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To my teachers and good people Isaak Perchenok and Rustem Valeyev. To my friend Aleksey Kiselev whose intellectual paradigm underlies this work.

On the scope of diagonal argument and on contradictory equivalence.

Abstract:

Restriction on the scope of diagonal argument will be set using two absolutely different proof techniques. One of the proof techniques will analyze contradictory equivalence ($R \in R \leftrightarrow R \notin R$) in a rather unconventional way.

Cantor's paradox.

Cantor's paradox is based on two things: the first is Cantor's theorem and the second one is the definition of U (universal set) as a set containing "everything" – all sets and all elements.

Cantor's theorem states that, for any set A, the set of all subsets of A (the power set of A) has a strictly greater cardinality than A itself.

When we try to answer the question whose cardinality is larger, U or the power set of U, we can see from Cantor's theorem that **|power set of U|>|U|** but according to the definition of U **|U|≥|any other set|** thus **|U|≥|power set of U|**. Of course, **|power set of U|>|U| and |U|≥|power set of U|** is a contradiction.

The scope of **Cantor's theorem** will be analyzed in both parts of the paper.

Before starting the analysis I will give the plan of it:

1. At first **Cantor's theorem** in its general form will be presented here.
2. Then an example of applying it to a more concrete thing – proof of uncountablity of power set of natural numbers will be presented.
3. Then the diagonal argument will be analyzed in the context of Cantor's paradox.

4. Final conclusion on the scope of **Cantor's theorem** will be made then (**it will be confirmed in part two of this paper using another proof technique!**).

Cantor's theorem in its general form

Formulation: For any set A, the set of all subsets of A (the power set of A) has a strictly greater cardinality than A itself.

We say that two sets have the same cardinality if and only if there can be one-to-one correspondence between them (bijection).

When we compare cardinality of two sets they are either of the same cardinality or one of them has larger cardinality than cardinality of another one.

Diagonal argument part of Cantor's theorem

This is to prove that there can be no surjection from any set A to its power set (farther denoted by **P(A)**) thus their cardinalities cannot be equal.

We can show that given any set *A* no function *f* from A into power set of A, can provide us with such a surjection - we will show the existence of at least one subset of P(A) that is not an element of the image of A under *f*.

Let us assume we have such a function. Let us denote this function by $f(x)$.

Then let us construct a set B.

$B=\{x \in A: x \notin f(x)\}$

This means, by definition, that for all x in A, $x \in B$ if and only if $x \notin f(x)$.

B is a subset of A, an element of P(A). Then, under the assumption that $f(x)$ exists, **B** should be paired (by $f(x)$) with some x, an element of A.

Now we will try to answer a question whether this element *x* is included in set B.

If x is included in the set B it is not included in the set B (by the definition of the set B).

If x is not included in the set B it is included in the set B (by the definition of the set B).

$x \in B \leftrightarrow x \notin B$ This **equivalence** is **contradictory** (proposition equivalent to its negation).

So, from supposing there can be such function $f(x)$ that gives us a surjection from a set to its power set we get a contradiction. That means there cannot be such function $f(x)$.

We use the proof that no surjection is possible for two purposes:

1. To prove there is no bijection possible (as bijection is just a special case of surjection), thus proving A and P(A) **cannot be equal**.

2. To exclude the possibility that A can "cover" ("surject") P(A) so that, after proving that P(A) can "cover" ("surject") A, we could state that P(A)'s cardinality is larger.

The intrinsic mechanism of the diagonal argument

We intentionally construct set B in such a way that it differs from any of the images (under *f*(x)) of elements of A. So the set B cannot be amongst elements of P(A) paired with elements of A**. Set B belongs to P (A) but not to the elements of P(A) paired with elements of A.** That means not all elements of P(A) can be paired with elements of A. That means that the surjection is impossible.

Non-diagonal part of Cantor's theorem

This is to exclude the possibility that P(A) cannot "cover" A thus leaving the only possibility that |A| < |P(A)|.

P(A) cannot be "smaller" than A, since it includes all of the singletons of A, thus, it contains an element of the form { a }, { b }, { c }, ... for each element a, b, c, ... in A.

Consequently **|A| < |P(A)|**.

Example: Proof of uncountablity of power set of natural numbers P(N)

Diagonal argument part

Suppose we have function *f*(x) pairing natural numbers with power set of natural numbers.

Now let us construct a "proof" set that is one of the elements (sets) of P(N) but not one of those paired with elements of N by *f*(x).

B={x ∈ N: x ∉ *f*(x)}

So we do it using the "diagonal" – for every **paired** element (set) of P(N), if a natural number is included in the set it is not included in our "proof" set and vice versa. This way we make sure our "proof" set differs from any paired element (set) of P(N). At the same time this set is definitely one of those included in P(N). If we suppose that set B is one of paired elements (sets) of P(N) we will get the contradictory equivalence (see "Cantor's theorem in its general form" section). **This is how the intrinsic mechanism of diagonal argument works.**

N	1*	2	3	4	5	6	...	Elements (sets) of P(N)
1**	0***	0	1	0	0	1	...	<-set A
2	1	1	0	0	1	0	...	<-set B

3	1	0	1	0	1	0	…	<-set C
4	1	0	0	1	0	0	…	<-set D
5	1	1	1	1	0	1	…	<-set E
6	0	1	0	1	1	0	…	<-set F
…	…	…	…	…	…	…	…	…
	1	0	0	0	1	1	…	<-"proof" set

*column heads – natural numbers to code presence\absence of the corresponding natural number in paired sets

**row heads – natural numbers paired with elements (sets) of P(N).

***0 – the corresponding natural number is not included in the element(set) of P(N)
1 – the corresponding natural number is included in the element(set) of P(N)

Function *f*(x) could be any function.

Now we can see there is no function *f*(x) pairing natural numbers with the power set of natural numbers.

Non-diagonal part

P(N) can "cover" N since it includes all of the singletons of N, thus, it contains an element of the form { 1 }, { 2 }, { 3 }, ... for each element 1, 2, 3, ... in N.

Consequently **|N| < |P(N)|**.

Analysis of diagonal argument in the context of Cantor's paradox

First thing I will analyze **usability of diagonal argument** for a rather weird but at the same time very simple task. This is to demonstrate an important aspect of the diagonal argument when applying it to some special cases.

I will try to use it **to investigate the possibility of a surjection from a replica (exact copy) of a set of sets to the same set of sets**.

Of course **it does not make any sense** to use the diagonal argument to investigate the possibility of a surjection from a set of sets to itself **if it is a "usual" set of sets** because, as a result of such diagonalization, we will get a set that **is obviously not included in this set of sets**. Please see below for better visualizing of this fact.

	A*	B	C	D	Elements (sets) of some arbitrary set of sets
A**	0***	0	1	0	<-set A

B	1	1	0	0	<-set B
C	1	0	1	0	<-set C
D	1	0	0	1	<-set D
	1	0	0	0	<-**not** a "proof" set

*column heads – set names to code presence\absence of the corresponding set in a paired set(here in itself).

**row heads – sets paired with themselves.

***0 – the corresponding set is not included in a paired set(here in itself)
1 – the corresponding natural number is included in a paired set(here in itself)

We constructed a set different form all the paired sets of but **it proves nothing** regarding the possibility of surjection from a replica (exact copy) of the set of sets to the same set of sets. It is **just another set,** not included in the set of sets whose possibility to be "surjected" we tried to investigate**. That is why we cannot apply a standard way of reasoning like, say, in the case of comparing N to P(N).**

Things may be not so obvious with an "unusual" set though.

Let us apply diagonal argument to investigate the possibility of surjection from the replica of the set of all sets(S replica) to the set of all sets itself.

	A	B	C	D	all possible sets …	Elements (sets) of the set of all sets
A	0	0	1	0	…	<-set A
B	1	1	0	0	…	<-set B
C	1	0	1	0	…	<-set C
D	1	0	0	1	…	<-set D
all possible sets …	…	…	…	…	…	all possible sets …
	1	0	0	0	…	<-**fake** "proof" set

Suppose we have a function *f*(x) that can provide us with surjection from the replica of the set of all sets to the set of all sets itself.

Then let us construct a "proof" set.

"proof" set ={x ∈ $S_{replica}$: x ∉ *f*(x)}

This means, by definition, that for all *x* in $S_{replica}$, *x* ∈ ["proof" set] if and only if *x* ∉ *f*(*x*).

On the picture above the "proof" set is a Russell's set actually. Though this is just a special case of all possible pairings, of course.

This is one of the key points of the whole paper:

we either have to consider the "proof" set as a real (existing) set and, thus, an element of the set of all sets

or

admit the fact that it is different from all possible sets and, thus, is not an element of the set of all sets, that is, does not exist.

Let us carefully analyze both options:

1. The "proof" set is a real (existing) set and, thus, an element of the set of all sets.

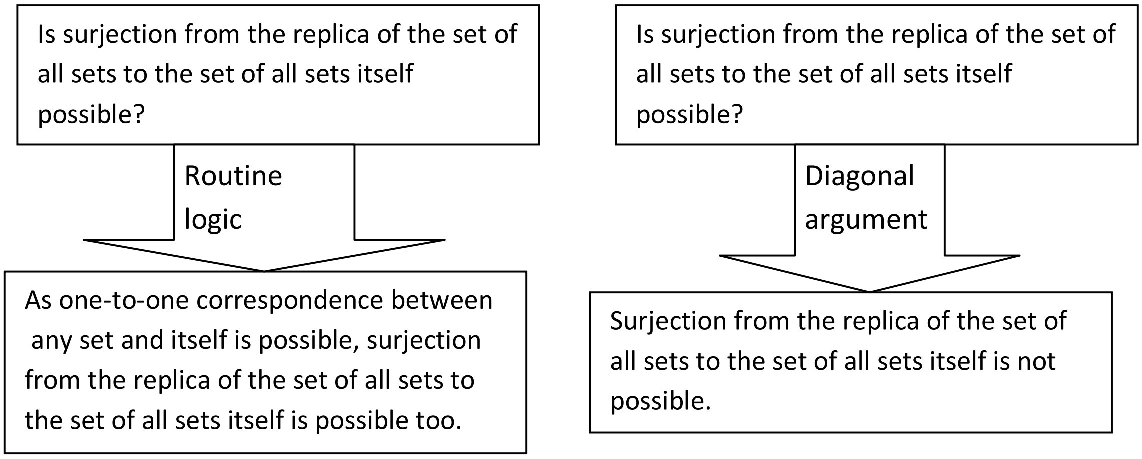
As I noted just above if we consider the "proof" set as a real existing set it consequently should be considered as an element of the set of all sets. This immediately leads to conclusion that we managed to prove there is no possible surjection from the replica of the set of all sets to the set of all sets itself (we constructed a set different from all paired sets though it belongs (**under option number one!**) to the set of all sets). **We did it with diagonal argument**.

I guess it is quite clear that reasoning like "if this 'proof' set is included in set of all sets then it is included in its replica" has nothing to do with what is being discussed here. For better understanding that this sort of reasoning is irrelevant here just remember, that after constructing the "proof" set in diagonal argument, we only state that this set is not paired, not that there is no element like it in the pairing set. If this does clear the case please remember that if someone says something like "this 'proof' set has a pairing copy in the replica of set of all sets" we can respond with making one more diagonal and so on.

By the way, even if you do not agree that choosing option number one implies proving there is no possible surjection from the replica of the set of all sets to the set of all sets itself (I think it does!), ***it only means that for you option one is no different from option two when considering the scope of the diagonal argument – for you*** *choosing any option of the two does not prove anything about possibility of surjection from the replica of the set of all sets to the set of all sets itself.*

We cannot blame the concept of the set of all sets for this weird contradictory result. If we use routine logic that any set of elements can be paired one-to-one (thus giving the surjection for sure) by its replica we can conclude that set of all sets can be paired by its replica too. **Thus it is not the concept of the set of all sets that caused this contradictory result.**

The diagram below will better demonstrate the point made above



It is the diagonal argument applied to this case that allowed us to make such a contradictory conclusion (absurdity naturally).

Please note that option number one is internally contradictory (contradictory within itself) – choosing it means that we consider the resulting "proof" set (which I prefer to call a fake "proof" set) to be both included in the set of all sets and, at the same time, different from all its elements (not from all other elements but from ALL its elements – this is important), thus not included in the set of all sets. Of course such a contradictory approach can only lead to the absurdity.

2. The "proof" set is different from all possible sets and, thus, is not an element of set of all sets, that is, does not exist.

In this case we **did not manage** to prove that surjection from the replica of the set of all sets to the set of all sets itself is not possible. I consider this option as the only correct one (**! please see part two of the paper for another way of reasoning with the same conclusion**) as the other one results in absurdity (due to the obvious internal contradiction mentioned above).

Let us generalize what we saw from the previous two examples (cross-through diagonalization of a usual set of sets and the set of all sets). Let us set a restriction on the usage of diagonal argument for investigating the possibility of surjection from one set to another:

Proposition1 . When investigating the possibility of surjection from one set to another, cases when it is known that *all* elements of a set, whose possibility to be surjected is under investigation, can be mapped with the elements of another set, whose possibility to surject is under investigation, are out of scope of diagonal argument.

Proof. When **all** elements of a set, whose possibility to be surjected is under investigation, can be mapped with the elements of another set, whose possibility to surject is under investigation, diagonalization crosses through **the whole** set (whose possibility to be surjected is under investigation), thus, **the intrinsic mechanism of diagonal argument fails**. We construct a set (a fake non-existent set, when diagonalization crosses through the set of all sets, is just an extreme case of Proposition1) that is not amongst paired elements but **it is not amongst ALL elements of the WHOLE surjected set either**.

!In part two of this paper the problem of non-existence of such fake sets like Russell's (using Russell's set itself as an example) will be thoroughly investigated in a rather unconventional way.

As you could see, it is when we do not pay attention to the fact that **the intrinsic mechanism of diagonal argument fails** when the whole set is mapped, or even **intentionally** consider some "set" to be both existent and not a member of the set of all sets (**thus, voluntary breaking the law of non-contradiction), ** we get the absurdity like "surjection from the replica of the set of all sets to the set of all sets itself is not possible".

The time has come to move on to **usability of diagonal argument for analyzing the possibility of a surjection from U to P(U).**

Obviously, if we take U for a set that contains all sets and all elements ("everything"), P(U) is the set of all sets.

As P(U) is a subset of U, for any element of P(U) there is an element of U thus surjection from U to P(U) is possible.

	A	B	C	D	all possible sets …	Elements (sets) of P(U) – the set of all sets
A, all elements(not sets)	0	0	1	0	…	<-set A
B	1	1	0	0	…	<-set B
C	1	0	1	0	…	<-set C
D	1	0	0	1	…	<-set D
all possible sets …	…	…	…	…	…	all possible sets …
	1	0	0	0	…	<-same **fake** "proof" set

For example, we can pair each element of P(U) with the same element of U ("first" element of P(U) can be mapped with the same element of U and all elements of U which are not sets).

The situation is very similar to the previous example (set of all sets against itself). In the context of studying the scope of diagonal argument the situation is no different at all. Same "**cross-through**" diagonalization. **Just a special case (one of extreme cases) of Proposition1**. The previous example (set of all sets against itself) was used only because it is ridiculously obvious. (I would call the previous example "**reductio ad amplius absurdum"** – which I consider a rather useful proof technique to set a restriction on this particular proof technique (diagonalization method as a mathematical object). "**Amplius**"(bigger) here means "less obscured", of course).

Just in case: constructing the above surjection (which is definitely not a bijection) to demonstrate that the case is out of scope of diagonal argument (whose final target is to prove that no bijection is possible) is not a contradictory approach. Just remember that for such cases to prove the possibility of bijection means to prove two sets can "cover" each other (see below the final conclusion). Thus, in such cases, by demonstrating that diagonal argument does not disprove the possibility of surjection we demonstrate it does not disprove the possibility of bijection either.

Final conclusion on the scope of Cantor's theorem

As clearly seen from the above investigation, cardinality comparison between U and P(U) is out of scope of diagonal part of Cantor's theorem.

As every element in P(U) has a corresponding element in U, injection from P(U) to U is possible. As every element in U has a corresponding element (singleton) in P(U), injection from U to P(U) is possible. That is cardinality comparison between U and P(U) is **under the scope of Cantor–Bernstein–Schroeder theorem**.

Their cardinalities must be equal.

Russell's paradox

Russell's paradox is based on two things: a naive approach to defining sets and a particular application of this naive approach.

According to this naive approach, for every ϕ (formula or statement), there should exist a set X such that, for all x, $x \in X$ if and only if ϕ(x) is true.

If so, let *R* be the set of all sets that are not members of themselves $R = \{x: x \notin x\}$. If it is a member of itself it is not a member of itself. If it is not a member of itself it is a member of itself. We encounter **contradictory equivalence** - proposition equivalent to its negation.

$x \in R \leftrightarrow x \notin x$. We instantiate x with R. As a result $R \in R \leftrightarrow R \notin R$.

Informal presentation: Suppose a librarian gets an order from her boss to make a catalog of all catalogs that don't list themselves. The librarian cannot include it in its own listing, because then it would not belong in the catalog. However, if the librarian leaves it out, the catalog will be incomplete. Either way, she cannot execute the order of her contradictive boss.

Again, before starting the analysis I will give the plan of it:

1. At first conventional aspects of the Russell's paradox will be given
2. Analysis of contradictory equivalence will be presented then

Conventional aspects of the Russell's paradox

There are basically three "paradoxical" aspects of the Russell's set

1. Contradictory equivalence (proposition equivalent to its negation)

The following two result from the first one:

2. Incompliance with the law of excluded middle
3. Incompliance with the law of non-contradiction (especially for those who consider the law of excluded middle as not applicable here)

Contradictory equivalence (proposition equivalent to its negation)

$R \in R \leftrightarrow R \notin R$

Incompliance with the law of excluded middle.

On the one hand R is a set. On the other R is a set of sets. So, it is either a member of itself or not.

Suppose R is a member of itself. If so, it is not a member of itself. We got a contradiction from supposing R is a member of itself. **Consequently "R is a member of itself" is wrong**.

Suppose R is not a member of itself. If so, it is a member of itself. We got a contradiction from supposing R is not a member of itself. **Consequently "R is not a member of itself" is wrong**.

Incompliance with the law of non-contradiction

From the definition of R, we have that $R \in R \leftrightarrow \neg(R \in R)$. Then $R \in R \rightarrow \neg(R \in R)$ (biconditional elimination). But also $R \in R \rightarrow R \in R$ (the law of identity), thus $R \in R \rightarrow (R \in R \land \neg(R \in R))$. Then, the law of non-contradiction tells us $\neg(R \in R \land \neg(R \in R))$. Thus, by modus tollens, **we can conclude $\neg(R \in R)$**. But since $R \in R \leftrightarrow \neg(R \in R)$, we have that $\neg(R \in R) \rightarrow R \in R$, and so **we can conclude $R \in R$** by modus ponens.

Analysis of contradictory equivalence

Before starting the analysis I have to mention **a very important logical difference between the informal presentation of Russell's paradox (the one mentioned above!) and Russell's paradox itself**.

When we talk about a librarian trying to complete the task **we do not state that she already did it** and we just need to guess whether she included the catalog of all catalogs that do not include themselves in it or she did not. We just state that she cannot execute the order of her boss. Though when we define the set R, **we state that R is formed**, it exists, and all we need is just to understand whether it includes itself or not.

Now let us move on to the analysis of the contradictory equivalence in Russell's paradox.

If R includes itself, R is not **that** $R = \{x: x \notin x\}$. What I mean is that **under the assumption "$R \in R$"** R includes a member that **is** included in itself (R itself is such a member). Thus under this assumption R is not a set of all sets that are not members of themselves - there is no logical ground to conclude that R (as an element) is not included in R (as a set). We could state that a set that includes itself is not included in R if R were like $\{x: x \notin x\}$ but **under the assumption "$R \in R$"** $R \neq \{x: x \notin x\}$ so, **under this assumption,** we cannot state that a particular set that includes itself is not included in R.

If R does not include itself, R is not **that** $R = \{x: x \notin x\}$ either. **Under the assumption "$R \notin R$"** R does not include a member that **is not** included in itself (R itself is such a member). Thus under this assumption R is not a set of **all** sets that are not members of themselves - there is no logical ground to conclude that R (as an element) is included in R (as a set). We could state that a set that does not include itself is included in R if R were like $\{x: x \notin x\}$ but **under the assumption "$R \notin R$"** $R \neq \{x: x \notin x\}$ so, **under this assumption,** we cannot state that a particular set that does not include itself is included in R.

Or, if you, for some reason, prefer seeing things through formulae:
$R = \{x: x \notin x\} \Rightarrow R \in R \leftrightarrow R \notin R$: under both assumptions ($R \in R$ or $R \notin R$) **the left part is not true** thus the right part cannot be derived from it.

If we use the following formula: $x \in R \leftrightarrow x \notin x \Rightarrow R \in R \leftrightarrow R \notin R$, we have to understand that, in spite of the fact that under both assumptions ($R \in R$ or $R \notin R$) the left part "becomes" (under instantiation) like $R \in R \leftrightarrow R \notin R$, ***this cannot be considered as "what we get" as this is a condition****. As the condition is contradictive the right part of the formula cannot be derived from it either. We can also consider $x \in R \leftrightarrow x \notin x$ the same way as $\{x: x \notin x\}$, just saying it is wrong under both assumptions.*

I guess it is obvious that questions of a sort like "if $R \neq \{x: x \notin x\}$ under these assumptions then what is it like under these assumptions?" are irrelevant here. The reasoning above is only on the absence of the logical ground for contradictory equivalence. As you will see from the below reasoning (proof of the next proposition) such questions are not relevant at all. ***Just in case: in spite of the fact that there must be such a set that includes all sets that are not members of themselves but not itself and such a set that includes all sets that are not members of themselves and itself – these two sets have nothing to do with our set R under the above assumptions.***

Please note that whatever words/phrases we use when reasoning on whether $R \in R$ or $R \notin R$ - "is included", "qualifies", "eligible", "should be included" or "must be included" etc. or their negations - we have to consider R as "formed" (of course not as $R = \{x: x \notin x\}$ but as $R \neq \{x: x \notin x\}$ – see above) and never as "forming" (**visualizing correspondingly in mind** (if visualizing anything but the letter "R" at all – it depends on the way of one's thought process): as "formed", not as "forming" – **this is important!**). That is for two reasons:

1. If we consider (**and visualize!**) R (as a set) as "forming" we have to consider R (as an element) the same way – not formed "yet" (as it is one and the same R). In this case (if we consider R as "forming") we do not have any logical ground to reason whether R (as an element) should be inside R (as a set) or not, because if R (as an element) is "forming" but not "formed", the value of its characteristic property is not defined then.

2. Though it might sound strange, considering/visualizing R as both formed (as an element) and forming (as a set) **at the same time,** is breaking the law of non-contradiction on the level of consideration/visualization. That is so because "formed" means here $R \neq \{x: x \notin x\}$ while "forming" means here $R = \{x: x \notin x\}$.

*"Visualizing" here corresponds to R (as a set) containing/not containing R (as an element), not to trying to imagine some whole weird fake set, of course (I am talking about **visualization of logic behind the reasoning** only – I guess it is clear). Please also note that it has nothing to do with visualizing diagonalization like on the pictures in the part one of this paper.*

By the way, for some words/phrases, like "qualifies" or "must be included", one and the same word can be used in the opposite meanings. One person may mean R is included in itself by "qualifies", another person may say it is not included but "qualifies" meaning its characteristic property (as an element, of course). It is no matter at all. The point is to consider/visualize R as "formed" (one way or another –included in itself or not).

Please also note that reasoning like "if it is included in itself it should not have been included in itself" is not only unjustified (as the opposite option – "not included in itself" also results in a set not like $\{x: x \notin x\}$) but also does not give us any contradiction – the way something is and the way something should have been are about different things.

What do we get from the above reasoning?

Proposition2. $x \in R \leftrightarrow x \notin x \Rightarrow R \in R \leftrightarrow R \notin R$ is an **unjustified (scholastic) instantiation**.

Proof. See the above reasoning.

Corollary. There is no **justified** contradictory equivalence in Russell's set problem.

Though what do we have instead?

1. $R = \{x: x \notin x\} \Rightarrow R \in R \vee R \notin R$.
2. $R \in R \Rightarrow R \neq \{x: x \notin x\}$.

3. R ∉ R ⇒ R ≠ {x: x∉x}.
4. From 2. and 3. R∈ R ∨ R ∉ R ⇒ R ≠ {x: x∉x}.
5. From 1. and 4. R = {x: x∉x} ⇒ R ≠ {x: x∉x}.

Obviously, R = {x: x∉x} ⇒ R ≠ {x: x∉x} cannot be considered as a sort of incompliance with the law of non-contradiction. If R = {x: x∉x} leads to contradiction then we can state it is wrong, thus the opposite - R ≠ {x: x∉x} is correct. Naive approach to defining sets is simply incorrect.

It is like, for example, with the proof that the square root of 2 is not rational. Suppose we take the statement "all square roots are rational" for an axiom. We derive "the square root of 2 is like n = a/b where a and b have no common factors" from this axiom. Then we get the well-known contradiction: "the square root of 2 is like n = a/b where a and b have no common factors" ⇒**…**⇒"the square root of 2 is not like n = a/b where a and b have no common factors". Of course we do not consider this as a sort of incompliance with the law of non-contradiction. Instead we just say "the square root of 2 is like n = a/b where a and b have no common factors" is a wrong statement, "the square root of 2 is not like n = a/b where a and b have no common factors" is correct, the axiom "all square roots are rational" is just wrong. No paradoxical thing with this.

I am sure it is clear, what the difference between the above reasoning on square root of 2 and reasoning on Russell's set is. Though just in case: reasoning on square root of 2 is about some property of the object, not about the existence of the object. That is ***in contrast to the reasoning on Russell's set in which defining it is logically equivalent to stating that it exists (no matter what philosophical approach is used: "a set is created when it is defined" or "we just describe a set that 'already' exists")****.*

No paradoxical thing with Russell's set either. Instead we have **an ordinary (without such a paradoxical equivalence like R ∈ R ↔ R ∉ R)** proof by contradiction that R = {x: x∉x} is wrong. As R is just an arbitrary identifier, we can be sure we proved that **Russell's set does not exist** (**Proposition3**).

Corollary. As Russell's set is just a special case of a "proof" set (which I prefer to name a fake "proof" set) when applying diagonal argument to cardinality comparison between U and P(U), we have **an additional evidence** (besides the fact that this "set" is defined as not a member of the set of all sets (**it differs from any possible set** – it differs from those which include themselves by not including them and vice versa – it differs from those which do not include themselves by including them) thus, defined with voluntary breaking the law of non-contradiction – something cannot be both a set and not a set - this aspect was discussed in the first part of the paper in detail) that this set is a fake non-existent set - **an extreme case of Proposition1**.

After investigating the scholastic mechanism of contradictory equivalence I would like to analyze it in the context of the laws of logic. Not the result of applying it to the Russell's set – that was discussed earlier (conventional aspects of the Russell's paradox), but the scholastic instantiation mechanism itself. Namely, in the context of **the law of identity.**

First thing we can notice is obvious from the proof of Proposition2 – when we reason like "let R = {x: x∉x} then R ∈ R ↔ R ∉ R" we mean different things by R**.** At first (when we state "let R = {x: x∉x}") we mean {x: x∉x} by R and then (when we state "if R is a member/not a member of itself") another R ≠ {x: x∉x}.

Within the proof of Proposition2 itself I mean different things by R too. I cannot help this happening as this is how our communication works – I have to use one and the same identifier R in the proof of Proposition2 (talking about R = {x: x∉x} and R ≠ {x: x∉x}) to let you know that R under the corresponding assumptions is not the "right" R (like I have in this very sentence).

There is a great difference between the usage of identifier R in the scholastic instantiation (R ∈ R ↔ R ∉ R) and in the proof of Proposition2 though: in the proof I state **explicitly** that R is not the "right" R under "is a member/ is not a member" assumptions. That is the **law of identity** is not broken in the proof of Proposition2 but is broken in the scholastic instantiation in which R ∈ R ⇒ R ∉ R and vice versa as if R were R = {x: x∉x} under such assumptions**.**

So, when reasoning on/considering/visualizing R we mainly have two options:

1. While understanding that R (as an element) and R (as a set) is one and the same R, to consider/visualize R (as an element) as "formed" but R (as a set) as "forming". Though this consideration/visualization algorithm comes from "common life" (in common life we put ripe ("formed") berries into a basket which is "forming" that way), when applied to R, **it becomes a conciseness bug** – under option 1. we consider/visualize R as both R = {x: x∉x}("forming") and R ≠ {x: x∉x}("formed") - **contradiction**. After we realize what happened we can see that **the law of identity** was broken that way.
2. Consider/visualize R as "formed" like R ≠ {x: x∉x} thus understanding that **the contradictive equivalence has no logical ground**.

Earlier we proved that there is no **justified** contradictory equivalence in Russell's set problem. Now let us look into the contradictory equivalence in a diagonal argument in its general form (diagonal part of Cantor's theorem). I will use the same identifiers as in part one of the paper.

The reasoning on the scholastic instantiation and its corollary "There is no justified contradictory equivalence in a diagonal argument" is quite similar.

There is a rather big difference though: the "proof" set of the diagonal argument no doubt exists (**if it is not a fake "proof" set like in comparison of U to P(U) –** a Russell's set is just a special case of such a fake set). The point is that instead of proving non-existence of the "proof" set itself, we prove the non-existence of such a "proof" set that is paired with some element x of A.

1. $B = \{x \in A: x \notin f(x)\} \wedge$ "B is the image of some x" $\Rightarrow x \in B \vee x \notin B$.
2. $x \in B \Rightarrow \neg$ ($B = \{ x \in A: x \notin f(x)\} \wedge$ "B is the image of some x").
3. $x \notin B \Rightarrow \neg$ ($B = \{ x \in A: x \notin f(x)\} \wedge$ "B is the image of some x").
4. From 2. and 3. $x \in B \vee x \notin B \Rightarrow \neg$ ($B = \{ x \in A: x \notin f(x)\} \wedge$ "B is the image of some x").
5. From 1. and 4. $B = \{x \in A: x \notin f(x)\}$ And "B is the image of some x"$\Rightarrow \neg$ ($B = \{ x \in A: x \notin f(x)\}$ $\wedge$ "B is the image of some x").

We proved non-existence of such a set constructed like $B = \{ x \in A: x \notin f(x)\}$ **and, at the same time, paired with some element x in A** (though, of course, it exists as a non-paired set in P(A), thus, proving the impossibility of the surjection from A to P(A)). This is equivalent to saying *"...set B cannot be amongst elements of P(A) paired with elements of A. Set B belongs to P (A) but not to the elements of P(A) paired with elements of A."* – see part one of the paper – "the intrinsic mechanism of the diagonal argument" section. **When we got rid of scholastic instantiation we got to the substantive meaning (intrinsic mechanism) of diagonal argument.** It is not some happenstance, of course. It demonstrates the role of scholastic instantiation – to obscure things (besides the role of being logically incorrect and widely accepted at the same time). It is actually scholastic.

Now let us summarize:

Possible sequences of events in diagonal argument for cardinality comparison:

After constructing set $B = \{x \in A: x \notin f(x)\}$ we have two options:

1) To state that we constructed a counterexample set to the assumption "A can 'cover' P(A)" (it differs from all **paired** elements of P(A)) and conclude that the assumption "A can 'cover' P(A)" is wrong (otherwise something could be both a paired set and not amongst all paired sets).

2) To state that, under the assumption "A can 'cover' P(A)", set B should be paired to some element of A. We have two options then:

   A. After breaking **the law of identity –** through **scholastic instantiation** – we get **a contradictory equivalence.** After that we state that our assumption "A can cover P(A)" is wrong.

   B. Instead of A. we prove that a set constructed like $B = \{ x \in A: x \notin f(x)\}$ **and paired with some element x in A** does not exist (though, of course, it exists as a non-paired set in P(A), thus, proving that our assumption "A can cover P(A)" is wrong).

Possible sequences of events in Russell's paradox problem:

After constructing a definition of a set like $R = \{x: x \notin x\}$ we have two options:

1) To state that we constructed a definition of a set that is a counterexample to a naive approach to defining sets (according to the definition, the set it defines **differs from any possible set** – it differs from those which include themselves by not including them and vice versa – it differs from those which do not include themselves by including them) and conclude that the naive approach to defining sets is wrong (otherwise something could be both a set and not amongst all possible sets).

2) To state that according to a naive approach to defining sets, $R = \{x: x \notin x\}$ is a real set of sets, and, so, it should be either a member of itself or not. We have two options then:

   A. After breaking **the law of identity –** through **scholastic instantiation** – we get **a contradictory equivalence** $R \in R \leftrightarrow R \notin R$**.** After that we have the following ways:

      1. State that our assumption "any meaningful formula can give us a set" is wrong (there is no such set like $R = \{x: x \notin x\}$).

      2. State that our assumption "any meaningful formula can give us a set" is contradictive (set $R = \{x: x \notin x\}$ is a contradictive set).

   B. Instead of A. we prove that the set defined by such a counterexample definition does not exist, thus, proving that the naive approach to defining sets is wrong.

Conclusions

Of course, using non-axiomatic approach to set theory means we encounter problems like Russell's or a Cantor's one and have to deal with these some way or another. What I am going to state next might seem controversial to you at first. I am absolutely sure that the possibility to run into paradoxical (or those that seem to be like that) problems is **not the minus of non-axiomatic approach but its great plus**. It is like a military range to study forms of thinking (erroneous forms of thinking in particular). The point I want to make here is that we can employ it this way only by careful investigations of the problems we encounter. We cannot use non-axiomatic approach like this though, if we choose the way to **only** prohibit everything that leads to paradoxes, without looking into them.

References:

1. Wikipedia.
2. Nationmaster encyclopedia.