On the well-posedness of the semi-relativistic Schrödinger-Poisson system

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Abstract. We show global existence and uniqueness of strong solutions for the Schrödinger-Poisson system in the repulsive Coulomb case with relativistic kinetic energy.

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1 Introduction

In this article, we study the global well-posedness of the semi-relativistic Schrödinger-Poisson system on a finite domain. This system is relevant to the description of many-body semi-relativistic quantum particles in the mean-field limit (for instance, in heated plasma), when the particles move with extremely high velocities. Consider semi-relativistic quantum particles confined in domain $\Omega \subset \mathbb{R}^3$ which is an open, finite volume set with a C^2 boundary. The particles interact by the electrostatic field they collectively generate. In the mean-field limit, the density matrix that describes the *mixed* state of the system satisfies the Hartree-von Neumann equation

$$\begin{cases} i\partial_t \rho(t) = [H_V, \rho(t)], \quad x \in \Omega, \quad t \geq 0 \\ -\Delta V = n(t,x), \quad n(t,x) = \rho(t,x,x), \ \rho(0) = \rho_0 \end{cases}, \tag{1.1}$$

satisfying Dirichlet boundary conditions, $\rho(t,x,y) = 0$ if x or $y \in \partial\Omega$, for $t \geq 0$. The Hamiltonian is given by

$$H_V := T_m + V(t,x) \tag{1.2}$$

where the relativistic kinetic energy operator $T_m := \sqrt{-\Delta + m^2} - m$ is defined via the spectral calculus. Here, Δ denotes the Dirichlet Laplacian on $L^2(\Omega)$, and $m > 0$ is the particle mass; see [3, 2] for a derivation of this system of equations in the *non-relativistic* case. Since $\rho(t)$ is a positive, self-adjoint trace-class operator acting on $L^2(\Omega)$, its kernel can, for every $t \in \mathbb{R}_+$, be decomposed with respect to an orthonormal basis of $L^2(\Omega)$. The kernel of the initial data ρ_0 can be represented in the form

$$\rho_0(x,y) = \sum_{k\in\mathbb{N}} \lambda_k \psi_k(x)\overline{\psi_k(y)} \tag{1.3}$$

where $\{\psi_k\}_{k\in\mathbb{N}}$ denotes an orthonormal basis of $L^2(\Omega)$, with $\psi_k|_{\partial\Omega} = 0$ for all $k \in \mathbb{N}$, and coefficients

$$\underline{\lambda} := \{\lambda_k\}_{k\in\mathbb{N}} \in \ell^1 \ , \ \ \lambda_k \geq 0 \ , \ \ \sum_k \lambda_k = 1. \tag{1.4}$$

As shown below, there exists a one-parameter family of complete orthonormal bases of $L^2(\Omega)$, $\{\psi_k(t)\}_{k\in\mathbb{N}}$, with $\psi_k(t)|_{\partial\Omega} = 0$ for all $k \in \mathbb{N}$, and for $t \in \mathbb{R}_+$, such that the kernel of the solution $\rho(t)$ to (1.1) can be represented as

$$\rho(t,x,y) = \sum_{k\in\mathbb{N}} \lambda_k \psi_k(t,x)\overline{\psi_k(t,y)}. \tag{1.5}$$

Notably, the coefficients $\underline{\lambda}$ are *independent* of t, and thus the same as those in ρ_0. Substituting (1.5) in (1.1), the one-parameter family of orthonormal vectors $\{\psi_k(t)\}_{k\in\mathbb{N}}$ is seen to satisfy the semi-relativistic Schrödinger-Poisson system

$$i\frac{\partial \psi_k}{\partial t} = T_m\psi_k + V\psi_k, \quad k \in \mathbb{N} \tag{1.6}$$

$$-\Delta V[\Psi] = n[\Psi], \quad \Psi := \{\psi_k\}_{k=1}^{\infty}, \tag{1.7}$$

$$n[\Psi(x,t)] = \sum_{k=1}^{\infty} \lambda_k |\psi_k|^2, \tag{1.8}$$

with initial data $\{\psi_k(0)\}_{k=1}^{\infty}$. The potential function $V[\Psi]$ solves the Poisson equation (1.7). On both $V[\Psi]$ and $\psi_k(t)$, for all $k \in \mathbb{N}$, we impose Dirichlet boundary conditions

$$\psi_k(t,x)\ ,\ \ V(x,t) = 0,\ t \geq 0,\ \forall x \in \partial\Omega. \tag{1.9}$$

As we show in Lemma 6, below, solutions of (1.6)-(1.8) preserve the orthonormality of $\{\psi_k(t)\}_{k\in\mathbb{N}}$.

The state space for the Schrödinger-Poisson system is given by

$$\mathcal{L} := \{(\Psi, \underline{\lambda}) \mid \Psi = \{\psi_k\}_{k=1}^{\infty} \subset H_0^{\frac{1}{2}}(\Omega) \cap H^1(\Omega) \ \textit{is a complete orthonormal system in } L^2(\Omega),$$

$$\underline{\lambda} = \{\lambda_k\}_{k=1}^{\infty} \in \ell^1, \quad \lambda_k \geq 0,\ k \in \mathbb{N}, \quad \sum_{k=1}^{\infty} \lambda_k \int_{\Omega} |\nabla \psi_k|^2 dx < \infty\}.$$

For fixed $\underline{\lambda} \in \ell^1$, $\lambda_k > 0$, and for sequences of square integrable functions $\Phi := \{\phi_k\}_{k=1}^{\infty}$ and $\Psi := \{\psi_k\}_{k=1}^{\infty}$, we define the inner product

$$(\Phi, \Psi)_{\mathcal{L}^2_{\underline{\lambda}}(\Omega)} := \sum_{k=1}^{\infty} \lambda_k (\phi_k, \psi_k)_{L^2(\Omega)},$$

which induces the norm

$$\|\Phi\|_{\mathcal{L}^2_{\underline{\lambda}}(\Omega)} := (\sum_{k=1}^{\infty} \lambda_k \|\phi_k\|^2_{L^2(\Omega)})^{\frac{1}{2}},$$

and we introduce the corresponding Hilbert space

$$\mathcal{L}^2_{\underline{\lambda}}(\Omega) := \{\Phi = \{\phi_k\}_{k=1}^{\infty} \mid \phi_k \in L^2(\Omega),\ \forall\ k \in \mathbb{N},\ \|\Phi\|_{\mathcal{L}^2_{\underline{\lambda}}(\Omega)} < \infty\}.$$

Our main result is as follows.

Theorem 1. *For every initial state $(\Psi(x,0), \underline{\lambda}) \in \mathcal{L}$, there is a unique mild solution $\Psi(x,t)$, $t \in [0,\infty)$, of (1.6)-(1.8) with $(\Psi(x,t), \underline{\lambda}) \in \mathcal{L}$, which is also a unique strong global solution in $\mathcal{L}^2_{\underline{\lambda}}(\Omega)$.*

Establishing the global well-posedness of the Schrödinger-Poisson system plays a crucial role in proving the existence and nonlinear stability of stationary states, i.e. the nonlinear bound states of the Schrödinger-Poisson system, which was done in the nonrelativistic case in [4, 6]. The problem in one dimension was treated in [8]. The semiclassical limit of the Schrödinger-Poisson system with the relativistic kinetic energy was studied in the recent article [1]. Global well-posedness for a single semi-relativistic Hartree equation in $\mathbb{R}^3$ was established in [5]. In the present work, we deal with the infinite system of equations in a finite volume set with Dirichlet boundary conditions, and, as distinct from [5], we do not use the regularization of the Poisson equation. Moreover, both the results of [5] and Theorem 1 above do not rely on Strichartz type estimates.

2 Proof of global well-posedness

We make a fixed choice of $\underline{\lambda} = \{\lambda_k\}_{k=1}^{\infty} \in \ell^1$, with $\lambda_k > 0$ and $\sum \lambda_k = 1$, denoting the sequence of coefficients determined by the initial data ρ_0 of the Hartree-von Neumann equation (1.1) via (1.5), for $t = 0$. We note that we require all $\lambda_k > 0$ to be positive for the subsequent analysis. This does not lead to any loss of generality since by density arguments, any ρ_0 (and likewise $\rho(t)$) can be approximated arbitrarily well by an expansion of the form (1.3), respectively (1.5), with $\lambda_k > 0$.

We introduce inner products $(\cdot,\cdot)_{\mathcal{H}^{1/2}_{\underline{\lambda}}(\Omega)}$ and $(\cdot,\cdot)_{\mathcal{H}^{1}_{\underline{\lambda}}(\Omega)}$ which induce the generalized inhomogenous Sobolev norms

$$\|\Phi\|_{\mathcal{H}^{1/2}_{\underline{\lambda}}(\Omega)} := (\sum_{k=1}^{\infty} \lambda_k \|\phi_k\|^2_{H^{\frac{1}{2}}(\Omega)})^{\frac{1}{2}} \ \text{ and } \ \|\Phi\|_{\mathcal{H}^{1}_{\underline{\lambda}}(\Omega)} := (\sum_{k=1}^{\infty} \lambda_k \|\phi_k\|^2_{H^{1}(\Omega)})^{\frac{1}{2}},$$

and define the corresponding Hilbert spaces

$$\mathcal{H}^{1/2}_{\underline{\lambda}}(\Omega) := \{\Phi = \{\phi_k\}_{k=1}^{\infty} \mid \phi_k \in H_0^{\frac{1}{2}}(\Omega), \ \forall\, k \in \mathbb{N}, \ \|\Phi\|_{\mathcal{H}^{1/2}_{\underline{\lambda}}(\Omega)} < \infty\}$$

and

$$\mathcal{H}^{1}_{\underline{\lambda}}(\Omega) := \{\Phi = \{\phi_k\}_{k=1}^{\infty} \mid \phi_k \in H_0^{\frac{1}{2}}(\Omega) \cap H^1(\Omega), \ \forall\, k \in \mathbb{N}, \ \|\Phi\|_{\mathcal{H}^{1}_{\underline{\lambda}}(\Omega)} < \infty\}$$

respectively. We also introduce the generalized homogenous Sobolev norms

$$\|\Phi\|_{\dot{\mathcal{H}}^{1/2}_{\underline{\lambda}}(\Omega)} := (\sum_{k=1}^{\infty} \lambda_k \||p|^{\frac{1}{2}}\phi_k\|^2_{L^2(\Omega)})^{\frac{1}{2}} \ \textit{and} \ \|\Phi\|_{\dot{\mathcal{H}}^{1}_{\underline{\lambda}}(\Omega)} := (\sum_{k=1}^{\infty} \lambda_k \|\nabla\phi_k\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

Here, $|p|$ stands for the operator $\sqrt{-\Delta}$, and has the meaning of the relativistic kinetic energy of a particle with zero mass. We note the following equivalence of norms.

Lemma 2. *For $\Phi \in \mathcal{H}^{1/2}_{\underline{\lambda}}(\Omega)$, the norms $\|\Phi\|_{\mathcal{H}^{1/2}_{\underline{\lambda}}(\Omega)}$ and $\|\Phi\|_{\dot{\mathcal{H}}^{1/2}_{\underline{\lambda}}(\Omega)}$ are equivalent. If $\Phi \in \mathcal{H}^{1}_{\underline{\lambda}}(\Omega)$, then $\|\Phi\|_{\mathcal{H}^{1}_{\underline{\lambda}}(\Omega)}$ is equivalent to $\|\Phi\|_{\dot{\mathcal{H}}^{1}_{\underline{\lambda}}(\Omega)}$.*

Proof. Clearly

$$\|\Phi\|_{\dot{\mathcal{H}}^{1/2}_{\underline{\lambda}}(\Omega)} \leq (\sum_{k=1}^{\infty} \lambda_k \{\|\phi_k\|^2_{L^2(\Omega)} + \||p|^{\frac{1}{2}}\phi_k\|^2_{L^2(\Omega)}\})^{\frac{1}{2}} = (\sum_{k=1}^{\infty} \lambda_k \|\phi_k\|^2_{H^{\frac{1}{2}}(\Omega)})^{\frac{1}{2}} = \|\Phi\|_{\mathcal{H}^{1/2}_{\underline{\lambda}}(\Omega)}.$$

We will make use of the Poincaré inequality

$$\int_{\Omega} |\nabla \phi_k|^2 dx \geq c_p \int_{\Omega} |\phi_k|^2 dx \tag{2.1}$$

with the constant $c_p > 0$ dependent upon the domain Ω with Dirichlet boundary conditions. Thus

$$\||p|^{\frac{1}{2}}\phi_k\|^2_{L^2(\Omega)} \geq \sqrt{c_p}\|\phi_k\|^2_{L^2(\Omega)},$$

which enables us to estimate

$$\|\Phi\|_{\mathcal{H}^{1/2}_{\underline{\lambda}}(\Omega)} = (\sum_{k=1}^{\infty} \lambda_k \{\|\phi_k\|^2_{L^2(\Omega)} + \||p|^{\frac{1}{2}}\phi_k\|^2_{L^2(\Omega)}\})^{\frac{1}{2}} \leq$$

$$\leq \sqrt{1 + \frac{1}{\sqrt{c_p}}} (\sum_{k=1}^{\infty} \lambda_k \||p|^{\frac{1}{2}}\phi_k\|^2_{L^2(\Omega)})^{\frac{1}{2}} = C\|\Phi\|_{\dot{\mathcal{H}}^{1/2}_{\underline{\lambda}}(\Omega)}.$$

Let us compare the remaining two norms. Clearly,

$$\|\Phi\|_{\dot{\mathcal{H}}^{1}_{\underline{\lambda}}(\Omega)} \leq (\sum_{k=1}^{\infty} \lambda_k \|\phi_k\|^2_{H^1(\Omega)})^{\frac{1}{2}} = \|\Phi\|_{\mathcal{H}^{1}_{\underline{\lambda}}(\Omega)}.$$

On the other hand, by means of the Poincaré inequality (2.1),

$$\|\Phi\|_{\mathcal{H}^{1}_{\underline{\lambda}}(\Omega)} = (\sum_{k=1}^{\infty} \lambda_k \{\|\phi_k\|^2_{L^2(\Omega)} + \|\nabla\phi_k\|^2_{L^2(\Omega)}\})^{\frac{1}{2}} \leq$$

$$\leq \sqrt{1 + \frac{1}{c_p}} (\sum_{k=1}^{\infty} \lambda_k \|\nabla\phi_k\|^2_{L^2(\Omega)})^{\frac{1}{2}} = \|\Phi\|_{\dot{\mathcal{H}}^{1}_{\underline{\lambda}}(\Omega)}.$$

□

Let $\Psi = \{\psi_m\}_{m=1}^{\infty}$ be a wave function and the relativistic kinetic energy operator acts on it $T_m\Psi = (\sqrt{-\Delta + m^2} - m)\psi$ componentwise. We have the following two lemmas.

Lemma 3. *The domain of the kinetic energy operator is given by* $D(T_m) = \mathcal{H}^1_{\underline{\lambda}}(\Omega) \subseteq \mathcal{L}^2_{\underline{\lambda}}(\Omega)$.

Proof. Let $\Psi \in \mathcal{H}^1_{\underline{\lambda}}(\Omega)$. Then

$$\sum_{m=1}^{\infty} \lambda_m \|\psi_m\|^2_{H^1(\Omega)} = \sum_{m=1}^{\infty} \lambda_m \{\|\psi_m\|^2_{L^2(\Omega)} + \|\nabla\psi_m\|^2_{L^2(\Omega)}\} \geq \sum_{m=1}^{\infty} \lambda_m \|\psi_m\|^2_{L^2(\Omega)},$$

and also, $\|\Psi\|_{\mathcal{L}^2_{\underline{\lambda}}(\Omega)} < \infty$. We estimate

$$\|T_m\psi_k\|^2_{L^2(\Omega)} = ((-\Delta + m^2)\psi_k, \psi_k)_{L^2(\Omega)} + m^2\|\psi_k\|^2_{L^2(\Omega)} - 2m(\sqrt{-\Delta + m^2}\psi_k, \psi_k)_{L^2(\Omega)} \leq$$

$$\leq \|\nabla\psi_k\|^2_{L^2(\Omega)} + 2m^2\|\psi_k\|^2_{L^2(\Omega)} \leq c(m)\|\psi_k\|^2_{H^1(\Omega)},$$

where $c(m)$ is a mass dependent constant. Hence

$$\|T_m\Psi\|^2_{\mathcal{L}^2_{\underline{\lambda}}(\Omega)} = \sum_{k=1}^{\infty} \lambda_k \|T_m\psi_k\|^2_{L^2(\Omega)} \leq c(m) \sum_{k=1}^{\infty} \lambda_k \|\psi_k\|^2_{H^1(\Omega)} < \infty.$$

□

Lemma 4. *The operator T_m generates the group $e^{-iT_m t}$, $t \in \mathbb{R}$, of unitary operators on $\mathcal{L}^2_{\underline{\lambda}}(\Omega)$.*

Proof. For $\alpha, \beta \in \mathcal{L}^2_{\underline{\lambda}}(\Omega)$ we compute the inner product

$$(e^{-iT_m t}\alpha, e^{-iT_m t}\beta)_{\mathcal{L}^2_{\underline{\lambda}}(\Omega)} = \sum_{k=1}^{\infty} \lambda_k (e^{-iT_m t}\alpha_k, e^{-iT_m t}\beta_k)_{L^2(\Omega)} = \sum_{k=1}^{\infty} \lambda_k (\alpha_k, \beta_k)_{L^2(\Omega)} = (\alpha, \beta)_{\mathcal{L}^2_{\underline{\lambda}}(\Omega)}.$$

□

We rewrite the Schrödinger-Poisson system for $x \in \Omega$ into the form

$$\Psi_t = -iT_m\Psi + F[\Psi(x,t)], \ \textit{where} \ F[\Psi] := i^{-1}V[\Psi]\Psi, \tag{2.2}$$

$$-\Delta V[\Psi] = n[\Psi], \ \textit{where} \ V|_{\partial\Omega} = 0,$$

$$n[\Psi] = \sum_{k=1}^{\infty} \lambda_k |\psi_k|^2$$

and prove the following auxiliary result.

Lemma 5. *The map defined in (2.2) $F : \mathcal{H}^1_{\underline{\lambda}}(\Omega) \to \mathcal{H}^1_{\underline{\lambda}}(\Omega)$ is locally Lipschitz continuous.*

Proof. Let $\Psi,\ \Phi \in \mathcal{H}^1_{\underline{\lambda}}(\Omega)$ with $\Psi = \{\psi_k\}_{k=1}^{\infty},\ \Phi = \{\phi_k\}_{k=1}^{\infty}$ and $t \in [0, T]$. Then,

$$\|F[\Psi]-F[\Phi]\|_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)} = \|i^{-1}V[\Psi]\Psi - i^{-1}V[\Phi]\Phi\|_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)} = \|V[\Psi](\Psi-\Phi)+(V[\Psi]-V[\Phi])\Phi\|_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)}.$$

This can be easily estimated above by means of Lemma 2 by

$$C\|V[\Psi](\Psi-\Phi)\|_{\dot{\mathcal{H}}^1_{\underline{\lambda}}(\Omega)} + C\|(V[\Psi]-V[\Phi])\Phi\|_{\dot{\mathcal{H}}^1_{\underline{\lambda}}(\Omega)},$$

which equals

$$C(\sum_{k=1}^{\infty}\lambda_k\|\nabla(V[\Psi](\psi_k-\phi_k))\|^2_{L^2(\Omega)})^{\frac{1}{2}} + C(\sum_{k=1}^{\infty}\lambda_k\|\nabla((V[\Psi]-V[\Phi])\phi_k)\|^2_{L^2(\Omega)})^{\frac{1}{2}}. \tag{2.3}$$

Here, C denotes a finite, positive, universal constant. Clearly, we have

$$\|\nabla(V[\Psi](\psi_k-\phi_k))\|^2_{L^2(\Omega)} \le 2\|(\nabla V[\Psi])(\psi_k-\phi_k)\|^2_{L^2(\Omega)} + 2\|V[\Psi]\nabla(\psi_k-\phi_k)\|^2_{L^2(\Omega)}.$$

By means of the Schwarz inequality this can be bounded above by

$$C\|\nabla V[\Psi]\|^2_{L^4(\Omega)}\|\psi_k-\phi_k\|^2_{L^6(\Omega)} + 2\|V[\Psi]\|^2_{L^\infty(\Omega)}\|\nabla(\psi_k-\phi_k)\|^2_{L^2(\Omega)}.$$

By applying the Sobolev embedding theorems to these expressions, we arrive at

$$C\|\Delta V[\Psi]\|^2_{L^2(\Omega)}\|\nabla(\psi_k-\phi_k)\|^2_{L^2(\Omega)} \le C\|V[\Psi]\|^2_{H^2(\Omega)}\|\nabla(\psi_k-\phi_k)\|^2_{L^2(\Omega)}.$$

To estimate the remaining term in (2.3), we use

$$\|\nabla((V[\Psi]-V[\Phi])\phi_k)\|^2_{L^2(\Omega)} \le 2\|\nabla(V[\Psi]-V[\Phi])\phi_k\|^2_{L^2(\Omega)} + 2\|(V[\Psi]-V[\Phi])\nabla\phi_k\|^2_{L^2(\Omega)}.$$

The Schwarz inequality yields

$$2\|\nabla(V[\Psi]-V[\Phi])\|^2_{L^4(\Omega)}\|\phi_k\|^2_{L^4(\Omega)} + 2\|(V[\Psi]-V[\Phi])\|^2_{L^\infty(\Omega)}\|\nabla\phi_k\|^2_{L^2(\Omega)}.$$

Applying the Sobolev embedding theorem along with the Hölder inequality to these expressions, we find

$$C\|\Delta(V[\Psi]-V[\Phi])\|^2_{L^2(\Omega)}\|\phi_k\|^2_{L^6(\Omega)} + C\|\Delta(V[\Psi]-V[\Phi])\|^2_{L^2(\Omega)}\|\nabla\phi_k\|^2_{L^2(\Omega)}.$$

From the Sobolev inequality used in the first of the two terms above we deduce the upper bound

$$C\|V[\Psi]-V[\Phi]\|^2_{H^2(\Omega)}\|\nabla\phi_k\|^2_{L^2(\Omega)}.$$

Therefore, for the norm of the difference $\|F[\psi]-F[\Phi]\|_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)}$ we have the estimate from above as

$$C\|V[\Psi]\|_{H^2(\Omega)}(\sum_{k=1}^{\infty}\lambda_k\|\nabla(\psi_k-\phi_k)\|^2_{L^2(\Omega)})^{\frac{1}{2}} + C\|V[\Psi]-V[\Phi]\|_{H^2(\Omega)}(\sum_{k=1}^{\infty}\lambda_k\|\nabla\phi_k\|^2_{L^2(\Omega)})^{\frac{1}{2}},$$

which obviously equals to

$$C\|V[\Psi]\|_{H^2(\Omega)}\|\Psi-\Phi\|_{\dot{\mathcal{H}}^1_{\underline{\lambda}}(\Omega)}+C\|V[\Psi]-V[\Phi]\|_{H^2(\Omega)}\|\Phi\|_{\dot{\mathcal{H}}^1_{\underline{\lambda}}(\Omega)}.$$

Let us apply the Poincaré and the Schwarz inequalities to estimate the Sobolev norm of the potential function as

$$\|V[\Psi]\|_{H^2(\Omega)}\le C\|\Delta V\|_{L^2(\Omega)}=C\|n[\Psi]\|_{L^2(\Omega)}.$$

Hence, our goal is to estimate the appropriate norm of the particle concentration. From the Schwarz inequality,

$$\|n[\Psi]\|^2_{L^2(\Omega)}=\sum_{k,l=1}^{\infty}\lambda_k\lambda_l(|\psi_k|^2,|\psi_l|^2)_{L^2(\Omega)}\le(\sum_{k=1}^{\infty}\lambda_k\|\psi_k\|^2_{L^4(\Omega)})^2.$$

and using the Hölder inequality along with the Sobolev inequality,

$$\|n[\Psi]\|_{L^2(\Omega)}\le C\sum_{k=1}^{\infty}\lambda_k\|\psi_k\|^2_{L^6(\Omega)}\le C\sum_{k=1}^{\infty}\lambda_k\|\nabla\psi_k\|^2_{L^2(\Omega)}.$$

Hence, we arrive at the estimates for the particle concentration and the norms on the potential function,

$$\|n[\Psi]\|_{L^2(\Omega)}\le C\|\Psi\|^2_{\dot{\mathcal{H}}^1_{\underline{\lambda}}(\Omega)},\quad \|V[\Psi]\|_{H^2(\Omega)}\le C\|\Psi\|^2_{\dot{\mathcal{H}}^1_{\underline{\lambda}}(\Omega)}$$

with $\|\cdot\|_{\dot{\mathcal{H}}^1_{\underline{\lambda}}(\Omega)}$ and $\|\cdot\|_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)}$ equivalent via Lemma 2. Evidently,

$$W:=V[\Psi]-V[\Phi]$$

satisfies the Poisson equation,

$$-\Delta W=n[\Psi]-n[\Phi],\quad W|_{\partial\Omega}=0,$$

and Dirichlet boundary conditions. Applying the Poincaré inequality along with the Schwarz inequality, we arrive at

$$\|W\|^2_{H^2(\Omega)}\le C\|\Delta W\|^2_{L^2(\Omega)},$$

such that

$$\|W\|_{H^2(\Omega)}\le C\|n[\Psi]-n[\Phi]\|_{L^2(\Omega)}.$$

We will use the trivial inequality

$$|n[\Psi]-n[\Phi]|\le\sum_{k=1}^{\infty}\lambda_k(|\psi_k|+|\phi_k|)|\psi_k-\phi_k|.$$

The Schwarz inequality applied twice yields

$$\|n[\Psi]-n[\Phi]\|^2_{L^2(\Omega)} \le \left(\sum_{k=1}^{\infty}\lambda_k\sqrt{\int_\Omega(|\psi_k|+|\phi_k|)^2|\psi_k-\phi_k|^2dx}\right)^2 \le$$

$$\le (\sum_{k=1}^{\infty}\lambda_k\||\psi_k|+|\phi_k|\|_{L^4(\Omega)}\|\psi_k-\phi_k\|_{L^4(\Omega)})^2 \le (\sum_{k=1}^{\infty}\lambda_k(\|\psi_k\|_{L^4(\Omega)}+\|\phi_k|\|_{L^4(\Omega)})\|\psi_k-\phi_k\|_{L^4(\Omega)})^2,$$

and using it again gives

$$\sum_{k=1}^{\infty}\lambda_k(\|\psi_k\|_{L^4(\Omega)}+\|\phi_k|\|_{L^4(\Omega)})^2\sum_{s=1}^{\infty}\lambda_s\|\psi_s-\phi_s\|^2_{L^4(\Omega)}.$$

Applying the Hölder and Sobolev inequalities, we arrive at

$$C\sum_{k=1}^{\infty}\lambda_k(\|\nabla\psi_k\|^2_{L^2(\Omega)}+\|\nabla\phi_k\|^2_{L^2(\Omega)})\sum_{s=1}^{\infty}\lambda_s\|\nabla\psi_s-\nabla\phi_s\|^2_{L^2(\Omega)}.$$

This quantity can be easily estimated above by

$$C\left(\sum_{k=1}^{\infty}\lambda_k\|\psi_k\|^2_{H^1(\Omega)}+\sum_{l=1}^{\infty}\lambda_l\|\phi_l\|^2_{H^1(\Omega)}\right)\sum_{s=1}^{\infty}\lambda_s\|\psi_s-\phi_s\|^2_{H^1(\Omega)},$$

which clearly equals to

$$C(\|\Psi\|^2_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)}+\|\Phi\|^2_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)})\|\Psi-\Phi\|^2_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)}.$$

Therefore,

$$\|n[\Psi]-n[\Phi]\|_{L^2(\Omega)} \le C(\|\Psi\|_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)}+\|\Phi\|_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)})\|\Psi-\Phi\|_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)}$$

and

$$\|V[\Psi]-V[\Phi]\|_{H^2(\Omega)} \le C(\|\Psi\|_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)}+\|\Phi\|_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)})\|\Psi-\Phi\|_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)}.$$

Collecting the estimates above, we arrive at

$$\|F[\Psi]-F[\Phi]\|_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)} \le C(\|\Psi\|^2_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)}+\|\Phi\|^2_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)})\|\Psi-\Phi\|_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)},$$

which completes the proof of the lemma. □

From standard arguments (see for instance Theorem 1.7 of [7]) thus follows that the above Schrödinger-Poisson system admits a unique mild solution (Ψ, n, V) in $\mathcal{H}^1_{\underline{\lambda}}(\Omega)$ on a time interval $[0,T)$, for some $T>0$, satisfying the integral equation

$$\Psi(t) = e^{-iT_m t}\Psi(0) + e^{-iT_m t}\int_0^t e^{iT_m s}F[\Psi(s)]ds \tag{2.4}$$

in $\mathcal{H}^1_{\underline{\lambda}}(\Omega)$. Moreover,

$$\lim_{t\nearrow T}\|\Psi(t)\|_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)} = \infty$$

if T is finite. We also note that Ψ is a unique strong solution in $\mathcal{L}^2_{\underline{\lambda}}(\Omega)$. We shall next prove that this solution is in fact global in time. First we prove the following lemma.

Lemma 6. *Suppose for the unique mild solution* (2.4) *of the Schrödinger-Poisson system (1.6)-(1.8) that* $\{\psi_k(x,0)\}_{k=1}^{\infty}$ *at* $t=0$ *forms a complete orthonormal system in* $L^2(\Omega)$*. Then, for any* $t\in[0,T)$*, the set* $\{\psi_k(x,t)\}_{k=1}^{\infty}$ *remains a complete orthonormal system in* $L^2(\Omega)$*. Moreover, the* $\mathcal{L}^2_{\underline{\lambda}}(\Omega)$*-norm is preserved,* $\|\Psi(x,t)\|_{\mathcal{L}^2_{\underline{\lambda}}(\Omega)} = \|\Psi(x,0)\|_{\mathcal{L}^2_{\underline{\lambda}}(\Omega)},\ t\in[0,T)$*.*

Proof. Given the solution $\Psi(t)$ of the Schrödinger-Poisson system on $[0,T)$, we obtain the time-dependent one-particle Hamiltonian

$$H_{V_\Psi}(t) = T_m + V_\Psi(t,x)$$

where the potential V_Ψ solves $-\Delta V_\Psi(t,x) = n[\Psi(t)]$ with Dirichlet boundary conditions, see (1.2). Accordingly, the components of $\Psi(t)$ solve the *linear, non-autonomous* Schrödinger equation $i\partial_t\psi_k(t,x) = H_{V_\Psi}(t)\psi_k(t,x)$, for $k\in\mathbb{N}$, on the time interval $[0,T)$. We thus have, for $t\in[0,T)$,

$$\psi_k(x,t) = (e^{-i\int_0^t H_{V_\Psi}(\tau)d\tau}\psi_k)(x,0),\ k\in\mathbb{N}, \tag{2.5}$$

and therefore

$$(\psi_k(x,t),\psi_l(x,t))_{L^2(\Omega)} = (e^{-i\int_0^t H_{V_\Psi}(\tau)d\tau}\psi_k(x,0), e^{-i\int_0^t H_{V_\Psi}(\tau)d\tau}\psi_l(x,0))_{L^2(\Omega)} =$$

$$= (\psi_k(x,0),\psi_l(x,0))_{L^2(\Omega)} = \delta_{k,l}, \quad k,l\in\mathbb{N},$$

where $\delta_{k,l}$ stands for the Kronecker symbol. Obviously, for $k\in\mathbb{N}$,

$$\|\psi_k(x,t)\|^2_{L^2(\Omega)} = \|\psi_k(x,0)\|^2_{L^2(\Omega)},$$

such that for $t\in[0,T)$, the $\mathcal{L}^2_{\underline{\lambda}}(\Omega)$-norm is conserved,

$$\|\Psi(x,t)\|_{\mathcal{L}^2_{\underline{\lambda}}(\Omega)} = (\sum_{k=1}^{\infty}\lambda_k\|\psi_k(x,t)\|^2_{L^2(\Omega)})^{\frac{1}{2}} = (\sum_{k=1}^{\infty}\lambda_k\|\psi_k(x,0)\|^2_{L^2(\Omega)})^{\frac{1}{2}} = \|\Psi(x,0)\|_{\mathcal{L}^2_{\underline{\lambda}}(\Omega)}.$$

Let us consider an arbitrary function $f(x)\in L^2(\Omega)$. Clearly, we have the expansion

$$f(x) = \sum_{k=1}^{\infty}(f(y),\psi_k(y,0))_{L^2(\Omega)}\psi_k(x,0)$$

and similarly

$$e^{i\int_0^t H_{V_\Psi}(\tau)d\tau} f(x) = \sum_{k=1}^{\infty} (e^{i\int_0^t H_{V_\Psi}(\tau)d\tau} f(y), \psi_k(y,0))_{L^2(\Omega)} \psi_k(x,0).$$

Thus, by means of (2.5) we arrive at the expansion

$$f(x) = \sum_{k=1}^{\infty} (f(y), \psi_k(y,t))_{L^2(\Omega)} \psi_k(x,t)$$

for $t \in [0,T)$. □

Furthermore, we have conservation of energy for solutions to the Schrödinger-Poisson system in the following sense.

Lemma 7. *For the unique mild solution* (2.4) *of the Schrödinger-Poisson system (1.6)-(1.8) and for any value of time* $t \in [0,T)$ *we have the identity*

$$\|\Psi(x,t)\|^2_{\dot{\mathcal{H}}^{1/2}_{\underline{\lambda}}(\Omega)} + \frac{1}{2}\|\nabla V[\Psi(x,t)]\|^2_{L^2(\Omega)} = \|\Psi(x,0)\|^2_{\dot{\mathcal{H}}^{1/2}_{\underline{\lambda}}(\Omega)} + \frac{1}{2}\|\nabla V[\Psi(x,0)]\|^2_{L^2(\Omega)}.$$

Proof. Complex conjugation of the Schrödinger-Poisson system (1.6) yields

$$-i\frac{\partial \bar{\psi}_k}{\partial t} = T_m \bar{\psi}_k + V[\psi]\bar{\psi}_k, \quad k \in \mathbb{N}. \tag{2.6}$$

Adding the k-th equation of the original system (1.6) multiplied by $\frac{\partial \bar{\psi}_k}{\partial t}$, and the k-th equation in (2.6) multiplied by $\frac{\partial \psi_k}{\partial t}$, we obtain

$$\frac{\partial}{\partial t}\|T_m^{\frac{1}{2}}\psi_k\|^2_{L^2(\Omega)} + \int_\Omega V[\psi]\frac{\partial}{\partial t}|\psi_k|^2 dx = 0, \quad k \in \mathbb{N}.$$

Thus, multiplying by λ_k, and summing over k, we find

$$\frac{\partial}{\partial t}\|\Psi(x,t)\|^2_{\dot{\mathcal{H}}^{1/2}_{\underline{\lambda}}(\Omega)} + \int_\Omega V[\Psi(x,t)]\frac{\partial}{\partial t} n[\Psi(x,t)] dx = 0. \tag{2.7}$$

One can easily verify the identity

$$\frac{\partial}{\partial t}\|\nabla V[\Psi(x,t)]\|^2_{L^2(\Omega)} = 2\int_\Omega V[\Psi(x,t)]\frac{\partial}{\partial t} n[\Psi(x,t)] dx,$$

which we substitute in (2.7) to complete the proof of the lemma. □

With the auxiliary statements proven above at our disposal, we may now prove our main result, Theorem 1.

Proof of Theorem 1. The proof follows from the blow-up alternative and conservation laws. It follows from Lemma 7 that $\|\Psi(t)\|_{\dot{\mathcal{H}}^{1/2}_{\underline{\lambda}}(\Omega)}$ is bounded from above uniformly in time,

$$\|\Psi(t)\|^2_{\dot{\mathcal{H}}^{1/2}_{\underline{\lambda}}(\Omega)} \leq \|\Psi(t)\|^2_{\dot{\mathcal{H}}^{1/2}_{\underline{\lambda}}(\Omega)} + \frac{1}{2}\|\nabla V[\Psi(t)]\|^2_{L^2(\Omega)} = \|\Psi(0)\|^2_{\dot{\mathcal{H}}^{1/2}_{\underline{\lambda}}(\Omega)} + \frac{1}{2}\|\nabla V[\Psi(0)]\|^2_{L^2(\Omega)}.$$

We need to bound $\|\Psi(t)\|_{\dot{\mathcal{H}}^1_{\underline{\lambda}}(\Omega)}$. We recall the mild solution of the Schrödinger-Poisson system (1.6)-(1.8), given by

$$\Psi(t) = e^{-iT_m t}\Psi(0) + e^{-iT_m t}\int_0^t e^{iT_m s} F[\Psi(s)]ds, \tag{2.8}$$

which implies

$$\|\Psi(t)\|_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)} \leq \|\Psi(0)\|_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)} + \int_0^t \|F[\Psi(s)]\|_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)}.$$

From Lemma 2, we have

$$\begin{aligned}\|F[\Psi]\|_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)} &= \|V[\Psi]\Psi\|_{\mathcal{H}^1_{\underline{\lambda}}(\Omega)} \leq C\|V[\Psi]\Psi\|_{\dot{\mathcal{H}}^1_{\underline{\lambda}}(\Omega)} \\ &\leq C\left(\sum_{k=1}^{\infty}\lambda_k\|\nabla(V[\Psi]\psi_k)\|^2_{L^2(\Omega)}\right)^{1/2}.\end{aligned}$$

Now,

$$\begin{aligned}\|\nabla(V[\psi]\psi)\|^2_{L^2(\Omega)} &\leq \|\nabla V[\Psi]\psi_k\|^2_{L^2(\Omega)} + \|V[\Psi]\nabla\psi_k\|^2_{L^2(\Omega)} \\ &\leq \|\nabla V[\Psi]\|^2_{L^6(\Omega)}\|\psi_k\|^2_{L^3(\Omega)} + \|V[\Psi]\|^2_{L^\infty(\Omega)}\|\nabla\psi_k\|^2_{L^2(\Omega)} \\ &\leq \|\nabla V[\Psi]\|^2_{L^6(\Omega)}\|\psi_k\|^2_{H^{1/2}(\Omega)} + \|V[\Psi]\|^2_{L^\infty(\Omega)}\|\psi_k\|^2_{H^1(\Omega)},\end{aligned}$$

where we have used Hölder's inequality in the second line and the Sobolev inequality

$$\|f\|_{L^{\frac{6}{3-2p}}(\Omega)} \leq C\|f\|_{H^p(\Omega)}$$

in the last line. To evaluate $\|\nabla V[\Psi]\|_{L^6(\Omega)}$, recall that $\Delta V[\Psi] = -n[\Psi]$. Applying Hölder's and Sobolev inequalities, we get

$$\begin{aligned}\|\nabla V[\Psi]\|^2_{L^6(\Omega)} &\leq C\|\nabla V[\Psi]\|^2_{H^1(\Omega)} \leq C\|n[\Psi]\|^2_{L^2(\Omega)} \\ &\leq C\sum_{k,l=1}^{\infty}\lambda_k\lambda_l(|\psi_k|^2, |\psi_l|^2)_{L^2(\Omega)} \leq C\sum_{k,l=1}^{\infty}\lambda_k\lambda_l\|\psi_k\psi_l\|^2_{L^2(\Omega)} \\ &\leq C\sum_{k,l=1}^{\infty}\lambda_k\lambda_l\|\psi_k\|^2_{L^6(\Omega)}\|\psi_l\|^2_{L^3(\Omega)} \leq C(\sum_{k=1}^{\infty}\lambda_k\|\psi_k\|^2_{H^1(\Omega)})(\sum_{l=1}^{\infty}\lambda_l\|\psi_l\|^2_{H^{1/2}(\Omega)}) \\ &\leq C\|\Psi\|^2_{\dot{\mathcal{H}}^1_{\underline{\lambda}}(\Omega)}\|\Psi\|^2_{\dot{\mathcal{H}}^{1/2}_{\underline{\lambda}}(\Omega)}.\end{aligned}$$

We now estimate $\|V[\Psi]\|_{L^\infty(\Omega)}$. The Sobolev inequality implies

$$\|V[\Psi]\|^2_{L^\infty(\Omega)} \le C\||p|^{-1/2}n[\Psi]\|^2_{L^2(\Omega)}.$$

We claim that $\||p|^{-1/2}n[\Psi]\|_{L^2(\Omega)}$ is controlled by $\|\Psi\|_{\dot{\mathcal{H}}^{1/2}_{\underline{\lambda}}(\Omega)}$.

$$\begin{aligned}\||p|^{-1/2}n[\Psi]\|^2_{L^2(\Omega)} &= (n[\Psi], |p|^{-1}n[\Psi])_{L^2(\Omega)} \le \|n[\Psi]\|_{L^{3/2}(\Omega)}\||p|^{-1}n[\Psi]\|_{L^3(\Omega)} \\ &\le C\|\Psi\|^2_{L^3(\Omega)}\||p|^{-1}n[\Psi]\|_{H^{1/2}(\Omega)} \le C\|\Psi\|^2_{H^{1/2}(\Omega)}\||p|^{-1/2}n[\Psi]\|_{L^2(\Omega)},\end{aligned}$$

where we have used Hölder's inequality in the first line, and the Sobolev inequality in the second line. It follows that

$$\||p|^{-1/2}n[\Psi]\|_{L^2(\Omega)} \le C\|\Psi\|^2_{\dot{\mathcal{H}}^{1/2}_{\underline{\lambda}}(\Omega)},$$

and hence

$$\|V[\Psi]\|^2_{L^\infty(\Omega)} \le C\|\Psi\|^4_{\dot{\mathcal{H}}^{1/2}_{\underline{\lambda}}(\Omega)}.$$

Combining the above estimates yields

$$\|F[\Psi]\|_{\dot{\mathcal{H}}^1_{\underline{\lambda}}(\Omega)} \le C\|\Psi\|^2_{\dot{\mathcal{H}}^{1/2}_{\underline{\lambda}}(\Omega)}\|\Psi\|_{\dot{\mathcal{H}}^1_{\underline{\lambda}}(\Omega)}.$$

This implies

$$\|\Psi(t)\|_{\dot{\mathcal{H}}^1_{\underline{\lambda}}(\Omega)} \le \|\Psi(0)\|_{\dot{\mathcal{H}}^1_{\underline{\lambda}}(\Omega)} + \int_0^t C_0\|\Psi(s)\|_{\dot{\mathcal{H}}^1_{\underline{\lambda}}(\Omega)},$$

where C_0 is a constant proportional to the initial energy $\|\Psi(0)\|^2_{\dot{\mathcal{H}}^{1/2}_{\underline{\lambda}}(\Omega)} + \frac{1}{2}\|\nabla V[\Psi(0)]\|^2_{L^2(\Omega)}$. By Gronwall's lemma,

$$\|\Psi(t)\|_{\dot{\mathcal{H}}^1_{\underline{\lambda}}(\Omega)} \le C_1 e^{C_2 t}, \quad t > 0.$$

By the blow-up alternative, this implies that the Schrödinger-Poisson system is globally well-posed in $\mathcal{H}^1_{\underline{\lambda}}(\Omega)$. □

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