C^1-rigidity of circle diffeomorphisms with breaks for almost all rotation numbers

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Abstract

We prove that for almost all rotation numbers, every two $C^{2+\alpha}$-smooth circle diffeomorphisms with a break point, with the same irrational rotation number and the same size of the break, are C^1-smoothly conjugate to each other, provided that the corresponding renormalizations approach asymptotically each other with an exponential rate.

1 Introduction

The problem of smoothness of a conjugacy to a linear rotation for smooth diffeomorphisms of a circle $\mathbb{T}^1$ is a classical problem in one-dimensional dynamics. It has been proved by Arnol'd [1] that every analytic circle diffeomorphism with a Diophantine rotation number, i.e., with a rotation number ρ for which there exists $C > 0$ and $\beta \geq 0$ such that $|\rho - p/q| > C/q^{2+\beta}$, for every $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, sufficiently close to the linear rigid rotation $R_\rho : x \mapsto x+\rho$, is analytically conjugate to R_ρ. Arnol'd also conjectured that the result remains true if the requirement of closeness to the rigid rotation is removed. The global C^∞-version of this result has been proved by Herman [4], and is the subject of classical Herman's theory [4, 15, 5, 13, 10]. Arnol'd also proved that his local result cannot be extended to all rotation numbers [1]. He constructed examples of analytic circle diffeomorphisms with

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irrational rotation numbers for which the invariant measure is singular, which implies that the conjugacy to the rigid rotation is not absolutely continuous.

The crucial step in establishing the smoothness of conjugation is C^1-smoothness. We use the term *rigidity* for the phenomenon that any two maps, within a given equivalence class determined by their topological conjugacy, are, in fact, C^1-smoothly conjugate to each other. It follows from the results of Arnol'd and Herman that, in the case of circle diffeomorphisms, rigidity is guaranteed only when rotation numbers satisfy a certain Diophantine condition. It was discovered recently that in the presence of singular points, this rigidity may actually be stronger, i.e., valid for a "larger" set of rotation numbers. The singular points refer either to the points where the derivative vanishes (critical points) or where it has a jump discontinuity (break points). In the case of critical circle maps, i.e., diffeomorphisms of a circle with a single critical point, the rigidity is especially strong. It has been proved by Khanin and Teplinsky [9] that any two analytic critical circle maps with the same rotation number and the same odd integer order of the critical point are C^1-smoothly conjugate to each other. This phenomenon, when rigidity holds without any Diophantine-type conditions, is called *robust rigidity.* This result relies on the exponential convergence of renormalizations proved by de Faria and de Melo for C^∞-smooth critical circle maps and rotation numbers of bounded type [2, 3], and extended, in the analytic setting, to all rotation numbers by Yampolsky [14]. Though robust rigidity is believed to be present in the general case of non-analytic critical circle maps, there is currently no proof of the exponential "convergence" of renormalizations in this case.

The above results for critical circle maps suggested that the rigidity may also be robust in the case of circle diffeomorphisms with a break point. Partial results in this direction were obtained in [6], where rigidity was established for a countable set of rotation numbers, and in [11], for a set of rotation numbers of zero Lebesgue measure. However, as was shown by two of us [7], the above conjecture is false — robust rigidity does not hold for circle maps with breaks. We proved in [7] that there are irrational rotation numbers, and pairs of analytic circle diffeomorphisms with breaks, with the same rotation number and the same size of the break (see below), for which any conjugacy between them is not even Lipschitz continuous. The question whether rigidity holds for typical rotation numbers, however, remained open. The following theorem, which is the main result of this paper, provides an affirmative answer to this question.

To be precise, every circle diffeomorphism with a break $T : \mathbb{T}^1 \to \mathbb{T}^1$, is defined uniquely by a function $\mathcal{T} : \mathbb{R} \to \mathbb{R}$ that is continuous and strictly increasing on $\mathbb{R}$, with $\mathcal{T}(0) \in [0, 1)$, and satisfies $\mathcal{T}(x + 1) = \mathcal{T}(x) + 1$, for every $x \in \mathbb{R}$. It is assumed that there exists a point $x_{br} \in [0, 1)$ such that $\mathcal{T}(x) \in C^r$, $r \geq 2$, on $[x_{br}, x_{br} + 1]$, and $\mathcal{T}'(x)$ is bounded from below by a positive constant for every $x \in [x_{br}, x_{br} + 1]$. The square root of the ratios of the one sided derivatives at x_{br},

$$\sqrt{\frac{\mathcal{T}'_-(x_{br})}{\mathcal{T}'_+(x_{br})}} = c \neq 1, \tag{1.1}$$

is called the size of the break.

Theorem 1.1 *There exists a set of rotation numbers $A_1 \subset [0, 1]$ of full Lebesgue measure such that any two $C^{2+\alpha}$-smooth, $\alpha \in (0, 1)$, circle diffeomorphisms with breaks T and $\widetilde{T}$, with the same irrational rotation number $\rho \in A_1$, and the same size of the break $c \in \mathbb{R}^+ \backslash \{1\}$, are C^1-smoothly conjugate to each other. Namely, there exists a C^1-smooth diffeomorphism $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$, such that*

$$\varphi \circ T \circ \varphi^{-1} = \widetilde{T}. \tag{1.2}$$

Remark 1 The proof of this theorem uses exponential "convergence" of renormalizations for circle diffeomorphisms with breaks: If f_n and $\widetilde{f}_n$ are renormalizations (see Section 2) of two circle diffeomorphisms with breaks T and $\widetilde{T}$ with the same irrational rotation number, respectively, then there exist $\lambda \in (0, 1)$, and $C > 0$, such that $\|f_n - \widetilde{f}_n\|_{C^2} \leq C\lambda^n$, for every $n \in \mathbb{N}$. It has been shown in [11], that the renormalizations approach exponentially fast to each other for a set of rotation numbers of zero Lebesgue measure. The full proof of the exponential approach of the renormalizations, for all irrational rotation numbers, will be given in [8].

Remark 2 The set A_1 consists of all irrational rotation numbers $\rho \in (0, 1)$ whose subsequence of partial quotients k_{n+1} for even n, if $0 < c < 1$, or odd n, if $c > 1$, satisfies the bound $k_{n+1} \leq C_1 \lambda_1^{-n}$, for some $\lambda_1 \in (\lambda, 1)$, and $C_1 > 0$. One can see that A_1 contains some strongly Liouville numbers. This is one manifestation of the remnants of robust rigidity still present in the case of circle diffeomorphisms with breaks. The difference between the cases of odd and even n is related to a difference in the behavior of the renormalizations (which is opposite for maps with $0 < c < 1$ and $c > 1$). This will be explained in more details in the next section.

The paper is organized as follows. In Section 2, we introduce the general renormalization setting for circle homeomorphisms and formulate regularity conditions and a rigidity theorem (Theorem 2.2) which follows from them. In Section 3, we formulate a criterion of smoothness of the conjugacy in terms of ratios of the lengths of the corresponding intervals of dynamical partitions. In the same section, we obtain necessary estimates on these ratios on a fundamental interval and prove Theorem 2.2 by spreading them to the whole circle and using the criterion of smoothness. Theorem 1.1 is then proved by simply verifying that the regularity conditions of Theorem 2.2 hold true in the case of circle diffeomorphisms with breaks.

2 Renormalizations of circle homeomorphisms and a formulation of a rigidity theorem

2.1 Renormalizations of circle homeomorphisms

It has been known since Poincaré that for every orientation-preserving homeomorphism T of the circle $\mathbb{T}^1 = \mathbb{R}\backslash\mathbb{Z}$ there is a unique rotation number ρ, which is given by the x-independent limit $\rho = \lim_{n\to\infty} \mathcal{T}^n(x)/n \mod 1$, where $\mathcal{T}$ is a lift of T to $\mathbb{R}$. The rotation number $\rho \in (0,1]$ can be expressed in the form of a *continued fraction expansion*

$$\rho = \cfrac{1}{k_1 + \frac{1}{k_2 + \frac{1}{k_3+\ldots}}}, \tag{2.1}$$

that we write as $\rho = [k_1, k_2, k_3, \ldots]$. The sequence of integers k_n, called *partial quotients*, is infinite and defined uniquely if and only if ρ is irrational. Every infinite sequence of partial quotients defines uniquely an irrational number ρ as the limit of the sequence of *rational convergents* $p_n/q_n = [k_1, k_2, \ldots, k_n]$. It is well-known that this sequence forms the sequence of best rational approximates of ρ, i.e. there are no rational numbers with denominators smaller or equal to q_n, that are closer to ρ than p_n/q_n. The rational convergents can also be defined recursively as $p_n = k_n p_{n-1} + p_{n-2}$ and $q_n = k_n q_{n-1} + q_{n-2}$, starting with $p_0 = 0$, $q_0 = 1$, $p_{-1} = 1$, $q_{-1} = 0$.

To define the renormalizations, we start with a *marked point* $x_0 \in \mathbb{T}^1$, and consider the *marked trajectory* $x_i = T^i x_0$, with $i \geq 0$. The subsequence x_{q_n}, $n \geq 0$, indexed by the denominators of the sequence of rational convergents of the rotation number ρ, will be called the sequence of *dynamical convergents*. We define $x_{q_{-1}} = x_0 - 1$. The combinatorial equivalence of all circle homeomorphisms with the same irrational rotation number implies that the order of the dynamical convergents of T is the same as the order of the dynamical convergents for the rigid rotation $T_\rho : x \mapsto x + \rho$. The well-known arithmetic properties of the rational convergents now imply that dynamical convergents alternate their order in the following way:

$$x_{q_{-1}} < x_{q_1} < x_{q_3} < \cdots < x_0 < \cdots < x_{q_2} < x_{q_0}. \tag{2.2}$$

The intervals $[x_{q_n}, x_0]$, for n odd, and $[x_0, x_{q_n}]$, for n even, will be denoted by $\Delta_0^{(n)}$, and called the n-th *renormalization segments.* The n-th renormalization segment associated to the marked point x_i will be denoted by $\Delta_i^{(n)}$. We will also define $\bar{\Delta}_0^{(n)} = \Delta_0^{(n)} \cup \Delta_0^{(n+1)}$, and $\check{\Delta}_0^{(n)} = \Delta_0^{(n)} \backslash \Delta_0^{(n+2)}$. In addition to the property (2.2), we also have the following important property: the only points of the trajectory $\{x_i : 0 < i \leq q_{n+2}\}$ that belong to $\Delta_0^{(n)}$ are $\{x_{q_n + i q_{n+1}} : 0 \leq i \leq k_{n+2}\}$.

Images of $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$ under T^{q_n} and $T^{q_{n-1}}$, respectively, until they return to $\bar{\Delta}_0^{(n-1)}$, cover the whole circle without overlapping beyond the end points, thus forming the n-th *dynamical partition* of $\mathbb{T}^1$,

$$\mathcal{P}_n = \{T^i \Delta_0^{(n-1)} : 0 \le i < q_n\} \cup \{T^i \Delta_0^{(n)} : 0 \le i < q_{n-1}\}. \tag{2.3}$$

The iterates of T^{q_n} and $T^{q_{n-1}}$ restricted to $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$, respectively, are the two continuous components of the first return map for T on the interval $\bar{\Delta}_0^{(n-1)}$. The endpoints of the segments from $\mathcal{P}_n$ form the set

$$\Xi_n = \{x_i : 0 \le i < q_{n-1} + q_n\}. \tag{2.4}$$

We also define the extended partition $\mathcal{P}_n^* = \mathcal{P}_n \cup \{T^{q_n}\Delta_0^{(n-1)}, T^{q_{n-1}}\Delta_0^{(n)}\}$ and the extended set $\Xi_n^* = \Xi_n \cup \{x_{q_{n-1}+q_n}\}$.

The following simple lemma follows directly from the properties of the continued fraction expansion of the rotation number.

Lemma 2.1 *For every $m > n$, we have the following decomposition*

$$\Xi_m \cap \check{\Delta}_0^{(n-1)} = \bigcup_{x_l \in \Xi_m \cap \Delta_0^{(n)} \setminus \{x_{q_n}\}} \ \bigcup_{0 \le i < k_{n+1}} \{x_{l+q_{n-1}+iq_n}\}. \tag{2.5}$$

Furthermore, for every $x_l \in \Xi_m \cap \Delta_0^{(n)} \setminus \{x_{q_n}\}$, we have $x_{l+q_{n-1}+k_{n+1}q_n} = x_{l+q_{n+1}} \in \Xi_m^* \cap \bar{\Delta}_0^{(n)}$.

The n-th *renormalization* of an orientation-preserving homeomorphism T of the circle $\mathbb{T}^1$, with rotation number $\rho = [k_1, k_2, k_3, \ldots]$, with respect to the marked point $x_0 \in \mathbb{T}^1$, is a function $f_n : [-1, 0] \to \mathbb{R}$ obtained from T^{q_n}, by rescaling the coordinates. More precisely, if τ_n is the affine change of coordinates that maps $x_{q_{n-1}}$ to -1 and x_0 to 0, then

$$f_n = \tau_n \circ T^{q_n} \circ \tau_n^{-1}. \tag{2.6}$$

If we identify x_0 with zero, then τ_n is exactly a multiplication by $(-1)^n/|\Delta_0^{(n-1)}|$. Here and in what follows, we use $|\cdot|$ to denote the length of an interval. Definition (2.6) is valid for all $n \ge 0$ if and only if ρ is irrational; otherwise, n must be less than the length of the continued fraction expansion of ρ or can be equal to it if $x_{q_{n-1}} \neq x_0$.

2.2 Renormalizations of circle diffeomorphisms with breaks

In the case of a circle diffeomorphism with a break, we will use the break point x_{br} as the marked point x_0.

It is well known [12] that renormalizations f_n of circle diffeomorphisms with a break of size $c \in \mathbb{R}_+ \setminus \{1\}$ approach exponentially fast in C^2-norm to a particular family of linear functional transformations

$$F_{a^{(n)}, v^{(n)}, c^{(n)}} : z \mapsto \frac{a^{(n)} + c^{(n)} z}{1 - v^{(n)} z}, \tag{2.7}$$

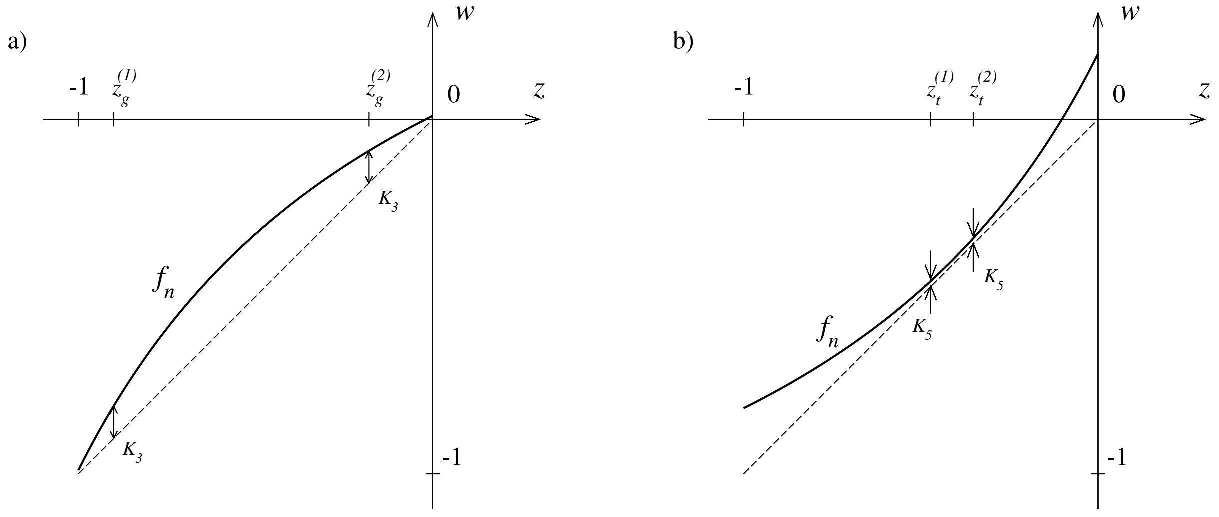

Figure 1: The graph of a renormalized map f_n for sufficiently large n and large k_{n+1}: a) Case $0 < c < 1$ and n even, or $c > 1$ and n odd; b) Case $0 < c < 1$ and n odd, or $c > 1$ and n even.

where $c^{(n)} = c$ if n is even, $c^{(n)} = c^{-1}$ if n is odd, and

$$a^{(n)} = \frac{|\Delta_0^{(n)}|}{|\Delta_0^{(n-1)}|}, \quad v^{(n)} = \frac{c^{(n)} - a^{(n)} - b^{(n)}}{b^{(n)}}, \quad b^{(n)} = \frac{|\Delta_0^{(n-1)}| - |\Delta_{q_{n-1}}^{(n)}|}{|\Delta_0^{(n-1)}|}. \tag{2.8}$$

The following estimates have also been proved in [12]. For every circle diffeomorphism with a break T, there exist constants $\mathcal{C} > 0$ and $\lambda \in (0,1)$, such that, for all $n \in \mathbb{N}$:

(A) $|\ln(T^{q_n})'(x)| \le \mathcal{C}$, for all $x \in \mathbb{T}^1$ (at points where the derivative has a break, both left and right derivatives are considered),

(B) $\|f_n - F_{a^{(n)},v^{(n)},c^{(n)}}\|_{C^2} \le \mathcal{C}\lambda^n$.

As already mentioned in Remark 2, for maps with breaks, the graphs of the renormalizations f_n look different in the cases of odd and even n (see Figure 1).

If $c > 1$, the map f_n is concave for sufficiently large odd n. Moreover, as $k_{n+1} \to \infty$, the graph of f_n approaches the diagonal $w = z$ at the end points $z = -1$ and $z = 0$. Below, we call the small intervals containing these end points the *gates* (the intervals $[-1, z_g^{(1)}]$ and $[z_g^{(2)}, 0]$ on Figure 1 (a)). On the contrary, if n is even and sufficiently large, the map f_n is convex. It approaches the diagonal as $k_{n+1} \to \infty$ at a single point of almost-tangency, strictly between -1 and 0. We will later call an interval containing this point of almost-tangency the *tunnel* (the interval $[z_t^{(1)}, z_t^{(2)}]$ on Figure 1 (b)).

If $0 < c < 1$, the behavior is the opposite, i.e., f_n is convex for n odd, and concave for n even. The restriction on k_{n+1} in Remark 2 is related to the concave case.

2.3 Regularity conditions and a rigidity theorem

A sequence of functions $g_n : [-1,0] \to \mathbb{R}$, with $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, will be called K-regular, if it satisfies the following conditions for some vector $K = (K_1, K_2, K_3, K_4, K_5, K_6)$, with positive components:

(i) $\|g_n\|_{C^2} \le K_1$ on $[-1, 0]$,

(ii) $g_n'(z) > K_2$ for every $z \in [-1, 0]$,

(iii) for all odd n, the set $B_{g_n,K_3} = \{z \in [-1, 0] : g_n(z) - z < K_3\}$ is either empty or consists of one or the union of two half-open intervals each of which contains an end point (we refer to these intervals as the *gates*),

(iv) for all odd n, $g_n''(z) < -K_4$, for $z \in B_{g_n,K_3}$,

(v) for all even n, the set B_{g_n,K_5} is either an open interval or empty (we refer to this interval as the *tunnel*; since the points -1 and 0 are outside of the tunnel, this implies $g_n(-1) \ge K_5 - 1$ and $g_n(0) \ge K_5$),

(vi) for all even n, $g_n''(z) > K_6$, for $z \in B_{g_n,K_5}$.

A system of nested partitions $\mathcal{P}_n$, i.e., a sequence of partitions such that each element of a partition $\mathcal{P}_{n+1}$ is contained in an element of a partition $\mathcal{P}_n$, is called refining if the maximal length of elements of partition $\mathcal{P}_n$ approaches zero as $n \to \infty$. It is called exponentially refining if there exist $C_{ref} > 0$ and $0 < \lambda_{ref} < 1$, such that $|I_m| \le C_{ref}\lambda_{ref}^{m-n}|I_n|$, for any $I_n \in \mathcal{P}_n$ and $I_m \in \mathcal{P}_m$, with $I_m \subset I_n$.

Theorem 2.2 *Let T and $\widetilde{T}$ be two $C^{2+\alpha}$-smooth, $\alpha > 0$, orientation-preserving circle diffeomorphisms with breaks that satisfy the following conditions:*

(a) $\rho(T) = \rho(\widetilde{T}) \in A_1$;

(b) there exists a vector $K = (K_1, K_2, K_3, K_4, K_5, K_6) \in \mathbb{R}_+^6$, *such that the sequences of renormalizations* $g_n = f_{n+1}$ *and* $\widetilde{g}_n = \widetilde{f}_{n+1}$, $n \in \mathbb{N}_0$, *if* $0 < c < 1$, *and the sequences* $g_n = f_n$ *and* $\widetilde{g}_n = \widetilde{f}_n$, $n \in \mathbb{N}$, *if* $c > 1$, *are* K*-regular;*

(c) the systems of dynamical partitions $\mathcal{P}_n$ *and* $\widetilde{\mathcal{P}}_n$ *are exponentially refining;*

(d) $\|f_n - \tilde{f}_n\|_{C^2} \le C\lambda^n$, *for some* $\lambda \in (0, 1)$ *and* $C > 0$.

Then, there exists a C^1*-smooth orientation-preserving circle diffeomorphism* φ *such that*

$$\varphi \circ T \circ \varphi^{-1} = \widetilde{T}. \tag{2.9}$$

Remark 3 Though the theorem has been formulated for circle maps with breaks, the result is valid for any two circle homeomorphisms which satisfy the assumptions (a), (c) and (d) of the theorem and whose renormalizations f_n are $C^{2+\alpha}$-smooth, and such that either the sequences $g_n = f_n, \widetilde{g}_n = \widetilde{f}_n$ or the sequences $g_n = f_{n+1}, \widetilde{g}_n = \widetilde{f}_{n+1}$ satisfy the above regularity conditions.

Remark 4 Condition (c) of the theorem implies that T and $\widetilde{T}$ are topologically conjugate to each other. It is easy to see that in the case of circle diffeomorphisms with breaks, the conjugacy φ can be C^1-smooth only in the case when it maps the break point x_0 of T into the break point $\widetilde{x}_0$ of $\widetilde{T}$. This condition defines φ uniquely, and below we only consider the case of this particular conjugacy.

A similar theorem has been proved in [9] under different regularity conditions (valid for renormalizations of critical circle maps) which require a simpler analysis.

3 A criterion of smoothness and the proof of the main theorem

3.1 A criterion of smoothness

We will use the following criterion of smoothness of φ. It is inspired by a similar criterion in [2] called the "coherence property". The same criterion was also used in [9]. For a segment $I \subset \mathbb{T}^1$, we define

$$\sigma(I) = \frac{|\varphi(I)|}{|I|}, \tag{3.1}$$

where $|\cdot|$ is the length of an interval.

Proposition 3.1 [9] *Suppose that the system of partitions $\mathcal{P}_n$ of the circle is refining, and that there exist constants $\bar{C} > 0$ and $\bar{\lambda} \in (0, 1)$ such that for any two segments $I, I' \in \mathcal{P}_n$ which are either adjacent or $I, I' \subset J$ for some $J \in \mathcal{P}_{n-1}$ the following estimate holds*

$$|\ln \sigma(I) - \ln \sigma(I')| \le \bar{C}\bar{\lambda}^n. \tag{3.2}$$

Then, $\varphi \in C^1(\mathbb{T}^1)$ and $\varphi' > 0$.

Proof. We present the proof below for completeness of the argument. Let φ_n be a homeomorphism of $\mathbb{T}^1$ that equals φ on Ξ_n and is linear on each of the segments $I \subset \mathcal{P}_n$. Let further $(\varphi_n)'_+$ be the right derivative of φ_n. It follows from (3.2) that the sequence of differences $\ln((\varphi_n)'_+(x))$ is a Cauchy sequence, uniformly on $\mathbb{T}^1$, and thus converges to some $h(x)$. To see this, notice first that over each $I \subset \mathcal{P}_n$ without the right endpoint,

$(\varphi_n)'_+(x) = \sigma(I)$, and that (3.2), for any two intervals $I, I' \subset J$ for some $J \in \mathcal{P}_{n-1}$, implies that

$$|\ln \sigma(I) - \ln \sigma(J)| \leq \bar{C}\bar{\lambda}^n. \tag{3.3}$$

Now, it is easy to show using (3.2) for adjacent intervals $I, I' \in \mathcal{P}_n$ that the function h is continuous on $\mathbb{T}^1$. Taking the limit $n \to \infty$ of $\varphi_n(x) = \int_0^x (\varphi_n)'_+(z)\,dz$, we get $\varphi(x) = \int_0^x e^{h(z)}\,dz$. Thus, $\varphi' = e^h$ is continuous and positive on $\mathbb{T}^1$. **QED**

We will also use the ratios of the corresponding rescaled intervals:

$$\mathfrak{s}_n(I) = \frac{|\widetilde{\tau}_n(\varphi(I))|}{|\tau_n(I)|}. \tag{3.4}$$

3.2 Renormalization graphs concave inside the gates

In this section we consider dynamics of a subsequence of renormalizations f_n and $\widetilde{f}_n$ of maps T and $\widetilde{T}$ for even n, if $0 < c < 1$, or odd n, if $c > 1$. The graphs of these renormalizations are concave inside the gates.

The following proposition gives the control of derivatives of renormalizations inside the narrow gates.

Proposition 3.2 *There exists $\varepsilon > 0$ and $B > 1$, such that for all even n, if $0 < c < 1$, or all odd n, if $c > 1$, either $f_n(-1) + 1 > K_3/2$ or $f'_n(z) > B$ for $z \in [-1, -1 + \varepsilon]$ and either $f_n(0) > K_3/2$ or $f'_n(z) < B^{-1}$ for $z \in [-\varepsilon, 0]$.*

Proof. It follows directly from the regularity conditions of Section 2.3 that if $f_n(-1)+1 \leq K_3/2$, then $f'_n(-1) > 1 + K_3/2$ and if $f_n(0) \leq K_3/2$, then $f'_n(0) = 1 - K_3/2$. Since the second derivative of f_n is bounded, in these cases, there exists $\varepsilon > 0$ and $B > 1$ such that $f'_n(z) > B$ for $z \in [-1, -1 + \varepsilon]$ and $f'_n(z) < B^{-1}$ for $z \in [-\varepsilon, 0]$. **QED**

The next proposition will be used repeatedly.

Proposition 3.3 *Let b_n, $n \in \mathbb{N}_0$, be a sequence of positive numbers such that $b_i > B > 1$ for $i \geq 1$, and*

$$s_n = \sum_{j=0}^{n} \prod_{i=0}^{j} b_i. \tag{3.5}$$

Then, there exists $A > 0$ such that $\prod_{i=0}^{n} b_i > A s_n$, for all $n \in \mathbb{N}$.

Proof. We can assume without loss of generality that $b_0 = 1$. The claim is proved by simple induction. For $n = 1$, the claim is obviously true for any $A < \frac{B}{1+B}$. Assume that the claim is true for some $n \in \mathbb{N}$, with $A < 1 - \frac{1}{B}$. Then,

$$\prod_{i=0}^{n+1} b_i > As_n b_{n+1} = Ab_{n+1}(s_{n+1} - \prod_{i=0}^{n+1} b_i), \tag{3.6}$$

and thus

$$\prod_{i=0}^{n+1} b_i > A\frac{b_{n+1}}{1 + Ab_{n+1}} s_{n+1} > As_{n+1}. \tag{3.7}$$

QED

Below λ is the exponent for the exponential approach of renormalizations (see Remark 1), and λ_1^{-1} is the exponent of the maximal growth of a subsequence of partial quotients (see Remark 2). It is assumed that $0 < \lambda < \lambda_1 < 1$.

Proposition 3.4 *Let $\lambda_3 \in (\sqrt{\lambda/\lambda_1}, 1)$. There exists $C_2 > 0$ such that for all even n, if $0 < c < 1$, or all odd n, if $c > 1$, we have*

$$\mathfrak{s}_n(\Delta_{q_{n-1}}^{(n)}), (\mathfrak{s}_n(\Delta_{q_{n-1}}^{(n)}))^{-1} \leq 1 + C_2\lambda_3^n. \tag{3.8}$$

Proof. Let λ_2 be a number in $[\lambda/\lambda_3, 1)$, whose choice will be specified later on. Notice first that if either $|\widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}}^{(n)})| \geq \lambda_2^n$ or $|\tau_n(\Delta_{q_{n-1}}^{(n)})| \geq \lambda_2^n$, then the claim follows directly from the closeness of renormalizations, since $||\widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}}^{(n)})| - |\tau_n(\Delta_{q_{n-1}}^{(n)})|| = |\widetilde{f}_n(-1) - f_n(-1)| \leq C\lambda^n$. In the case when $|\widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}}^{(n)})| < \lambda_2^n$ and $|\tau_n(\Delta_{q_{n-1}}^{(n)})| < \lambda_2^n$, we will prove the claim by contradiction. To prove the first inequality, let us assume that $\mathfrak{s}_n(\Delta_{q_{n-1}}^{(n)}) > 1 + \lambda_3^n$ (the proof of the second inequality is analogous, by assuming $(\mathfrak{s}_n(\Delta_{q_{n-1}}^{(n)}))^{-1} > 1 + \lambda_3^n$). Then, for $1 \leq j \leq k_{n+1}$ such that $\tau_n(\Delta_{q_{n-1}+(j-1)q_n}^{(n)}), \widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}+(j-1)q_n}^{(n)}) \subset [-1, -1 + \lambda_2^n]$, we have

$$\begin{aligned}
\mathfrak{s}_n(\Delta_{q_{n-1}+jq_n}^{(n)}) &= \prod_{i=1}^{j} \left(1 + \frac{\widetilde{f}_n'(\widetilde{\zeta}_i) - f_n'(\zeta_i)}{f_n'(\zeta_i)}\right) \mathfrak{s}_n(\Delta_{q_{n-1}}^{(n)}) \\
&\geq \prod_{i=1}^{j} \left(1 - \frac{|\widetilde{f}_n'(\widetilde{\zeta}_i) - f_n'(\widetilde{\zeta}_i)| + |f_n'(\widetilde{\zeta}_i) - f_n'(\zeta_i)|}{|f_n'(\zeta_i)|}\right) \mathfrak{s}_n(\Delta_{q_{n-1}}^{(n)}) \\
&\geq \left(1 - K_2^{-1}(C\lambda^n + K_1\lambda_2^n)\right)^j \mathfrak{s}_n(\Delta_{q_{n-1}}^{(n)}).
\end{aligned} \tag{3.9}$$

Here $\zeta_i \in \tau_n(\Delta_{q_{n-1}+(i-1)q_n}^{(n)})$, and $\widetilde{\zeta}_i \in \widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}+(i-1)q_n}^{(n)})$. Since $j \leq k_{n+1} \leq C_1\lambda_1^{-n}$, for every $\epsilon_1 > 0$ and sufficiently large n, we have

$$\mathfrak{s}_n(\Delta_{q_{n-1}+jq_n}^{(n)}) > 1 + (1 - \epsilon_1)\lambda_3^n, \tag{3.10}$$

if $\lambda \leq \lambda_2 < \lambda_1\lambda_3$.

This estimate implies that if j_{λ_2} is the index j of the last interval $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})$ that belongs to the interval $[-1,-1+\lambda_2^n]$ then we have

$$\begin{aligned}\widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+j_{\lambda_2}q_n}) - \tau_n(x_{q_{n-1}+j_{\lambda_2}q_n}) &= \sum_{j=0}^{j_{\lambda_2}-1} |\widetilde{\tau}_n(\tilde{\Delta}^{(n)}_{q_{n-1}+jq_n})| - \sum_{j=0}^{j_{\lambda_2}-1} |\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})| \\ &> \frac{(1-\epsilon_1)\lambda_3^n}{1+(1-\epsilon_1)\lambda_3^n} \sum_{j=0}^{j_{\lambda_2}-1} |\widetilde{\tau}_n(\tilde{\Delta}^{(n)}_{q_{n-1}+jq_n})| > C_3\lambda_2^n\lambda_3^n,\end{aligned} \tag{3.11}$$

for some $C_3 > 0$, since $\sum_{j=0}^{j_{\lambda_2}-1} |\widetilde{\tau}_n(\tilde{\Delta}^{(n)}_{q_{n-1}+jq_n})|$ is of the order of λ_2^n by Proposition 3.3. Here, we have also used Proposition 3.2 and that $\lambda_2^n < \varepsilon$, for sufficiently large n.

We now have

$$\begin{aligned}\widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+jq_n}) - \tau_n(x_{q_{n-1}+jq_n}) &= \widetilde{f}_n(\widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+(j-1)q_n})) - f_n(\tau_n(x_{q_{n-1}+(j-1)q_n})) \\ &\geq \widetilde{f}'_n(\xi_j)(\widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+(j-1)q_n}) - \tau_n(x_{q_{n-1}+(j-1)q_n})) - C\lambda^n \\ &\geq (\widetilde{f}'_n(\xi_j) - \epsilon_2)(\widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+(j-1)q_n}) - \tau_n(x_{q_{n-1}+(j-1)q_n})),\end{aligned} \tag{3.12}$$

if $\epsilon_2(\widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+(j-1)q_n}) - \tau_n(x_{q_{n-1}+(j-1)q_n})) \geq C\lambda^n$, for some small $\epsilon_2 > 0$. Here, ξ_j is a point in the interval $(\tau_n(x_{q_{n-1}+(j-1)q_n}), \widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+(j-1)q_n}))$. If $\lambda < \lambda_2\lambda_3$, then this condition is satisfied for $j = j_{\lambda_2}$ and sufficiently large n, by condition (3.11). By iterating the estimate (3.12), we see that the estimate (3.11) gets only better as j is increased from j_{λ_2}, as long as $\widetilde{f}'_n(\xi_j) > 1 + \epsilon_2$, for some $\epsilon_2 > 0$. This is satisfied with $\epsilon_2 = B - 1$, where B is the constant from Proposition 3.2, as long as the intervals $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})$ lie inside $[-1,-1+\varepsilon]$.

From the first inequality in (3.12), we have

$$\widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+jq_n}) - \tau_n(x_{q_{n-1}+jq_n}) \geq K_2(\widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+(j-1)q_n}) - \tau_n(x_{q_{n-1}+(j-1)q_n})) - C\lambda^n, \tag{3.13}$$

which can be iterated a constant number of times, if $\lambda < \lambda_2\lambda_3$, to obtain

$$\widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+jq_n}) - \tau_n(x_{q_{n-1}+jq_n}) \geq C_4\lambda_2^n\lambda_3^n, \tag{3.14}$$

for some constant $C_4 > 0$, and all j such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}) \cap (-1+\varepsilon, -\varepsilon) \neq \emptyset$.

We will now prepare the setting to estimate the same difference, by starting from the other end of the interval $[-1,0]$. Notice that

$$
\begin{aligned}
\mathfrak{s}_n(\Delta_0^{(n)}) &= \mathfrak{s}_n(\Delta_{q_{n-1}}^{(n)}) \frac{f'_{n-1}(\tau_{n-1} \circ \tau_n^{-1}(\zeta_0))}{\widetilde{f}'_{n-1}(\widetilde{\tau}_{n-1} \circ \widetilde{\tau}_n^{-1}(\widetilde{\zeta}_0))} \\
&= \mathfrak{s}_n(\Delta_{q_{n-1}}^{(n)}) \frac{f'_{n-1}(-f_{n-1}(0)\zeta_0)}{\widetilde{f}'_{n-1}(-\widetilde{f}_{n-1}(0)\widetilde{\zeta}_0)} \\
&= \mathfrak{s}_n(\Delta_{q_{n-1}}^{(n)}) \left(1 + \frac{f'_{n-1}(-f_{n-1}(0)\zeta_0) - \widetilde{f}'_{n-1}(-\widetilde{f}_{n-1}(0)\widetilde{\zeta}_0)}{\widetilde{f}'_{n-1}(-\widetilde{f}_{n-1}(0)\widetilde{\zeta}_0)}\right) \\
&\geq \mathfrak{s}_n(\Delta_{q_{n-1}}^{(n)}) \left(1 - K_2^{-1}(C\lambda^{n-1} + K_1^2 C\lambda^{n-1} + K_1^2|\zeta_0 - \widetilde{\zeta}_0|)\right)
\end{aligned}
\tag{3.15}
$$

where $\zeta_0 \in \tau_n(\Delta_0^{(n)})$ and $\widetilde{\zeta}_0 \in \widetilde{\tau}_n(\widetilde{\Delta}_0^{(n)})$. We next estimate $|\zeta_0 - \widetilde{\zeta}_0|$. Since

$$
\mathfrak{s}_n(\Delta_{q_{n-1}}^{(n)}) = \frac{1 + \tilde{f}_n(-1)}{1 + f_n(-1)} \leq 1 + \frac{C\lambda^n}{|\tau_n(\Delta_{q_{n-1}}^{(n)})|}, \tag{3.16}
$$

and, by assumption $\mathfrak{s}_n(\Delta_{q_{n-1}}^{(n)}) > 1 + \lambda_3^n$, we have $|\tau_n(\Delta_{q_{n-1}}^{(n)})| < C\lambda_2^n$, if $\lambda < \lambda_2\lambda_3$, which we assume. Furthermore, since the length of $\tau_n(\Delta_{q_{n-1}}^{(n)})$ is of the same order as $f_n(0)$, we have $f_n(0) \leq C_5\lambda_2^n$, for some $C_5 > 0$. This implies that $|\zeta_0 - \widetilde{\zeta}_0| \leq f_n(0) + C\lambda^n \leq C_5\lambda_2^n + C\lambda^n$. Using this estimate and the last inequality in (3.15), we obtain, that for some $\epsilon_3 > 0$ and sufficiently large n,

$$
\mathfrak{s}_n(\Delta_0^{(n)}) \geq 1 + (1 - \epsilon_3)\lambda_3^n. \tag{3.17}
$$

Notice, further, that

$$
\frac{\mathfrak{s}_n(\Delta_{q_{n+1}}^{(n)})}{\mathfrak{s}_n(\Delta_0^{(n)})} = \frac{\tilde{\tau}_n(\tilde{x}_{q_{n+1}+q_n}) - \tilde{\tau}_n(\tilde{x}_{q_{n+1}})}{\tau_n(x_{q_{n+1}+q_n}) - \tau_n(x_{q_{n+1}})} \cdot \frac{1}{\mathfrak{s}_n(\Delta_0^{(n)})} = \frac{\tilde{f}_{n+1}(0) - \tilde{f}_{n+1}(-1)}{f_{n+1}(0) - f_{n+1}(-1)}, \tag{3.18}
$$

and that the right hand side is bounded from below by $1 - C_6\lambda^n$, for some $C_6 > 0$. Together with (3.17), this implies that, for sufficiently large n,

$$
\mathfrak{s}_n(\Delta_{q_{n+1}}^{(n)}) > 1 + (1 - 2\epsilon_3)\lambda_3^n. \tag{3.19}
$$

For sufficiently large n, we also have

$$
\mathfrak{s}_n(\Delta_0^{(n+1)}) = \mathfrak{s}_n(\Delta_0^{(n)}) \frac{\widetilde{f}_{n+1}(0)}{f_{n+1}(0)} > (1 + (1 - \epsilon_3)\lambda_3^n) \left(1 - \frac{C\lambda^n}{K_5}\right) > 1 + (1 - 2\epsilon_3)\lambda_3^n. \tag{3.20}
$$

Here, we have used that $f_{n+1}(0)$ is bounded from below by K_5.

We can now perform backward iterations of the intervals $\tau_n(\Delta^{(n)}_{q_{n+1}})$ and $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n+1}})$. The estimates below are similar to the estimates for forward iterations that were used above. For $1 \le j \le k_{n+1}$ such that $\tau_n(\Delta^{(n)}_{q_{n-1}+(j-1)q_n}), \widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+(j-1)q_n}) \subset [-\lambda_2^n, 0]$, we have

$$\begin{aligned}
\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}) &= \prod_{i=j+1}^{k_{n+1}} \left(1 + \frac{f_n'(\zeta_i) - \widetilde{f}_n'(\widetilde{\zeta}_i)}{\widetilde{f}_n'(\widetilde{\zeta}_i)}\right) \mathfrak{s}_n(\Delta^{(n)}_{q_{n+1}}) \\
&= \prod_{i=j+1}^{k_{n+1}} \left(1 + \frac{f_n'(\zeta_i) - \widetilde{f}_n'(\zeta_i) + \widetilde{f}_n'(\zeta_i) - \widetilde{f}_n'(\widetilde{\zeta}_i)}{\widetilde{f}_n'(\widetilde{\zeta}_i)}\right) \mathfrak{s}_n(\Delta^{(n)}_{q_{n+1}}) \\
&\ge \prod_{i=j+1}^{k_{n+1}} \left(1 - \frac{|f_n'(\zeta_i) - \widetilde{f}_n'(\zeta_i)| + |\widetilde{f}_n'(\zeta_i) - \widetilde{f}_n'(\widetilde{\zeta}_i)|}{|\widetilde{f}_n'(\widetilde{\zeta}_i)|}\right) \mathfrak{s}_n(\Delta^{(n)}_{q_{n+1}}) \\
&\ge \left(1 - K_2^{-1}(C\lambda^n + K_1\lambda_2^n)\right)^{k_{n+1}-j} \mathfrak{s}_n(\Delta^{(n)}_{q_{n+1}}).
\end{aligned} \tag{3.21}$$

As before, $\zeta_i \in \tau_n(\Delta^{(n)}_{q_{n-1}+(i-1)q_n})$, and $\widetilde{\zeta}_i \in \widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+(i-1)q_n})$. Since $j \le k_{n+1} \le C_1\lambda_1^{-n}$, using (3.19), for sufficiently large n, we obtain

$$\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}) > 1 + (1-3\epsilon_3)\lambda_3^n, \tag{3.22}$$

if $\lambda \le \lambda_2 < \lambda_1\lambda_3$.

If $j_{-\lambda_2}$ is the smallest index j of such that the interval $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})$ is a subset of $[-\lambda_2^n, 0]$, then we have

$$\begin{aligned}
\tau_n(x_{q_{n-1}+j_{-\lambda_2}q_n}) - \widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+j_{-\lambda_2}q_n}) &= |\widetilde{\tau}_n(\tilde{\Delta}^{(n+1)}_0)| - |\tau_n(\Delta^{(n+1)}_0)| \\
&\quad + \sum_{j=j_{-\lambda_2}}^{k_{n+1}-1} |\widetilde{\tau}_n(\tilde{\Delta}^{(n)}_{q_{n-1}+jq_n})| - \sum_{j=j_{-\lambda_2}}^{k_{n+1}-1} |\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})| \\
&> \sum_{j=j_{-\lambda_2}}^{k_{n+1}-1} |\widetilde{\tau}_n(\tilde{\Delta}^{(n)}_{q_{n-1}+jq_n})| - \sum_{j=j_{-\lambda_2}}^{k_{n+1}-1} |\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})| \\
&> \frac{(1-3\epsilon_3)\lambda_3^n}{1+(1-3\epsilon_3)\lambda_3^n} \sum_{j=j_{-\lambda_2}}^{k_{n+1}-1} |\widetilde{\tau}_n(\tilde{\Delta}^{(n)}_{q_{n-1}+jq_n})| > C_7\lambda_2^n\lambda_3^n,
\end{aligned} \tag{3.23}$$

for some $C_7 > 0$, since $\sum_{j=-\lambda_2}^{k_{n+1}-1} |\widetilde{\tau}_n(\tilde{\Delta}^{(n)}_{q_{n-1}+jq_n})|$ is of the order of λ_2^n. In the first of these inequalities, we have also used the estimate (3.20).

We also have

$$\begin{aligned}
\tau_n(x_{q_{n-1}+jq_n}) - \widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+jq_n}) &\ge (\widetilde{f}_n'(\xi_{j+1}))^{-1}(\tau_n(x_{q_{n-1}+(j+1)q_n}) - \widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+(j+1)q_n}) - C\lambda^n) \\
&\ge ((\widetilde{f}_n'(\xi_{j+1}))^{-1} - \epsilon_4)(\tau_n(x_{q_{n-1}+(j+1)q_n}) - \widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+(j+1)q_n})),
\end{aligned} \tag{3.24}$$

if $\epsilon_4(\tau_n(x_{q_{n-1}+(j+1)q_n}) - \widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+(j+1)q_n})) \geq (\widetilde{f}_n'(\xi_{j+1}))^{-1} C\lambda^n$, for some small $\epsilon_4 > 0$. As before, ξ_{j+1} is a point in the interval $(\tau_n(x_{q_{n-1}+jq_n}), \widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+jq_n}))$. If $\lambda < \lambda_2\lambda_3$, then this condition is satisfied for $j = j_{-\lambda_2}$ and sufficiently large n, by condition (3.23). We can iterate this estimate, as long as $\widetilde{(f_n'(\xi_{j+1}))}^{-1} > 1 + \epsilon_4$, for some $\epsilon_4 > 0$. This is certainly the case as long as the intervals $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})$ lie inside $[-\varepsilon, 0]$ as follows from Proposition 3.2. Iterating again the first estimate in (3.24) a constant number of times, we obtain an estimate opposite to (3.14), i.e.

$$\tau_n(x_{q_{n-1}+jq_n}) - \widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+jq_n}) \geq C_8\lambda_2^n\lambda_3^n, \tag{3.25}$$

for some $C_8 > 0$, and all j such that $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}) \cap (-1+\varepsilon, -\varepsilon) \neq \emptyset$. By considering this estimate for any such j, we get a contradiction with (3.14). **QED**

The next lemma deals with the iteration of "long" intervals, i.e., intervals whose lengths are at least of the order of λ_2^n.

Let η and $\widetilde{\eta}$ be the left or right ends of the intervals $I \subset \Delta_0^{(n-1)}$ and $\widetilde{I} \subset \widetilde{\Delta}_0^{(n-1)}$, and let $r_n(I) = \frac{|\widetilde{\tau}_n(\widetilde{\eta}) - \tau_n(\eta)|}{|\tau_n(I)|}$. Let $I_i = f_n^i(\tau_n(I))$ and $\widetilde{I}_i = \widetilde{f}_n^i(\widetilde{\tau}_n(\widetilde{I}))$.

In what follows, we assume that λ_2 is chosen such that $\lambda_2 \in (\frac{\lambda}{\lambda_3}, \lambda_1\lambda_3)$.

Lemma 3.5 *Assume that there exist $C_9, C_{10}, C_{11} > 0$ such that $I_i \subset \Delta_0^{(n-1)}, \widetilde{I}_i \subset \widetilde{\Delta}_0^{(n-1)}$ and $|\tau_n(I_i)| \geq C_9\lambda_2^n$, for all $0 \leq i \leq N_n$, where $N_n \leq C_{10}n$. Assume further that $r_n(I) \leq C_{11}\lambda_3^n$. Then, there exists $C_{12} > 0$ such that $r_n(I_i) \leq C_{12}\lambda_3^n$ for all $0 \leq i \leq N_n$.*

Proof. We will assume that η and $\widetilde{\eta}$ are the left ends. For the right ends, the proof does not change. The lemma is proved by induction. For $i = 0$, the claim is true. Assume that for all $0 \leq i \leq j$, $r_n(I_i) \leq C_{12}\lambda_3^n < 1$, for some $C_{12} > 0$ and $n \geq n_0$ specified below. Clearly,

$$\begin{aligned} r_n(I_{i+1}) &\leq \frac{f_n'(\xi_{i+1})|\widetilde{\tau}_n(\widetilde{\eta}_i) - \tau_n(\eta_i)| + C\lambda^n}{f_n'(\bar{\zeta}_{i+1})|\tau_n(I_i)|} \\ &\leq \left(1 + \frac{|f_n'(\xi_{i+1}) - f_n'(\bar{\zeta}_{i+1})|}{f_n'(\bar{\zeta}_{i+1})}\right) r_n(I_i) + \frac{C\lambda^n}{K_2C_9\lambda_2^n} \\ &\leq \left(1 + K_1K_2^{-1}(1 + r_n(I_i))|\tau_n(I_i)|\right) r_n(I_i) + \frac{C\lambda^n}{K_2C_9\lambda_2^n}, \end{aligned} \tag{3.26}$$

where $\xi_{i+1} \in (\tau_n(\widetilde{\eta}_i), \tau_n(\eta_i))$ and $\bar{\zeta}_{i+1} \in \tau_n(I_i)$. Applying this inequality recursively from

$i = j$ down to $i = 0$, we find

$$
\begin{aligned}
r_n(I_{j+1}) \leq & \prod_{i=0}^{j} \left(1 + 2K_1K_2^{-1}|\tau_n(I_i)|\right) r_n(I) + CC_9^{-1}K_2^{-1}(\lambda/\lambda_2)^n \\
& \cdot \left(1 + \sum_{k=1}^{j-1} \prod_{i=1}^{k} \left(1 + 2K_1K_2^{-1}|\tau_n(I_i)|\right)\right) \\
\leq & e^{2K_1K_2^{-1}} r_n(I) + CC_9^{-1}K_2^{-1}(\lambda/\lambda_2)^n(1 + C_{10}ne^{2K_1K_2^{-1}}) \\
\leq & C_{12}\lambda_3^n,
\end{aligned}
\tag{3.27}
$$

where $C_{12} = C_{11}e^{2K_1K_2^{-1}} + CC_9^{-1}K_2^{-1} + CC_9^{-1}K_2^{-1}C_{10}e^{2K_1K_2^{-1}} \max_{n\in\mathbb{N}} \left(n\left(\frac{\lambda}{\lambda_2\lambda_3}\right)^n\right)$. For $n_0 \in \mathbb{N}$ such that $C_{12}\lambda_3^{n_0} < 1$, we thus have $r_n(I_{j+1}) \leq C_{12}\lambda_3^n$, for all $n \geq n_0$. **QED**

In the next proposition, we again use $k_{n+1} \leq C_1\lambda_1^{-n}$, for those n considered here.

Proposition 3.6 *Assume that there exists $C_2 > 0$ such that for all even n, if $0 < c < 1$, or all odd n, if $c > 1$, (3.8) is valid. Then, there exists $C_{13} > 0$ such that for all such n and all $1 \leq j \leq k_{n+1}$, we have*

$$
\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}), (\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}))^{-1} \leq 1 + C_{13}\lambda_3^n. \tag{3.28}
$$

Proof. Let λ_2 again be chosen such that $\lambda < \lambda_2\lambda_3 < \lambda_2 < \lambda_1\lambda_3 < 1$. For $1 \leq j \leq k_{n+1}$ such that both intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+(j-1)q_n}), \widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+(j-1)q_n}) \subset [-1, -1 + \lambda_2^n]$, we have

$$
\begin{aligned}
\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}) &= \prod_{i=1}^{j} \left(1 + \frac{\widetilde{f}'_n(\widetilde{\zeta}_i) - f'_n(\zeta_i)}{f'_n(\zeta_i)}\right) \mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}}) \\
&= \prod_{i=1}^{j} \left(1 + \frac{\widetilde{f}'_n(\widetilde{\zeta}_i) - f'_n(\widetilde{\zeta}_i) + f'_n(\widetilde{\zeta}_i) - f'_n(\zeta_i)}{f'_n(\zeta_i)}\right) \mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}}) \\
&\leq \prod_{i=1}^{j} \left(1 + \frac{|\widetilde{f}'_n(\widetilde{\zeta}_i) - f'_n(\widetilde{\zeta}_i)| + |f'_n(\widetilde{\zeta}_i) - f'_n(\zeta_i)|}{|f'_n(\zeta_i)|}\right) \mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}}) \\
&\leq \left[1 + K_2^{-1}(C\lambda^n + K_1\lambda_2^n)\right]^j \mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}}).
\end{aligned}
\tag{3.29}
$$

As in the proof of Proposition 3.4, $\zeta_i \in \tau_n(\Delta^{(n)}_{q_{n-1}+(i-1)q_n})$ and $\widetilde{\zeta}_i \in \widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+(i-1)q_n})$. Similarly, one can obtain the estimate

$$
\left(\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n})\right)^{-1} \leq \left[1 + K_2^{-1}(C\lambda^n + K_1\lambda_2^n)\right]^j \left(\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}})\right)^{-1}. \tag{3.30}
$$

Since $j \leq k_{n+1} \leq C_1\lambda_1^{-n}$ and $\lambda < \lambda_2 < \lambda_1\lambda_3$, by using these inequalities and Proposition 3.4, there exists a constant $C_{14} > 0$, such that

$$|\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}) - 1| \leq C_{14}\lambda_3^n. \tag{3.31}$$

Let j_{λ_2} be the largest index j such that both intervals $\Delta^{(n)}_{q_{n-1}+jq_n}$ and $\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n}$ are contained inside the interval $[-1, -1+\lambda_2^n]$, if such an index j exists. To be specific, let us assume that $\widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+j_{\lambda_2}q_n}) \geq \tau_n(x_{q_{n-1}+j_{\lambda_2}q_n})$. In the opposite case, the proof is analogous. For all j such that $1 \leq j \leq j_{\lambda_2}$, we obtain

$$\begin{aligned} |\widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+jq_n}) - \tau_n(x_{q_{n-1}+jq_n})| &= \left| \left(\mathfrak{s}_n \left(\bigcup_{i=0}^{j-1} \Delta^{(n)}_{q_{n-1}+iq_n} \right) \right)^{-1} - 1 \right| \sum_{i=0}^{j-1} |\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})| \\ &\leq \max_{0 \leq i \leq j-1} |(\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+iq_n}))^{-1} - 1| \sum_{i=0}^{j-1} |\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})| \\ &\leq C_{15}\lambda_2^n\lambda_3^n, \end{aligned} \tag{3.32}$$

for some $C_{15} > 0$. In the last inequality, we have used the estimate (3.31). The same estimate for $j = j_{\lambda_2}$ implies that $|\tau_n(\Delta^{(n)}_{q_{n-1}+j_{\lambda_2}q_n})|$ is of the order of λ_2^n since, by Proposition 3.3, $|\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+j_{\lambda_2}q_n})|$ is of that order. If j_{λ_2} is not defined, then $|\widetilde{\tau}_n(\Delta^{(n)}_{q_{n-1}})|$ and $|\tau_n(\Delta^{(n)}_{q_{n-1}})|$ are at least of the order of λ_2^n.

We are now preparing the setting to extend these estimates to j such that both intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+(j-1)q_n}), \widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+(j-1)q_n}) \subset [-\lambda_2^n, 0]$. Using the first three equalities in (3.15), we obtain

$$\begin{aligned} |\mathfrak{s}_n(\Delta_0^{(n)}) - 1| \leq |\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}}) - 1| \left[1 + K_2^{-1}(C\lambda^{n-1} + K_1^2 C\lambda^{n-1} + K_1^2|\zeta_0 - \widetilde{\zeta}_0|) \right] \\ + K_2^{-1}(C\lambda^{n-1} + K_1^2 C\lambda^{n-1} + K_1^2|\zeta_0 - \widetilde{\zeta}_0|) \end{aligned} \tag{3.33}$$

where $\zeta_0 \in \tau_n(\Delta_0^{(n)})$ and $\widetilde{\zeta}_0 \in \widetilde{\tau}_n(\widetilde{\Delta}_0^{(n)})$. Since $|\zeta_0 - \widetilde{\zeta}_0| \leq f_n(0) + C\lambda^n$, if $f_n(0) \leq C_{16}\lambda_2^n$, $C_{16} > 0$, then, for some $C_{17} > 0$, we have

$$|\mathfrak{s}_n(\Delta_0^{(n)}) - 1| \leq C_{17}\lambda_3^n. \tag{3.34}$$

In that case, since

$$\mathfrak{s}_n(\Delta_0^{(n+1)}) = \mathfrak{s}_n(\Delta_0^{(n)}) \frac{\widetilde{f}_{n+1}(0)}{f_{n+1}(0)}, \tag{3.35}$$

we also have

$$\begin{aligned} |\mathfrak{s}_n(\Delta_0^{(n+1)}) - 1| &\leq \left| \mathfrak{s}_n(\Delta_0^{(n)}) - 1 \right| \frac{\widetilde{f}_{n+1}(0)}{f_{n+1}(0)} + \left| \frac{\widetilde{f}_{n+1}(0)}{f_{n+1}(0)} - 1 \right| \\ &\leq C_{17}\lambda_3^n(1 + K_5^{-1}C\lambda^{n+1}) + K_5^{-1}C\lambda^{n+1} \leq C_{18}\lambda_3^n, \end{aligned} \tag{3.36}$$

for some $C_{18} > 0$. If, on the other hand, $f_n(0) > C_{16}\lambda_2^n$, it is easy to show that the estimates (3.34) and (3.36) are still valid, since $|\tau_n(\Delta_0^{(n+1)})|$ is of the same order as $f_n(0)$.

Since

$$\mathfrak{s}_n(\Delta_{q_n}^{(n+1)}) = \mathfrak{s}_{n+1}(\Delta_{q_n}^{(n+1)})\mathfrak{s}_n(\Delta_0^{(n)}) = \frac{\widetilde{f}_{n+1}(-1)+1}{f_{n+1}(-1)+1}\mathfrak{s}_n(\Delta_0^{(n)}), \tag{3.37}$$

we further have that there exists $C_{19} > 0$, such that

$$|\mathfrak{s}_n(\Delta_{q_n}^{(n+1)}) - 1| \le \frac{C\lambda^{n+1}}{K_5}(1 + C_{17}\lambda_3^n) + C_{17}\lambda_3^n \le C_{19}\lambda_3^n. \tag{3.38}$$

Using the estimates (3.34), (3.36) and (3.38), and the fact that $|\Delta_{q_{n+1}}^{(n)}| = |\Delta_0^{(n)}| + |\Delta_0^{(n+1)}| - |\Delta_{q_n}^{(n+1)}|$, we obtain, for some $C_{20} > 0$,

$$\begin{aligned} |\mathfrak{s}_n(\Delta_{q_{n+1}}^{(n)}) - 1| &\le \max\{C_{17}, C_{18}, C_{19}\}\lambda_3^n \frac{|\tau_n(\Delta_0^{(n)})| + |\tau_n(\Delta_0^{(n+1)})| + |\tau_n(\Delta_{q_n}^{(n+1)})|}{|\tau_n(\Delta_0^{(n)})| + |\tau_n(\Delta_0^{(n+1)})| - |\tau_n(\Delta_{q_n}^{(n+1)})|} \\ &\le \max\{C_{17}, C_{18}, C_{19}\}\lambda_3^n \left(1 + 2\frac{|\tau_n(\Delta_0^{(n)})|}{|\tau_n(\Delta_0^{(n+1)})|}\right) \\ &\le \max\{C_{17}, C_{18}, C_{19}\}\lambda_3^n \left(1 + \frac{2}{K_5}\right) \le C_{20}\lambda_3^n. \end{aligned} \tag{3.39}$$

We can now perform backward iterations of the intervals $\tau_n(\Delta_{q_{n+1}}^{(n)})$ and $\widetilde{\tau}_n(\widetilde{\Delta}_{q_{n+1}}^{(n)})$. For $1 \le j \le k_{n+1}$ such that $\tau_n(\Delta_{q_{n-1}+(j-1)q_n}^{(n)}), \widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}+(j-1)q_n}^{(n)}) \subset [-\lambda_2^n, 0]$, we have

$$\begin{aligned} \mathfrak{s}_n(\Delta_{q_{n-1}+jq_n}^{(n)}) &\le \prod_{i=j+1}^{k_{n+1}} \left(1 + \frac{|f_n'(\zeta_i) - \widetilde{f}_n'(\zeta_i)| + |\widetilde{f}_n'(\zeta_i) - \widetilde{f}_n'(\widetilde{\zeta}_i)|}{|\widetilde{f}_n'(\widetilde{\zeta}_i)|}\right) \mathfrak{s}_n(\Delta_{q_{n+1}}^{(n)}) \\ &\le \left[1 + K_2^{-1}(C\lambda^n + K_1\lambda_2^n)\right]^{k_{n+1}-j} \mathfrak{s}_n(\Delta_{q_{n+1}}^{(n)}). \end{aligned} \tag{3.40}$$

Since $j \le k_{n+1} \le C_1\lambda_1^{-n}$, for sufficiently large n, we obtain

$$\mathfrak{s}_n(\Delta_{q_{n-1}+jq_n}^{(n)}) \le 1 + C_{21}\lambda_3^n, \tag{3.41}$$

for some $C_{21} > 0$. Similarly, one can obtain

$$\left(\mathfrak{s}_n(\Delta_{q_{n-1}+jq_n}^{(n)})\right)^{-1} \le 1 + C_{22}\lambda_3^n, \tag{3.42}$$

with $C_{22} > 0$, which together with estimate (3.41) gives us

$$|\mathfrak{s}_n(\Delta_{q_{n-1}+jq_n}^{(n)}) - 1| \le C_{23}\lambda_3^n, \tag{3.43}$$

for some $C_{23} > 0$, and $j_{-\lambda_2} \le j \le k_{n+1}$. Here, $j_{-\lambda_2}$ is the smallest index j such that both intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})$ and $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+jq_n})$ are subsets of $[-\lambda_2^n, 0]$, if such a j exists.

We further have

$$
\begin{aligned}
|\tau_n(x_{q_{n-1}+j_{-\lambda_2}q_n}) - \widetilde{\tau}_n(\widetilde{x}_{q_{n-1}+j_{-\lambda_2}q_n})| \le C_{24}\lambda_3^n \Big(\sum_{j=j_{-\lambda_2}}^{k_{n+1}-1} |\widetilde{\tau}_n(\tilde{\Delta}^{(n)}_{q_{n-1}+jq_n})| \\
+ |\widetilde{\tau}_n(\tilde{\Delta}_0^{(n+1)})| \Big) \le C_{25}\lambda_2^n\lambda_3^n,
\end{aligned}
\tag{3.44}
$$

for some $C_{24}, C_{25} > 0$, since $\sum_{j=-\lambda_2}^{k_{n+1}-1} |\widetilde{\tau}_n(\tilde{\Delta}^{(n)}_{q_{n-1}+jq_n})|$ is of the order of λ_2^n. In the first of these inequalities, we have also used the estimate (3.36). As before, we can conclude that the lengths of both intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+j_{-\lambda_2}q_n})$ and $\widetilde{\tau}_n(\tilde{\Delta}^{(n)}_{q_{n-1}+j_{-\lambda_2}q_n})$ are of the order of λ_2^n. If $j_{-\lambda_2}$ is not defined, the lengths of these intervals are either again of this order or larger.

In order to prove the desired estimate for $j_{\lambda_2} < j < j_{-\lambda_2}$ (if j_{λ_2} is not defined, we formally set $j_{\lambda_2} = 0$ here; if $j_{-\lambda_2}$ is not defined, we formally set $j_{\lambda_2} = k_{n+1}$), we can apply Lemma 3.5, since the lengths of the corresponding intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})$ and $\widetilde{\tau}_n(\tilde{\Delta}^{(n)}_{q_{n-1}+jq_n})$ are at least of the order of λ_2^n. The remaining assumptions of this Lemma are satisfied by (3.32) and (3.44), and the fact that the number of these indices is at most of the order of n. The desired estimates then follow from this lemma. **QED**

3.3 Renormalization graphs which are convex inside the tunnels

In this section, we focus on the dynamics of a subsequence of renormalizations f_n and $\widetilde{f}_n$ of maps T and $\widetilde{T}$ for odd n, if $0 < c < 1$, or even n, if $c > 1$. The graphs of these renormalizations are convex inside the tunnels.

If B_{f_n,K_5} is not empty, let ζ_n^* be a point such that $f_n'(\zeta_n^*) = 1$. Similarly, if $\widetilde{B}_{\widetilde{f}_n,K_5/2}$ is not empty, let $\widetilde{\zeta}_n^*$ be a point such that $\widetilde{f}_n'(\widetilde{\zeta}_n^*) = 1$.

Lemma 3.7 *There exists $C_{26} > 0$ such that for all $j = 1, \dots, k_{n+1}$, we have*

$$
|\tilde{\tau}_n(\tilde{x}_{q_{n-1}+jq_n}) - \tau_n(x_{q_{n-1}+jq_n})| \le C_{26}\lambda^{n/2}. \tag{3.45}
$$

Proof. To simplify the notation let $z_i = \tau_n(x_{q_{n-1}+iq_n})$ and $\tilde{z}_i = \tilde{\tau}_n(\tilde{x}_{q_{n-1}+iq_n})$. Notice that first pair of points satisfies the desired bound since

$$
\tilde{z}_1 - z_1 = |\tilde{f}_n(-1) - f_n(-1)| \le C\lambda^n. \tag{3.46}
$$

The same is true for the last pair since

$$
\tilde{z}_{k_{n+1}} - z_{k_{n+1}} = |\tilde{f}_{n+1}(0)\tilde{f}_n(0) - f_{n+1}(0)f_n(0)| \le K_1 C(1+\lambda)\lambda^n. \tag{3.47}
$$

Let ξ_i be a point between z_i and $\tilde{z}_i$ such that $|f_n(\tilde{z}_i) - f_n(z_i)| = f_n'(\xi_i)|\tilde{z}_i - z_i|$. Then,

$$\begin{aligned}|\tilde{z}_{i+1} - z_{i+1}| &\leq f_n'(\xi_i)|\tilde{z}_i - z_i| + C\lambda^n, \\ |\tilde{z}_{i-1} - z_{i-1}| &\leq (f_n'(\xi_{i-1}))^{-1}(|\tilde{z}_i - z_i| + C\lambda^n).\end{aligned} \tag{3.48}$$

By iterating these two inequalities we obtain

$$|\tilde{z}_j - z_j| \;\leq\; |\tilde{z}_1 - z_1| \prod_{i=1}^{j-1} f_n'(\xi_i) + C\lambda^n \left(1 + \sum_{k=2}^{j-1} \prod_{i=k}^{j-1} f_n'(\xi_i)\right), \tag{3.49}$$

$$|\tilde{z}_{k_{n+1}-j} - z_{k_{n+1}-j}| \;\leq\; |\tilde{z}_{k_{n+1}} - z_{k_{n+1}}| \prod_{i=k_{n+1}-j}^{k_{n+1}-1} (f_n'(\xi_i))^{-1} + C\lambda^n \sum_{k=k_{n+1}-j}^{k_{n+1}-1} \prod_{i=k}^{k_{n+1}-1} (f_n'(\xi_i))^{-1}.$$

We can now apply these estimates for all $j \in \mathbb{N}$ smaller or equal to $J = [1/K_5] + 1$, obtaining $|\tilde{z}_j - z_j| \leq C_{27}\lambda^n$, and $|\tilde{z}_{k_{n+1}-j} - z_{k_{n+1}-j}| \leq C_{27}\lambda^n$, for some $C_{27} > 0$. If $k_{n+1} \leq 2J$, then the claim is proved. Otherwise, all the remaining points z_i and $\tilde{z}_i$ belong to $B_{f_n,K_5} \cap B_{\tilde{f}_n,K_5}$, ζ_n^* and $\widetilde{\zeta_n^*}$ are well-defined and $|\widetilde{\zeta_n^*} - \zeta_n^*| < C\lambda^n$.

The objective now is to apply the inequalities (3.49) to obtain the desired estimate for all the remaining points. We will first make at most $L_n = [\lambda^{-n/2}] + 1$ steps from both ends. More precisely, we will make at most L_n steps from the left end, but stop when $\max\{z_j, \tilde{z}_j\} > \zeta_n^*$. From the first of the inequalities (3.49), we obtain $|\tilde{z}_j - z_j| \leq C_{28}\lambda^{n/2}$, for some $C_{28} > 0$, and all $J < j \leq L_l$, where $L_l = \min\{L_n, \min\{k : \max\{z_k, \tilde{z}_k\} > \zeta_n^*\}\}$. Here, we have used that the products of derivatives in (3.49) are smaller than 1, since all points ξ_i now belong to $B_{f_n,K_5} \cap B_{\tilde{f}_n,K_5}$, and satisfy $\xi_i \leq \zeta_n^*$. The same estimate is obtained for $k_{n+1} - L_r \leq j < k_{n+1} - J$, such that $L_r = \min\{L_n, k_{n+1} - \max\{k : \min\{z_k, \tilde{z}_k\} < \zeta_n^*\}\}$, by applying the second inequality in (3.49).

If an early stop did not occur in the previous iteration, i.e., if $L_l = L_r = L_n$, then for the rest of the points we have $|z_j - \zeta_n^*| \leq C_{29}L_n^{-1} \leq C_{29}\lambda^{n/2}$, and $|\tilde{z}_j - \widetilde{\zeta_n^*}| \leq C_{29}\lambda^{n/2}$, for some $C_{29} > 0$. This follows from the asymptotic estimates for iterates under a non-degenerate tangency [9]. Together with $|\widetilde{\zeta_n^*} - \zeta_n^*| < C\lambda^n$, this completes the proof, in this case. If both the forward and the backward iterations were stopped earlier at L_l and L_r, respectively, then the interval between the leftmost and the rightmost of the points $z_{L_l}, \tilde{z}_{L_l}, z_{L_r}, \tilde{z}_{L_r}$ has a length bounded by $2C_{28}\lambda^{n/2}$, and contains all the other points. If the iteration in one direction was stopped earlier, while the other was not, then the two arguments above can be easily combined to complete the proof. **QED**

Lemma 3.8 *There exists $C_{30} > 0$, such that for all $0 \leq j \leq k_{n+1}$, we have*

$$\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}), (\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}))^{-1} \leq 1 + C_{30}\lambda_4^n, \tag{3.50}$$

with $\lambda_4 = \lambda^{\frac{(1+\alpha)\alpha}{8(2+\alpha)}}$.

Remark 5 This lemma holds without conditions on the growth rate of k_{n+1}. Below we prove it with the assumption $k_{n+1} \leq \lambda^{-n/8}$, for every $\lambda_4 \in [\lambda^{1/8}, 1)$. We treat this case differently, since for such k_{n+1} we have a stronger statement given in the corollary that follows. For $k_{n+1} > \lambda^{-n/8}$, one needs to perform a more detailed analysis of the asymptotic behavior of iterates near a non-degenerate tangency. Such an analysis, in two different asymptotic regimes, is carried out in [9] (see sections 3.2.2 and 3.2.3 therein), which we reference for the completion of the proof.

Proof of Lemma 3.8. For intervals corresponding to $0 \leq j \leq \min\{J, k_{n+1}\}$ and $k_{n+1} - \min\{J, k_{n+1}\} \leq j < k_{n+1}$, the estimates (3.50) follow directly from Lemma 3.7, since the length of each of these intervals is bounded below by a positive constant.

If $k_{n+1} > 2J$, we can continue to propagate this estimate up to $j \leq \min\{J + \lambda_5^{-n}, k_{n+1}\}$, obtaining

$$\begin{aligned} \mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}) &= \prod_{i=J+1}^{j} \left(1 + \frac{\widetilde{f}'_n(\widetilde{\zeta}_i) - f'_n(\zeta_i)}{f'_n(\zeta_i)}\right) \mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+Jq_n}) \\ &\leq \left(1 + C_{31}\lambda^{n/2}\lambda_5^{-2n}\right)^{j-J} (1 + C_{26}\lambda^{n/2}) \leq (1 + C_{32}\lambda^{n/2}\lambda_5^{-3n}). \end{aligned} \tag{3.51}$$

Here, $\zeta_i \in \tau_n(\Delta^{(n)}_{q_{n-1}+(i-1)q_n})$ and $\widetilde{\zeta}_i \in \widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+(i-1)q_n})$, and $C_{31}, C_{32} > 0$. We have used the estimate

$$\begin{aligned} |\widetilde{f}'_n(\widetilde{\zeta}_i) - f'_n(\zeta_i)| &= \left| \frac{\int_{\widetilde{z}_{i-1}}^{\widetilde{z}_i} \widetilde{f}'_n(z)\,dz}{\widetilde{z}_i - \widetilde{z}_{i-1}} - \frac{\int_{z_{i-1}}^{z_i} f'_n(z)\,dz}{z_i - z_{i-1}} \right| \leq \left| \frac{\int_{\widetilde{z}_{i-1}}^{\widetilde{z}_i} (\widetilde{f}'_n(z) - f'_n(z))\,dz}{\widetilde{z}_i - \widetilde{z}_{i-1}} \right| \\ &+ \left| \frac{\int_{z_{i-1}}^{\widetilde{z}_i} f'_n(z)\,dz - \int_{z_i}^{\widetilde{z}_i} f'_n(z)\,dz}{\widetilde{z}_i - \widetilde{z}_{i-1}} \right| + f'_n(\zeta_i) \left| \frac{z_i - z_{i-1}}{\widetilde{z}_i - \widetilde{z}_{i-1}} - 1 \right| \\ &\leq C\lambda^n + C_{33}\frac{\lambda^{n/2}}{\lambda_5^{2n}} \leq C_{34}\frac{\lambda^{n/2}}{\lambda_5^{2n}}, \end{aligned} \tag{3.52}$$

where $C_{33}, C_{34} > 0$. The last inequality follows by using the asymptotic estimates $|\widetilde{z}_i - \widetilde{z}_{i-1}| \geq C_{35}\lambda_5^{2n}$, with $C_{35} > 0$, for iterates under a non-degenerate tangency [9]. Similarly, we can propagate the desired estimate from $j = k_{n+1} - J$ backwards to all $j \geq \max\{0, k_{n+1} - J - \lambda_5^{-n}\}$. Taking $\lambda_5 = \lambda^{1/8}$, we obtain the estimate

$$\mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n}) \leq 1 + C_{32}\lambda^{n/8}, \tag{3.53}$$

which implies (3.50) for the considered indices j. This proves the first part of the claim, if $k_{n+1} \leq \lambda_5^{-n}$.

The proof of the second estimate can be obtained in the same way. **QED**

Corollary 3.9 *If $k_{n+1} \leq \lambda^{-n/8}$, it follows from Lemma 3.7 and Lemma 3.8 that, for all $1 \leq j \leq k_{n+1}$, we have*

$$|\tilde{\tau}_n(\tilde{x}_{q_{n-1}+jq_n}) - \tau_n(x_{q_{n-1}+jq_n})| \leq C_{35}^{-1}\lambda_4^n |\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})|. \tag{3.54}$$

Proof. It follows from Lemma 3.7 and Lemma 3.8 using the estimate $|\tau_n(\Delta^{(n)}_{q_{n-1}+jq_n})| > C_{35}\lambda^{n/4}$. **QED**

3.4 The estimates on the fundamental intervals

Lemma 3.10 *There exists $C_{36} > 0$ such that for all $n \in \mathbb{N}$,*

$$\left|\mathfrak{s}_n(\Delta_0^{(n)}) - 1\right| \leq C_{36}\lambda_3^n. \tag{3.55}$$

Moreover, if $C_{36} > 0$ has been chosen sufficiently large, then for all odd n, if $0 < c < 1$, or all even n, if $c > 1$, we have

$$\left|\mathfrak{s}_n(\Delta_0^{(n)}) - 1\right| \leq C_{36}\lambda^n. \tag{3.56}$$

Proof. For all even n, if $0 < c < 1$, or all odd n, if $c > 1$, the first estimate follows from inequality (3.34) in Proposition 3.6, and an analogous inequality for $(\mathfrak{s}_n(\Delta_0^{(n)}))^{-1}$. The improved estimate for all odd n, if $0 < c < 1$, or all even n, if $c > 1$, follows directly from

$$\left|\mathfrak{s}_n(\Delta_0^{(n)}) - 1\right| = \left|\frac{\tilde{f}_n(0) - f_n(0)}{f_n(0)}\right| \leq K_5^{-1}C\lambda^n, \tag{3.57}$$

since for such an n, $f_n(0) \geq K_5$. **QED**

Lemma 3.11 *There exists $\sigma_\infty > 0$ and $C_{37} > 0$ such that for all $n \in \mathbb{N}$, we have*

$$\left|\frac{|\widetilde{\Delta}_0^{(n)}|}{|\Delta_0^{(n)}|} - \sigma_\infty\right| \leq C_{37}\lambda_3^n. \tag{3.58}$$

Proof. Let $\sigma_n = \frac{|\widetilde{\Delta}_0^{(n)}|}{|\Delta_0^{(n)}|}$. It follows from Lemma 3.10, that

$$\left|\frac{\sigma_n}{\sigma_{n-1}} - 1\right| \leq C_{36}\lambda_3^n, \tag{3.59}$$

and thus $|\ln\sigma_n - \ln\sigma_{n-1}| = \epsilon_{n-1}$, where $0 \leq \epsilon_{n-1} \leq C_{38}\lambda_3^n$, for some $C_{38} > 0$. Since the sequence ϵ_n decreases exponentially, the sequence $\ln\sigma_n$, $n \in \mathbb{N}$, is a Cauchy sequence and converges to some $\ell_\infty = \lim_{n\to\infty}\ln\sigma_n$. The sequence σ_n, thus, converges to $\sigma_\infty = e^{\ell_\infty} > 0$.

Furthermore, since

$$|\ln \sigma_\infty - \ln \sigma_n| \leq \sum_{m=n}^{\infty} \epsilon_m \leq C_{38} \frac{\lambda_3^n}{1-\lambda_3}, \tag{3.60}$$

we have

$$\left| \frac{\sigma_\infty}{\sigma_n} - 1 \right| \leq C_{39} \lambda_3^n, \tag{3.61}$$

for some $C_{39} > 0$. **QED**

Without loss of generality, we may assume that $\sigma_\infty = 1$. This follows from the following simple lemma.

Lemma 3.12 *There exists an arbitrary smooth conjugation $\hat{T}$ of $\widetilde{T}$ and $C_{40} > 0$ such that, for all $n \in \mathbb{N}$, we have $\sigma_\infty(\hat{T}) = 1$, i.e.,*

$$\left| \frac{|\hat{\Delta}_0^{(n)}|}{|\Delta_0^{(n)}|} - 1 \right| \leq C_{40} \lambda_3^n. \tag{3.62}$$

Proof. It is enough to rescale the intervals of $\widetilde{\Delta}_0^{(n)}$ by means of a smooth change of coordinates affine in a neighborhood of the break point of $\widetilde{T}$. Assume that $\sigma_\infty < 1$. Let ψ be a smooth orientation-preserving diffeomorphism of $\mathbb{T}^1$, which is affine on a neighborhood of $\widetilde{x}_0$, with factor σ_∞^{-1}. Let $\hat{T} = \psi \circ \widetilde{T} \circ \psi^{-1}$. This change of $\widetilde{T}$ will not affect the renormalizations $\widetilde{f}_n$, $n \geq 2$, but σ_∞ corresponding to $\hat{T}$ and T will be equal to 1. A similar argument works in the case $\sigma_\infty > 1$. **QED**

3.5 Estimates on the intervals of the partition $\mathcal{P}_m$, inside $\Delta_0^{(n-1)}$, with n a constant fraction of m

Assume that $k_{n+1} \leq C_1 \lambda_1^{-n}$, for all n odd, if $0 < c < 1$, or for all n even, if $c > 1$. Let $\lambda_6 = \max\{\lambda_3, \lambda_4\}$, and let $S_1 = \max\{C_{13}, C_{30}\}$.

Proposition 3.13 *Assume that for all intervals $\mathcal{P}_m \ni I \subset \Delta_{q_{n-1}}^{(n)}$, with $m > n$, and the corresponding intervals $\widetilde{\mathcal{P}}_m \ni \widetilde{I} \subset \widetilde{\Delta}_{q_{n-1}}^{(n)}$, we have $|\mathfrak{s}_n(I) - 1| < C_{41} \lambda_6^n$. Then, there exist $C_{42} > 0$ such that for all $0 \leq i < k_{n+1}$, $\tau_n(I_i) = f_n^i(\tau_n(I))$ and $\widetilde{\tau}_n(\widetilde{I}_i) = \widetilde{f}_n^i(\widetilde{\tau}_n(\widetilde{I}))$, we have $|\mathfrak{s}_n(I_i) - 1| < C_{42} \lambda_6^n$.*

Proof. Let $y_i, \widetilde{y}_i$ and $z_i, \widetilde{z}_i$ the left and the right ends of the intervals $\tau_n(I_i)$ and $\widetilde{\tau}_n(\widetilde{I}_i)$. If $z_i - y_i \le \max\{|\widetilde{y}_i - y_i|, |\widetilde{z}_i - z_i|\}$, then

$$\begin{aligned} |\mathfrak{s}_n(I_{i+1}) - 1| &= \left| \frac{\widetilde{f}_n'(\widetilde{\zeta}_i)|\widetilde{z}_i - \widetilde{y}_i|}{f_n'(\zeta_i)|z_i - y_i|} - 1 \right| \\ \le \left(C\lambda^n K_2^{-1} + 3K_1K_2^{-1}\max\{|\widetilde{y}_i - y_i|, |\widetilde{z}_i - z_i|\}\right) &\frac{|\widetilde{z}_i - \widetilde{y}_i|}{|z_i - y_i|} + \left| \frac{|\widetilde{z}_i - \widetilde{y}_i|}{|z_i - y_i|} - 1 \right| \\ &\le \left(2C\lambda^n K_2^{-1} + 6K_1K_2^{-1}\max\{|\widetilde{y}_i - y_i|, |\widetilde{z}_i - z_i|\}\right) + |\mathfrak{s}_n(I_i) - 1|, \end{aligned} \tag{3.63}$$

if $|\mathfrak{s}_n(I_i)| < 2$. If $z_i - y_i > \max\{|\widetilde{y}_i - y_i|, |\widetilde{z}_i - z_i|\}$, then

$$\begin{aligned} |\mathfrak{s}_n(I_{i+1}) - 1| &= \left| \frac{\int_{\widetilde{y}_i}^{\widetilde{z}_i} \widetilde{f}_n'(\zeta)d\zeta}{\int_{y_i}^{z_i} f_n'(\zeta)d\zeta} - 1 \right| = \left| \frac{\widetilde{f}_n'(\widetilde{z}_i)(\widetilde{z}_i - \widetilde{y}_i) - \int_{\widetilde{y}_i}^{\widetilde{z}_i} \widetilde{f}_n''(\zeta)(\zeta - \widetilde{y}_i)d\zeta}{f_n'(z_i)(z_i - y_i) - \int_{y_i}^{z_i} f_n''(\zeta)(\zeta - y_i)d\zeta} - 1 \right| \\ &= \left| \frac{\frac{\widetilde{f}_n'(\widetilde{z}_i)}{f_n'(z_i)}\frac{\widetilde{z}_i - \widetilde{y}_i}{z_i - y_i} - 1 - \frac{1}{f_n'(z_i)(z_i - y_i)}\left(\int_{\widetilde{y}_i}^{\widetilde{z}_i} \widetilde{f}_n''(\zeta)(\zeta - \widetilde{y}_i)d\zeta - \int_{y_i}^{z_i} f_n''(\zeta)(\zeta - y_i)d\zeta\right)}{1 - \frac{1}{f_n'(z_i)(z_i - y_i)}\int_{y_i}^{z_i} f_n''(\zeta)(\zeta - y_i)d\zeta} \right|, \end{aligned} \tag{3.64}$$

and, thus,

$$\begin{aligned} |\mathfrak{s}_n(I_{i+1}) - 1| \le \Bigg[&2CK_2^{-1}\lambda^n \left(1 + \frac{z_i - y_i}{4}\right) + \frac{3}{2}K_1K_2^{-1}|\widetilde{y}_i - y_i| \\ &+ 4K_1K_2^{-1}|\widetilde{z}_i - z_i| + |\mathfrak{s}_n(I_i) - 1| \Bigg] \cdot (1 + K_1K_2^{-1}|z_i - y_i|). \end{aligned} \tag{3.65}$$

Here, we have used that

$$\left| \frac{\widetilde{f}_n'(\widetilde{z}_i)}{f_n'(z_i)} \frac{\widetilde{z}_i - \widetilde{y}_i}{z_i - y_i} - 1 \right| \le 2(CK_2^{-1}\lambda^n + K_1K_2^{-1}|\widetilde{z}_i - z_i|) + |\mathfrak{s}_n(I_i) - 1|\,, \tag{3.66}$$

if $\frac{\widetilde{z}_i - \widetilde{y}_i}{z_i - y_i} < 2$,

$$\begin{aligned} \frac{1}{f_n'(z_i)(z_i - y_i)} &\left| \int_{\widetilde{y}_i}^{\widetilde{z}_i} \widetilde{f}_n''(\zeta)(\zeta - \widetilde{y}_i)d\zeta - \int_{y_i}^{z_i} f_n''(\zeta)(\zeta - y_i)d\zeta \right| \\ &\le \frac{1}{K_2(z_i - y_i)} \Bigg[C\lambda^n \frac{(z_i - y_i)^2}{2} + K_1|\widetilde{y}_i - y_i||z_i - y_i| \\ &\qquad + K_1\frac{(\widetilde{y}_i - y_i)^2}{2} + 2K_1|z_i - y_i||\widetilde{z}_i - z_i| \Bigg] \\ &\le \frac{1}{K_2}\left[C\lambda^n \frac{(z_i - y_i)}{2} + \frac{3}{2}K_1|\widetilde{y}_i - y_i| + 2K_1|\widetilde{z}_i - z_i| \right], \end{aligned} \tag{3.67}$$

and

$$\frac{1}{f_n'(z_i)(z_i-y_i)}\int_{y_i}^{z_i}|f_n''(\zeta)|(\zeta-y_i)d\zeta \le \frac{1}{2}K_1K_2^{-1}|z_i-y_i|. \tag{3.68}$$

We have also used that $(1-x)^{-1}\le 1+2|x|$, for $x<1/2$, and assumed $K_1K_2^{-1}|z_i-y_i|<1$.

Therefore, in either case we have

$$\begin{aligned}|\mathfrak{s}_n(I_{i+1})-1| \le \Bigg[&2CK_2^{-1}\lambda^n\left(1+\frac{z_i-y_i}{4}\right)+6K_1K_2^{-1}\max\{|\widetilde{y}_i-y_i|,|\widetilde{z}_i-z_i|\}\\ &+|\mathfrak{s}_n(I_i)-1|\Bigg]\cdot(1+K_1K_2^{-1}|z_i-y_i|).\end{aligned} \tag{3.69}$$

Using the estimate

$$\max\{|\widetilde{y}_i-y_i|,|\widetilde{z}_i-z_i|\}\le\left(C_{43}S_1\lambda_6^n+\max_{\mathcal{P}_m\ni I_i\subset\Delta^{(n)}_{q_{n-1}+iq_n}}|\mathfrak{s}_n(I_i)-1|\right)|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|, \tag{3.70}$$

where $C_{43}>0$, we obtain

$$\begin{aligned}|\mathfrak{s}_n(I_{i+1})-1| \le \Bigg[&4CK_2^{-1}\lambda^n+6K_1K_2^{-1}\left(C_{43}S_1\lambda_6^n+\max_{\mathcal{P}_m\ni I_i\subset\Delta^{(n)}_{q_{n-1}+iq_n}}|\mathfrak{s}_n(I_i)-1|\right)\\ &\cdot|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|+\max_{\mathcal{P}_m\ni I_i\subset\Delta^{(n)}_{q_{n-1}+iq_n}}|\mathfrak{s}_n(I_i)-1|\Bigg](1+K_1K_2^{-1}|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|).\end{aligned} \tag{3.71}$$

In the estimate (3.70), we have used the fact that for n odd, if $0<c<1$, and n even, if $c>1$ (corresponding to renormalization graphs concave inside the gates), the distance of the corresponding endpoints of the intervals $\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})$ and $\widetilde{\tau}_n(\widetilde{\Delta}^{(n)}_{q_{n-1}+iq_n})$ is bounded from above by $C_{43}S_1\lambda_6^n|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|$. The same estimate is valid for n even, if $0<c<1$, and n odd if $c>1$ (corresponding to renormalization graphs convex inside the tunnels), if $i\le[\lambda^{-n/8}]$ or $i\ge k_{n+1}-[\lambda^{-n/8}]$ (assuming $k_{n+1}>[\lambda^{-n/8}]$). This follows from Corollary 3.9. In this case, we will consider first only $i\le[\lambda^{-n/8}]$.

Taking the maximum of the left hand side of (3.71) over all I_{i+1}, such that $\mathcal{P}_m\ni I_{i+1}\subset\Delta^{(n)}_{q_{n-1}+(i+1)q_n}$, we obtain the inequality

$$M_{i+1}\le P_i+Q_iM_i, \tag{3.72}$$

where $M_i=\max_{\mathcal{P}_m\ni I_i\subset\Delta^{(n)}_{q_{n-1}+iq_n}}|\mathfrak{s}_n(I_i)-1|$, and

$$\begin{aligned}P_i&=\left(4CK_2^{-1}\lambda^n+6K_1K_2^{-1}C_{43}S_1\lambda_6^n|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|\right)(1+K_1K_2^{-1}|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|),\\ Q_i&=\left(1+6K_1K_2^{-1}|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|\right)(1+K_1K_2^{-1}|\tau_n(\Delta^{(n)}_{q_{n-1}+iq_n})|).\end{aligned} \tag{3.73}$$

By iterating this inequality from $i=j$ down to $i=0$, we obtain

$$M_{j+1} \leq P_j + \sum_{k=0}^{j-1} P_k \prod_{l=k+1}^{j} Q_l + M_0 \prod_{l=0}^{j} Q_l, \tag{3.74}$$

and, thus,

$$|\mathfrak{s}_n(I_{j+1}) - 1| \leq e^{7K_1K_2^{-1}} \left[4CK_2^{-1}\lambda^n(j+1) + 6K_1K_2^{-1}C_{43}S_1\lambda_6^n + \max_{\mathcal{P}_m \ni I \subset \Delta_{q_{n-1}}^{(n)}} |\mathfrak{s}_n(I) - 1| \right]. \tag{3.75}$$

Here, we have used that $\sum_{i=0}^{j} |\tau_n(\Delta_{q_{n-1}+iq_n}^{(n)})| < 1$ and the inequality $1+x < e^x$, for $x>0$. To complete the proof in the case of renormalization graphs concave inside the gates, with our standing assumption $k_{n+1} \leq C_1\lambda_1^{-n}$, or in the case of renormalization graphs convex inside tunnels if $k_{n+1} \leq [\lambda^{-n/8}]$, we can proceed by induction in j. Let for some j, $|\mathfrak{s}_n(I_j) - 1| < M\lambda_6^n$, where

$$M = e^{7K_1K_2^{-1}}(4CK_2^{-1}C_1 + 6K_1K_2^{-1}C_{43}S_1 + C_{41}), \tag{3.76}$$

and n is large enough such that $M\lambda_6^n < 1$. Then, the inequality (3.75) implies the same bound for $j+1$. Furthermore, for $j=0$ the inequality is true due to our initial assumption.

In the following, we consider the case of such n and assume that $k_{n+1} > [\lambda^{-n/8}]$.

Notice first that there exist $\zeta_1 \in \tau_n(I_i)$, $\widetilde{\zeta}_1 \in \widetilde{\tau}_n(\widetilde{I}_i)$, $\zeta_2, \zeta_3 \in \tau_n(\Delta_{q_{n-1}+iq_n}^{(n)})$, $\widetilde{\zeta}_2, \widetilde{\zeta}_3 \in \widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}+iq_n}^{(n)})$, such that

$$\begin{aligned} \left| \ln\left(\frac{\mathfrak{s}_n(I_{i+1})}{\mathfrak{s}_n(I_i)} \frac{\mathfrak{s}_n(\Delta_{q_{n-1}+iq_n}^{(n)})}{\mathfrak{s}_n(\Delta_{q_{n-1}+(i+1)q_n}^{(n)})} \right) \right| &= |\ln \widetilde{f}_n'(\widetilde{\zeta}_1) - \ln f_n'(\zeta_1) - \ln \widetilde{f}_n'(\widetilde{\zeta}_2) + \ln f_n'(\zeta_2)| \\ &= |(\ln \widetilde{f}')'(\widetilde{\zeta}_3)(\widetilde{\zeta}_2 - \widetilde{\zeta}_1) - (\ln f')'(\zeta_3)(\zeta_2 - \zeta_1)| \leq \frac{K_1}{K_2}|\widetilde{\zeta}_2 - \widetilde{\zeta}_1| + \frac{K_1}{K_2}|\zeta_2 - \zeta_1|. \end{aligned} \tag{3.77}$$

Summing these inequalities from $i=[\lambda^{-n/8}]$ to $j-1$, for some $[\lambda^{-n/8}] < j \leq k_{n+1} - [\lambda^{-n/8}]$, we obtain

$$\begin{aligned} \left| \ln\left(\frac{\mathfrak{s}_n(I_j)}{\mathfrak{s}_n(I_{[\lambda^{-n/8}]})} \frac{\mathfrak{s}_n(\Delta_{q_{n-1}+[\lambda^{-n/8}]q_n}^{(n)})}{\mathfrak{s}_n(\Delta_{q_{n-1}+jq_n}^{(n)})} \right) \right| &\leq \frac{K_1}{K_2} \left(\sum_{i=[\lambda^{-n/8}]}^{j-1} |\widetilde{\tau}_n(\widetilde{\Delta}_{q_{n-1}+iq_n}^{(n)})| \right. \\ &\left. + \sum_{i=[\lambda^{-n/8}]}^{j-1} |\tau_n(\Delta_{q_{n-1}+iq_n}^{(n)})| \right) \leq C_{44}\frac{K_1}{K_2}\lambda^{n/8}, \end{aligned} \tag{3.78}$$

for some $C_{44} > 0$. The last inequality follows from Lemma 5 of [9]. Using Lemma 3.8, we find that there exists $C_{45} > 0$, such that

$$\begin{aligned}|\ln \mathfrak{s}_n(I_j)| &\leq |\ln \mathfrak{s}_n(I_{[\lambda^{-n/8}]})| + |\ln \mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+[\lambda^{-n/8}]q_n})| + |\ln \mathfrak{s}_n(\Delta^{(n)}_{q_{n-1}+jq_n})| + C_{44}\frac{K_1}{K_2}\lambda^{n/8} \\ &\leq |\ln \mathfrak{s}_n(I_{[\lambda^{-n/8}]})| + C_{45}\lambda_4^n,\end{aligned}$$

and thus

$$|\mathfrak{s}_n(I_j) - 1| < 2(|\mathfrak{s}_n(I_{[\lambda^{-n/8}]}) - 1| + C_{45}\lambda_4^n). \tag{3.79}$$

For $j = 0, \ldots, [\lambda^{-n/8}]$, the desired bound $|\mathfrak{s}_n(I_j) - 1| < C_{42}\lambda_6^n$, for some $C_{42} > 0$, follows from (3.75). For $[\lambda^{-n/8}] < j \leq k_{n+1} - [\lambda^{-n/8}]$, the estimate (3.79) then proves this bound. Finally, by iterating the inequality (3.72) from $i = j - 1$ down to $i = k_{n+1} - [\lambda^{-n/8}]$, we obtain the desired bound for $j = k_{n+1} - [\lambda^{-n/8}], \ldots, k_{n+1}$, with $C_{42} = e^{7K_1K_2^{-1}}(4CK_2^{-1}C_1 + 6K_1K_2^{-1}C_{43}S_1 + 2M + 2C_{45})$. **QED**

Proposition 3.14 *For every $\lambda_7 \in (\lambda_6, 1)$, there exists $\nu > 0$ and $C_{46} > 0$, such that*

$$|\sigma(I) - 1| \leq C_{46}\lambda_7^m, \tag{3.80}$$

for all $I \in \mathcal{P}_m$ such that $I \subset \bar{\Delta}_0^{(m-[\nu m])}$.

Proof. It is easy to derive from Proposition 3.13 by induction, starting with Proposition 3.6 and Lemma 3.8, that for all $I \in \mathcal{P}_m$ such that $I \subset \bar{\Delta}_0^{(n)}$,

$$|\mathfrak{s}_n(I) - 1| \leq C_{47}(C_{48}C_{42})^{m-n}\lambda_6^n, \tag{3.81}$$

with $C_{47}, C_{48} > 0$. The C_{48} comes from the mapping of the intervals from $\tau_n(\Delta_0^{(i)})$ into $\tau_n(\Delta_0^{(i-1)})$, by the map f_{i-1}, for $i = m-2, \ldots, n+1$. The claim follows after we choose $\nu > 0$ such that $(C_{48}C_{42})^\nu\lambda_6^{1-\nu} \leq \lambda_7$, and rescale the intervals, using (3.58) with $\sigma_\infty = 1$. **QED**

3.6 Spreading the estimates to the whole circle

Proof of Theorem 2.2. In order to prove the claim, we will use Proposition 3.1. To verify the assumptions of Proposition 3.1, we need to spread the estimates (3.2) on the intervals $I, I' \subset \mathcal{P}_m$ which are either adjacent or belong to the same element of partition $\mathcal{P}_{m-1}$, from $\bar{\Delta}_0^{(m-[\nu m])}$ to the whole circle $\mathbb{T}^1$. Proposition 3.14 implies the estimate

$$|\ln\sigma(I) - \ln\sigma(I')| \leq 4C_{46}\lambda_7^m, \tag{3.82}$$

for sufficiently large m and all pairs of such intervals I, I' which are both contained in $\bar{\Delta}_0^{(m-[\nu m])}$. We will now spread such an estimate from $\bar{\Delta}_0^{(n)}$ to $\bar{\Delta}_0^{(n-1)}$ in $m-[\nu m]$ steps, starting with $n = m - [\nu m]$, and counting down to $n = 1$. In each step, the new intervals for which we need to show such an estimate appear in threads $I_i = T^{iq_n} I_0$ and $I'_i = T^{iq_n} I'_0$, for $0 \leq i < k_{n+1}$. Let us fix the order of the pairs in such a way that I'_0 is closer to x_0 than I_0. This implies that $I_0 \subset T^{q_{n-1}}(\Delta_0^{(n)})$ and that either I'_0 belongs to $T^{q_{n-1}}(\Delta_0^{(n)})$ as well or is adjacent to it.

We will prove that there is a constant $C_{49} > 0$, such that for all $0 \leq i < k_{n+1}$, we have

$$|\ln \mathfrak{s}_n(I_i) - \ln \mathfrak{s}_n(I'_i)| \leq |\ln \mathfrak{s}_n(I_{k_{n+1}}) - \ln \mathfrak{s}_n(I'_{k_{n+1}})| + C_{49}\lambda_{ref}^{m-n}. \tag{3.83}$$

This estimate will follow from the estimates below on

$$\delta_i = |\ln|\tau_n(I_{i+1})| - \ln|\tau_n(I_i)| - \ln|\tau_n(I'_{i+1})| + \ln|\tau_n(I'_i)||, \tag{3.84}$$

and similar estimates on $\widetilde{\delta}_i$ corresponding to $\widetilde{T}$.

Clearly, there exist $\bar{\zeta}_i \in \tau_n(I_i)$, $\bar{\zeta}'_i \in \tau_n(I'_i)$ and $\zeta_i \in [\bar{\zeta}_i, \bar{\zeta}'_i]$, such that

$$\delta_i = \left|\frac{f''_n(\zeta_i)}{f'_n(\zeta_i)}\right| |\bar{\zeta}'_i - \bar{\zeta}_i|. \tag{3.85}$$

There are two kinds of threads that we need to consider. First, if I_0 and I'_0 belong to the same element J_0 of $\mathcal{P}_{m-1}$, then there is a thread $J_i = T^{iq_n} J_0 \in \mathcal{P}_{m-1}$, with $0 \leq i < k_{n+1}$, such that $I_i \cup I'_i \subset J_i \subset T^{q_{n-1}+iq_n}\Delta_0^{(n)}$. Then, $\delta_i \leq K_1 K_2^{-1}|\tau_n(J_i)| \leq K_1 K_2^{-1} C_{ref}\lambda_{ref}^{m-1-n}|\tau_n(\Delta_{q_{n-1}+iq_n}^{(n)})|$. Since the sum of $|\tau_n(\Delta_{q_{n-1}+iq_n}^{(n)})|$ is smaller than 1, the bound (3.83) follows using

$$|\ln \mathfrak{s}_n(I_i) - \ln \mathfrak{s}_n(I'_i)| - |\ln \mathfrak{s}_n(I_{i+1}) - \ln \mathfrak{s}_n(I'_{i+1})| \leq \delta_i + \widetilde{\delta}_i. \tag{3.86}$$

Second, if I_0 and I'_0 are adjacent and belong to different elements of $\mathcal{P}_{m-1}$, then we similarly have $\delta_i \leq 2K_1 K_2^{-1} C_{ref}\lambda_{ref}^{m-n}|\tau_n(\Delta_{q_{n-1}+iq_n}^{(n)})|$, and the bound (3.83) follows again.

Therefore, using the estimate (3.82) for $n = m - [\nu m]$, we obtain

$$|\ln \sigma(I) - \ln \sigma(I')| \leq 4C_{46}\lambda_7^m + C_{49}\sum_{n=1}^{m-[\nu m]} \lambda_{ref}^{m-n} \leq C_{50}(\lambda_7^m + \lambda_{ref}^{\nu m}), \tag{3.87}$$

for all pairs of $I, I' \in \mathcal{P}_m$, as in Proposition 3.1. Hence, (3.2) holds with $\bar{\lambda} = \max\{\lambda_7, \lambda_{ref}^{\nu}\}$, and the claim is proven. **QED**

3.7 The proof of Theorem 1.1

To prove Theorem 1.1, we need to verify that the conditions of Theorem 2.2 hold true in the case of circle diffeomorphisms with a break point.

Condition (a) of Theorem 2.2 is an assumption of Theorem 1.1. To verify the condition (b), we need to check that the renormalization sequences for circle diffeomorphisms with breaks are regularized. The first regularity condition is proved in [11]. The other regularity conditions follow from the estimate (B) in Section 2.2 and the explicit form of the fractional-linear transformations (2.7). Condition (c) follows from the estimate (A) in Section 2.2. Condition (d) is proved in [8].

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References

[1] V. I. Arnol'd, Small denominators I: On the mapping of a circle into itself, *Izv. Akad. Nauk. Math. Serie* **25** (1961), 21–86; *Transl. A.M.S. Serie 2* **46** (1965).

[2] E. de Faria, W. de Melo, Rigidity of critical circle maps I, *J. Eur. Math. Soc.* **1** 4 (1999), 339–392.

[3] E. de Faria, W. de Melo, Rigidity of critical circle maps II, *J. Am. Math. Soc.* **13** 2 (2000), 343–370.

[4] M. R. Herman, Sur la conjugasion differentiable des difféomorphismes du cercle a de rotations, *Publ. Math. Inst. Hautes Etudes Sci.* **49** (1979), 5–234.

[5] Y. Katznelson, D. Orstein, The differentiability of conjugation of certain diffeomorphisms of the circle, *Ergod. Th. & Dynam. Sys.* **9** (1989), 643–680.

[6] K. Khanin, D. Khmelev, Renormalizations and rigidity theory for circle homeomorphisms with singularities of break type, *Commun. Math. Phys.* **235** 1 (2003), 69–124.

[7] K. Khanin, S. Kocić, Examples of non-rigidity for circle diffeomorphisms with breaks, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, *to appear* (2011).

[8] K. Khanin, S. Kocić, Exponential approach of renormalization for circle diffeomorphisms with a break point, *in preparation* (2011).

[9] K. Khanin, A. Teplinsky, Robust rigidity for circle diffeomorphisms with singularities, *Invent. Math.* **169** (2007), 193–218.

[10] K. Khanin, A. Teplinsky, Herman's theory revisited, *Invent. Math.* **178** (2009), 333–344.

[11] K. Khanin, A. Teplinsky, Rigidity theory for circle diffeomorphisms with breaks, *in preparation* (2010).

[12] K. M. Khanin, E. B. Vul, Circle Homeomorphisms with Weak Discontinuities, *Advances in Sov. Math.* **3** (1991), 57–98.

[13] Ya. G. Sinai, K. M. Khanin, Smoothness of conjugacies of diffeomorphisms of the circle with rotations, *Uspekhi Mat. Nauk* **44** 1 (1989), 57–82.

[14] M. Yampolsky, Hyperbolicity of renormalization of critical circle maps, *Publ. Math. Inst. Hautes Etudes Sci.* **96** (2002), 1–41.

[15] J.-C. Yoccoz, Conjugaison differentiable des difféomorphismes du cercle donc le nombre de rotation vérifie une condition Diophantienne, *Ann. Sci. Ec. Norm. Sup.* **17** (1984), 333–361.