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On $q-$Gevrey asymptotics for singularly perturbed $q-$difference-differential problems with an irregular singularity

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Abstract

We study a $q-$analog of a singularly perturbed Cauchy problem with irregular singularity in the complex domain which generalizes a previous result by S. Malek in [11]. First, we construct solutions defined in open $q-$spirals to the origin. By means of a $q-$Gevrey version of Malgrange-Sibuya theorem we show the existence of a formal power series in the perturbation parameter which turns out to be the $q-$Gevrey asymptotic expansion (of certain type) of the actual solutions.

Key words: $q-$Laplace transform, Malgrange-Sibuya theorem, $q-$Gevrey asymptotic expansion, formal power series. 2010 MSC: 35C10, 35C20.

1 Introduction

We study a family of q-difference-differential equations of the following form

$$(1) \qquad \epsilon t\partial_z^S X(\epsilon, qt, z) + \partial_z^S X(\epsilon, t, z) = \sum_{k=0}^{S-1} b_k(\epsilon, z)(t\sigma_q)^{m_{0,k}}(\partial_z^k X)(\epsilon, t, zq^{-m_{1,k}}),$$

where $q \in \mathbb{C}$ such that $|q| > 1$, $m_{0,k}, m_{1,k}$ are positive integers, $b_k(\epsilon, z)$ are polynomials in z with holomorphic coefficients in ϵ on some neighborhood of 0 in $\mathbb{C}$ and σ_q is the dilation operator given by $(\sigma_q X)(\epsilon, t, z) = X(\epsilon, qt, z)$. As in previous works [12], [14], [9], the map $(t, z) \mapsto (q^{m_{0,k}}t, zq^{-m_{1,k}})$ is assumed to be a volume shrinking map, meaning that the modulus of the Jacobian determinant $|q|^{m_{0,k}-m_{1,k}}$ is less than 1, for every $0 \leq k \leq S-1$.

In [11], the second author studies a similar singularly perturbed Cauchy problem. In this previous work, the polynomial $b_k(\epsilon, z) := \sum_{s\in I_k} b_{ks}(\epsilon)z^s$ is such that, for all $0 \leq k \leq S-1$, I_k is a finite subset of $\mathbb{N} = \{0, 1, ...\}$ and $b_{ks}(\epsilon)$ are bounded holomorphic functions on some disc $D(0, r_0)$ in $\mathbb{C}$ which verify that the origin is a zero of order at least $m_{0,k}$. The main point on these flatness conditions on the coefficients in $b_k(\epsilon, z)$ is that the method used by M. Canalis-Durand, J. Mozo-Fernández and R. Schäfke in [3] could be adapted so that the initial singularly perturbed problem turns into an auxiliary regularly perturbed $q-$difference$-$differential equation with an irregular singularity at $t = 0$, preserving holomorphic coefficients b_{ks} (we refer to [11] for the details). These constricting conditions on the flatness of $b_k(\epsilon, z)$ is now omitted, so that previous result is generalized. In the present work we will not only make use of the procedure considered in [3] but also of the methodology followed in [13]. In that work, the second author considers a family of singularly perturbed nonlinear partial differential equations such that the coefficients appearing possess poles with respect to ϵ at the origin after the change of variable $t \mapsto t/\epsilon$. This scenary fits our problem.

In both, the present work and [13], the procedure for locating actual solutions relies on the research of certain appropriate Banach spaces. The ones appearing here may be regarded as $q-$analogs of the ones in [13].

In order to fix ideas we first settle a brief summary of the procedure followed. We consider a finite family of discrete $q-$spirals $(U_I q^{-\mathbb{N}})_{I\in\mathcal{I}}$ in such a way that it provides a good covering at 0 (Definition 3).

We depart from a finite family, with indices belonging to a set $\mathcal{I}$, of perturbed Cauchy problems (33)+(34). Let $I \in \mathcal{I}$ be fixed. Firstly, by means of a non-discrete $q-$analog of Laplace transform introduced by C. Zhang in [21] (for details on classical Laplace transform we refer to [1],[5]), we are able to transform our initial problem into auxiliary equation (9) (or (21)).

The transformed problem fits into certain Cauchy auxiliary problem such as (9)+(10) which is considered in Section 2. Here, its solution is found in the space of formal power series in z with coefficients belonging to the space of holomorphic functions defined in the product of discrete $q-$spirals to the origin in the variable ϵ (this domain corresponds to $U_I q^{-\mathbb{N}}$ in the auxiliary transformed problem) times a continuous $q-$spiral to infinity in the variable τ ($V_I q^{\mathbb{R}_+}$ for the auxiliary equation). Moreover, for any fixed ϵ and regarding our auxiliary equation, one can deduce that the coefficients, as functions in the variable τ, belong to the Banach space of holomorphic functions in $V_I q^{\mathbb{R}_+}$ subject to $q-$Gevrey bounds

$$|W^I_\beta(\epsilon,\tau)| \le C_1 \beta! H^\beta e^{M\log^2|\tau/\epsilon|} \left|\frac{\tau}{\epsilon}\right|^{C\beta} |q|^{-A_1\beta^2}, \quad \tau \in V_I q^{\mathbb{R}_+}$$

for positive constants $C_1, C, M, H, A_1 > 0$, where the index of the coefficient considered is β (see Theorem 1).

Also, the transformed problem fits into the auxiliary problem (21)+(22), studied in detail in Section 3. In this case, the solution is found in the space of formal power series in z with coefficients belonging to the space of holomorphic functions defined in the product of a punctured disc at 0 in the variable ϵ times a punctured disc at the origin in τ. For a fixed ϵ, the coefficients belong to the Banach space of holomorphic functions in $D(0,\rho_0)\setminus\{0\}$ such that

$$|W^I_\beta(\epsilon,\tau)| \le C_1 \beta! H^\beta e^{M\log^2|\tau/\epsilon|} |\epsilon|^{-C\beta} |q|^{-A_1\beta^2}, \quad \tau \in D(0,\rho_0)\setminus\{0\}$$

for positive constants $C_1, C, M, H, A_1 > 0$ when β is the index of the coefficient considered (see Theorem 2).

From these results, we get a sequence $(W^I_\beta)_{\beta\in\mathbb{N}}$ consisting of holomorphic functions in the variable τ so that $q-$Laplace transform can be applied to its elements. In addition, the function

$$X_I(\epsilon,t,z) := \sum_{\beta\ge 0} \mathcal{L}^{\lambda_I}_{q;1} W^I_\beta(\epsilon,\epsilon t)\frac{z^\beta}{\beta!} \tag{2}$$

turns out to be a holomorphic function defined in $U_I q^{-\mathbb{N}} \times \mathcal{T} \times \mathbb{C}$ which is a solution of the initial problem. Here, $\mathcal{T}$ is an adequate open half $q-$spiral to 0 and λ_I corresponds to certain $q-$directions for the $q-$Laplace transform (see Proposition 1). The way to proceed is also followed by the authors in [6] and [7] when studying asymptotic properties of analytic solutions of $q-$difference equations with irregular singularities.

It is worth pointing out that the choice of a continuous summation procedure unlike the discrete one in [11] is due to the requirement of Cauchy's theorem on the way.

At this point we own a finite family $(X_I)_{I\in\mathcal{I}}$ of solutions of (33)+(34). The main goal is to study its asymptotic behavior at the origin in some sense. Let $\rho>0$. One observes (Theorem 3) that whenever the intersection $U_I\cap U_{I'}$ is not empty we have

$$|X_I(\epsilon,t,z)-X_{I'}(\epsilon,t,z)|\le C_1e^{-\frac{1}{A}\log^2|\epsilon|} \tag{3}$$

for positive constants C_1,A and for every $(\epsilon,t,z)\in(U_Iq^{-\mathbb{N}}\cap U_{I'}q^{-\mathbb{N}})\times\mathcal{T}\times D(0,\rho)$. Equation (3) implies that the difference of two solutions of (33)+(34) admits $q-$Gevrey null expansion of type $A>0$ at 0 in $U_I\cap U_{I'}$ as a function with values in the Banach space $\mathbb{H}_{\mathcal{T},\rho}$ of holomorphic bounded functions defined in $\mathcal{T}\times D(0,\rho)$ endowed with the supremum norm. Flatness condition (3) allows us to establish the main result of the present work (Theorem 7): the existence of a formal power series

$$\hat{X}(\epsilon)=\sum_{k\ge0}\frac{X_k}{k!}\epsilon^k\in\mathbb{H}_{\mathcal{T},\rho}[[\epsilon]],$$

formal solution of (1), such that for every $I\in\mathcal{I}$, each of the actual solutions (2) of the problem (33)+(34) admits $\hat{X}$ as its $q-$Gevrey expansion of a certain type in the corresponding domain of definition.

The main result heavily rests on a Malgrange-Sibuya type theorem involving $q-$Gevrey bounds, which generalizes a result in [11] where no precise bounds on the asymptotic appears. In this step, we make use of Whitney-type extension results in the framework of ultradifferentiable functions. Whitney-type extension theory is widely studied in literature under the framework of ultradifferentiable functions subject to bounds of their derivatives (see for example [4], [2]) and also it is a useful tool taken into account on the study of continuity of ultraholomorphic operators (see [19],[20],[10]). It is also worth saying that, although $q-$Gevrey bounds have been achieved in the present work, the type involved might be increased when applying an extension result for ultradifferentiable functions from [2].

The paper is organized as follows.
In Section 2 and Section 3, we introduce Banach spaces of formal power series and solve auxiliary Cauchy problems involving these spaces. In Section 2, this is done when the variables rely in a product of a discrete $q-$spiral to the origin times a $q-$spiral to infinity, while in Section 3 it is done when working on a product of a punctured disc at 0 times a disc at 0.

In Section 4 we first recall definitions and some properties related to $q-$Laplace transform appearing in [21], firstly developed by C. Zhang. In this section we also find actual solutions of the main Cauchy problem (33)+(34) and settle a flatness condition on the difference of two of them so that, when regarding the difference of two solutions in the variable ϵ, we are able to give some information on its asymptotic behavior at 0. Finally, in Section 6 we conclude with the existence of a formal power series in ϵ with coefficients in an adequate Banach space of functions which solves in a formal sense the problem considered. The procedure heavily rests on a $q-$Gevrey version of Malgrange-Sibuya theorem, developed in Section 5.

2 A Cauchy problem in weighted Banach spaces of Taylor series

$M,A_1,C>0$ are fixed positive real numbers throughout the whole paper.

Let U,V be nonempty bounded open sets in $\mathbb{C}^\star:=\mathbb{C}\setminus\{0\}$ and let $q\in\mathbb{C}^\star$ such that $|q|>1$. We define

$$Uq^{-\mathbb{N}}=\{\epsilon q^{-n}\in\mathbb{C}:\epsilon\in U,n\in\mathbb{N}\}\quad,\quad Vq^{\mathbb{R}_+}=\{\tau q^l\in\mathbb{C}:\tau\in V,l\in\mathbb{R},l\ge0\}.$$

We assume there exists $M_1 > 0$ such that $|\tau + 1| > M_1$ for all $\tau \in Vq^{\mathbb{R}_+}$ and also that the distance from the set V to the origin is positive.

Definition 1 *Let $\epsilon \in Uq^{-\mathbb{N}}$ and $\beta \in \mathbb{N}$. $E_{\beta,\epsilon,Vq^{\mathbb{R}_+}}$ denotes the vector space of functions $v \in \mathcal{O}(Vq^{\mathbb{R}_+})$ such that*

$$||v(\tau)||_{\beta,\epsilon,Vq^{\mathbb{R}_+}} := \sup_{\tau\in Vq^{\mathbb{R}_+}} \left\{ \frac{|v(\tau)|}{e^{M\log^2\left|\frac{\tau}{\epsilon}\right|}} \left|\frac{\tau}{\epsilon}\right|^{-C\beta} \right\} |q|^{A_1\beta^2}$$

is finite.

Let $\delta > 0$. $H(\epsilon, \delta, Vq^{\mathbb{R}_+})$ denotes the complex vector space of all formal series $v(\tau, z) = \sum_{\beta\geq 0} v_\beta(\tau)z^\beta/\beta!$ belonging to $\mathcal{O}(Vq^{\mathbb{R}_+})[[z]]$ such that

$$||v(\tau, z)||_{(\epsilon,\delta,Vq^{\mathbb{R}_+})} := \sum_{\beta\geq 0} ||v_\beta(\tau)||_{\beta,\epsilon,Vq^{\mathbb{R}_+}} \frac{\delta^\beta}{\beta!} < \infty.$$

It is straightforward to check that the pair $(H(\epsilon, \delta, Vq^{\mathbb{R}_+}), ||\cdot||_{(\epsilon,\delta,Vq^{\mathbb{R}_+})})$ is a Banach space.

We consider the formal integration operator ∂_z^{-1} defined on $\mathcal{O}(Vq^{\mathbb{R}_+})[[z]]$ by

$$\partial_z^{-1}(v(\tau, z)) := \sum_{\beta\geq 1} v_{\beta-1}(\tau)\frac{z^\beta}{\beta!} \in \mathcal{O}(Vq^{\mathbb{R}_+})[[z]].$$

Lemma 1 *Let $s, k, m_1, m_2 \in \mathbb{N}$, $\delta > 0$, $\epsilon \in Uq^{-\mathbb{N}}$. We assume that the following conditions hold:*

$$m_1 \leq C(k + s) \quad , \quad m_2 \geq 2(k + s)A_1. \tag{4}$$

Then, there exists a constant $C_1 = C_1(s, k, m_1, m_2, V, U, C, A_1)$ (not depending on ϵ nor δ) such that

$$\left|\left| z^s \left(\frac{\tau}{\epsilon}\right)^{m_1} \partial_z^{-k} v(\tau, zq^{-m_2})\right|\right|_{(\epsilon,\delta,Vq^{\mathbb{R}_+})} \leq C_1\delta^{k+s} ||v(\tau, z)||_{(\epsilon,\delta,Vq^{\mathbb{R}_+})}, \tag{5}$$

for every $v \in H(\epsilon, \delta, Vq^{\mathbb{R}_+})$.

Proof Let $v(\tau, z) = \sum_{\beta\geq 0} v_\beta(\tau)\frac{z^\beta}{\beta!} \in \mathcal{O}(Vq^{\mathbb{R}_+})[[z]]$. We have that

$$\left|\left| z^s \left(\frac{\tau}{\epsilon}\right)^{m_1} \partial_z^{-k} v(\tau, zq^{-m_2})\right|\right|_{(\epsilon,\delta,Vq^{\mathbb{R}_+})} = \left|\left| \sum_{\beta\geq k+s} \left(\frac{\tau}{\epsilon}\right)^{m_1} v_{\beta-(k+s)}(\tau)\frac{\beta!}{(\beta - s)!}\frac{1}{q^{m_2(\beta-s)}}\frac{z^\beta}{\beta!}\right|\right|_{(\epsilon,\delta,Vq^{\mathbb{R}_+})}$$

$$= \sum_{\beta\geq k+s} \left|\left| \left(\frac{\tau}{\epsilon}\right)^{m_1} v_{\beta-(k+s)}(\tau)\frac{\beta!}{(\beta - s)!}\frac{1}{q^{m_2(\beta-s)}}\right|\right|_{\beta,\epsilon,Vq^{\mathbb{R}_+}} \frac{\delta^\beta}{\beta!} \tag{6}$$

Taking into account the definition of the norm $||\cdot||_{\beta,\epsilon,Vq^{\mathbb{R}_+}}$, we get

$$\left|\left| \left(\frac{\tau}{\epsilon}\right)^{m_1} v_{\beta-(k+s)}(\tau)\frac{\beta!}{(\beta - s)!}\frac{1}{q^{m_2(\beta-s)}}\right|\right|_{\beta,\epsilon,Vq^{\mathbb{R}_+}} = \frac{\beta!}{(\beta - s)!}|q|^{A_1(\beta-(k+s))^2}|q|^{p(\beta)}$$

$$\sup_{\tau\in Vq^{\mathbb{R}_+}} \left\{ \frac{|v_{\beta-(k+s)}(\tau)|}{e^{M\log^2\left|\frac{\tau}{\epsilon}\right|}} \left|\frac{\tau}{\epsilon}\right|^{-C(\beta-(k+s))} \left|\frac{\epsilon}{\tau}\right|^{C(k+s)-m_1} \right\}, \tag{7}$$

with $p(\beta) = A_1\beta^2 - A_1(\beta-(k+s))^2 - m_2(\beta - s)$. From (4) we derive $|\epsilon/\tau|^{C(k+s)-m_1} \leq (C_U/C_V)^{C(k+s)-m_1}$ for every $\epsilon \in Uq^{-\mathbb{N}}$ and $\tau \in Vq^{\mathbb{R}_+}$, where $0 < C_V := \min\{|\tau| : \tau \in V\}$ and $0 < C_U := \max\{|\epsilon| : \epsilon \in U\}$. Moreover,

$$p(\beta) = (2(k+s)A_1 - m_2)\beta - (k+s)^2A_1 + m_2 s,$$

for every $\beta \in \mathbb{N}$. Regarding condition (4) we obtain the existence of $C_1 > 0$ such that

$$\left|\frac{\epsilon}{\tau}\right|^{C(k+s)-m_1} |q|^{p(\beta)} \leq C_1, \tag{8}$$

for every $\tau \in Vq^{\mathbb{R}_+}$ and $\beta \in \mathbb{N}$. Inequality (5) follows from (6), (7) and (8):

$$\begin{aligned}\left\|z^s\left(\frac{\tau}{\epsilon}\right)^{m_1}\partial_z^{-k}v(\tau,zq^{-m_2})\right\|_{(\epsilon,\delta,Vq^{\mathbb{R}_+})} &\leq C_1\sum_{\beta\geq k+s}\left\|v_{\beta-(k+s)}(\tau)\right\|_{\beta-(k+s),\epsilon,Vq^{\mathbb{R}_+}}\frac{\beta!}{(\beta-s)!}\frac{\delta^\beta}{\beta!}\\ &\leq C_1\delta^{k+s}\sum_{\beta\geq k+s}\left\|v_{\beta-(k+s)}(\tau)\right\|_{\beta-(k+s),\epsilon,Vq^{\mathbb{R}_+}}\frac{\delta^{\beta-(k+s)}}{(\beta-(k+s))!}.\end{aligned}$$

□

Lemma 2 *Let $F(\epsilon,\tau)$ be a holomorphic and bounded function defined on $Uq^{-\mathbb{N}} \times Vq^{\mathbb{R}_+}$. Then, there exists a constant $C_2 = C_2(F,U,V) > 0$ such that*

$$\|F(\epsilon,\tau)v_\epsilon(\tau,z)\|_{(\epsilon,\delta,Vq^{\mathbb{R}_+})} \leq C_2\,\|v_\epsilon(\tau,z)\|_{(\epsilon,\delta,Vq^{\mathbb{R}_+})}$$

for every $\epsilon \in Uq^{-\mathbb{N}}$, every $\delta > 0$ and all $v_\epsilon \in H(\epsilon,\delta,Vq^{\mathbb{R}_+})$.

Proof Direct calculations regarding the definition of the elements in $H(\epsilon,\delta,Vq^{\mathbb{R}_+})$ allow us to conclude when taking $C_2 := \max\{|F(\epsilon,\tau)| : \epsilon \in Uq^{-\mathbb{N}}, \tau \in Vq^{\mathbb{R}_+}\}$. □

Let $S \geq 1$ be an integer. For all $0 \leq k \leq S-1$, let $m_{0,k}, m_{1,k}$ be positive integers and $b_k(\epsilon,z) = \sum_{s\in I_k} b_{ks}(\epsilon)z^s$ be a polynomial in z, where I_k is a finite subset of $\mathbb{N}$ and $b_{ks}(\epsilon)$ are holomorphic bounded functions on $D(0,r_0)$. We assume $\overline{Uq^{-\mathbb{N}}} \subseteq D(0,r_0)$).

We consider the following functional equation

$$\partial_z^S W(\epsilon,\tau,z) = \sum_{k=0}^{S-1}\frac{b_k(\epsilon,z)}{(\tau+1)\epsilon^{m_{0,k}}}\tau^{m_{0,k}}(\partial_z^k W)(\epsilon,\tau,zq^{-m_{1,k}}) \tag{9}$$

with initial conditions

$$(\partial_z^j W)(\epsilon,\tau,0) = W_j(\epsilon,\tau) \quad , \quad 0 \leq j \leq S-1, \tag{10}$$

where the functions $(\epsilon,\tau) \mapsto W_j(\epsilon,\tau)$ belong to $\mathcal{O}(Uq^{-\mathbb{N}} \times Vq^{\mathbb{R}_+})$ for every $0 \leq j \leq S-1$.

We make the following

Assumption (A) For every $0 \leq k \leq S-1$ and $s \in I_k$, we have

$$m_{0,k} \leq C(S-k+s) \quad , \quad m_{1,k} \geq 2(S-k+s)A_1.$$

Theorem 1 *Let Assumption (A) be fulfilled. We also make the following assumption on the initial conditions in (10): there exist a constant $\Delta > 0$ and $0 < \tilde{M} < M$ such that for every $0 \leq j \leq S-1$*

$$|W_j(\epsilon,\tau)| \leq \Delta e^{\tilde{M}\log^2|\frac{\tau}{\epsilon}|}, \tag{11}$$

for all $\tau \in Vq^{\mathbb{R}_+}$, $\epsilon \in Uq^{-\mathbb{N}}$. Then, there exists $W(\epsilon,\tau,z) \in \mathcal{O}(Uq^{-\mathbb{N}} \times Vq^{\mathbb{R}_+})[[z]]$ solution of (9)+(10) such that if $W(\epsilon,\tau,z) = \sum_{\beta\geq 0} W_\beta(\epsilon,\tau)\frac{z^\beta}{\beta!}$, then there exist $C_2 > 0$ and $0 < \delta < 1$ such that

$$|W_\beta(\epsilon,\tau)| \leq C_2\beta!\left(\frac{|q|^{2A_1S}}{\delta}\right)^\beta \left|\frac{\tau}{\epsilon}\right|^{C\beta} e^{M\log^2|\frac{\tau}{\epsilon}|}|q|^{-A_1\beta^2}, \qquad \beta \geq 0 \tag{12}$$

for every $\epsilon \in Uq^{-\mathbb{N}}$ and $\tau \in Vq^{\mathbb{R}_+}$.

Proof Let $\epsilon \in Uq^{-\mathbb{N}}$. We define the map $\mathcal{A}_\epsilon$ from $\mathcal{O}(Vq^{\mathbb{R}_+})[[z]]$ into itself by

$$\mathcal{A}_\epsilon(\tilde{W}(\tau,z)) := \sum_{k=0}^{S-1} \frac{b_k(\epsilon,z)}{(\tau+1)\epsilon^{m_{0,k}}}\tau^{m_{0,k}}\left[(\partial_z^{k-S}\tilde{W})(\tau,zq^{-m_{1,k}}) + \partial_z^k w_\epsilon(\tau,zq^{-m_{1,k}})\right], \tag{13}$$

where $w_\epsilon(\tau,z) := \sum_{j=0}^{S-1} W_j(\epsilon,\tau)\frac{z^j}{j!}$. In the following lemma, we show the restriction of $\mathcal{A}_\epsilon$ to a neighborhood of the origin in $H(\epsilon,\delta,Vq^{\mathbb{R}_+})$ is a Lipschitz shrinking map for an appropriate choice of $\delta > 0$.

Lemma 3 *There exist $R > 0$ and $\delta > 0$ (not depending on ϵ) such that:*

1. $\left\|\mathcal{A}_\epsilon(\tilde{W}(\tau,z))\right\|_{(\epsilon,\delta,Vq^{\mathbb{R}_+})} \leq R$ *for every $\tilde{W}(\tau,z) \in B(0,R)$. $B(0,R)$ denotes the closed ball centered at 0 with radius R in $H(\epsilon,\delta,Vq^{\mathbb{R}_+})$.*

2. $$\left\|\mathcal{A}_\epsilon(\tilde{W}_1(\tau,z)) - \mathcal{A}_\epsilon(\tilde{W}_2(\tau,z))\right\|_{(\epsilon,\delta,Vq^{\mathbb{R}_+})} \leq \frac{1}{2}\left\|\tilde{W}_1(\tau,z) - \tilde{W}_2(\tau,z)\right\|_{(\epsilon,\delta,Vq^{\mathbb{R}_+})}$$
   for every $\tilde{W}_1, \tilde{W}_2 \in B(0,R)$.

Proof Let $R > 0$ and $0 < \delta < 1$.

For the first part we consider $\tilde{W}(\tau,z) \in B(0,R) \subseteq H(\epsilon,\delta,Vq^{\mathbb{R}_+})$. Lemma 1 and Lemma 2 can be applied so that

$$\left\|\mathcal{A}_\epsilon(\tilde{W}(\tau,z))\right\|_{(\epsilon,\delta,Vq^{\mathbb{R}_+})}$$

$$\leq \sum_{k=0}^{S-1}\sum_{s\in I_k}\frac{M_{ks}}{M_1}\left[C_1\delta^{S-k+s}\left\|\tilde{W}(\tau,z)\right\|_{(\epsilon,\delta,Vq^{\mathbb{R}_+})} + \left\|z^s\left(\frac{\tau}{\epsilon}\right)^{m_{0,k}}\partial_z^k w_\epsilon(\tau,zq^{-m_{1,k}})\right\|_{(\epsilon,\delta,Vq^{\mathbb{R}_+})}\right], \tag{14}$$

with $M_{ks} = \sup_{\epsilon\in Uq^{-\mathbb{N}}}|b_{ks}(\epsilon)| < \infty$, $s \in I_k$, $0 \leq k \leq S-1$. Taking into account the definition of $H(\epsilon,\delta,Vq^{\mathbb{R}_+})$ and (11) we have

$$\left\|z^s\left(\frac{\tau}{\epsilon}\right)^{m_{0,k}}\partial_z^k w_\epsilon(\tau,zq^{-m_{1,k}})\right\|_{(\epsilon,\delta,Vq^{\mathbb{R}_+})} = \left\|\sum_{j=0}^{S-1-k}\left(\frac{\tau}{\epsilon}\right)^{m_{0,k}}W_{j+k}(\epsilon,\tau)\frac{z^{j+s}}{j!q^{m_{1,k}j}}\right\|_{(\epsilon,\delta,Vq^{\mathbb{R}_+})}$$

$$= \sum_{j=0}^{S-1-k} \sup_{\tau \in Vq^{\mathbb{R}_+}} \left\{ \frac{|W_{j+k}(\epsilon,\tau)|}{e^{M\log^2|\frac{\tau}{\epsilon}|}} \left|\frac{\tau}{\epsilon}\right|^{m_{0,k}-C(j+s)} \right\} |q|^{A_1(j+s)^2} \frac{\delta^{j+s}}{j!|q|^{m_{1,k}j}}$$

(15)

$$\leq \Delta \sum_{j=0}^{S-1-k} \frac{|q|^{A_1(j+s)^2}\delta^{j+s}}{j!|q|^{m_{1,k}j}} \max\{e^{-(M-\tilde{M})\log^2(x)}x^{m_{0,k}-C(j+s)} : x>0, 0\leq j+k\leq S-1, s\in I_k\}$$

$$\leq \Delta C_2',$$

for a positive constant C_2'.

We conclude this first part from an appropriate choice of R and $\delta>0$.

For the second part we take $\tilde{W}_1, \tilde{W}_2 \in B(0,R) \subseteq H(\epsilon,\delta,Vq^{\mathbb{R}_+})$. Similar arguments as before yield

$$\left\|\mathcal{A}_\epsilon(\tilde{W}_1)-\mathcal{A}_\epsilon(\tilde{W}_2)\right\|_{(\epsilon,\delta,Vq^{\mathbb{R}_+})} \leq \sum_{k=0}^{S-1}\sum_{s\in I_k} \frac{M_{ks}}{M_1} C_1 \delta^{S-k+s} \left\|\tilde{W}_1-\tilde{W}_2\right\|_{(\epsilon,\delta,Vq^{\mathbb{R}_+})}.$$

An adequate choice for $\delta>0$ allows us to conclude the proof. □

We choose constants R, δ as in the previous lemma.
From Lemma 3 and taking into account the shrinking map theorem on complete metric spaces, we guarantee the existence of $\tilde{W}_\epsilon(\tau,z) \in H(\epsilon,\delta,Vq^{\mathbb{R}_+})$ which is a fixed point for $\mathcal{A}_\epsilon$ in $B(0,R)$, it is to say, $\|\tilde{W}_\epsilon(\tau,z)\|_{(\epsilon,\delta,Vq^{\mathbb{R}_+})} \leq R$ and $\mathcal{A}_\epsilon(\tilde{W}_\epsilon(\tau,z)) = \tilde{W}_\epsilon(\tau,z)$.

Let us define

$$W_\epsilon(\tau,z) := \partial_z^{-S}\tilde{W}_\epsilon(\tau,z) + w_\epsilon(\tau,z). \tag{16}$$

If we write $\tilde{W}_\epsilon(\tau,z) = \sum_{\beta\geq 0}\tilde{W}_{\beta,\epsilon}(\tau)\frac{z^\beta}{\beta!}$ and $W_\epsilon(\tau,z) = \sum_{\beta\geq 0} W_{\beta,\epsilon}(\tau)\frac{z^\beta}{\beta!}$, then we have that $W_{\beta+S,\epsilon} \equiv \tilde{W}_{\beta,\epsilon}$ for $\beta \geq 0$ and $W_{j,\epsilon}(\tau) = W_j(\epsilon,\tau)$, $0\leq j\leq S-1$.

From $\|\tilde{W}_\epsilon(\tau,z)\|_{(\epsilon,\delta,Vq^{\mathbb{R}_+})} \leq R$ we arrive at $\|\tilde{W}_{\beta,\epsilon}\|_{\beta,\epsilon,Vq^{\mathbb{R}_+}} \leq R\beta!\left(\frac{1}{\delta}\right)^\beta$ for every $\beta\geq 0$. This implies

$$|\tilde{W}_{\beta,\epsilon}(\tau)| \leq R\beta!\left(\frac{1}{\delta}\right)^\beta \left|\frac{\tau}{\epsilon}\right|^{C\beta} e^{M\log^2|\frac{\tau}{\epsilon}|}|q|^{-A_1\beta^2},$$

for every $\beta\geq 0$ and $\tau\in Vq^{\mathbb{R}_+}$.

This is valid for every $\epsilon \in Uq^{-\mathbb{N}}$. We define $W(\epsilon,\tau,z) := W_\epsilon(\tau,z)$ and $W_\beta(\epsilon,\tau) := W_{\beta,\epsilon}(\tau)$ for every $(\epsilon,\tau)\in Uq^{-\mathbb{N}}\times Vq^{\mathbb{R}_+}$, $z\in\mathbb{C}$ and $\beta\geq S$. From (16), it is straightforward to prove that $W(\epsilon,\tau,z) = \sum_{\beta\geq 0} W_\beta(\epsilon,\tau)\frac{z^\beta}{\beta!}$ is a solution of (9)+(10).

Moreover, holomorphy of W_β in $Uq^{-\mathbb{N}}\times Vq^{\mathbb{R}_+}$ for every $\beta\geq 0$ can be deduced from the recursion formula verified by the coefficients:

$$\frac{W_{h+S}(\epsilon,\tau)}{h!} = \sum_{k=0}^{S-1}\sum_{h_1+h_2=h, h_1\in I_k} \frac{b_{kh_1}(\epsilon)\tau^{m_{0,k}}}{(\tau+1)\epsilon^{m_{0,k}}} \frac{W_{h_2+k}(\epsilon,\tau)}{h_2!q^{m_{1,k}h_2}}, \quad h\geq 0. \tag{17}$$

This implies $W_\beta(\epsilon,\tau)$ is holomorphic in $Uq^{-\mathbb{N}}\times Vq^{\mathbb{R}_+}$ for every $\beta\in\mathbb{N}$.

It only rests to prove (12). Upper and lower bounds for the modulus of the elements in $Uq^{-\mathbb{N}}$ and $Vq^{\mathbb{R}_+}$ respectively and usual calculations lead us to assure the existence of a positive constant $R_1>0$ such that

$$|W_\beta(\epsilon,\tau)| = |\tilde{W}_{\beta-S,\epsilon}(\tau)| \leq R_1\beta!\left(\frac{|q|^{2A_1S}}{\delta}\right)^\beta \left|\frac{\tau}{\epsilon}\right|^{C\beta} e^{M\log^2|\frac{\tau}{\epsilon}|}|q|^{-A_1\beta^2}, \tag{18}$$

for every $\beta \geq S$, and for every $\epsilon \in Uq^{-\mathbb{N}}$ and $\tau \in Vq^{\mathbb{R}_+}$. This concludes the proof for $\beta \geq S$.

Hypothesis (11) leads us to obtain (18) for $0 \leq k \leq S-1$. □

Remark: If $s > 0$ for every $s \in I_k$, $0 \leq k \leq S-1$, then for every $R > 0$, there exists small enough $\delta > 0$ in such a way that Lemma 3 holds.

3 Second Cauchy problem in a weighted Banach space of Taylor series

This section is devoted to the study of the same equation as in the previous section when the initial conditions are of a different nature. Proofs will only be sketched not to repeat calculations.

Let $1 < \rho_0$ and $U \subseteq \mathbb{C}^\star$ a bounded and open set with positive distance to the origin. $\dot{D}_{\rho_0}$ stands for $D(0,\rho_0) \setminus \{0\}$ in this section. M, A_1, C remain the same positive constants as in the previous section.

Definition 2 *Let $r_0 > 0$, $\epsilon \in D(0,r_0) \setminus \{0\}$ and $\beta \in \mathbb{N}$. $E^2_{\beta,\epsilon,\dot{D}_{\rho_0}}$ denotes the vector space of functions $v \in \mathcal{O}(\dot{D}_{\rho_0})$ such that*

$$|v(\tau)|_{\beta,\epsilon,\dot{D}_{\rho_0}} := \sup_{\tau \in \dot{D}_{\rho_0}} \{|v(\tau)| \frac{|\epsilon|^{C\beta}}{e^{M\log^2|\tau/\epsilon|}}\} |q|^{A_1\beta^2},$$

is finite. Let $\delta > 0$. $H_2(\epsilon,\delta,\dot{D}_{\rho_0})$ stands for the vector space of all formal series $v(\tau,z) = \sum_{\beta \geq 0} v_\beta(\tau) z^\beta/\beta!$ belonging to $\mathcal{O}(\dot{D}_{\rho_0})[[z]]$ such that

$$|v(\tau,z)|_{(\epsilon,\delta,\dot{D}_{\rho_0})} := \sum_{\beta \geq 0} |v_\beta(\tau)|_{\beta,\epsilon,\dot{D}_{\rho_0}} \frac{\delta^\beta}{\beta!} < \infty.$$

It is straightforward to check that the pair $(H_2(\epsilon,\delta,\dot{D}_{\rho_0}), |\cdot|_{(\epsilon,\delta,\dot{D}_{\rho_0})})$ is a Banach space.

Lemma 4 *Let $s,k,m_1,m_2 \in \mathbb{N}$, $\delta > 0$ and $\epsilon \in D(0,r_0) \setminus \{0\}$. We assume that the following conditions hold:*

$$m_1 \leq C(k+s) \quad , \quad m_2 \geq 2(k+s)A_1. \tag{19}$$

Then, there exists a constant $C_1 = C_1(s,k,m_1,m_2,\dot{D}_{\rho_0},U)$ (not depending on ϵ nor δ) such that

$$\left| z^s \left(\frac{\tau}{\epsilon}\right)^{m_1} \partial_z^{-k} v(\tau, zq^{-m_2}) \right|_{(\epsilon,\delta,\dot{D}_{\rho_0})} \leq C_1 \delta^{k+s} |v(\tau,z)|_{(\epsilon,\delta,\dot{D}_{\rho_0})}, \tag{20}$$

for every $v \in H_2(\epsilon,\delta,\dot{D}_{\rho_0})$.

Proof Let $v(\tau,z) \in \mathcal{O}(\dot{D}_{\rho_0})[[z]]$. The proof follows similar steps as in Lemma 1. We have

$$\left| z^s \left(\frac{\tau}{\epsilon}\right)^{m_1} \partial_z^{-k} v(\tau, zq^{-m_2}) \right|_{(\epsilon,\delta,\dot{D}_{\rho_0})} = \sum_{\beta \geq k+s} \left| \left(\frac{\tau}{\epsilon}\right)^{m_1} v_{\beta-(k+s)}(\tau) \frac{\beta!}{(\beta-s)!} \frac{1}{q^{m_2(\beta-s)}} \right|_{\beta,\epsilon,\dot{D}_{\rho_0}} \frac{\delta^\beta}{\beta!}.$$

From the definition of the norm $|\cdot|_{\beta,\epsilon,\dot{D}_{\rho_0}}$, we get

$$\left| \left(\frac{\tau}{\epsilon}\right)^{m_1} v_{\beta-(k+s)}(\tau) \frac{\beta!}{(\beta-s)!} \frac{1}{q^{m_2(\beta-s)}} \right|_{\beta,\epsilon,\dot{D}_{\rho_0}} \leq \frac{\beta!}{(\beta-s)!} |q|^{A_1(\beta-(k+s))^2} |q|^{p(\beta)}$$

$$\times \sup_{\tau \in \dot{D}_{\rho_0}} \left\{ \frac{|v_{\beta-(k+s)}(\tau)|}{e^{M\log^2|\tau/\epsilon|}} |\epsilon|^{C(\beta-(k+s))} \right\} \rho_0^{m_1} |\epsilon|^{C(k+s)-m_1},$$

with $p(\beta) = A_1\beta^2 - A_1(\beta-(k+s))^2 - m_2(\beta - s)$. Identical arguments as in Lemma 1 allow us to conclude. □

Lemma 5 *Let $F(\epsilon,\tau)$ be a holomorphic and bounded function defined on $(D(0,r_0)\setminus\{0\})\times\dot{D}_{\rho_0}$. Then, there exists a constant $C_2 = C_2(F) > 0$ such that*

$$|F(\epsilon,\tau)v_\epsilon(\tau,z)|_{(\epsilon,\delta,\dot{D}_{\rho_0})} \leq C_2\,|v_\epsilon(\tau,z)|_{(\epsilon,\delta,\dot{D}_{\rho_0})}$$

for every $\epsilon \in D(0,r_0)\setminus\{0\}$, every $\delta > 0$ and every $v_\epsilon \in H_2(\epsilon,\delta,\dot{D}_{\rho_0})$.

Let $S, r_0, m_{0,k}, m_{1,k}$ and b_k as in Section 2 and $\rho_0 > 0$. We consider the Cauchy problem

$$\partial_z^S W(\epsilon,\tau,z) = \sum_{k=0}^{S-1} \frac{b_k(\epsilon,z)}{(\tau+1)\epsilon^{m_{0,k}}} \tau^{m_{0,k}} (\partial_z^k W)(\epsilon,\tau,zq^{-m_{1,k}}) \tag{21}$$

with initial conditions

$$(\partial_z^j W)(\epsilon,\tau,0) = W_j(\epsilon,\tau) \quad , \quad 0 \leq j \leq S-1, \tag{22}$$

where the functions $(\epsilon,\tau)\mapsto W_j(\epsilon,\tau)$ belong to $\mathcal{O}((D(0,r_0)\setminus\{0\})\times\dot{D}_{\rho_0})$ for every $0 \leq j \leq S-1$.

Theorem 2 *Let Assumption (A) be fulfilled. We make the following assumption on the initial conditions (22): there exist constants $\Delta > 0$ and $0 < \tilde{M} < M$ such that*

$$|W_j(\epsilon,\tau)| \leq \Delta e^{\tilde{M}\log^2|\frac{\tau}{\epsilon}|}, \tag{23}$$

for every $\tau \in \dot{D}_{\rho_0}$, $\epsilon \in D(0,r_0)\setminus\{0\}$ and $0 \leq j \leq S-1$. Then, there exists $W(\epsilon,\tau,z) \in \mathcal{O}((D(0,r_0)\setminus\{0\})\times\dot{D}_{\rho_0})[[z]]$ solution of (21)+(22) such that if $W(\epsilon,\tau,z) = \sum_{\beta\geq 0} W_\beta(\epsilon,\tau)\frac{z^\beta}{\beta!}$, then there exist $C_3 > 0$ and $0 < \delta < 1$ such that

$$|W_\beta(\epsilon,\tau)| \leq C_3\beta! \left(\frac{|q|^{2A_1S}}{\delta}\right)^\beta |\epsilon|^{-C\beta} e^{M\log^2|\frac{\tau}{\epsilon}|} |q|^{-A_1\beta^2}, \qquad \beta \geq 0, \tag{24}$$

for every $\epsilon \in D(0,r_0)\setminus\{0\}$ and $\tau \in \dot{D}_{\rho_0}$.

Proof The proof of Theorem 1 can be adapted here so details will be omitted.

Let $\epsilon \in D(0,r_0)\setminus\{0\}$ and $0 < \delta < 1$. We consider the map $\mathcal{A}_\epsilon$ from $\mathcal{O}(\dot{D}_{\rho_0})[[z]]$ into itself defined as in (13) and construct $w_\epsilon(\tau,z)$ as above. From (23) we derive

$$\left| z^s \left(\frac{\tau}{\epsilon}\right)^{m_{0,k}} \partial_z^k w_\epsilon(\tau, zq^{-m_{1,k}}) \right|_{(\epsilon,\delta,\dot{D}_{\rho_0})}$$

$$= \sum_{j=0}^{S-1-k} \sup_{\tau\in\dot{D}_{\rho_0}} |W_{j+k}(\epsilon,\tau)| \frac{|\epsilon|^{C(j+s)}}{e^{M\log^2|\frac{\epsilon}{\tau}|}} \left|\frac{\tau}{\epsilon}\right|^{m_{0,k}} |q|^{A_1(j+s)^2} \frac{\delta^{j+s}}{j!|q|^{m_{1,k}j}}$$

$$\leq \Delta C_3', \tag{25}$$

for a positive constant C_3' not depending on ϵ nor δ.

Lemma 4, Lemma 5 and (25) allow us to affirm that one can find $R > 0$ and $\delta > 0$ such that the restriction of $\mathcal{A}_\epsilon$ to the disc $D(0,R)$ in $H_2(\epsilon,\delta,\dot{D}_{\rho_0})$ is a Lipschitz shrinking map. Moreover, there exists $\tilde{W}_\epsilon(\tau,z) \in H_2(\epsilon,\delta,\dot{D}_{\rho_0})$ which is a fixed point for $\mathcal{A}_\epsilon$ in $B(0,R)$.

If we put $\tilde{W}_\epsilon(\tau,z) = \sum_{\beta\geq 0} \tilde{W}_{\beta,\epsilon}(\tau)\frac{z^\beta}{\beta!}$, then one gets $|\tilde{W}_{\beta,\epsilon}|_{\beta,\epsilon,\dot{D}_{\rho_0}} \leq R\beta!\left(\frac{1}{\delta}\right)^\beta$ for $\beta \geq 0$. This implies

$$|\tilde{W}_{\beta,\epsilon}(\tau)| \leq R\beta!\left(\frac{1}{\delta}\right)^\beta |\epsilon|^{-C\beta} e^{M\log^2|\frac{\tau}{\epsilon}|}|q|^{-A_1\beta^2}, \quad \beta \geq 0, \tau \in \dot{D}_{\rho_0}.$$

The formal power series

$$W(\epsilon,\tau,z) := \sum_{\beta\geq S} \tilde{W}_{\beta-S,\epsilon}(\tau)\frac{z^\beta}{\beta!} + w_\epsilon(\tau,z) := \sum_{\beta\geq 0} W_\beta(\epsilon,\tau)\frac{z^\beta}{\beta!}$$

turns out to be a solution of (21)+(22) verifying that $W_\beta(\epsilon,\tau)$ is a holomorphic function in $(D(0,r_0)\setminus\{0\})\times\dot{D}_{\rho_0}$ and the estimates (24) hold for $\beta \geq 0$.

□

4 Analytic solutions in a small parameter of a singularly perturbed problem

4.1 A $q-$analog of the Laplace transform and $q-$asymptotic expansion

In this subsection, we recall the definition and several results related to the Jacobi Theta function and also a $q-$analog of the Laplace transform which was firstly developed by C. Zhang in [21].

Let $q \in \mathbb{C}$ such that $|q| > 1$.

The Jacobi Theta function is defined in $\mathbb{C}^\star$ by

$$\Theta(x) = \sum_{n\in\mathbb{Z}} q^{-n(n-1)/2}x^n, \quad x \in \mathbb{C}^\star.$$

From the fact that the Jacobi Theta function satisfies the functional equation $xq\Theta(x) = \Theta(qx)$, for $x \neq 0$, we have

$$\Theta(q^m x) = q^{\frac{m(m+1)}{2}} x^m \Theta(x), \quad x \in \mathbb{C}, x \neq 0 \tag{26}$$

for every $m \in \mathbb{Z}$. The following lower bounds for the Jacobi Theta function will be useful in the sequel.

Lemma 6 *Let $\delta > 0$. There exists $C > 0$ (not depending on δ) such that*

$$|\Theta(x)| \geq C\delta e^{\frac{\log^2|x|}{2\log|q|}}|x|^{\frac{1}{2}}, \tag{27}$$

for every $x \in \mathbb{C}^\star$ such that $|1 + xq^k| > \delta$ for all $k \in \mathbb{Z}$.

Proof Let $\delta > 0$. From Lemma 5.1.6 in [18] we get the existence of a positive constant C_1 such that $|\Theta(x)| \geq C_1\delta\Theta_{|q|}(|x|)$ for every $x \in \mathbb{C}^\star$ such that $|1 + xq^k| > \delta$ for all $k \in \mathbb{Z}$. Now,

$$\Theta_{|q|}(|x|) = \sum_{n\in\mathbb{Z}} |q|^{-\frac{n(n-1)}{2}}|x|^n \geq \max_{n\in\mathbb{Z}} |q|^{-\frac{n(n-1)}{2}}|x|^n.$$

Let us fix $|x|$. The function

$$f(t)=\exp\left(-\frac{1}{2}t(t-1)\log|q|+t\log|x|\right)$$

takes its maximum value at $t_0=\frac{\log|x|}{\log|q|}+\frac{1}{2}$ with $f(t_0)=C_2\exp(\frac{\log^2|x|}{2\log|q|})|x|^{1/2}$, for certain $C_2>0$. Taking into account that

$$\max_{n\in\mathbb{Z}}|q|^{-\frac{n(n-1)}{2}}|x|^n\geq f(\lfloor t_0\rfloor)=f(t_0)|q|^{-\frac{(\lfloor t_0\rfloor-t_0)^2}{2}}\geq f(t_0)|q|^{-\frac{1}{2}},$$

one can conclude the result. Here $\lfloor\cdot\rfloor$ stands for the entire part. □

Corollary 1 *Let $\delta>0$. For any $\xi\in(0,1)$ there exists $C_\xi=C_\xi(\delta)>0$ such that*

$$|\Theta(x)|\geq C_\xi e^{\frac{\xi\log^2|x|}{2\log|q|}}, \tag{28}$$

for every $x\in\mathbb{C}^\star$ such that $|1+xq^k|>\delta$, for all $k\in\mathbb{Z}$.

From now on, $(\mathbb{H},\|\cdot\|_{\mathbb{H}})$ stands for a complex Banach space.

For any $\lambda\in\mathbb{C}$ and $\delta>0$

$$\mathcal{R}_{\lambda,q,\delta}:=\{z\in\mathbb{C}^\star:|1+\frac{\lambda}{zq^k}|>\delta,\forall k\in\mathbb{R}\}.$$

The following definition corresponds to a $q-$analog of Laplace transform and can be found in [21] when working with sectors in the complex plane.

Proposition 1 *Let $\delta>0$ and $\rho_0>0$. We fix an open and bounded set V in $\mathbb{C}^\star$ such that $D(0,\rho_0)\cap V\neq\emptyset$. Let $\lambda\in D(0,\rho_0)\cap V$ and f be a holomorphic function defined in $\dot{D}_{\rho_0}$ with values in $\mathbb{H}$ such that can be extended to a function F defined in $\dot{D}_{\rho_0}\cup Vq^{\mathbb{R}_+}$ and*

$$\|F(x)\|_{\mathbb{H}}\leq C_1e^{\overline{M}\log^2|x|},\qquad x\in\dot{D}_{\rho_0}\cup Vq^{\mathbb{R}_+}, \tag{29}$$

for positive constants $C_1>0$ and $0<\overline{M}<\frac{1}{2\log|q|}$.

Let $\pi_q=\log(q)\prod_{n\geq0}(1-q^{-n-1})^{-1}$ and put

$$\mathcal{L}^{\lambda}_{q;1}F(z)=\frac{1}{\pi_q}\int_0^{\infty\lambda}\frac{F(\xi)}{\Theta(\frac{\xi}{z})}\frac{d\xi}{\xi}, \tag{30}$$

where the path $[0,\infty\lambda]$ is given by $t\in(-\infty,\infty)\mapsto q^t\lambda$. Then, $\mathcal{L}^{\lambda}_{q;1}F$ defines a holomorphic function in $\mathcal{R}_{\lambda,q,\delta}$ and it is known as the $q-$Laplace transform of f following direction $[\lambda]$.

Proof

Let $K\subseteq\mathcal{R}_{\lambda,q,\delta}$ be a compact set and $z\in K$. From the parametrization of the path $[0,\infty\lambda]$ we have

$$\int_0^{\infty\lambda}\frac{F(\xi)}{\Theta\left(\frac{\xi}{z}\right)}\frac{d\xi}{\xi}=\log(q)\int_{-\infty}^{\infty}\frac{F(q^t\lambda)}{\Theta\left(\frac{q^t\lambda}{z}\right)}dt.$$

Let $0 < \xi_1 < 1$ such that $0 < \overline{M} < \frac{\xi_1}{2\log|q|}$ and let $t \in \mathbb{R}$. We have $w = \frac{q^t\lambda}{z}$ satisfies $|1 + q^k w| > \delta$ for every $k \in \mathbb{Z}$. Corollary 1 and (29) yield

$$\int_{-\infty}^{\infty} \left\| \frac{F(q^t\lambda)}{\Theta\left(\frac{q^t\lambda}{z}\right)} \right\|_{\mathbb{H}} dt \leq \int_{-\infty}^{\infty} \frac{C_1 e^{\overline{M}\log^2|q^t\lambda|}}{C_{\xi_1} e^{\frac{\xi_1}{2\log|q|}\log^2|q^t\lambda/z|}} dt \leq L_1 \int_{-\infty}^{\infty} |q^t\lambda|^{\frac{\xi_1 \log|z|}{\log|q|}} e^{(\overline{M} - \frac{\xi_1}{2\log|q|})\log^2|q^t\lambda|} dt,$$

for a positive constant L_1. There exist $0 < A < B$ such that $A \leq |z| \leq B$ for every $z \in K$, so that the last term in the chain of inequalities above is upper bounded by

$$L_1 \int_{-\infty}^{-\log|\lambda|/\log|q|} |q^t\lambda|^{\frac{\xi_1 \log A}{\log|q|}} e^{(\overline{M} - \frac{\xi_1}{2\log|q|})\log^2|q^t\lambda|} dt$$
$$+ L_1 \int_{-\log|\lambda|/\log|q|}^{\infty} |q^t\lambda|^{\frac{\xi_1 \log B}{\log|q|}} e^{(\overline{M} - \frac{\xi_1}{2\log|q|})\log^2|q^t\lambda|} dt.$$

The result follows from this last expression. □

Remark: If we let $\overline{M} = \frac{1}{2\log|q|}$, then $\mathcal{L}^{\lambda}_{q;1}F$ will only remain holomorphic in $\mathcal{R}_{\lambda,q,\delta} \cap D(0, r_1)$ for certain $r_1 > 0$.

In the next proposition, we recall a commutation formula for the $q-$Laplace transform and the multiplication by a polynomial.

Proposition 2 *Let V be an open and bounded set in $\mathbb{C}^\star$ and $D(0,\rho_0)$ such that $V \cap D(0,\rho_0) \neq \emptyset$. Let ϕ a holomorphic function on $Vq^{\mathbb{R}_+} \cup \dot{D}_{\rho_0}$ with values in the Banach space $(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$ which satisfies the following estimates: there exist $C_1 > 0$ and $0 < \overline{M} < \frac{1}{2\log|q|}$ such that*

$$\|\phi(x)\|_{\mathbb{H}} < C_1 e^{\overline{M}\log^2|x|}, \qquad x \in \dot{D}_{\rho_0} \cup Vq^{\mathbb{R}_+}. \tag{31}$$

Then, the function $m\phi(\tau) = \tau\phi(\tau)$ is holomorphic on $Vq^{\mathbb{R}_+} \cup \dot{D}_{\rho_0}$ and satisfies estimates in the shape above. Let $\lambda \in V \cap D(0,\rho_0)$ and $\delta > 0$. We have the following equality

$$\mathcal{L}^{\lambda}_{q;1}(m\phi)(t) = t\mathcal{L}^{\lambda}_{q;1}\phi(qt)$$

for every $t \in \mathcal{R}_{\lambda,q,\delta}$.

Proof It is direct to prove that $m\phi$ is a holomorphic function in $Vq^{\mathbb{R}_+} \cup \dot{D}_{\rho_0}$ and also that $m\phi$ verifies bounds as in (31). From (26) we have $\Theta(x) = x\Theta(x/q)$, $x \in \mathbb{C}^\star$, so

$$\mathcal{L}^{\lambda}_{q;1}(m\phi)(t) = \frac{1}{\pi_q}\int_0^{\infty\lambda} \frac{(m\phi)(\xi)}{\Theta(\frac{\xi}{t})}\frac{d\xi}{\xi} = \frac{1}{\pi_q}\int_0^{\infty\lambda} \frac{\phi(\xi)}{\Theta(\frac{\xi}{t})} d\xi$$
$$= \frac{1}{\pi_q}\int_0^{\infty\lambda} \frac{\phi(\xi)}{\frac{\xi}{t}\Theta(\frac{\xi}{qt})} d\xi = t\mathcal{L}^{\lambda}_{q;1}(\phi)(qt),$$

for every $t \in \mathcal{R}_{\lambda,q,\delta}$. □

4.2 Analytic solutions in a parameter of a singularly perturbed Cauchy problem

The following definition of a good covering firstly appeared in [18], p. 36.

Definition 3 *Let $I = (I_1, I_2)$ be a pair of open intervals in $\mathbb{R}$ each one of length smaller than $1/4$ and let U_I be the corresponding open bounded set in $\mathbb{C}^\star$ defined by*

$$U_I = \{e^{2\pi ui}q^v \in \mathbb{C}^\star : u \in I_1, v \in I_2\}.$$

Let $\mathcal{I}$ be a finite family of tuple I as above verifying

1. *$\cup_{I\in\mathcal{I}}(U_I q^{-\mathbb{N}}) = \nu \setminus \{0\}$, where ν is a neighborhood of 0 in $\mathbb{C}$, and*
2. *the open sets $U_I q^{-\mathbb{N}}$, $I \in \mathcal{I}$ are four by four disjoint.*

Then, we say $(U_I q^{-\mathbb{N}})_{I\in\mathcal{I}}$ is a good covering.

Definition 4 *Let $(U_I q^{-\mathbb{N}})_{I\in\mathcal{I}}$ be a good covering. Let $\delta > 0$. We consider a family of open bounded sets $\{(V_I)_{I\in\mathcal{I}}, \mathcal{T}\}$ in $\mathbb{C}^\star$ such that:*

1. *There exists $1 < \rho_0$ with $V_I \cap D(0, \rho_0) \neq \emptyset$, for all $I \in \mathcal{I}$.*
2. *For every $I \in \mathcal{I}$ and $\tau \in V_I q^{\mathbb{R}}$, $|\tau + 1| > \delta$.*
3. *For every $I \in \mathcal{I}$, $t \in \mathcal{T}$, $\epsilon_u \in U_I$ and $\lambda_v \in V_I \cap D(0, \rho_0)$, we have*

$$|1 + \frac{\lambda_v}{\epsilon_u t q^r}| > \delta,$$

   for every $r \in \mathbb{R}$.

4. *$|t| \leq 1$ for every $t \in \mathcal{T}$.*

We say the family $\{(V_I)_{I\in\mathcal{I}}, \mathcal{T}\}$ is associated to the good covering $(U_I q^{-\mathbb{N}})_{I\in\mathcal{I}}$.

Let $S \geq 1$ be an integer. For every $0 \leq k \leq S - 1$, let $m_{0,k}, m_{1,k}$ be positive integers and $b_k(\epsilon, z) = \sum_{s\in I_k} b_{ks}(\epsilon)z^s$ be a polynomial in z, where I_k is a subset of $\mathbb{N}$ and $b_{ks}(\epsilon)$ are bounded holomorphic functions on some disc $D(0, r_0)$ in $\mathbb{C}$, $0 < r_0 \leq 1$. Let $(U_I q^{-\mathbb{N}})_{I\in\mathcal{I}}$ be a good covering such that $U_I q^{-\mathbb{N}} \subseteq D(0, r_0)$ for every $I \in \mathcal{I}$.

Assumption (B):

$$M < \frac{1}{2\log|q|}.$$

Definition 5 *Let $\rho_0 > 1$ such that $V \cap D(0, \rho_0) \neq \emptyset$. Let $\Delta, \tilde{M} > 0$ such that $\tilde{M} < M$ and $(\epsilon, \tau) \mapsto W(\epsilon, \tau)$ a bounded holomorphic function on $(D(0, r_0) \setminus \{0\}) \times \dot{D}_{\rho_0}$ verifying*

$$|W(\epsilon, \tau)| \leq \Delta e^{\tilde{M}\log^2|\tau/\epsilon|},$$

for every $(\epsilon, \tau) \in (D(0, r_0) \setminus \{0\}) \times \dot{D}_{\rho_0}$. Assume moreover that $W(\epsilon, \tau)$ can be extended to an analytic function $(\epsilon, \tau) \mapsto W_{UV}(\epsilon, \tau)$ on $Uq^{-\mathbb{N}} \times (Vq^{\mathbb{R}_+} \cup \dot{D}_{\rho_0})$ and

$$|W_{UV}(\epsilon, \tau)| \leq \Delta e^{\tilde{M}\log^2|\tau/\epsilon|}, \quad (32)$$

for every $(\epsilon, \tau) \in Uq^{-\mathbb{N}} \times (Vq^{\mathbb{R}_+} \cup \dot{D}_{\rho_0})$. We say that the set $\{W, W_{UV}, \rho_0\}$ is admissible.

Let $\mathcal{I}$ be a finite family of indices. For every $I \in \mathcal{I}$, we consider the following singularly perturbed Cauchy problem

$$\epsilon t\partial_z^S X_I(\epsilon,qt,z) + \partial_z^S X_I(\epsilon,t,z) = \sum_{k=0}^{S-1} b_k(\epsilon,z)(t\sigma_q)^{m_{0,k}}(\partial_z^k X_I)(\epsilon,t,zq^{-m_{1,k}}) \tag{33}$$

with b_k as in (9), and with initial conditions

$$(\partial_z^j X_I)(\epsilon,t,0) = \phi_{I,j}(\epsilon,t) \quad , \quad 0 \le j \le S-1, \tag{34}$$

where the functions $\phi_{I,j}(\epsilon,t)$ are constructed as follows. Let $\{(V_I)_{I\in\mathcal{I}},\mathcal{T}\}$ be a family of open sets associated to the good covering $(U_I q^{-\mathbb{N}})_{I\in\mathcal{I}}$. For every $0 \le j \le S-1$ and $I \in \mathcal{I}$, let $\{W_j, W_{U_I,V_I,j},\rho_0\}$ be an admissible set. Let λ_I be a complex number in $V_I \cap D(0,\rho_0)$. We can assume that $r_0 < 1 < |\lambda_I|$. If not, we diminish r_0 as desired. We put

$$\phi_{I,j}(\epsilon,t) := \mathcal{L}_{q;1}^{\lambda_I}(\tau \mapsto W_{U_I,V_I,j}(\epsilon,\tau))(\epsilon,\epsilon t).$$

Lemma 7 *The function $(\epsilon,t) \mapsto \phi_{I,j}(\epsilon,t)$, constructed as above, turns out to be holomorphic and bounded on $U_I q^{-\mathbb{N}} \times \mathcal{T}$ for every $I \in \mathcal{I}$ and all $0 \le j \le S-1$.*

Proof Let $I \in \mathcal{I}$ and $0 \le j \le S-1$. From (32), one has

$$|W_{U_I,V_I,j}(\epsilon,\tau)| \le \Delta e^{\tilde{M}\log^2|\tau/\epsilon|} = \Delta e^{\tilde{M}\log^2|\epsilon|}|\tau|^{-2\tilde{M}\log|\epsilon|}e^{\tilde{M}\log^2|\tau|}, \tag{35}$$

for every $(\epsilon,\tau) \in U_I q^{-\mathbb{N}} \times (V_I q^{\mathbb{R}_+} \cup \dot{D}_{\rho_0})$. Let $\epsilon \in U_I q^{-\mathbb{N}}$ and $\tilde{M} < \tilde{M}_2 < \frac{1}{2\log|q|}$. Then, (35) can be upper bounded by $\tilde{\Delta}\exp(\tilde{M}_2\log^2|\tau|)$, for some $\tilde{\Delta} = \tilde{\Delta}(\epsilon) > 0$. Estimates in (29) holds so that Proposition 1 can be applied here. The third item in Definition 4 derives holomorphy of $\phi_{I,j}$ on $U_I q^{-\mathbb{N}} \times \mathcal{T}$.

We now prove boundness of $\phi_{I,j}$ in its domain of definition. One has

$$|\phi_{I,j}(\epsilon,t)| = \left|\mathcal{L}_{q;1}^{\lambda_I} W_{U_I,V_I,j}(\epsilon,\epsilon t)\right| \le \left|\mathcal{L}_{q;1,+}^{\lambda_I} W_{U_I,V_I,j}(\epsilon,\epsilon t)\right| + \left|\mathcal{L}_{q;1,-}^{\lambda_I} W_{U_I,V_I,j}(\epsilon,\epsilon t)\right|,$$

for every $(\epsilon,t) \in U_I q^{-\mathbb{N}} \times \mathcal{T}$, where

$$\mathcal{L}_{q;1,+}^{\lambda_I} W_{U_I,V_I,j}(\epsilon,\epsilon t) = \frac{\log(q)}{\pi_q}\int_0^\infty \frac{W_{U_I,V_I,j}(\epsilon,q^s\lambda_I)}{\Theta(\frac{q^s\lambda_I}{\epsilon t})}ds,$$

$$\mathcal{L}_{q;1}^{\lambda_I} W_{U_I,V_I,j,-}(\epsilon,\epsilon t) = \frac{\log(q)}{\pi_q}\int_{-\infty}^0 \frac{W_{U_I,V_I,j}(\epsilon,q^s\lambda_I)}{\Theta(\frac{q^s\lambda_I}{\epsilon t})}ds.$$

We only give bounds for the first integral. The estimates for the second one can be deduced following a similar procedure.

Let $0 < \xi < 1$ such that $\tilde{M} < \frac{\xi}{2\log|q|}$. From Corollary 1 and (32) we deduce

$$\left|\mathcal{L}_{q;1,+}^{\lambda_I} W_{U_I,V_I,j}(\epsilon,\epsilon t)\right| \le \frac{|\log(q)|}{|\pi_q|}\int_0^\infty \left|\frac{W_{U_I,V_I,j}(\epsilon,q^s\lambda_I)}{\Theta(\frac{q^s\lambda_I}{\epsilon t})}\right| ds \le \frac{|\log(q)|\Delta}{|\pi_q|C_\xi}\int_0^\infty \frac{e^{\tilde{M}\log^2|q^s\lambda_I/\epsilon|}}{e^{\frac{\xi\log^2|\frac{q^s\lambda_I}{\epsilon t}|}{2\log|q|}}}ds$$

$$\le \frac{|\log(q)|\Delta}{|\pi_q|C_\xi}e^{(\tilde{M}-\frac{\xi}{2\log|q|})\log^2|\frac{\lambda_I}{\epsilon}|}e^{-\frac{\xi\log^2|t|}{2\log|q|}}e^{\frac{\xi\log|\lambda_I/\epsilon|\log|t|}{\log|q|}}$$

$$\times\int_0^\infty e^{2(\tilde{M}-\frac{\xi}{2\log|q|})\log^2|q|s^2}e^{(\tilde{M}-\frac{\xi}{2\log|q|})\log|q|\log|\lambda_I/\epsilon|s}e^{\xi\log|t|s}ds \le C_j,$$

for some $C_j > 0$ which does not depend on ϵ nor t. □

The following assumption is related to technical reasons appearing in the proof of Lemma 7 and Theorem 3.

Assumption (C): There exist $a_1, a_2 > 0$, $0 < \xi, \overline{\xi} < 1$ such that

$$\text{(C.1)} \qquad M < \frac{\xi}{2\log|q|},$$

$$\text{(C.2)} \qquad \frac{\xi}{2} - M\log|q| - \frac{Ca_1}{2a_2} > 0,$$

$$\text{(C.3)} \qquad \frac{Ca_2}{2a_1} + \frac{C^2}{4\overline{\xi}\log|q|\left(\frac{\xi}{2\log|q|} - M\right)} < A_1.$$

Next remark clarifies availability of these constants for a posed problem.

Remark: Assumptions (A), (B) and (C) strongly depend on the choice of q whose modulus must rest near 1. For example, these assumptions on the constants are verified when taking $\log|q| = 1/16$, $M = 1$, $A_1 = 5$, $C = 1$, $\xi = 1/2$, $\overline{\xi} = 1/2$, $a_1 = 1$, $a_2 = 4$. Then, next theorem provides a solution for the equation

$$\epsilon t\partial_z^2 X_I(\epsilon, qt, z) + \partial_z^2 X_I(\epsilon, t, z) = \big(b_{00}(\epsilon) + b_{01}(\epsilon)z\big)t^2 X_I(\epsilon, q^2 t, zq^{-30}) + b_{10}(\epsilon)t\partial_z X_I(\epsilon, qt, zq^{-10}),$$

with b_{00}, b_{01}, b_{10} being holomorphic functions near the origin.

Theorem 3 *Let Assumption (A) be fulfilled by the integers $m_{0,k}, m_{1,k}$, for $0 \le k \le S-1$ and also assumptions (B) and (C) for M, A_1, C. We consider the problem (33)+(34) where the initial conditions are constructed as above. Then, for every $I \in \mathcal{I}$, the problem (33)+(34) has a solution $X_I(\epsilon, t, z)$ which is holomorphic and bounded in $U_I q^{-\mathbb{N}} \times \mathcal{T} \times \mathbb{C}$.*

Moreover, for every $\rho > 0$, if $I, I' \in \mathcal{I}$ are such that $U_I q^{-\mathbb{N}} \cap U_{I'} q^{-\mathbb{N}} \neq \emptyset$ then there exists a positive constant $C_1 = C_1(\rho) > 0$ such that

$$|X_I(\epsilon, t, z) - X_{I'}(\epsilon, t, z)| \le C_1 e^{-\frac{1}{A}\log^2|\epsilon|}, \quad (\epsilon, t, z) \in (U_I q^{-\mathbb{N}} \cap U_{I'} q^{-\mathbb{N}}) \times \mathcal{T} \times D(0, \rho),$$

with $\frac{1}{A} = (1-\overline{\xi})(\frac{\xi}{2\log|q|} - M)$ with $\xi, \overline{\xi}$ chosen as in Assumption (C).

Proof Let $\delta > 0$ and $I \in \mathcal{I}$. We consider the Cauchy problem (21) with initial conditions $(\partial_z^j W)(\epsilon, \tau, 0) = W_j(\epsilon, \tau)$ for $0 \le j \le S-1$. From Theorem 2 we obtain the existence of a unique formal solution $W(\epsilon, \tau, z) = \sum_{\beta \ge 0} W_\beta(\epsilon, \tau)\frac{z^\beta}{\beta} \in \mathcal{O}((D(0, r_0) \setminus \{0\}) \times \dot{D}_{\rho_0})[[z]]$ and positive constants $C_3 > 0$ and $0 < \delta_1 < 1$ such that

$$\text{(36)} \qquad |W_\beta(\epsilon, \tau)| \le C_3 \beta! \Big(\frac{|q|^{2A_1 S}}{\delta_1}\Big)^\beta |\epsilon|^{-C\beta} e^{M\log^2|\frac{\tau}{\epsilon}|} |q|^{-A_1\beta^2}, \quad \beta \ge 0,$$

for $(\epsilon, \tau) \in (D(0, r_0) \setminus \{0\}) \times \dot{D}_{\rho_0}$.

Moreover, from Theorem 1 we get that the coefficients $W_\beta(\epsilon, \tau)$ can be extended to holomorphic functions defined in $U_I q^{-\mathbb{N}} \times V_I q^{\mathbb{R}_+}$ and also the existence of positive constants C_2 and $0 < \delta_2 < 1$ such that

$$\text{(37)} \qquad |W_\beta(\epsilon, \tau)| \le C_2 \beta! \Big(\frac{|q|^{2A_1 S}}{\delta_2}\Big)^\beta \left|\frac{\tau}{\epsilon}\right|^{C\beta} e^{M\log^2|\frac{\tau}{\epsilon}|} |q|^{-A_1\beta^2}, \quad \beta \ge 0,$$

for $(\epsilon,\tau) \in U_I q^{-\mathbb{N}} \times V_I q^{\mathbb{R}_+}$.

We choose $\lambda_I \in V_I \cap D(0,\rho_0)$. In the following estimates we will make use of the fact that $|\epsilon| \leq |\lambda_I|$ for every $\epsilon \in D(0,r_0 \setminus \{0\})$. Proposition 1 allows us to calculate the $q-$Laplace transform of W_β with respect to τ for every $\beta \geq 0$, $\mathcal{L}_{q;1}^{\lambda_I}(W_\beta)(\epsilon,\tau)$. It defines a holomorphic function in $U_I q^{-\mathbb{N}} \times \mathcal{R}_{\lambda_I,q,\delta}$. From the fact that $\{(V_I)_{I\in\mathcal{I}},\mathcal{T}\}$ is chosen to be a family associated to the good covering $(U_I q^{-\mathbb{N}})_{I\in\mathcal{I}}$ we derive that the function

$$(\epsilon,t) \mapsto \mathcal{L}_{q;1}^{\lambda_I}(W_\beta)(\epsilon,\epsilon t)$$

is a holomorphic and bounded function defined in $U_I q^{-\mathbb{N}} \times \mathcal{T}$. We can define, at least formally,

$$X_I(\epsilon,t,z) := \sum_{\beta\geq 0} \mathcal{L}_{q;1}^{\lambda_I}(W_\beta)(\epsilon,\epsilon t)\frac{z^\beta}{\beta!}, \tag{38}$$

in $\mathcal{O}(U_I q^{-\mathbb{N}} \times \mathcal{T})[[z]]$. If $X_I(\epsilon,t,z)$ were a holomorphic function in $U_I q^{-\mathbb{N}} \times \mathcal{T} \times \mathbb{C}$, then Proposition 2 would allow us to affirm that (38) is an actual solution of (33)+(34). In order to end the first part of the proof it rests to demonstrate that (38) defines in fact a bounded holomorphic function in $U_I q^{-\mathbb{N}} \times \mathcal{T} \times \mathbb{C}$. Let $(\epsilon,t) \in U_I q^{-\mathbb{N}} \times \mathcal{T}$ and $\beta \geq 0$. We have

$$|\mathcal{L}_{q;1}^{\lambda_I} W_\beta(\epsilon,\epsilon t)| \leq |\mathcal{L}_{q;1,+}^{\lambda_I} W_\beta(\epsilon,\epsilon t)| + |\mathcal{L}_{q;1,-}^{\lambda_I} W_\beta(\epsilon,\epsilon t)|,$$

where

$$\mathcal{L}_{q;1,+}^{\lambda_I} W_\beta(\epsilon,\epsilon t) = \frac{\log(q)}{\pi_q}\int_0^\infty \frac{W_\beta(\epsilon,q^s\lambda_I)}{\Theta(\frac{q^s\lambda_I}{\epsilon t})}ds, \quad \mathcal{L}_{q;1,-}^{\lambda_I} W_\beta(\epsilon,\epsilon t) = \frac{\log(q)}{\pi_q}\int_{-\infty}^0 \frac{W_\beta(\epsilon,q^s\lambda_I)}{\Theta(\frac{q^s\lambda_I}{\epsilon t})}ds.$$

We now establish bounds for both integrals.

$$|\mathcal{L}_{q;1,+}^{\lambda_I} W_\beta(\epsilon,\epsilon t)| \leq \frac{|\log q|}{|\pi_q|}\int_0^\infty \left|\frac{W_\beta(\epsilon,q^s\lambda_I)}{\Theta(\frac{q^s\lambda_I}{\epsilon t})}\right| ds.$$

Let $0<\xi<1$ as in Assumption (C). From (37) and (28), the previous integral is bounded by

$$\frac{|\log q|}{|\pi_q|}\int_0^\infty \frac{C_2\beta!\left(\frac{|q|^{2A_1S}}{\delta_2}\right)^\beta \left|\frac{q^s\lambda_I}{\epsilon}\right|^{C\beta} e^{M\log^2\left|\frac{q^s\lambda_I}{\epsilon}\right|}|q|^{-A_1\beta^2}}{C_\xi \exp(\frac{\xi\log^2|\frac{q^s\lambda_I}{\epsilon t}|}{2\log|q|})}ds$$

$$\leq \frac{|\log q|}{|\pi_q|}\frac{C_2}{C_\xi}\beta!\left(\frac{|q|^{2A_1S}}{\delta_2}\right)^\beta \left|\frac{\lambda_I}{\epsilon}\right|^{C\beta}|q|^{-A_1\beta^2}\int_0^\infty \frac{|q|^{Cs\beta}e^{M\log^2\left|\frac{q^s\lambda_I}{\epsilon}\right|}}{\exp(\frac{\xi\log^2|\frac{q^s\lambda_I}{\epsilon t}|}{2\log|q|})}ds.$$

Let a_1,a_2 as in Assumption (C.2) and (C.3).

From $(a_1s-a_2\beta)^2\geq 0$ and 4. in Definition 4, the previous inequality is upper bounded by

$$\mathcal{A}\int_0^\infty |q|^{-Bs^2}e^{(M-\frac{\xi}{2\log|q|})\log^2|\lambda_I/\epsilon|}e^{((2M\log|q|-\xi)\log|\lambda_I/\epsilon|+\xi\log|t|)s}ds, \tag{39}$$

where $0<B=\frac{\xi}{2}-M\log|q|-\frac{Ca_1}{2a_2}$ and

$$\mathcal{A} = \frac{|\log q|}{|\pi_q|}\frac{C_2}{C_\xi}\beta!\left(\frac{|q|^{2A_1S}}{\delta_2}\right)^\beta\left|\frac{\lambda_I}{\epsilon}\right|^{C\beta}|q|^{-A_1\beta^2+\frac{Ca_2\beta^2}{2a_1}}e^{-\frac{\xi\log^2|t|}{2\log|q|}}e^{\frac{\xi\log|\lambda_I/\epsilon|\log|t|}{\log|q|}}.$$

The previous integral is uniformly bounded for $\epsilon \in D(0,r_0)\setminus\{0\}$ and $t \in \mathcal{T}$ from hypotheses made on these sets. The expression in (39) can be bounded by

$$\frac{|\log q|}{|\pi_q|}\frac{C_2'}{C_\xi}\beta!\left(\frac{|q|^{2A_1S}}{\delta_2}\right)^{\beta}\left|\frac{\lambda_I}{\epsilon}\right|^{C\beta}e^{(M-\frac{\xi}{2\log|q|})\log^2|\lambda_I/\epsilon|}|q|^{-A_1\beta^2+\frac{Ca_2\beta^2}{2a_1}}e^{-\frac{\xi\log^2|t|}{2\log|q|}}e^{\frac{\xi\log|\lambda_I/\epsilon|\log|t|}{\log|q|}},$$

for an appropriate constant $C_2' > 0$.

The function $s \mapsto s^{\gamma\beta}e^{-\alpha\log^2(s)}$ takes its maximum at $s = e^{\gamma\beta/(2\alpha)}$ so each element in the image set is bounded by $e^{(\gamma\beta)^2/(4\alpha)}$. Taking this to the expression above we get

$$|\mathcal{L}_{q;1,+}^{\lambda_I}W_\beta(\epsilon,\epsilon t)| \le \frac{|\log q|}{|\pi_q|}\frac{C_2''}{C_\xi}\beta!\left(\frac{|q|^{2A_1S}}{\delta_2}\right)^{\beta}|q|^{-A_1\beta^2+\frac{Ca_2\beta^2}{2a_1}+\frac{C^2\beta^2}{4\log|q|(\xi/(2\log|q|)-M)}},$$

for certain $C_2'' > 0$.

Assumption (C.3) applied to the last term in the previous expression allows us to deduce that the sum

$$\sum_{\beta\ge 0}|\mathcal{L}_{q;1,+}^{\lambda_I}W_\beta(\epsilon,\epsilon t)|\frac{|z|^\beta}{\beta!} \tag{40}$$

converges in the variable z uniformly in the compact sets of $\mathbb{C}$.

We now study $\mathcal{L}_{q;1,-}^{\lambda_I}W_\beta(\epsilon,\epsilon t)$. We have

$$|\mathcal{L}_{q;1,-}^{\lambda_I}W_\beta(\epsilon,\epsilon t)| \le \frac{|\log q|}{|\pi_q|}\int_{-\infty}^{0}\left|\frac{W_\beta(\epsilon,q^s\lambda_I)}{\Theta(\frac{q^s\lambda_I}{\epsilon t})}\right|ds.$$

From (24) and (28) the previous integral is bounded by

$$\frac{|\log q|}{|\pi_q|}\int_{-\infty}^{0}\frac{C_3\beta!\left(\frac{|q|^{2A_1S}}{\delta_1}\right)^{\beta}|\epsilon|^{-C\beta}e^{M\log^2\left|\frac{q^s\lambda_I}{\epsilon}\right|}|q|^{-A_1\beta^2}}{C_\xi e^{\frac{\xi\log^2|\frac{q^s\lambda_I}{\epsilon t}|}{2\log|q|}}}ds.$$

Similar calculations as in the first part of the proof resting on Assumption (C) can be followed so that the series

$$\sum_{\beta\ge 0}\mathcal{L}_{q;1,-}^{\lambda_I}W_\beta(\epsilon,\epsilon t)\frac{z^\beta}{\beta!} \tag{41}$$

is uniformly convergent with respect to the variable z in the compact sets of $\mathbb{C}$, for $(\epsilon,t) \in U_Iq^{-\mathbb{N}} \times \mathcal{T}$. We will not enter into detail not to repeat calculations.

The estimates (40) and (41) imply convergence of the series in (38) for every $z \in \mathbb{C}$. Boundness of the $q-$Laplace transform with respect to ϵ is guaranteed so the first part of the result is achieved.

Let $I, I' \in \mathcal{I}$ such that $U_Iq^{-\mathbb{N}} \cap U_{I'}q^{-\mathbb{N}} \neq \emptyset$ and $\rho > 0$. For every $(\epsilon,t,z) \in (U_Iq^{-\mathbb{N}} \cap U_{I'}q^{-\mathbb{N}}) \times \mathcal{T} \times D(0,\rho)$ we have

$$|X_I(\epsilon,t,z) - X_{I'}(\epsilon,t,z)| \le \sum_{\beta\ge 0}|\mathcal{L}_{q;1}^{\lambda_I}W_\beta(\epsilon,\epsilon t) - \mathcal{L}_{q;1}^{\lambda_{I'}}W_\beta(\epsilon,\epsilon t)|\frac{\rho^\beta}{\beta!}. \tag{42}$$

We can write

$$\mathcal{L}_{q;1}^{\lambda_I} W_\beta(\epsilon,\epsilon t) - \mathcal{L}_{q;1}^{\lambda_{I'}} W_\beta(\epsilon,\epsilon t) = \frac{1}{\pi_q}\Big(\int_{\gamma_1}\frac{W_\beta(\epsilon,\xi)}{\Theta(\xi/\epsilon t)}\frac{d\xi}{\xi} - \int_{\gamma_2}\frac{W_\beta(\epsilon,\xi)}{\Theta(\xi/\epsilon t)}\frac{d\xi}{\xi} + \int_{\gamma_3-\gamma_4}\frac{W_\beta(\epsilon,\xi)}{\Theta(\xi/\epsilon t)}\frac{d\xi}{\xi}\Big) \quad (43)$$

where the path γ_1 is given by $s \in (0,\infty) \mapsto q^s\lambda_I$, γ_2 is given by $s \in (0,\infty) \mapsto q^s\lambda_{I'}$, γ_3 is $s \in (-\infty,0) \mapsto q^s\lambda_I$ and γ_4 is $s \in (-\infty,0) \mapsto q^s\lambda_{I'}$.

Without loss of generality, we can assume that $|\lambda_I| = |\lambda_{I'}|$.

For the first integral we deduce

$$\Big|\int_{\gamma_1}\frac{W_\beta(\epsilon,\xi)}{\Theta(\xi/\epsilon t)}\frac{d\xi}{\xi}\Big| \le |\log(q)|\int_0^\infty \frac{|W_\beta(\epsilon,q^s\lambda_I)|}{|\Theta(\frac{q^s\lambda_I}{\epsilon t})|}ds.$$

Similar estimates as in the first part of the proof lead us to bound the right part of previous inequality by

$$\frac{C_2''''}{C_\xi}\beta!\Big(\frac{|q|^{2A_1S}}{\delta_2}\Big)^\beta\Big|\frac{\lambda_I}{\epsilon}\Big|^{C\beta}|q|^{-A_1\beta^2+\frac{Ca_2}{2a_1}\beta^2}e^{(M-\frac{\xi}{2\log|q|})\log^2|\lambda_I/\epsilon|},$$

for certain $C_2'''' > 0$. For any $\overline{\xi} \in (0,1)$ we have

$$\Big|\frac{\lambda_I}{\epsilon}\Big|^{C\beta}e^{\overline{\xi}(M-\frac{\xi}{2\log|q|})\log^2|\lambda_I/\epsilon|} \le e^{\frac{C^2\beta^2}{4\overline{\xi}(\frac{\xi}{2\log|q|}-M)}}, \quad \beta \ge 0.$$

This yields

(44)

$$\int_{\gamma_1}\Big|\frac{W_\beta(\epsilon,q^s\lambda_I)}{\Theta(\frac{q^s\lambda_I}{\epsilon t})}\Big|ds \le \frac{C_2''''}{C_\xi}\beta!\Big(\frac{|q|^{2A_1S}}{\delta_2}\Big)^\beta|q|^{(-A_1+\frac{Ca_2}{2a_1}+\frac{C^2}{4\overline{\xi}\log|q|(\frac{\xi}{2\log|q|}-M)})\beta^2}e^{(1-\overline{\xi})(M-\frac{\xi}{2\log|q|})\log^2|\lambda_I/\epsilon|}.$$

We choose $\overline{\xi}$ as in Assumption (C).

The integral corresponding to the path γ_2 can be bounded following identical steps.

We now give estimates concerning $\gamma_3-\gamma_4$. It is worth saying that the function in the integrand is well defined for $(\epsilon,\tau) \in (D(0,r_0)\setminus\{0\}) \times \dot{D}_{\rho_0}$ and does not depend on the index $I \in \mathcal{I}$. This fact and Cauchy Theorem allow us to write for any $n \in \mathbb{N}$

$$\int_{\Gamma_n}\frac{W_\beta(\epsilon,\xi)}{\Theta(\xi/\epsilon t)}\frac{d\xi}{\xi} = 0,$$

where $\Gamma_n = \gamma_{n,1}+\gamma_5-\gamma_{n,2}-\gamma_{n,3}$ is the closed path defined in the following way: $s \in [-n,0] \mapsto \gamma_{n,1}(s) = \lambda_I q^s$,$\gamma_5$ is the arc of circunference from λ_I to $\lambda_{I'}$, $s \in [-n,0] \mapsto \gamma_{n,2}(s) = \lambda_{I'}q^s$ and $\gamma_{n,3}$ is the arc of circunference from $\lambda_I q^{-n}$ to $\lambda_{I'}q^{-n}$. Taking $n \to \infty$ we derive

$$0 = \lim_{n\to\infty}\int_{\Gamma_n}\frac{W_\beta(\epsilon,\xi)}{\Theta(\xi/\epsilon t)}\frac{d\xi}{\xi} = \lim_{n\to\infty}\int_{\gamma_{n,1}+\gamma_5-\gamma_{n,2}}\frac{W_\beta(\epsilon,\xi)}{\Theta(\xi/\epsilon t)}\frac{d\xi}{\xi} - \lim_{n\to\infty}\int_{\gamma_{n,3}}\frac{W_\beta(\epsilon,\xi)}{\Theta(\xi/\epsilon t)}\frac{d\xi}{\xi}. \quad (45)$$

Usual estimates lead us to prove that

$$\lim_{n\to\infty}\int_{\gamma_{n,3}}\frac{W_\beta(\epsilon,\xi)}{\Theta(\xi/\epsilon t)}\frac{d\xi}{\xi} = 0. \quad (46)$$

Moreover,

$$\lim_{n\to\infty}\int_{\gamma_{n,1}+\gamma_5-\gamma_{n,2}}\frac{W_\beta(\epsilon,\xi)}{\Theta(\xi/\epsilon t)}\frac{d\xi}{\xi} = \int_{\gamma_3+\gamma_5-\gamma_4}\frac{W_\beta(\epsilon,\xi)}{\Theta(\xi/\epsilon t)}\frac{d\xi}{\xi}. \quad (47)$$

From (45), (46) and (47) we obtain

$$\int_{\gamma_3-\gamma_4}\frac{W_\beta(\epsilon,\xi)}{\Theta(\xi/\epsilon t)}\frac{d\xi}{\xi}=\int_{-\gamma_5}\frac{W_\beta(\epsilon,\xi)}{\Theta(\xi/\epsilon t)}\frac{d\xi}{\xi}=\int_{\theta_{I'}}^{\theta_I}\frac{W_\beta(\epsilon,|\lambda_I|e^{i\theta})}{\Theta(\frac{|\lambda_I|e^{i\theta}}{\epsilon t})}d\theta,$$

where $\theta_I=\arg(\lambda_I)$, $\theta_{I'}=\arg(\lambda_{I'})$. Taking into account Definition 4 and (36) we derive the modulus of the last term in the previous equality is bounded by

$$\frac{\mathrm{length}(\gamma_5)C_3}{C_\xi}\beta!\Big(\frac{|q|^{2A_1S}}{\delta_1}\Big)^\beta|\epsilon|^{-C\beta}\frac{e^{M\log^2\left|\frac{\lambda_I}{\epsilon}\right|}}{e^{\frac{\xi}{2\log|q|}\log^2\left|\frac{\lambda_I}{\epsilon t}\right|}}|q|^{-A_1\beta^2}$$
$$\leq C_3'\beta!\Big(\frac{|q|^{2A_1S}}{\delta_1}\Big)^\beta|\epsilon|^{-C\beta}e^{(M-\frac{\xi}{2\log|q|})\log^2\left|\frac{\lambda_I}{\epsilon}\right|}|q|^{-A_1\beta^2}$$
$$\leq C_3'\beta!\Big(\frac{|q|^{2A_1S}}{\delta_1}\Big)^\beta|\epsilon|^{-C\beta}e^{\bar{\xi}(M-\frac{\xi}{2\log|q|})\log^2|\epsilon|}|q|^{-A_1\beta^2}e^{(1-\bar{\xi})(M-\frac{\xi}{2\log|q|})\log^2|\epsilon|}.$$

for adequate positive constants C_3, C_3'. From standard estimates we achieve

$$\Big|\int_{\gamma_3-\gamma_4}\frac{W_\beta(\epsilon,\xi)}{\Theta(\xi/\epsilon t)}\frac{d\xi}{\xi}\Big|\leq C_3'\beta!\Big(\frac{|q|^{2A_1S}}{\delta_1}\Big)^\beta|q|^{-A_1\beta^2}e^{\frac{C^2}{4\bar{\xi}(\frac{\xi}{2\log|q|}-M)}\beta^2}e^{(1-\bar{\xi})(M-\frac{\xi}{2\log|q|})\log^2|\epsilon|}. \tag{48}$$

From (42), (43), (44), (48) and Assumption (C.3) we conclude the existence of a positive constant $C_1'>0$ such that

$$|X_I(\epsilon,t,z)-X_{I'}(\epsilon,t,z)|\leq C_1'\sum_{\beta\geq0}\beta!\Big(\frac{|q|^{2A_1S}}{\delta_0}\Big)^\beta|q|^{\left(-A_1+\frac{Ca_2}{2a_1}+\frac{C^2}{4\bar{\xi}\log|q|(\frac{\xi}{2\log|q|}-M)}\right)\beta^2}\times$$
$$\times e^{(1-\bar{\xi})(M-\frac{\xi}{2\log|q|})\log^2|\epsilon|}\frac{\rho^\beta}{\beta!}\leq C_1e^{(1-\bar{\xi})(M-\frac{\xi}{2\log|q|})\log^2|\epsilon|},$$

for every $(\epsilon,t,z)\in(U_Iq^{-\mathbb{N}}\cap U_{I'}q^{-\mathbb{N}})\times\mathcal{T}\times D(0,\rho)$, with $\delta_0=\min\{\delta_1,\delta_2\}$.

□

5 A $q-$Gevrey Malgrange-Sibuya type theorem

In this section we obtain a $q-$Gevrey version of the so called Malgrange-Sibuya theorem which allows us to reach our final main achievement: the existence of a formal series solution of problem (33)+(34) which asymptotically represents the actual solutions obtained in Theorem 3, meaning that for every $I\in\mathcal{I}$, X_I admits this formal solution as its $q-$Gevrey asymptotic expansion in the variable ϵ.

In [11], a Malgrange-Sibuya type theorem appears with similar aims as in this work. We complete the information there giving bounds on the estimates appearing for the $q-$asymptotic expansion. This mentioned work heavily rests on the theory developed by J-P. Ramis, J. Sauloy and C. Zhang in [18].

In the present work, although $q-$Gevrey bounds are achieved, the $q-$Gevrey type involved will not be preserved, suffering an increase on the way.

The nature of the proof relies in the one concerning classical Malgrange-Sibuya theorem for Gevrey asymptotics which can be found in [16].

Let $\mathbb{H}$ be a complex Banach space.

Definition 6 *Let U be a bounded open set in $\mathbb{C}^\star$ and $A>0$. We say a holomorphic function $f:Uq^{-\mathbb{N}}\to\mathbb{H}$ admits $\hat{f}=\sum_{n\ge0}f_n\epsilon^n\in\mathbb{H}[[\epsilon]]$ as its $q-$Gevrey asymptotic expansion of type A in $Uq^{-\mathbb{N}}$ if for every compact set $K\subseteq U$ there exist $C_1,H>0$ such that*

$$\left\|f(\epsilon)-\sum_{n=0}^{N}f_n\epsilon^n\right\|_{\mathbb{H}}\le C_1H^N|q|^{A\frac{N^2}{2}}\frac{|\epsilon|^{N+1}}{(N+1)!},\quad N\ge0,$$

for every $\epsilon\in Kq^{-\mathbb{N}}$.

The following proposition can be found, under slight modifications in Section 4 of [18].

Proposition 3 *Let $A>0$ and $U\subseteq\mathbb{C}^\star$ be an open and bounded set. Let $f:Uq^{-\mathbb{N}}\to\mathbb{H}$ be a holomorphic function that admits a formal power series $\hat{f}\in\mathbb{H}[[\epsilon]]$ as its $q-$Gevrey asymptotic expansion of type A in $Uq^{-\mathbb{N}}$. Then, if $\hat{f}^{(k)}$ stands for the $k-$th formal derivative of $\hat{f}$ for every $k\in\mathbb{N}$, we have that $f^{(k)}$ admits $\hat{f}^{(k)}$ as its $q-$Gevrey asymptotic expansion of type A in $Uq^{-\mathbb{N}}$.*

Proposition 4 *Let $A>0$ and $f:Uq^{-\mathbb{N}}\to\mathbb{H}$ a holomorphic function in $Uq^{-\mathbb{N}}$. Then,*

i) If f admits $\hat{0}$ as its $q-$Gevrey expansion of type A, then for every compact set $K\subseteq U$ there exists $C_1>0$ with

$$\|f(\epsilon)\|_{\mathbb{H}}\le C_1e^{-\frac{1}{\tilde{a}}\frac{1}{2\log|q|}\log^2|\epsilon|},$$

for every $\epsilon\in Kq^{-\mathbb{N}}$ and every $\tilde{a}>A$.

ii) If for every compact set $K\subseteq U$ there exists $C_1>0$ with

$$\|f(\epsilon)\|_{\mathbb{H}}\le C_1e^{-\frac{1}{A}\frac{1}{2\log|q|}\log^2|\epsilon|},$$

for every $\epsilon\in Kq^{-\mathbb{N}}$ then f admits $\hat{0}$ as its $q-$Gevrey asymptotic expansion of type $\tilde{a}$ in $Uq^{-\mathbb{N}}$, for every $\tilde{a}>A$.

Proof Let $C_1,H,A>0$ and $\epsilon\in\mathbb{C}^\star$. The function

$$G(x)=C_1\exp(\log(H)x+\frac{\log|q|A}{2}x^2+(x+1)\log|\epsilon|)$$

reaches its minimum for $x>0$ at $x_0=\frac{-\log(H)-\log|\epsilon|}{A\log|q|}$. We deduce both results from standard calculations. □

Definition 7 *Let $(U_Iq^{-\mathbb{N}})_{I\in\mathcal{I}}$ be a good covering at 0 (see Definition 3), and $g_{I,I'}:U_Iq^{-\mathbb{N}}\cap U_{I'}q^{-\mathbb{N}}\to\mathbb{H}$ a holomorphic function in $U_Iq^{-\mathbb{N}}\cap U_{I'}q^{-\mathbb{N}}$ for $I,I'\in\mathcal{I}$ when the intersection is not empty. The family $(g_{I,I'})_{(I,I')\in\mathcal{I}^2}$ is a $q-$Gevrey $\mathbb{H}-$cocycle of type $A>0$ attached to a good covering $(U_Iq^{-\mathbb{N}})_{I\in\mathcal{I}}$ if the following properties are satisfied:*

1. *$g_{I,I'}$ admits $\hat{0}$ as its $q-$Gevrey asymptotic expansion of type $A>0$ on $U_Iq^{-\mathbb{N}}\cap U_{I'}q^{-\mathbb{N}}$ for every $(I,I')\in\mathcal{I}$.*
2. *$g_{I,I'}(\epsilon)=-g_{I',I}(\epsilon)$ for every $(I,I')\in\mathcal{I}$, and $\epsilon\in U_Iq^{-\mathbb{N}}\cap U_{I'}q^{-\mathbb{N}}$.*
3. *We have $g_{I,I''}(\epsilon)=g_{I,I'}(\epsilon)+g_{I',I''}(\epsilon)$ for all $\epsilon\in U_Iq^{-\mathbb{N}}\cap U_{I'}q^{-\mathbb{N}}\cap U_{I''}q^{-\mathbb{N}}$, $I,I',I''\in\mathcal{I}$.*

Let $\rho > 0$ and $\mathcal{T} \subseteq \mathbb{C}^{\star}$ be an open and bounded set. $\mathbb{H}_{\mathcal{T},\rho}$ stands for the Banach space of holomorphic and bounded functions in $\mathcal{T} \times D(0,\rho)$ with the supremum norm.

Proposition 5 *Let $\rho > 0$. We consider the family $(X_I(\epsilon,t,z))_{I\in\mathcal{I}}$ constructed in Theorem 3. Then, the set of functions $(g_{I,I'}(\epsilon))_{(I,I')\in\mathcal{I}^2}$ defined by*

$$g_{I,I'}(\epsilon) := (t,z) \in \mathcal{T} \times D(0,\rho) \mapsto X_{I'}(\epsilon,t,z) - X_I(\epsilon,t,z)$$

for $I, I' \in \mathcal{I}$ is a $q-$Gevrey $\mathbb{H}_{\mathcal{T},\rho}$-cocycle of type $\tilde{A}$ for every

$$\tilde{A} > A := \frac{1}{(1-\overline{\xi})(\frac{\xi}{2\log|q|} - M)2\log|q|} = \frac{1}{(1-\overline{\xi})(\xi - 2M\log|q|)},$$

attached to the good covering $(U_I q^{-\mathbb{N}})_{I\in\mathcal{I}}$.

Proof The first property in Definition 7 directly comes from Theorem 3 and Proposition 4. The other two are verified by construction of the cocycle. □

We recall several definitions and an extension result from [2] which will be crucial in our work.

Definition 8 *A continuous increasing function $w : [0,\infty) \to [0,\infty)$ is a weight function if it satisfies*

- *(α) there exists $k \geq 1$ with $w(2t) \leq k(w(t)+1)$ for all $t \geq 0$,*
- *(β) $\int_0^{\infty} \frac{w(t)}{1+t^2}dt < \infty$,*
- *(γ) $\lim_{t\to\infty} \frac{\log t}{w(t)} = 0$,*
- *(δ) $\phi : t \mapsto w(e^t)$ is convex.*

The Young conjugate associated to ϕ, $\phi^{\star} : [0,\infty) \to \mathbb{R}$ is defined by

$$\phi^{\star}(y) := \sup\{xy - \phi(x) : x \geq 0\}.$$

Definition 9 *Let K be a nonempty compact set in $\mathbb{R}^2$. A jet on K is a family $F = (f^{\alpha})_{\alpha\in\mathbb{N}^2}$ where $f^{\alpha} : K \to \mathbb{C}$ is a continuous function on K for each $\alpha \in \mathbb{N}^2$.*

Let w be a weight function. A jet $F = (f^{\alpha})_{\alpha\in\mathbb{N}^2}$ on K is said to be a $w-$Whitney jet (of Roumieu type) on K if there exist $m > 0$ and $M > 0$ such that

$$\|f\|_{K,1/m} := \sup_{x\in K,\alpha\in\mathbb{N}^2} |f^{\alpha}(x)| \exp(-\frac{1}{m}\phi^{\star}(m|\alpha|)) \leq M,$$

and for every $l \in \mathbb{N}$, $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq l$ and $x, y \in K$ one has

$$|(R_x^l F)_{\alpha}(y)| \leq M \frac{|x-y|^{l+1-|\alpha|}}{(l+1-|\alpha|)!} \exp(\frac{1}{m}\phi^{\star}(m(l+1))),$$

where $(R_x^l F)_{\alpha}(y) := f^{\alpha}(y) - \sum_{|\alpha+\beta|\leq l} \frac{1}{\beta!} f^{\alpha+\beta}(x)(y-x)^{\beta}$.

$\mathcal{E}_{\{w\}}(K)$ denotes the linear space of $w-$Whitney jets on K.

Definition 10 *Let $K \subseteq \mathbb{R}^2$ be a nonempty compact set and w a weight function in K. A continuous function $f : K \to \mathbb{C}$ is $w - \mathcal{C}^\infty$ in the sense of Whitney in K if there exists a $w-$Whitney jet on K, $(f^\alpha)_{\alpha\in\mathbb{N}^2}$ such that $f^{(0,0)} = f$.*

For an open set $\Omega \in \mathbb{R}^2$ we define

$$\mathcal{E}_{\{w\}}(\Omega) := \{f \in \mathcal{C}^\infty(\Omega) : \forall K \subseteq \Omega, K \text{ compact }, \exists m > 0, \|f\|_{K,1/m} < \infty\}.$$

The following result establishes conditions on a weight function so that a jet in $\mathcal{E}_{\{w\}}(K)$ can be extended to an element in $\mathcal{E}_{\{w\}}(\mathbb{R}^2)$.

Theorem 4 (Corollary 3.10, [2]) *For a given weight function w, the following statements are equivalent:*

1. *For every nonempty closed set K in $\mathbb{R}^2$ the restriction map sending a function $f \in \mathcal{E}_{\{w\}}(\mathbb{R}^2)$ to the family of derivatives of f in K, $(f^{(\alpha)}|_K)_{\alpha\in\mathbb{N}^2} \in \mathcal{E}_{\{w\}}(K)$ is a surjective map.*

2. *w is a strong weight function, it is to say,*

$$\lim_{\epsilon\to 0^+} \overline{\lim_{t\to\infty}} \frac{\epsilon w(t)}{w(\epsilon t)} = 0.$$

Let $k_1 = \frac{1}{4\log|q|}$. We consider the weight function defined by $w_0(t) = k_1 \log^2(t)$ for $t \geq 1$ and $w_0(t) = 0$ for $0 \leq t \leq 1$. As the authors write in [2], the value of a weight function near the origin is not relevant for the space of functions generated in the sequel.

The following lemma can be easily verified.

Lemma 8 *w_0 is a weight function.*

Under this definition of w_0 we have

$$\phi^\star_{w_0}(y) = \sup\{xy - \phi_{w_0}(x) : x \geq 0\} = \sup\{xy - \frac{x^2}{4\log|q|} : x \geq 0\} = \log|q| y^2, \quad y \geq 0.$$

The spaces appearing in Definition 9 concerning this weight function are the following: for any nonempty compact set $K \subseteq \mathbb{R}^2$, $\mathcal{E}_{\{w_0\}}(K)$ is the set of w_0-Whitney jets on K, which consists of every jet $F = (f^\alpha)_{\alpha\in\mathbb{N}^2}$ on K such that there exist $m \in \mathbb{N}$, $M > 0$ with

$$|f^\alpha(x)| \leq M|q|^{m|\alpha|^2}, \quad x \in K, \alpha \in \mathbb{N}^2$$

and such that for every $l \in \mathbb{N}$ and $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq l$ we have

$$|(R^l_x F)_\alpha(y)| \leq M\frac{|x-y|^{l+1-|\alpha|}}{(l+1-|\alpha|)!}|q|^{m(l+1)^2}, \quad x, y \in K.$$

We derive that $\mathcal{E}_{\{w_0\}}(K)$ consists of the Whitney jets on K such that there exist $C_1, H > 0$ with

$$|f^\alpha(x)| \leq C_1 H^{|\alpha|}|q|^{A\frac{|\alpha|^2}{2}}, \quad x \in K, \alpha \in \mathbb{N}^2, \tag{49}$$

and for every $x, y \in K$ and all $l \in \mathbb{N}, \alpha \in \mathbb{N}^2$ with $|\alpha| \leq l$

$$|(R^l_x F)_\alpha(y)| \leq C_1 H^l |q|^{A\frac{l^2}{2}} \frac{|x-y|^{l+1-|\alpha|}}{(l+1-|\alpha|)!}. \tag{50}$$

Theorem 5 *w_0 is a strong weight function so that Theorem 4 holds.*

Proof

$$\lim_{\epsilon\to 0^+}\lim_{t\to\infty}\frac{\epsilon w(t)}{w(\epsilon t)} = \lim_{\epsilon\to 0^+}\lim_{t\to\infty}\frac{\epsilon k_1\log^2(t)}{k_1\log^2(\epsilon t)} = \lim_{\epsilon\to 0^+}\epsilon = 0.$$

□

Remark: A continuous function f which is $w_0-\mathcal{C}^\infty$ in the sense of Whitney on a compact set K is indeed $\mathcal{C}^\infty$ in the usual sense in $\mathrm{Int}(K)$ and verifies $q-$Gevrey bounds of the same type. Moreover, we have

$$f^k(x,y) = \partial_x^{k_1}\partial_y^{k_2}f(x,y),$$

for every $k=(k_1,k_2)\in\mathbb{N}^2$ and $(x,y)\in\mathrm{Int}(K)$.

Next result is an adaptation of Lemma 4.1.2 in [18]. Here, we need to determine bounds in order to achieve a $q-$Gevrey type result.

Lemma 9 *Let U be an open set in $\mathbb{C}^\star$ and $f: Uq^{-\mathbb{N}}\to\mathbb{H}$ a holomorphic function with $\hat{f}=\sum_{h\geq 0}a_h\epsilon^h\in\mathbb{H}[[\epsilon]]$ being its q-Gevrey asymptotic expansion of type $A>0$ in $Uq^{-\mathbb{N}}$. Then, for any $n\in\mathbb{N}$, the family $\partial_\epsilon^n f(\epsilon)$ of $n-$complex derivatives of f satisfies that for every compact set $K\subseteq U$ and $k,m\in\mathbb{N}$ with $k\leq m$, there exist $C_1,H>0$ such that*

$$\left\|\partial_\epsilon^k f(\epsilon_a) - \sum_{h=0}^{m-k}\frac{\partial_\epsilon^{k+h}f(\epsilon_b)}{h!}(\epsilon_a-\epsilon_b)^h\right\|_{\mathbb{H}} \leq C_1 H^m |q|^{A\frac{m^2}{2}}\frac{|\epsilon_a-\epsilon_b|^{m+1-k}}{(m+1-k)!}, \tag{51}$$

for every $\epsilon_a,\epsilon_b\in Kq^{-\mathbb{N}}\cup\{0\}$. Here, we write $\partial_\epsilon^l f(0)=l!a_l$ for $l\in\mathbb{N}$.

Proof We will first state the result when $\epsilon_b=0$. Indeed, we prove in this first step that the family of functions with $q-$Gevrey asymptotic expansion of type $A>0$ in a fixed $q-$spiral is closed under derivation. Proposition 3 turns out to be a particular case of this result.
Let $m\in\mathbb{N}$, K be a compact set in U and consider another compact set K_1 such that $K\subseteq K_1\subseteq U$. We define

$$R_m(\epsilon) := \epsilon^{-m-1}\Big(f(\epsilon) - \sum_{h=0}^{m}\frac{\partial_\epsilon^h f(0)}{h!}\epsilon^h\Big),\quad \epsilon\in Kq^{-\mathbb{N}},$$

where $\partial_\epsilon^h f(0)$ denotes the limit of $\partial_\epsilon^h f(\epsilon)$ for $\epsilon\in Kq^{-\mathbb{N}}$ tending to 0. Then we have that

$$\partial_\epsilon f(\epsilon) = \sum_{h=1}^{m}\frac{\partial_\epsilon^h f(0)}{h!}h\epsilon^{h-1} + (\partial_\epsilon R_m(\epsilon))\epsilon^{m+1} + (m+1)R_m(\epsilon)\epsilon^m. \tag{52}$$

Moreover, from Definition 6, there exist $C,H>0$ such that $\|R_m(\epsilon)\|\leq CH^m\frac{|q|^{A\frac{m^2}{2}}}{(m+1)!}$ for every $\epsilon\in K_1q^{-\mathbb{N}}$.

Lemma 10 (Lemma 4.4.1 [18]) *There exists $\rho>0$ such that $\overline{D}(\epsilon,\rho|\epsilon|)\subseteq K_1q^{-\mathbb{N}}$ for every $\epsilon\in Kq^{-\mathbb{N}}$.*

Cauchy's integral formula and $q-$Gevrey expansion of f guarantee the existence of a positive constant $C_2>0$ such that

$$\|\partial_\epsilon R_m(\epsilon)\|_{\mathbb{H}} \leq C_2 H^m\frac{|q|^{A\frac{m^2}{2}}}{(m+1)!}\frac{1}{\rho|\epsilon|},\quad \epsilon\in Kq^{-\mathbb{N}},$$

This yields the existence of $C_3 > 0$ such that

$$\left\| \epsilon^{-m}(\partial_\epsilon f(\epsilon) - \sum_{h=0}^{m-1} \frac{\partial_\epsilon^{h+1} f(0)}{h!}\epsilon^h) \right\|_{\mathbb{H}} \le \|\partial_\epsilon R_m(\epsilon)\|_{\mathbb{H}} |\epsilon| + (m+1)\|R_m(\epsilon)\|_{\mathbb{H}}$$

$$\le C_2 A_1^m \frac{|q|^{A\frac{m^2}{2}}}{m!}, \quad \epsilon \in Kq^{-\mathbb{N}}.$$

An induction reasoning is sufficient to conclude the proof for every $m \ge 0$.

We now study the case where $\epsilon_b \neq 0$ and only give details for $k = 0$. For $k \ge 1$ one only has to take into account that the derivatives of f also admit $q-$Gevrey asymptotic expansion of type A and consider the function $\partial_\epsilon^k f$.

If $\epsilon_b \neq 0$ we treat two cases:

If $|\epsilon_a - \epsilon_b| \le \rho|\epsilon_b|$, then $[\epsilon_a, \epsilon_b]$ is contained in $K_1 q^{-\mathbb{N}}$ and we conclude from Cauchy's integral formula.

If $|\epsilon_a - \epsilon_b| > \rho|\epsilon_b|$, then we bear in mind that the result is obvious when f is a polynomial and write $f(\epsilon) = \epsilon^{m+1}R_m(\epsilon) + p(\epsilon)$ where $p(\epsilon) = \sum_{h=0}^m \frac{\partial_\epsilon^h f(0)}{h!}\epsilon^h$. So, it is sufficient to prove (51) when $f(\epsilon) := \epsilon^{m+1}R_m(\epsilon)$. The result follows from $q-$Gevrey bounds for $\left\|\partial_\epsilon^k R_m\right\|_{\mathbb{H}}$, $k = 0, ..., n$ and usual estimates. □

The following lemma generalizes Lemma 6 in [11].

Lemma 11 *Let $f : Uq^{-\mathbb{N}} \to \mathbb{H}$ be a holomorphic function having $\hat{f}(\epsilon) = \sum_{h \ge 0} a_h \epsilon^h \in \mathbb{H}[[\epsilon]]$ as its $q-$Gevrey asymptotic expansion of type $A > 0$ on $Uq^{-\mathbb{N}}$. Let $K \subseteq U$ be a compact set. Then, the function $(\epsilon_1, \epsilon_2) \mapsto \phi(\epsilon_1 + i\epsilon_2) = f(\epsilon_1, \epsilon_2)$ is a $w_0 - \mathcal{C}^\infty$ function in the sense of Whitney on the compact set*

$$K' = \{(\epsilon_1, \epsilon_2) \in \mathbb{R}^2 : \epsilon_1 + i\epsilon_2 \in Kq^{-\mathbb{N}} \cup \{0\}\}.$$

Proof We consider the set of functions $(\phi^{(k_1,k_2)})_{(k_1,k_2)\in\mathbb{N}^2}$ defined by

$$\phi^{(k_1,k_2)} := i^{k_2}\partial_\epsilon^{k_1+k_2} f(\epsilon), \quad (k_1, k_2) \in \mathbb{N}^2, (\epsilon_1, \epsilon_2) \in K'. \tag{53}$$

From Lemma 9, function f satisfies bounds as in (51). Written in terms of the elements in $(\phi^{(k_1,k_2)})_{(k_1,k_2)\in\mathbb{N}^2}$ we have the existence of $C_1, H > 0$ such that for every $(k_1, k_2) \in \mathbb{N}^2$, $m \ge 0$

$$\left\| \frac{1}{i^{k_2}}\phi^{(k_1,k_2)}(x_1, y_1) - \sum_{p=0}^{m-|(k_1,k_2)|} \sum_{h_1+h_2=p} \frac{\phi^{(k_1+h_1,k_2+h_2)}(x_2, y_2)}{i^{k_2+h_2}p!} \right.$$

$$\left. \times \frac{p!}{h_1!h_2!}(x_1 - x_2)^{h_1} i^{h_2}(y_1 - y_2)^{h_2} \right\|_{\mathbb{H}} \le C_1 H^m |q|^{A\frac{m^2}{2}} \frac{\|(x_1 - x_2, y_1 - y_2)\|_{\mathbb{R}^2}^{m+1-|(k_1,k_2)|}}{(m+1-|(k_1,k_2)|)!}$$

for $(x_1, y_1), (x_2, y_2) \in K'$. Expression (49) can be directly checked from (53) and (51) for $\epsilon_b = 0$ and $m = k$. This yields the set $(\phi^{(k_1,k_2)})_{(k_1,k_2)\in\mathbb{N}^2}$ is an element in $\mathcal{E}_{\{w_0\}}(K')$ □

Next result allows us to glue together a finite number of jets in $\mathcal{E}_{\{w_0\}}(K)$, for a given compact set K.

Theorem 6 *[[8]. Theorem II.1.3] Let K_1, K_2 be compact sets in $\mathbb{R}^2$. The following statements are equivalent:*

i. The sequence

$$0 \longrightarrow \mathcal{E}_{\{w_0\}}(K_1 \cup K_2) \xrightarrow{\pi} \mathcal{E}_{\{w_0\}}(K_1) \oplus \mathcal{E}_{\{w_0\}}(K_2) \xrightarrow{\delta} \mathcal{E}_{\{w_0\}}(K_1 \cap K_2) \longrightarrow 0$$

is exact. $\pi(f) = (f|_{K_1}, f|_{K_2})$ *and* $\delta(f,g) = f|_{K_1\cap K_2} - g|_{K_1 \cap K_2}$.

ii. Let $f_1 \in \mathcal{E}_{\{w_0\}}(K_1)$ *and* $f_2 \in \mathcal{E}_{\{w_0\}}(K_2)$ *be such that* $f_1(x) = f_2(x)$ *for every* $x \in K_1 \cap K_2$. *The function* f *defined by* $f(x) = f_1(x)$ *if* $x \in K_1$ *and* $f(x) = f_2(x)$ *if* $x \in K_2$ *belongs to* $\mathcal{E}_{\{w_0\}}(K_1 \cup K_2)$.

iii. If $K_1 \cap K_2 \neq \emptyset$ *then there exist* $A_3, A_4 > 0$ *such that*

$$\overline{M}(A_3 dist(x, K_1 \cap K_2)) \leq A_4 \overline{M}(dist(x, K_2)),$$

for every $x \in K_1$. *Here,* $\overline{M}$ *denotes the function given by* $\overline{M}(0) = 0$ *and* $\overline{M}(t) = \inf_{n\in\mathbb{N}} t^n M_n$ *for* $t > 0$. $dist(x, K)$ *stands for the distance from* x *to the set* K.

Corollary 2 *[[18], Lemma 4.3.6] Given* $\tilde{K}_1, \tilde{K}_2$ *nonempty compact sets in* $\mathbb{C}^\star$, *if we put* $K_j := \{(\epsilon_1, \epsilon_2) \in \mathbb{R}^2 : \epsilon_1 + i\epsilon_2 \in \tilde{K}_j q^{-\mathbb{N}} \cup \{0\}\}$, $j = 1, 2$, *then the previous theorem holds for* K_1 *and* K_2.

As the authors remark in [18], condition *iii*) in the previous result is known as transversality condition which is more constricting than Łojasiewicz's condition (see [15]).

Next proposition is devoted to show that the cocycle constructed in Proposition 5 splits in the space of $w_0 - \mathcal{C}^\infty$ functions in the sense of Whitney. Whitney-type extension results on $\mathcal{E}_{\{w_0\}}(K)$ (Theorem 4 and Theorem 5) will play an important role in the following step.

Proposition 6 *Let* $(U_I q^{-\mathbb{N}})_{I\in\mathcal{I}}$ *be a good covering and let* $(g_{I,I'}(\epsilon))_{(I,I')\in\mathcal{I}^2}$ *be the* $q-$*Gevrey* $\mathbb{H}_{\mathcal{T},\rho}$*-cocycle of type* $\tilde{A}$ *constructed in Proposition 5. We choose a family of compact sets* $K_I \subseteq U_I$ *for* $I \in \mathcal{I}$, *with* $Int(K_I) \neq \emptyset$, *in such a way that* $\cup_{I\in\mathcal{I}}(K_I q^{-\mathbb{N}})$ *is* $\mathcal{U} \setminus \{0\}$, *where* $\mathcal{U}$ *is a neighborhood of 0 in* $\mathbb{C}$.

Then, for all $I \in \mathcal{I}$, *there exists a* $w_0 - \mathcal{C}^\infty$ *function* $f_I(\epsilon_1, \epsilon_2)$ *in the sense of Whitney on the compact set* $A_I = \{(\epsilon_1, \epsilon_2) \in \mathbb{R}^2 : \epsilon_1 + i\epsilon_2 \in K_I q^{-\mathbb{N}} \cup \{0\}\}$, *with values in the Banach space* $\mathbb{H}_{\mathcal{T},\rho}$, *such that*

$$g_{I,I'}(\epsilon_1 + i\epsilon_2) = f_{I'}(\epsilon_1, \epsilon_2) - f_I(\epsilon_1, \epsilon_2) \tag{54}$$

for all $I, I' \in \mathcal{I}$ *such that* $A_I \cap A_{I'} \neq \emptyset$ *and, for every* $(\epsilon_1, \epsilon_2) \in A_I \cap A_{I'}$.

Proof The proof follows similar arguments as Lemma 3.12 in [18] and it is an adaptation of Proposition 5 in [11] under $q-$Gevrey settings.

Let $I, I' \in \mathcal{I}$ such that $A_I \cap A_{I'} \neq \emptyset$. From Lemma 11, we have the function $(\epsilon_1, \epsilon_2) \mapsto g_{I,I'}(\epsilon_1 + i\epsilon_2)$ is a $w_0 - \mathcal{C}^\infty$ function in the sense of Whitney on $A_I \cap A_{I'}$. In the following we provide the construction of f_I for $I \in \mathcal{I}$ verifying (54).

Let us fix any $I \in \mathcal{I}$. We consider any $w_0 - \mathcal{C}^\infty$ function in the sense of Whitney on A_I. By definition of the good covering $(U_I q^{-\mathbb{N}})_{I\in\mathcal{I}}$ the following cases are possible:

Case 1: If there is at least one $I' \in \mathcal{I}$, $I \neq I'$, such that $A_I \cap A_{I'} \neq \emptyset$ but $A_I \cap A_{I'} \cap A_{I''} = \emptyset$ for every $I'' \in \mathcal{I}$ with $I'' \neq I' \neq I$, then we define $e_{I,I'}(\epsilon_1, \epsilon_2) = f_I(\epsilon_1, \epsilon_2) + g_{I,I'}(\epsilon_1 + i\epsilon_2)$ for every $(\epsilon_1, \epsilon_2) \in A_I \cap A_{I'}$. $e_{I,I'}$ is a $w_0 - \mathcal{C}^\infty$ function in the sense of Whitney in $A_I \cap A_{I'}$. From

Theorem 4 and Theorem 5, we can extend $e_{I,I'}$ to a $w_0-\mathcal{C}^\infty$ function in the sense of Whitney on $A_{I'}$. This extension is called $f_{I'}$. We have

$$g_{I,I'}(\epsilon_1+i\epsilon_2)=f_{I'}(\epsilon_1,\epsilon_2)-f_I(\epsilon_1,\epsilon_2),\quad (\epsilon_1,\epsilon_2)\in A_I\cap A_{I'}.$$

Case 2: There exist two different $I',I''\in\mathcal{I}$ with $I'\neq I\neq I''$ such that $A_I\cap A_{I'}\cap A_{I''}\neq\emptyset$. We first construct a $w_0-\mathcal{C}^\infty$ function in the sense of Whitney on $A_{I'}$, $f_{I'}(\epsilon_1,\epsilon_2)$, verifying

$$g_{I,I'}(\epsilon_1+i\epsilon_2)=f_{I'}(\epsilon_1,\epsilon_2)-f_I(\epsilon_1,\epsilon_2),\quad (\epsilon_1,\epsilon_2)\in A_I\cap A_{I'}. \tag{55}$$

We define $e_{I,I''}(\epsilon_1,\epsilon_2)=f_I(\epsilon_1,\epsilon_2)+g_{I,I''}(\epsilon_1+i\epsilon_2)$ for every $(\epsilon_1,\epsilon_2)\in A_I\cap A_{I''}$ and $e_{I',I''}(\epsilon_1,\epsilon_2)=f_{I'}(\epsilon_1,\epsilon_2)+g_{I',I''}(\epsilon_1+i\epsilon_2)$ whenever $(\epsilon_1,\epsilon_2)\in A_{I'}\cap A_{I''}$. From (55) we have $e_{I,I''}(\epsilon_1,\epsilon_2)=e_{I',I''}(\epsilon_1,\epsilon_2)$ for every $(\epsilon_1,\epsilon_2)\in A_I\cap A_{I'}\cap A_{I''}$. From this, we can define

$$e_{I''}(\epsilon_1,\epsilon_2):=\begin{cases} e_{I,I''}(\epsilon_1,\epsilon_2) & \text{if } (\epsilon_1,\epsilon_2)\in A_I\cap A_{I''}\\ e_{I',I''}(\epsilon_1,\epsilon_2) & \text{if } (\epsilon_1,\epsilon_2)\in A_{I'}\cap A_{I''}.\end{cases}$$

From Theorem 6 and Corollary 2 we deduce $e_{I''}(\epsilon_1,\epsilon_2)$ can be extended to a $w_0-\mathcal{C}^\infty$ function in the sense of Whitney in $A_{I''}$, $f_{I''}(\epsilon_1,\epsilon_2)$. It is straightforward to check, from the way $f_{I''}$ was constructed, that $f_{I''}(\epsilon_1,\epsilon_2)=f_I(\epsilon_1,\epsilon_2)+g_{I,I''}(\epsilon_1+i\epsilon_2)$ when $(\epsilon_1,\epsilon_2)\in A_I\cap A_{I''}$ and also $f_{I''}(\epsilon_1,\epsilon_2)=f_{I'}(\epsilon_1,\epsilon_2)+g_{I',I''}(\epsilon_1+i\epsilon_2)$ for $(\epsilon_1,\epsilon_2)\in A_{I'}\cap A_{I''}$.

These two cases solve completely the problem since nonempty intersection of four different compacts in $(A_I)_{I\in\mathcal{I}}$ is not allowed when working with a good covering. The functions in $(f_I)_{I\in\mathcal{I}}$ satisfy (54). □

6 Existence of formal series solutions and $q-$Gevrey expansions

In the current section we set the main result in this work. We establish the existence of a formal power series with coefficients belonging to $\mathbb{H}_{\mathcal{T},\rho}$ which asymptotically represents the actual solutions found in Theorem 3 for the problem (33)+(34). Moreover, each actual solution turns out to admit this formal power series as $q-$Gevrey expansion of a certain type in the $q-$spiral where the solution is defined.

The following lemma will be useful in the following. We only sketch its proof. For more details we refer to [17].

Lemma 12 *Let U be an open and bounded set in $\mathbb{R}^2$. We consider $h\in\mathcal{C}^\infty(U)$ (in the classical sense) verifying bounds as in (49) and (50) for every $(\epsilon_1,\epsilon_2)\in U$. Let g be the solution of the equation*

$$\partial_{\overline{\epsilon}}g(\epsilon_1,\epsilon_2):=\frac{1}{2}(\partial_{\epsilon_1}+i\partial_{\epsilon_2})g(\epsilon_1+i\epsilon_2)=h(\epsilon_1,\epsilon_2),\quad (\epsilon_1,\epsilon_2)\in U. \tag{56}$$

Then g also verifies bounds such as those in (49) and (50) for $(\epsilon_1,\epsilon_2)\in U$.

Proof Let h_1 be any extension of the function h to $\mathbb{R}^2$ with compact support which preserves bounds in (49) and (50) in $\mathbb{R}^2$. We have

$$g(\epsilon_1,\epsilon_2):=-\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{h_1(x)}{x-\epsilon}d\xi d\eta,\quad (\epsilon_1,\epsilon_2)\in U$$

solves (56). Here, $\epsilon = (\epsilon_1, \epsilon_2)$, $x = (\xi, \eta)$ and $d\xi d\eta$ stands for Lebesgue measure in $x-$plane. Bounds in (49) for the function g come out from

$$\frac{\partial^{\alpha_1+\alpha_2} g}{\partial \epsilon_1^{\alpha_1} \partial \epsilon_2^{\alpha_2}}(\epsilon_1, \epsilon_2) = -\frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\partial^{\alpha_1+\alpha_2} h_1}{\partial \epsilon_1^{\alpha_1} \partial \epsilon_2^{\alpha_2}}(x) \frac{1}{x-\epsilon} d\xi d\eta,$$

for every $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ and $(\epsilon_1, \epsilon_2) \in U$, and from the fact that the function $x = (x_1, x_2) \mapsto 1/|x|$ is Lebesgue integrable in any compact set containing 0.

On the other hand, g satisfies estimates in (50) from Taylor formula with integral remainder. □

We now give a decomposition result of the functions X_I constructed in Theorem 3. The procedure is adapted from [11] under $q-$Gevrey settings. For every $I \in \mathcal{I}$, we write $X_I(\epsilon) : U_I q^{-\mathbb{N}} \to \mathbb{H}_{\mathcal{T},\rho}$ for the holomorphic function given by $X_I(\epsilon) := (t, z) \mapsto X_I(\epsilon, t, z)$.

Proposition 7 *There exists a $w_0-\mathcal{C}^\infty$ function $u(\epsilon_1, \epsilon_2)$ and a holomorphic function $a(\epsilon_1+i\epsilon_2)$ defined on the neighborhood $Int(\cup_{I\in\mathcal{I}} A_I)$ of 0 such that*

$$X_I(\epsilon_1 + i\epsilon_2) = f_I(\epsilon_1, \epsilon_2) + u(\epsilon_1, \epsilon_2) + a(\epsilon_1 + i\epsilon_2), \quad (\epsilon_1, \epsilon_2) \in Int(A_I), \tag{57}$$

for every $I \in \mathcal{I}$.

Proof From the definition of the cocycle $(g_{I,I'})_{(I,I')\in\mathcal{I}^2}$ in Proposition 5 and from Proposition 6 we derive

$$X_I(\epsilon_1 + i\epsilon_2) - f_I(\epsilon_1, \epsilon_2) = X_{I'}(\epsilon_1 + i\epsilon_2) - f_{I'}(\epsilon_1, \epsilon_2), \quad (\epsilon_1, \epsilon_2) \in A_I \cap A_{I'} \setminus \{(0,0)\},$$

whenever $(I, I') \in \mathcal{I}^2$ and $A_I \cap A_{I'} \neq \emptyset$. The function $X - f$ given by

$$(X - f)(\epsilon_1, \epsilon_2) := X_I(\epsilon_1 + i\epsilon_2) - f_I(\epsilon_1, \epsilon_2), \quad (\epsilon_1, \epsilon_2) \in A_I \setminus \{(0,0)\}$$

is well defined on $W \setminus \{(0,0)\}$, where $W = \cup_{I\in\mathcal{I}} A_I$ is a closed neighborhood of $(0,0)$.

For every $I \in \mathcal{I}$, X_I is a holomorphic function on $U_I q^{-\mathbb{N}}$ so that Cauchy-Riemann equations hold:

$$\partial_{\overline{\epsilon}}(X_I)(\epsilon_1 + i\epsilon_2) = 0, \quad (\epsilon_1, \epsilon_2) \in A_I \setminus \{(0,0)\}.$$

This yields $\partial_{\overline{\epsilon}}(X - f)(\epsilon_1, \epsilon_2) = -\partial_{\overline{\epsilon}} f_I(\epsilon_1, \epsilon_2)$ for every $I \in \mathcal{I}$ and $(\epsilon_1, \epsilon_2) \in \text{Int}(A_I)$.

We have $-\partial_{\overline{\epsilon}} f_I(\epsilon_1, \epsilon_2)$ can be extended to a $w_0 - \mathcal{C}^\infty$ function in the sense of Whitney on A_I. This yields f_I is $w_0 - \mathcal{C}^\infty$ in the sense of Whitney on A_I. In fact, their $q-$Gevrey types coincide.

From this, we deduce that $\partial_{\overline{\epsilon}}(X - f)$ is a $w_0 - \mathcal{C}^\infty$ function in the sense of Whitney on A_I for every $I \in \mathcal{I}$ and also that $\partial_{\overline{\epsilon}} f_I(\epsilon_1, \epsilon_2) = \partial_{\overline{\epsilon}} f_{I'}(\epsilon_1, \epsilon_2)$ for every $(\epsilon_1, \epsilon_2) \in \text{Int}(A_I \cap A_{I'})$ and every $I, I' \in \mathcal{I}$ due to $g_{I,I'}(\epsilon)$ is a holomorphic function on $U_I q^{-\mathbb{N}} \cap U_{I'} q^{-\mathbb{N}}$. The previous equality is also true for $(\epsilon_1, \epsilon_2) \in A_I \cap A_{I'}$ from the fact that f_I is $w_0 - \mathcal{C}^\infty$ in the sense of Whitney on A_I.

From Theorem 6 and Corollary 2 we derive $\partial_{\overline{\epsilon}}(X - f)$ is a $w_0 - \mathcal{C}^\infty$ function in the sense of Whitney on $\cup_{I\in\mathcal{I}} A_I$.

Taking into account Lemma 12 we derive the existence of a $\mathcal{C}^\infty$ function $u(\epsilon_1, \epsilon_2)$ in the usual sense, defined in $\text{Int}(W)$ and verifying $q-$Gevrey bounds of a certain positive type, such that

$$\partial_{\overline{\epsilon}} u(\epsilon_1, \epsilon_2) = \partial_{\overline{\epsilon}}(X - f)(\epsilon_1, \epsilon_2), \quad (\epsilon_1, \epsilon_2) \in \text{Int}(W).$$

From this last expression we have $u(\epsilon_1, \epsilon_2) - (X - f)(\epsilon_1, \epsilon_2)$ defines a holomorphic function on $\text{Int}(W) \setminus \{(0,0)\}$.

For every $I \in \mathcal{I}$, X_I is a bounded $\mathbb{H}_{\mathcal{T},\rho}-$function in $\mathrm{Int}(W) \setminus \{(0,0)\}$, and so it is the function $u(\epsilon_1,\epsilon_2) - (X-f)(\epsilon_1,\epsilon_2)$. The origin turns out to be a removable singularity so the function $u(\epsilon_1,\epsilon_2) - (X-f)(\epsilon_1,\epsilon_2)$ can be extended to a holomorphic function defined on $\mathrm{Int}(W)$. The result follows from here. □

We are under conditions to enunciate the main result in the present work.

Theorem 7 *Under the same hypotheses as in Theorem 3, there exists a formal power series*

$$\hat{X}(\epsilon,t,z) = \sum_{k\geq 0} \frac{X_k(t,z)}{k!}\epsilon^k \in \mathbb{H}_{\mathcal{T},\rho}[[\epsilon]],$$

formal solution of

$$\epsilon t \partial_z^S \hat{X}(\epsilon,qt,z) + \partial_z^S \hat{X}(\epsilon,t,z) = \sum_{k=0}^{S-1} b_k(\epsilon,z)(t\sigma_q)^{m_{0,k}}(\partial_z^k \hat{X})(\epsilon,t,zq^{-m_{1,k}}). \tag{58}$$

Moreover, let $I \in \mathcal{I}$ and $\tilde{K}_I$ any compact subset of $\mathrm{Int}(K_I)$. There exists $B > 0$ such that the function $X_I(\epsilon,t,z)$ constructed in Theorem 3 admits $\hat{X}(\epsilon,t,z)$ as its $q-$Gevrey asymptotic expansion of type B in $\tilde{K}_I q^{-\mathbb{N}}$.

Proof Let $I \in \mathcal{I}$ and $\tilde{K}_I$ any compact subset of $\mathrm{Int}(K_I)$.

From Proposition 7 we can extend $X_I(\epsilon_1+i\epsilon_2)$ to a $w_0-\mathcal{C}^\infty$ function in the sense of Whitney on $\tilde{A}_I = \{(\epsilon_1,\epsilon_2) \in \mathbb{R}^2 : \epsilon_1 + i\epsilon_2 \in \tilde{K}_I q^{-\mathbb{N}} \cup \{0\}\} \subseteq \mathrm{Int}(A_I) \cup \{(0,0)\}$. Let us fix $I \in \mathcal{I}$. We consider the family $(X^{(h_1,h_2)}(\epsilon_1,\epsilon_2))_{(h_1,h_2)\in\mathbb{N}^2}$ associated to X_I by Definition 9. We have

$$X_I^{(h_1,h_2)}(\epsilon_1,\epsilon_2) = \partial_{\epsilon_1}^{h_1}\partial_{\epsilon_2}^{h_2} X_I(\epsilon_1+i\epsilon_2) = i^{h_2}\partial_\epsilon^{h_1+h_2} X_I(\epsilon), \quad (\epsilon_1,\epsilon_2) \in \tilde{A}_I \setminus \{(0,0)\},$$

due to $X_I(\epsilon)$ is holomorphic on $\mathrm{Int}(K_I)q^{-\mathbb{N}}$.

We have $X_I^{(h_1,h_2)}(\epsilon_1,\epsilon_2)$ is continuous at $(0,0)$ for every $(h_1,h_2) \in \mathbb{N}^2$ so we can define for every $k \geq 0$

$$X_{k,I} := \frac{X_I^{(h_1,h_2)}(0,0)}{i^{h_2}} \in \mathbb{H}_{\mathcal{T},\rho}, \tag{59}$$

whenever $h_1 + h_2 = k$. Estimates held by any $w_0 - \mathcal{C}^\infty$ function in the sense of Whitney (see Definition 9 for $\alpha = (0,0)$) lead us to the existence of positive constants $C_1, H, B > 0$ such that

$$\left\| X_I(\epsilon_1+i\epsilon_2) - \sum_{p=0}^m \frac{X_{p,I}}{p!}(\epsilon_1+i\epsilon_2)^p \right\|_{\mathbb{H}_{\mathcal{T},\rho}} \leq C_1 H^m |q|^{B\frac{m^2}{2}} \frac{|\epsilon_1+i\epsilon_2|^{m+1}}{(m+1)!},$$

for every $m \geq 0$ and $\epsilon_1 + i\epsilon_2 \in \tilde{K}_I q^{-\mathbb{N}}$. As a matter of fact, this shows that X_I admits $\hat{X}_I(\epsilon) = \sum_{k\geq 0} \frac{X_k}{k!}\epsilon^k$ as its $q-$Gevrey expansion of type $B > 0$ in $\tilde{K}_I q^{-\mathbb{N}}$.

The formal power series $\hat{X}_I$ does not depend on $I \in \mathcal{I}$. Indeed, from Theorem 3 we have that $X_I(\epsilon) - X_{I'}(\epsilon)$ admits both $\hat{0}$ and $\hat{X}_{I'} - \hat{X}_I$ as $q-$asymptotic expansion on $\tilde{K}_I q^{-\mathbb{N}} \cap \tilde{K}_{I'} q^{-\mathbb{N}}$ whenever this intersection is not empty. We put $\hat{X} := \hat{X}_I$ for any $I \in \mathcal{I}$. The function $X_{k,I} = X_{k,I}(t,z) \in \mathbb{H}_{\mathcal{T},\rho}$ does not depend on I for every $k \geq 0$. We write $X_k := X_{k,I}$ for $k \geq 0$. X_I admits $\hat{X} = \sum_{k\geq 0} \frac{X_k}{k!}\epsilon^k$ as its $q-$Gevrey asymptotic expansion of type $B > 0$ in $\tilde{K}_I q^{-\mathbb{N}}$ for all $I \in \mathcal{I}$.

In order to achieve the result, it only remains to prove that $\hat{X}(\epsilon,t,z)$ is a formal solution of (58). Let $l \geq 1$. If we derive l times with respect to ϵ in equation (58) we get that $\partial_\epsilon^l X_I(\epsilon,t,z)$ is a solution of

$$\epsilon t\partial_z^S\partial_\epsilon^l X_I(\epsilon,qt,z) + t\partial_z^S l\partial_\epsilon^{l-1}X_I(\epsilon,qt,z) + \partial_z^S\partial_\epsilon^l X_I(\epsilon,t,z) = \sum_{k=0}^{S-1}\sum_{l_1+l_2=l}\frac{l!}{l_1!l_2!}\partial_\epsilon^{l_1}b_k(\epsilon,z)\partial_\epsilon^{l_2}((t\sigma_q)^{m_{0,k}})\partial_z^k X_I)(\epsilon,t,zq^{-m_{1,k}}). \tag{60}$$

for every $l \geq 1, (t,z) \in \mathcal{T} \times D(0,\rho)$ and $\epsilon \in \tilde{K}_I q^{-\mathbb{N}}$. Letting ϵ tend to 0 in (60) we obtain

$$t\partial_z^S\frac{X_{l-1}(qt,z)}{(l-1)!} + \partial_z^S\frac{X_l(t,z)}{l!} = \sum_{k=0}^{S-1}\sum_{l_1+l_2=l}\frac{\partial_\epsilon^{l_1}b_k(\epsilon,z)|_{\epsilon=0}}{l_1!}\frac{((t\sigma_q)^{m_{0,k}}\partial_z^k X_{l_2})(t,zq^{-m_{1,k}})}{l_2!} \tag{61}$$

for every $l \geq 1, (t,z) \in \mathcal{T} \times D(0,\rho)$. Holomorphy of $b_k(\epsilon,z)$ with respect to ϵ at 0 implies

$$b_k(\epsilon,z) = \sum_{l\geq 0}\frac{\partial_\epsilon^l b_k(\epsilon,z)|_{\epsilon=0}}{l!}\epsilon^l, \tag{62}$$

for ϵ near 0 and for every $z \in \mathbb{C}$. Statements (60) and (61) conclude $\hat{X}(\epsilon,t,z) = \sum_{k\geq 0} X_k(t,z)\frac{\epsilon^k}{k!}$ is a formal solution of (58). □

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