infty (= \max_k |\xi_i|)$ and $|\xi|_2 (= \sqrt{\sum_{i=1}^d \xi_i^2})$ for $\xi \in \mathbb{Z}^d$, that says that for the purposes of proving the main theorems it is irrelevant which of these two norms are used to define $|\xi|$ in the definition of the operator H in equation (1.1).

Lemma 2.7. *Let* $h_r(\xi, x) = 2\rho \sum_{j=1}^{d} \cos x_j + |\xi|_r^k + 2d\rho,\ 0 \le \rho \le 1,\ k > 0,\ x_j \in \mathbb{T}$, $\xi \in \mathbb{Z}^d$ *and* $r = 2, \infty$. *Define*

$$\varphi_r(\lambda) = \frac{1}{(2\pi)^d} \sum \int_{h_r(\xi,x) \le \lambda} dx,\ r = 2, \infty.$$

Then $d^{-\frac{d}{2}} \varphi_\infty(\lambda) \le \varphi_2(\lambda) \le \varphi_\infty(\lambda)$ *for large* λ.

Before going in to Szegö theorem, we will introduce few definitions and theorems which can be found in [5].

Definition 2.8. *Let* f *be a positive function on the half line* $[0, \infty)$. *Let* S *denote the set of numbers* α *for which there exist numbers* M *and* R *such that* $f(tr) \le Mt^\alpha$ *for* $t \ge 1$ *and* $r \ge R$. *Then* $\alpha(f) = \inf S$ *is called The upper Matushevskaya index of* f.

Let G denote the set of numbers ζ for which there exist numbers M and R such that $f(tr) \geq Mt^{\zeta}$ for $t \geq 1$ and $r \geq R$. Then $\beta(f) = \sup G$ is called The lower Matushevskaya index of f.

Theorem 2.9. *([5], Theorem 2) Let $m > -1$. Assume that φ is positive measurable function on $[0, \infty)$ that does not vanish identically in any neighbourhood of infinity. Let $\Phi(r) = \int_0^\infty \frac{1}{(1+\frac{t}{r})^m} d\varphi(t)$. Then the functions φ and Φ have same growth at infinity if and only if $\beta(\varphi) > -1$ and $\alpha(\varphi) < m$.*

Definition 2.10. *A function φ is said to be multiplicatively continuous at infinity if it satisfies*

$$\lim_{\substack{r\to\infty \\ \tau\to 1}} \frac{\varphi(\tau r)}{\varphi(r)} = 1 \text{ and } \lim_{\substack{\tau\to 1 \\ r\to\infty}} \frac{\varphi(\tau r)}{\varphi(r)} = 1.$$

Theorem 2.11. *([5], Theorem 8) Let φ and ψ be positive functions on $[0, \infty)$ satisfying the following conditions:*

(1) *the functions φ and ψ do not vanish identically in any neighbourhood of infinity;*
(2) *the function φ is multiplicatively continuous at infinity and $\beta(\varphi) > -1$;*
(3) *the function ψ is increasing;*
(4) *at least one of the inequalities $\alpha(\varphi) < m$ and $\alpha(\phi) < m$ holds, where $m > -1$;*
(5) *the functions*
$$\Phi(r) = \int_0^\infty \frac{1}{(1+\frac{u}{r})^m} d\varphi(u) \text{ and } \Psi(r) = \int_0^\infty \frac{1}{(1+\frac{u}{r})^m} d\psi(u)$$
are finite and $\lim_{r\to\infty} \frac{\Psi(r)}{\Phi(r)} = 1$ then $\lim_{r\to\infty} \frac{\psi(r)}{\varphi(r)} = 1$.

The above theorem derives asymptotic behaviour of φ, ψ from the asymptotic behaviour of Φ, Ψ by assuming additional conditions on φ and ψ. we prove our main theorem by using theorem 2.11 as an important tool.

Lemma 2.12. *For $\xi \in \mathbb{Z}^d$, let $|\xi|$ denote either of $|\xi|_\infty, |\xi|_2$ and let $h(\xi, x), \phi(\lambda)$ denote the corresponding $h_r(\xi, x), \phi_r(\lambda)$, $r = 2, \infty$ given in Lemma 2.7 with $0 \leq \rho \leq 1$, $k > 0$. Then φ satisfies the following conditions:*

(1) *the function φ do not vanish identically in any neighbourhood of infinity;*

(2) *the function φ is multiplicatively continuous at infinity and $\beta(\varphi) > 1$;*

(3) $\alpha(\varphi) < m$, *where* $m \geq \frac{d}{k} + 1$.

Proof. We will give the proof for the case $r = \infty$ and the case $r = 2$ is similar. We have by the definition of ϕ, setting $c(x,\rho) = 2\rho\sum_{j=1}^{d}\cos x_j - 2d\rho$ for ease of writing,

$$(2\pi)^d)\phi(\lambda) = \sum \int_{V(\xi)\leq(\lambda - c(x,\rho))_+} dx.$$

Since $V(\xi) = |\xi|^k$ for large ξ, suppose it is so for $|\xi| > R$. Then the sum

$$\sum_{\{\xi: V(\xi)\leq(\lambda-c(x,\rho))_+\}} = O(R^d) + \sum_{R<|\xi|\leq(\lambda-c(x,\rho))_+^{\frac{1}{k}}} = \sum_{|\xi|\leq(\lambda-c(x,\rho))_+^{\frac{1}{k}}} + o(\lambda).$$

as λ goes to infinity. Therefore by direct computation,

$$\begin{aligned}
(2\pi)^d)\varphi(\lambda) &= \sum \int_{|\xi|\leq(\lambda-c(x,\rho))_+^{\frac{1}{k}}} dx + o(\lambda) \\
&= \int_{\mathbb{T}^d} (2[(\lambda - c(x,\rho))_+^{\frac{1}{k}}] + 1)^d dx + o(\lambda) \\
&= \int_{\mathbb{T}^d} (2(\lambda - c(x,\rho))_+^{\frac{1}{k}} - 2\{(\lambda - c(x,\rho))_+^{\frac{1}{k}}\} + 1)^d dx + o(\lambda) \\
&= \lambda^{\frac{d}{k}} \int_{\mathbb{T}^d} (2(1 - \frac{c(x,\rho)}{\lambda})_+^{\frac{1}{k}} - \frac{1}{\lambda}\{(\lambda - c(x,\rho))_+^{\frac{1}{k}}\} + \frac{1}{\lambda^{\frac{1}{k}}})^d dx + o(\lambda), \qquad (2.4)
\end{aligned}$$

where $[p]$ denote the greatest integer function and $\{p\}$ is the fractional part of p.

Since the integrand is bounded for large λ and the integration is over a compact set, it can be realised that $\varphi(\lambda)$ behaves like constant times $\lambda^{\frac{d}{k}}$. We need to show $\beta(\varphi) > 1$ and $\alpha(\varphi) < m$, where $m > -1$. It is enough to show φ and Φ have same growth at infinity. A straight forward computation gives

$$\lim_{r\to\infty} \frac{\Phi(r)}{\varphi(r)} = C \int_0^\infty \frac{u^{\frac{d}{k}}}{(1+u)^{m+1}} du.$$

We notice that $\dfrac{u^{\frac{d}{k}}}{(1+u)^{m+1}}$ converges if $m \geq \frac{d}{k} + 1$. So if we choose $m = \frac{d}{k} + 1$, we have $0 < \lim\limits_{r\to\infty} \dfrac{\Phi(r)}{\varphi(r)} < \infty$. Thus φ and Φ have same growth at infinity. $\square$

For the following corollary recall the definition of η_j given before equation (1.2) and take $h(\xi, x)$ to be either of the h_rs in the above lemma.

Corollary 2.13. *Let*

$$\psi(\lambda) = \sum_{\lambda_j \leq \lambda} \langle \eta_j, \eta_j \rangle, \ \varphi(\lambda) = \frac{1}{(2\pi)^d} \sum \int_{h(\xi,x)\leq\lambda} dx.$$

Then $\lim_{\lambda\to\infty} \frac{\psi(\lambda)}{\varphi(\lambda)} = 1$

Proof. By lemma 2.12, it follows that $\varphi(\lambda)$ satisfies all the assumptions of theorem 2.11. Also $\Psi(\lambda) = \mathrm{tr}\left((\mathrm{H}+\lambda)^{-\mathrm{m}}\right) < \infty$ and $\Phi(\lambda) = \mathrm{tr}\left([(\mathrm{h}(\xi,\mathrm{x}) + \lambda)^{-\mathrm{m}})]^{\mathrm{w}}\right) < \infty$. It follows from proposition 2.6 (ii) that $\lim_{r\to\infty} \frac{\Psi(r)}{\Phi(r)} = 1$ implying $\lim_{\lambda\to\infty} \frac{\psi(\lambda)}{\varphi(\lambda)} = 1$ □

3. Proof of Main Theorem

Our aim in this section is to prove the averaging theorem and then deduce the Szegö Theorem from it. Before proving the averaging theorem, we need to prove the following lemma.

Lemma 3.1. *Let A be a bounded, positive self adjoint operator and H be positive self adjoint operator with discrete spectrum. Let $E_H(\cdot)$ be the spectral measure of H. Then*

(i) $\mathrm{tr}\,(E_H(\cdot)A)$
(ii) $\mathrm{tr}\,(E_H(\cdot))$

are σ-finite measures.

Proof. The second item is obvious, the first item follows by writing $\mathrm{tr}\,(\mathrm{E_H}(\cdot)\mathrm{A})$ as $\mathrm{tr}\,(\mathrm{A}^{\frac{1}{2}}\mathrm{E_H}(\cdot)\mathrm{A}^{\frac{1}{2}})$ using the properties of trace and the positivity of A. □

Lemma 3.2. *Let a be a non-negative symbol in $S^0(\mathbb{Z}^d, \mathbb{T}^d)$ and A the associated (0th order) pseudo difference operator. Consider the operator H as in equation (1.1) and its symbol h given there. Then*

(i) $\phi(\lambda) = \sum \int_{\{(\xi,x):h(\xi,x)\leq\lambda\}} a(\xi,x)\ dx$

(ii) $\psi(\lambda) = \sum \int_{\{(\xi,x):h(\xi,x)\leq\lambda\}}\ dx$

are distribution functions of σ finite positive Borel measures.

Proof. We note that both ϕ and ψ are non-decreasing positive functions on $[0,\infty)$ with $\phi(0) = \psi(0) = 0$ and hence there are σ-finite measures with ϕ and ψ as distribution functions. □

Now we are in a position to prove the averaging theorem.

Theorem 3.3. *Let A be a bounded 0th order pseudo difference operator with symbol a and H the operator given in equation (1.1) with k fixed. Suppose $m \geq \frac{d}{k} + 1$ and*

$$\lim_{\lambda\to\infty} \frac{\mathrm{tr}\,(\mathrm{A}(\mathrm{H}+\lambda)^{-\mathrm{m}})}{\mathrm{tr}\,((\mathrm{H}+\lambda)^{-\mathrm{m}})}$$

exists (and nonzero) then the following limits exist and assumes the same value:

(i) $\lim_{\lambda\to\infty} \dfrac{\mathrm{tr}\,(\pi_\lambda \mathrm{A} \pi_\lambda)}{\mathrm{rank}\ \pi_\lambda}$

(ii) $\lim_{\lambda\to\infty} \dfrac{\sum \int_{h(\xi,x)\leq\lambda} a(\xi,x)\tilde{d}x}{vol(\{(\xi,x) : h(\xi,x) \leq \lambda\})}$ *where* $\tilde{d}x = \dfrac{dx}{2\pi}$.

Proof. We first note that since $a(\xi,x)$ is a bounded function, we can add a constant c so that $a(\xi,x) + c$ is positive and since the limits in items (i) (respectively (ii)) exist iff the corresponding limits exit with A replaced by $A+c$ (respectively $a(\xi,x)+c$), we can take without loss of generality $a(\xi,x)$ to be a positive function and hence A to be a positive self adjoint bounded pseudo difference operator in the argument below.

Assume that $\lim_{\lambda\to\infty} \frac{\mathrm{tr}\,(\mathrm{A}(\mathrm{H}+\lambda)^{-\mathrm{m}})}{\mathrm{tr}\,(\mathrm{H}+\lambda)^{-\mathrm{m}}}$ exists ($l \neq 0$). By writing the spectral theorem for H and using lemma 3.1, proposition 2.6 we have

$$\begin{aligned}
\lim_{\lambda\to\infty} \frac{\mathrm{tr}\ (A(H+\lambda)^{-m})}{\mathrm{tr}\ (H+\lambda)^{-m}} &= \lim_{\lambda\to\infty} \frac{\mathrm{tr}\ (A^{\frac{1}{2}}(H+\lambda)^{-m}A^{\frac{1}{2}})}{\mathrm{tr}\ (H+\lambda)^{-m}} \\
&= \lim_{\lambda\to\infty} \frac{\int_0^\infty \dfrac{1}{(x+\lambda)^m} d(\mathrm{tr}\ (\mathrm{A}^{\frac{1}{2}}\mathrm{E_H(x)}\mathrm{A}^{\frac{1}{2}})}{\int_0^\infty \dfrac{1}{(x+\lambda)^m} d(\mathrm{tr}\ (\mathrm{E_H(x)})} = l
\end{aligned}$$

Then by using theorem 2.11 we have

$$\lim_{\lambda\to\infty}\frac{\text{tr }(\pi_\lambda A\pi_\lambda)}{\sum\int_{h(\xi,x)\le\lambda}\tilde{d}x}=l. \tag{3.1}$$

Again using proposition 2.6 we have

$$\begin{aligned}
\lim_{\lambda\to\infty}\frac{\text{tr }(A(H+\lambda)^{-m})}{\text{tr }(H+\lambda)^{-m}} &= \lim_{\lambda\to\infty}\frac{\text{tr }(A[(h+\lambda)^{-m}]^W)}{\text{tr }[(h+\lambda)^{-m}]^W}\\
&= \lim_{\lambda\to\infty}\frac{\sum\int\frac{a(\xi,x)}{(h(\xi,x)+\lambda)^m}dx}{\sum\int\frac{1}{(h(\xi,x)+\lambda)^m}dx}\\
&= \lim_{\lambda\to\infty}\frac{\int_0^\infty\frac{1}{(u+\lambda)^m}aoh^{-1}(u)d(\overline{\mu}oh^{-1})(u)}{\int_0^\infty\frac{1}{(u+\lambda)^m}d(\overline{\mu}oh^{-1})(u)}=l,
\end{aligned}$$

where $\overline{\mu}$ is the product measure of counting measure on $\mathbb{Z}^d$ and the normalized invariant measure on $\mathbb{T}^d$. Again by using theorem 2.11 we have

$$\lim_{\lambda\to\infty}\frac{\sum\int_{h(\xi,x)\le\lambda}a(\xi,x)dx}{\sum\int_{h(\xi,x)\le\lambda}dx}=l. \tag{3.2}$$

By using equation (3.1) and (3.2), one has

$$\lim_{\lambda\to\infty}\frac{\text{tr }(\pi_\lambda A\pi_\lambda)}{\sum\int_{h(\xi,x)\le\lambda}a(\xi,x)dx}=1.$$

Then we have the equalities

$$\begin{aligned}
&\lim_{\lambda\to\infty}\frac{tr(\pi_\lambda A\pi_\lambda)}{\text{rank }\pi_\lambda}\\
=&\lim_{\lambda\to\infty}\frac{\text{tr }(\pi_\lambda \text{A}\pi_\lambda)}{\sum\int_{h(\xi,x)\le\lambda}a(\xi,x)dx}\frac{\sum\int_{h(\xi,x)\le\lambda}a(\xi,x)dx}{\text{rank }\pi_\lambda}\frac{\text{rank }\pi_\lambda}{\sum\int_{h(\xi,x)\le\lambda}dx}\\
=&\lim_{\lambda\to\infty}\frac{\sum\int_{h(\xi,x)\le\lambda}a(\xi,x)dx}{\sum\int_{h(\xi,x)\le\lambda}dx}\\
=&\lim_{\lambda\to\infty}\frac{\sum\int_{h(\xi,x)\le\lambda}a(\xi,x)\tilde{d}x}{vol(\{(\xi,x):h(\xi,x)\le\lambda\})}.
\end{aligned}$$

□

Corollary 3.4. *Let P be a polynomial on $\mathbb{R}$. Then*

$$\lim_{\lambda\to\infty}\frac{\text{tr }\pi_\lambda \text{P(A)}\pi_\lambda}{\text{rank }\pi_\lambda}=\lim_{\lambda\to\infty}\frac{1}{vol(h(\xi,x)\le\lambda)}\sum_{h(\xi,x)\le\lambda}\int P(a(\xi,x))\tilde{d}x$$

Proof. Notice that by the composition rule (theorem 2.4) of symbols, $P(A)$ is a pseudo difference operator with symbol $P(a(\xi,x)) + r_{-1}(\xi,x), r_{-1} \in S^{-1}(\mathbb{Z}^d, \mathbb{T}^d)$.

$$\lim_{\lambda\to\infty} \frac{\text{tr } \pi_\lambda \text{P(A)} \pi_\lambda}{\text{rank } \pi_\lambda}$$

$$= \lim_{\lambda\to\infty} \frac{1}{vol(h(\xi,x) \le \lambda)} \sum_{h(\xi,x)\le\lambda} \int P(a(\xi,x)) \tilde{d}x \tag{3.3}$$

$$+ \lim_{\lambda\to\infty} \frac{1}{vol(h(\xi,x) \le \lambda)} \sum_{h(\xi,x)\le\lambda} \int r(\xi,x) \tilde{d}x$$

Since any $r \in S^{-1}(\mathbb{Z}^d, \mathbb{T}^d)$ satisfies $|r(\xi,x)| \le C(\langle\xi\rangle)^{-1}$, we have

$$\frac{\sum_{\{\xi : h(\xi,x)\le\lambda\}} C(\langle\xi\rangle)^{-1}}{\int \sum_{\{\xi : h(\xi,x)\le\lambda\}} \tilde{d}x}.$$

The sum in the denominator has a higher growth as a function of λ than that in the numerator, so as λ goes to infinity this quantity goes to zero point wise in x and hence also its integral over $\mathbb{T}^d$. ☐

Corollary 3.5. *Let$f \in \mathcal{C}(\mathbb{R})$. Then* $\dfrac{\mu_\lambda(f)}{\text{rank } \pi_\lambda}$ *has a limit* $\mu(f)$ *as* $\lambda \to \infty$, *where* $\mu(f)$, $\mu_\lambda(f)$ *are defined in equation (1.2).*

Proof. We note that $rank(\pi_\lambda) = \text{tr}\,(\pi_\lambda)$. The eigen-values of $\pi_\lambda A \pi_\lambda$ are bounded by $\|A\|$ for all λ. Also the values of $a(\xi,x)$ are bounded by some constant C (say). Therefore any continuous function $f \in \mathcal{C}(\mathbb{R})$ can be approximated uniformly on $I = \{|x| \le \max(\|A\|, C)\}$ by a polynomial. Then

$$\begin{aligned}
\lim_{\lambda\to\infty} \frac{\mu_\lambda(f)}{\text{rank } \pi_\lambda} &= \lim_{\lambda\to\infty} \frac{\sum_{\lambda_j\le\lambda} (f(A)\eta_j, \eta_j)}{\text{rank } \pi_\lambda} \\
&= \lim_{n,\lambda\to\infty} \frac{\sum_{\lambda_j\le\lambda} (P_n(A)\eta_j, \eta_j)}{\text{rank } \pi_\lambda} \\
&= \lim_{n,\lambda\to\infty} \frac{1}{vol(h(\xi,x)\le\lambda)} \sum_{h(\xi,x)\le\lambda} \int P_n(a(\xi,x)) \tilde{d}x \\
&= \lim_{\lambda\to\infty} \frac{1}{vol(h(\xi,x)\le\lambda)} \sum_{h(\xi,x)\le\lambda} \int f(a(\xi,x)) \tilde{d}x.
\end{aligned}$$

☐

We now prove theorem 1.1 before taking up the proof of the main theorem.

Proof of Theorem 1.1:

The proof is almost identical to that of the Laptev-Safarov proof in [3] with mild modification to accommodate for the relative boundedness of $\tilde{T}$. We shall indicate the main steps of the proof.

Assume that $[S,T]$ is relatively bounded with respect to S . To prove the above result it is sufficient to estimate $\|(I-\pi_\lambda)T\pi_\lambda\|_{HS}$, where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm. Since

$$\|(I-\pi_\lambda)T\pi_\lambda\|_{HS}^2 \leq 2(\|(I-\pi_\lambda)T\pi_{\lambda-\epsilon}\|_{HS}^2 + \|(I-\pi_\lambda)T(\pi_\lambda-\pi_{\lambda-\epsilon})\|_{HS}^2)$$

and

$$\|(I-\pi_\lambda)T(\pi_\lambda-\pi_{\lambda-\epsilon}\|_{HS} \leq \|T\|^2 N_\epsilon(\lambda),$$

we need to estimate $\|(I-\pi_\lambda)T\pi_{\lambda-\epsilon}\|_{HS}$ only.

So by definition

$$\|(I-\pi_\lambda)T\pi_{\lambda-\epsilon}\|_{HS}^2 = \sum_{\lambda_k\geq\lambda}\sum_{\lambda_j<\lambda-\epsilon}|\langle Tf_j,f_k\rangle|^2.$$

Since

$$\langle Tf_j,f_k\rangle = (\lambda_k-\lambda_j)^{-1}(\lambda_j+1)^\sigma\langle\tilde{T}(S+I)^{-\sigma}f_j,f_k\rangle,$$

where $\tilde{T}=[T,S]$, we have,

$$\begin{aligned}
\|(I-\pi_\lambda)T\pi_{\lambda-\epsilon}\|_{HS}^2 &= \sum_{\lambda_k\geq\lambda}\sum_{\lambda_j<\lambda-\epsilon}|\langle Tf_j,f_k\rangle|^2\\
&= \sum_{\lambda_k\geq\lambda}\sum_{\lambda_j<\lambda-\epsilon}|(\lambda_k-\lambda_j)^{-2}(\lambda_j+1)^{2\sigma}||\langle\tilde{T}(S+I)^{-\sigma}f_j,f_k\rangle|^2\\
&\leq \sum_{k}\sum_{\lambda_j<\lambda-\epsilon}|(\lambda-\lambda_j)^{-2}(\lambda_j+1)^{2\sigma}]|\langle\tilde{T}(S+I)^{-\sigma}f_j,f_k\rangle|^2\\
&\leq \|\tilde{T}(S+I)^{-\sigma}\|^2(\lambda+1)^{2\sigma}\sum_{\lambda_j<\lambda-\epsilon}|(\lambda-\lambda_j)^{-2}|\\
&= \|\tilde{T}(S+I)^{-\sigma}\|^2(\lambda+1)^{2\sigma}\\
&\quad \sum_{k=1}^{K^*}\frac{1}{|\lambda-k\epsilon|^2}\#\{\lambda_j\in((k-1)\epsilon,k\epsilon)\}
\end{aligned}$$

where in the penultimate step we used $\lambda_k > \lambda$ to estimate the term and made a lower bound on the summand for each collection of λ_j's in intervals of length ϵ and we have taken K^* so that $\lambda - \frac{\epsilon}{2} \geq K^* > \lambda - \epsilon$. The maximum value of the counting measure as k varies is then estimated by $N_\epsilon(\lambda)$ and the remaining sum is uniformly bounded since it is essentially $\sum_k 1/k^2$ so the estimate becomes $N_\epsilon(\lambda)$. So,

$$\|(I - \pi_\lambda)T\pi_{\lambda-\epsilon}\|_{HS}^2 \leq C\|\tilde{T}(S+I)^{-\sigma}\|^2 N_{\frac{\epsilon}{2}}(\lambda)(\lambda+1)^{2\sigma}.$$

□

Proof of Theorem 1.2 :

We prove the theorem for the case $\rho = 1$, the other cases are similar. We take S to be H and T to be A for applying theorem 1.1, by using which we see that the theorem follows if we show

$\frac{N_\epsilon(\lambda)\lambda^{2\sigma}}{\text{rank } \pi_\lambda} \to 0$ as $\lambda \to \infty$, where N_ϵ is taken as in theorem 1.1. Denote $P_{H\leq\beta}$ to be orthogonal projection onto the spectral subspace $H \leq \beta$.

Then $N(\beta) = \text{tr P}_{\text{H}\leq\beta}$, We show that $P_{H\leq\beta} \leq P_{V\leq\beta}$ and $P_{P\leq\beta-\epsilon} \geq P_{V\leq\beta-4d-\epsilon}$. To show $P_{H\leq\beta} \leq P_{V\leq\beta}$ it is enough to show $\mathcal{H}_\beta := P_{H\leq\beta}\ell^2(\mathbb{Z}^d) \subset P_{V\leq\beta}\ell^2(\mathbb{Z}^d) := \tilde{\mathcal{H}}_\beta$.

Let $u \in \mathcal{H}_\beta$. Then $\langle Vu, u\rangle_{\mathbb{Z}^d} \leq \langle Hu, u\rangle_{\mathbb{Z}^d} \leq \beta\|u\|_{\mathbb{Z}^d}^2$. So $V \in \tilde{\mathcal{H}}_\beta$. Similarly $P_{P\leq\beta-\epsilon} \geq P_{V\leq\beta-4d-\epsilon}$ can be shown.

We get using equation (2.4) and Corollary 2.13 that for large β,

$$\begin{aligned}
\frac{\text{tr }(\text{P}_{\text{H}\leq\beta} - \text{P}_{\text{H}\leq\beta-\epsilon})}{\text{rank }(\pi_\lambda)} &\leq \frac{\text{tr }(\text{P}_{\text{H}\leq\beta}) - \text{tr }(\text{P}_{\text{V}\leq\beta-4\text{d}-\epsilon})}{\text{rank }(\pi_\lambda)} \\
&\leq \frac{\beta^{\frac{d}{k}} - (\beta - 4d - \epsilon)^{\frac{d}{k}}}{\text{rank }(\pi_\lambda)} \\
&= \frac{\beta^{\frac{d}{k}} - (\beta - 4d - \epsilon)^{\frac{d}{k}}}{\lambda^{\frac{d}{k}}} \\
&\leq C/\lambda,
\end{aligned}$$

where for the last estimate we use the binomial series to estimate the numerator to have growth atmost of the order of $\lambda^{d/k-1}$.

Then it follows that for $0 \le \sigma < \frac{1}{2}$,

$$\frac{N_\epsilon(\lambda)(\lambda+1)^{2\sigma}}{Tr(\pi_\lambda)} \to 0, \ \ as \ \lambda \to \infty.$$

□

Remark 3.6. The above limits are the same if A is replaced by $A+B$ for any compact operator B on $\ell^2(\mathbb{Z}^d)$.

Proof. To prove the above result, enough to show $\lim_{\lambda\to\infty} \dfrac{\mathrm{tr}\ (\pi_\lambda \mathrm{A^n} \pi_\lambda)}{\mathrm{tr}\ (\pi_\lambda)} = \lim_{\lambda\to\infty} \dfrac{\mathrm{tr}\ (\pi_\lambda (\mathrm{A+B})^{\mathrm{n}} \pi_\lambda)}{\mathrm{tr}\ (\pi_\lambda)}$ for any compact operator B on $\ell^2(\mathbb{Z}^d)$. Notice that $(A+B)^n = A^n +$ terms with factor $A^p B^{n-p}$ or $B^p A^{n-p}$ where $p \in \{1, 2 \cdots, n\}$. Since the class of compact operators form a two sided ideal of the class of bounded operators $(A+B)^n = A^n +$ a compact operator. We show that for any compact operator T, $\lim_{\lambda\to\infty} \frac{\mathrm{tr}\ (\mathrm{T(H+\lambda)^{-m}})}{\mathrm{tr}\ ((\mathrm{H+\lambda})^{-\mathrm{m}})} = 0$.

Since T is a compact operator, for given $\epsilon > 0$ there exist a finite rank operator T_n such that $\|T_n - T\| < \epsilon$ for $n \ge N_0$. Consider

$$\begin{aligned} \frac{\mathrm{tr}\ (\mathrm{T(H+\lambda)^{-m}})}{\mathrm{tr}\ ((\mathrm{H+\lambda})^{-\mathrm{m}})} &= \frac{\mathrm{tr}\ (\mathrm{T_n(H+\lambda)^{-m}})}{\mathrm{tr}\ ((\mathrm{H+\lambda})^{-\mathrm{m}})} + \frac{\mathrm{tr}\ ((\mathrm{T-T_n})(\mathrm{H+\lambda})^{-\mathrm{m}})}{\mathrm{tr}\ ((\mathrm{H+\lambda})^{-\mathrm{m}})} \\ &\le \frac{\mathrm{tr}\ (\mathrm{T_n(H+\lambda)^{-m}})}{\mathrm{tr}\ ((\mathrm{H+\lambda})^{-\mathrm{m}})} + \|T - T_n\| \\ &\le \frac{\mathrm{tr}\ (\mathrm{T_n(H+\lambda)^{-m}})}{\mathrm{tr}\ ((\mathrm{H+\lambda})^{-\mathrm{m}})} + \epsilon \end{aligned}$$

We will show that

$$\frac{\mathrm{tr}\ (\mathrm{T(H+\lambda)^{-m}})}{\mathrm{tr}\ ((\mathrm{H+\lambda})^{-\mathrm{m}})} \to 0$$

for each fixed finite rank T. Then the result follows.

Since T is a finite rank operator, its range is finite dimensional. Let $\{\psi_n : n = 1, \ldots, N\}$ be an orthonormal basis for the range of T. Then we have, using the positivity of H for the last inequality,

$$\begin{aligned} |\mathrm{tr}\ (\mathrm{T(H+\lambda)^{-m}})| &= |\textstyle\sum_{n=1}^N \langle (H+\lambda)^{-m})\psi_n, T\psi_n\rangle| \\ &= |\textstyle\sum_{n=1}^N \int_{Spec(H)} \frac{1}{(x+\lambda)^m}\ d\langle \psi_n, E_H(x) T\psi_n\rangle| \le \frac{N\|T\|}{\lambda^m}. \end{aligned}$$

So for such a finite rank T

$$
\begin{aligned}
\left|\frac{\mathrm{tr}\ (\mathrm{T(H+\lambda)^{-m}})}{\mathrm{tr}\ ((\mathrm{H+\lambda})^{-\mathrm{m}})}\right| &\leq CN\frac{1}{\lambda^m \mathrm{tr}\ ((\mathrm{H}+\lambda)^{-\mathrm{m}})} \\
&= CN\frac{1}{\lambda^m \sum_{i\in\mathbb{N}} \frac{1}{(\lambda_i+\lambda)^m}} \\
&\leq CN\frac{1}{\lambda^m \sum_{\lambda_i\leq\lambda} \frac{1}{(\lambda_i+\lambda)^m}} \\
&\leq \frac{CN2^m}{N(\lambda)}
\end{aligned}
$$

where $N(\lambda)$ is the total number of eigen values of H in $(0,\lambda)$ which goes to ∞ as λ goes to ∞, so this term goes to zero for each fixed N as $\lambda \to \infty$.

Therefore $\lim_{\lambda\to\infty} \frac{\mathrm{tr}\ (\mathrm{A(H+\lambda)^{-m}})}{\mathrm{tr}\ ((\mathrm{H+\lambda})^{-\mathrm{m}})} = \lim_{\lambda\to\infty} \frac{\mathrm{tr}\ (\tilde{\mathrm{A}}(\mathrm{H+\lambda})^{-\mathrm{m}})}{\mathrm{tr}\ ((\mathrm{H+\lambda})^{-\mathrm{m}})}$, where $\tilde{A} = A + B$. Now applying theorem lemma 3.1, proposition 2.6 and theorem 2.11 individually to both the limits, we have $\lim_{\lambda\to\infty} \frac{\mathrm{tr}\ (\pi_\lambda \mathrm{A^n} \pi_\lambda)}{\mathrm{tr}\ (\pi_\lambda)} = \lim_{\lambda\to\infty} \frac{\mathrm{tr}\ (\pi_\lambda (\tilde{\mathrm{A}})^{\mathrm{n}} \pi_\lambda)}{\mathrm{tr}\ (\pi_\lambda)}$. $\square$

Example 3.7. Let W be a bounded function on $\mathbb{Z}^d$. Let A be the operator of multiplication by the function W on $\ell^2(\mathbb{Z}^d)$ and $H = \rho\Delta + |\xi|^k, 0 \leq \rho \leq 1,\ k > 0$ with Δ defined as in the equation 1.1. Then it is clear that H is a positive operator and a simple estimate of the resolvent shows that it has purely discrete spectrum.

The commutator $[A, H] = [\Delta, V]$ is bounded and hence also $[A, H](H + I)^{-\sigma}$ for any $\sigma > 0$. Therefore in this case the conclusions of theorems 1.1 is valid. However without additional conditions on W the theorem 1.2 is not clear.

Example 3.8. Let W and Δ be defined as in the previous example. Let $A = \Delta + W$ and $H = \rho\Delta + |\xi|^k$ for $0 \leq \rho,\ 0 < k < 2$. Then the commutator $[A, H] = [A, \rho\Delta] + [A, |\xi|^k]$ turns out to be $[\Delta, |\xi|^k]$ up to an addition of a bounded operator. This term behaves like $C|\xi|^{k-1}$. If $k \leq 1$, this is bounded. On the other hand we have as operators $[A, H](H + I)^{-\sigma} \sim$

$|\xi|^{k-1}(|\xi|^k)^{-\sigma}$ + *Bounded operator*. Therefore the condition for the boundedness of this operator is that $k-1-k\sigma \leq 0$. Clearly if $k \in (0,2)$ we can find a $\sigma \in (0, \frac{1}{2})$ so that this condition is satisfied, while for $k \geq 2$ we cannot find any $\sigma \in (0, 1/2)$. This forces the condition on k.

Example 3.9. Let W be defined as before. Let $A = P(\Delta) + V$ and $H = \rho\Delta + |\xi|^k$ for $0 \leq \rho,\ 0 < k < 2$, where $P(\Delta)$ is a real polynomial in Δ. Then by using the previous argument we have $[A, H](H+I)^{-\sigma} \sim |\xi|^{k-1}(H+I)^{-\sigma}+$ is a bounded operator for some σ in $(0, \frac{1}{2})$ if $k \in (0, 2)$.

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