A Fourier Integral approach to solving the Navier-Stokes equations on an unbounded space domain

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Abstract: The symbol $\hat{\vec{U}}$ denotes the velocity or momentum (the mass multiplied by the velocity). Transform the Navier Stokes momentum and density equations into continuous families of ordinary differential and linear equations for the classical Fourier coefficients. Prove theorems on existence, uniqueness and smoothness of solutions of the Navier Stokes equations. Interpret these results for solutions of the Navier Stokes partial differential equations using the Fourier integral representation $\vec{U} = \hat{\hat{\vec{U}}}, P = \hat{\hat{P}}$.

Key Words: Fluid Mechanics, incompressible Navier-Stokes equations

1. Introduction

The main result in this paper can be stated as follows. If the data is smooth, spatially Schwartz, and the body force and its higher order time derivatives satisfy generalized sector conditions, then the (space) average of the kinetic energy of $\vec{U}^{(k)}$ is bounded for all forward time. A unique physical solution $(\vec{U}, P)$ exists which is smooth in $t \geq 0$. The solution is the extension of the unique regular (jointly smooth) short time solution determined by the data. Solutions are unique, separately smooth and bounded in time and smooth in space for all forward time.

In 1934 Leray (in [9]) formulated the regularity problem and related it to the smoothness problem. In the year 2000, Fefferman formulated the problem (in [5]). In that same year, Bardos wrote a monograph on the problem ([3]) which summarized the then literature. The author interprets the remarks in [3] Bardos to indicate that the problem of regularity/smoothness can be solved as formulated in (A) of [5] Fefferman.

What is new?

As far as the author knows all frequency domain formulas but one which appear in this paper are new. Cannone uses the Fourier transform to rewrite the variation of constants formula solving the Navier-Stokes evolution equations in the formula just prior to (27) on page 15 of *Harmonic Analysis Tools for Solving the Incompressible Navier-Stokes Equations*.

The vector form of the Navier-Stokes equations for spatially Schwartz data is

$$\vec{U}_t + \vec{U}\cdot\nabla\vec{U} = \eta\Delta\vec{U} - \nabla P + \vec{F}, \eta > 0, \vec{x}\in R^3, t\geq 0$$
$$\vec{U}\cdot\nabla = 0, \vec{x}\in R^3, t\geq 0 \quad (1\text{-}1)$$
$$\vec{U}(0,\vec{x}) = \vec{U}_0(\vec{x}), \vec{x}\in R^3.$$

The author seeks a unique solution of (1-1) given the data $\vec{U}_0, \vec{F}$ which is jointly smooth in space and time and bounded for all forward time.

In (1-1) $\vec{U}$ is the velocity vector field and ∇P is the pressure gradient to be determined, $\nabla\cdot\vec{U}$ is the divergence of the vector field and $\vec{U}\cdot\nabla$ is the tensor matrix $D_{\vec{x}}\vec{U}$. Equation (1-1) is equivalent to the formulation in [7] (7)-(11) Fefferman.

The body force $\vec{F}$ is smooth on $[0,\infty)\times R^3$ $\vec{F}$ and the initial function $\vec{U}_0$ are Schwartz (smooth) on R^3.

Since $\vec{U}_t$ is the acceleration, the momentum equation (first equation of (1-1)) can also be interpreted as Newton's second law for incompressible fluids since the net force appears on the left hand side if both sides are multiplied by the mass *m*. In fact the momentum equation is Newton's second law of motion for fluids combined with a dynamic version of Archimedes' law of hydrostatics. If $\vec{U}(t,\vec{x}) = \vec{0}$, the equation reduces to Archimedes' law $\nabla P(t,\vec{x}) = \vec{F}(t,\vec{x})$.

The second equation of (1-1) is called the equation of continuity. It is the reduction of the more general Navier-Stokes equation for the density which follows from the incompressibility of the fluid.

In this paper the following equations for the Fourier transform of the velocity are equivalent to (1-1).

$$\frac{d\hat{\vec{U}}(t,\vec{\omega})}{dt} = -\eta\,|\,\vec{\omega}|^2\,\hat{\vec{U}}(t,\vec{\omega}) - i\int_{\Phi^3}\hat{\vec{U}}(t,\vec{\omega}-\vec{\phi})(\vec{\omega}-\vec{\phi})^t\hat{\vec{U}}(t,\vec{\phi})d\vec{\phi} - i\vec{\omega}\hat{P}(t,\vec{\omega}) + \hat{\vec{F}}(t,\vec{\omega}),$$
$$t\geq 0, \vec{\omega}\in\Omega^3$$
$$\hat{\vec{U}}(t,\vec{\omega})\cdot\vec{\omega} = 0, \vec{\omega}\in\Omega^3, t\geq 0 \quad (1\text{-}2)$$
$$\hat{\vec{U}}(0,\vec{\omega}) = \hat{\vec{U}}_0(\vec{\omega}), \vec{\omega}\in\Omega^3$$
$$\vec{\omega}^{\vec{s}}\hat{\vec{U}} \in\vec{L}^p(\Omega^3), p = 1,2,3,..,\infty, \vec{s}\in W^3, t\geq 0.$$

In (1-2) the Fourier transforms of the velocity, the initial velocity, the pressure and the body force are denoted by $\hat{\vec{U}}, \hat{\vec{U}}_0, \hat{\vec{F}}, \hat{P}$. Also $i = \sqrt{-1}$ and W^3 is the set of whole number triples.

The law governing the average mechanical energy of an incompressible fluid

Theorem 2-4 establishes the existence of a unique smooth solution defined for all forward time. The following formulas provide a smooth generalization of Leray's mechanical energy law ([12] section 17 formula 3.4) for the Navier-Stokes equation with non zero body force.

$$\frac{1}{2}\int_{R^3}|\vec{U}^{(k)}|^2 d\vec{y}-\frac{1}{2}\int_{R^3}|\vec{U}_0^{(k)}|^2 d\vec{y}=-\eta\int_0^t\int_{R^3}|\nabla\vec{U}^{(k)}|^2 d\vec{y}ds+\int_0^t\int_{R^3}\vec{U}^{(k)}\cdot\vec{F}^{(k)}d\vec{y}ds,$$
$$\eta>0,t\geq 0,k=0,1,2,... \qquad (1\text{-}3)$$

Equations (1-3) state that the difference of the space average of the kinetic energy (at time $t>0$) minus that at time $t=0^+$ is equal to the viscosity times the potential energy minus the average work done by the body force acting on the incompressible fluid where $|\nabla\vec{U}|^2=\nabla u\cdot\nabla u+\nabla v\cdot\nabla v+\nabla w\cdot\nabla w,\vec{U}^t=(u,v,w)$. The energy formulas (1-3) are equivalent to the formulas

$$\frac{1}{(2\pi)^3}\int_{\Omega^3}|\hat{\vec{U}}^{(k)}|^2(t,\vec{\omega})d\vec{\omega}-\frac{1}{(2\pi)^3}\int_{\Omega^3}|\hat{\vec{U}}^{(k)}|^2(0,\vec{\omega})d\vec{\omega}=$$
$$-\eta\frac{1}{(2\pi)^3}\int_0^t\int_{\Omega^3}|\vec{\omega}|^2|\hat{\vec{U}}^{(k)}(s,\vec{\omega})|^2 d\vec{\omega}ds+\frac{1}{(2\pi)^3}\int_0^t\int_{\Omega^3}\hat{\vec{U}}^{(k)}(s,\vec{\omega})\cdot\hat{\vec{F}}^{(k)}(s,\vec{\omega})d\vec{\omega}ds,k=0,1,2,... \qquad (1\text{-}4)$$

established in theorem 2-2.

By Parseval's theorem, the quantities on the left side of (1-3) and (1-4) are both equal to the average kinetic energy of $\vec{U}^{(k)}$.

The problem of finite time blow up

In theorem 2-3 the author shows that finite time blow up of solutions of (1-1) is impossible given the conditions on the data, the equation of continuity, and the following conditions on the forcing function

$$\int_0^t\int_{R^3}\vec{U}^{(k)}\cdot\vec{F}^{(k)}d\vec{x}ds<\eta\int_0^t\int_{R^3}|\nabla\vec{U}^{(k)}|^2 d\vec{x}ds,t\geq 0,\eta>0,k=0,1,2,... \qquad (1\text{-}5)$$

In the frequency domain, inequalities (1-5) take the form

$$\int_0^t\int_{\Omega^3}\hat{\vec{U}}^{(k)}(s,\vec{\omega})\cdot\hat{\vec{F}}^{(k)}(s,\vec{\omega})d\vec{\omega}ds<\eta\int_0^t\int_{\Omega^3}|\vec{\omega}|^2|\hat{\vec{U}}^{(k)}|^2\ d\vec{\omega}ds,t\geq 0,\eta>0,k=0,1,2,... \qquad (1\text{-}6)$$

These conditions extend Leray's result of 1934 in [9] that a solution continuous in the time variable and weakly differentiable in the space variables exists for all forward time when $\vec{F} = \vec{0}$.

Under the conditions (1-3) on $\vec{F}$ the solutions of (1-1) are bounded. Absolute stability/boundedness extends the concept of Lyapunov stability/boundedness from homogeneous nonlinear systems to nonlinear systems with a forcing function.

2. Existence and Extension

The notation $\Omega^3 = (-\infty,\infty)^3$ denotes the set of all continuous frequency triples; W denotes the set of whole numbers.

Let $\vec{S}(R^3)$ denote the space of spatially Schwartz functions. Here the initial function is in $\vec{S}(R^3)$ i.e.

$$\sup_{\vec{x}\in R^3} |x^{p(1)}y^{p(2)}z^{p(3)}|\, D^k\vec{U}_0 | \le M_k, k = 0,1,2,..., \vec{p} \in W^3, \vec{x} = (x,y,z).$$

The notation

$$\hat{\vec{U}} \in \{\vec{C}^\infty([0,T]), \vec{x} \in R^3\} \cap \{\vec{S}(R^3), t \ge 0\}$$

denotes the space of functions which are separately smooth in time and space and Schwartz over space. Thus $\vec{U}$ is separately smooth on $(t,\vec{x}) \in [0,T]\times R^3$ if it is smooth as a function of t for each fixed $\vec{x} \in R^3$ and smooth as a function of $\vec{x}$ for each fixed $t \in [0,T]$.

The notation

$$\hat{\vec{U}} \in \{\vec{C}^\infty([0,T]), \vec{\omega} \in \Omega^3\} \cap \{\hat{\vec{S}}(\Omega^3), t \ge 0\}$$

used for functions of time and frequency in the Fourier transform domain is analogous.

Lemma 2-1. The Navier-Stokes ordinary differential equations for the Fourier transform of the solution of the spatially Schwartz problem (1-1b) are

$$\begin{aligned}
&\frac{d\hat{\vec{U}}(t,\vec{\omega})}{dt} = -\eta\,|\vec{\omega}|^2\,\hat{\vec{U}}(t,\vec{\omega}) - i\int_{\Phi^3}\hat{\vec{U}}(t,\vec{\omega}-\vec{\phi})\vec{\phi}\,'\hat{\vec{U}}(t,\vec{\phi})d\vec{\phi} - i\vec{\omega}\hat{P}(t,\vec{\omega}) + \hat{\vec{F}}(t,\vec{\omega}),\\
&t \ge 0, \vec{\omega} \in \Omega^3 \qquad\qquad (2\text{-}1)\\
&\vec{U}(t,\vec{\omega})\cdot\vec{\omega} = 0, \vec{\omega} \in \Omega^3, t \ge 0\\
&\hat{\vec{U}}(0,\vec{\omega}) = \hat{\vec{U}}_0(\vec{\omega}), \vec{\omega} \in \Omega^3
\end{aligned}$$

where the Fourier transform is defined by

$$\hat{\vec{U}}(t,\vec{\omega}) = \Im\{\vec{U}(t,\vec{x})\} = \frac{1}{(2\pi)^3}\int_{R^3}\vec{U}(t,x)e^{-i\vec{\omega}\cdot\vec{x}}d\vec{x}, \vec{\omega}\in\Omega^3, t\geq 0 \tag{2-2}$$

and similarly for $\hat{\vec{F}}(t,\vec{\omega}), \hat{P}(t,\vec{\omega})$.

PROOF

Note that the triple integral over R^3 in the definition of the Fourier transform (2-2) is well defined for all forward time since, by the Schwartz property in $\vec{x}$, $\vec{F},\vec{U}_0$ are continuous (in fact smooth) and integrable over $\vec{x}\in R^3$ and continuous and bounded on $t\in[0,\infty)$. Lemma 2-3 shows that the Fourier transform of the pressure, calculated below, is likewise well defined.

The terms $\frac{d\hat{\vec{U}}}{dt}, \vec{\omega}\cdot\hat{P}, \hat{\vec{F}}$ follow directly by the application of the Fourier transform to the terms of (1-1) and the differentiation property of Fourier transforms applied to the first partials in the case of the pressure gradient term.

The transform of the Laplacian term can be calculated integrating by parts twice. Since the three calculations are identical, it suffices to calculate the transform of the second partial derivative with respect to the first component of the vector of space variables x

$$\begin{aligned}\int_{R^3}\vec{U}_{xx}e^{-i\vec{\omega}\cdot\vec{x}}dxdydz &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\vec{U}_x e^{-i\omega(1)x}\,|_{x=-\infty}^{\infty}\; e^{-i[\omega(2)y+\omega(3)z]}dydz + \\ \omega_1\int_{R^3}\vec{U}_x e^{-i\vec{\omega}\cdot\vec{x}}d\vec{x} &= \omega_1^2\int_{R^3}\vec{U}e^{-i\vec{\omega}\cdot\vec{x}}d\vec{x} = -\omega_1^2\hat{\vec{U}}(t,\vec{\omega}), t\geq 0.\end{aligned} \tag{2-3}$$

It follows that

$$\Im\{\eta\Delta\vec{U}\} = -\eta\,|\,\vec{\omega}\,|^2\,\hat{\vec{U}}. \tag{2-4}$$

The Fourier transform of the Euler term is

$$\begin{aligned}\Im\{\vec{U}\cdot\nabla\vec{U}\} &= \int_{R^3}\vec{U}\cdot\nabla\vec{U}e^{-i\vec{\omega}\cdot\vec{x}}d\vec{x} = i[\hat{\vec{U}}(t,\vec{\omega})\vec{\omega}^t]*\hat{\vec{U}}(t,\vec{\omega}) = i\int_{\Phi^3}\hat{\vec{U}}(t,\vec{\omega}-\vec{\phi})(\vec{\omega}-\vec{\phi})^t\hat{\vec{U}}(t,\vec{\phi})d\vec{\phi} \\ &= i\int_{\Phi^3}\hat{\vec{U}}(t,\vec{\phi})\vec{\phi}^t\hat{\vec{U}}(t,\vec{\omega}-\vec{\phi})d\vec{\phi}\end{aligned} \tag{2-5}$$

The transform of the pressure gradient term is

$$\int_{R^3}\nabla Pe^{-i\vec{\omega}\cdot\vec{x}}d\vec{x} = i\vec{\omega}\hat{P}. \tag{2-6}$$

The transform of the equation of continuity is

$$\int_{R^3} (U_x + V_y + W_z) e^{-i\vec{\omega}\cdot\vec{x}} d\vec{x} = i[\omega_1 U + \omega_2 V + \omega_3 W] = i\vec{\omega}\cdot\vec{U} = 0, t \geq 0, \vec{\omega} \in \Omega^3. \qquad (2\text{-}7)$$

Divide both sides of (2-7) by i to obtain the third line of (2-1).

Since $\hat{\vec{U}} \in \hat{\vec{S}}(\Omega^3) \Rightarrow \hat{\vec{U}} \in \hat{\vec{L}}^p(\Omega^3), p = 1,2,3,...,\infty$ the Schwartz conditions of (2-1a) permit the Fourier integrals of $\vec{U}, P, \vec{F}$ and the terms in (2-1) involving them to be well defined.

END PROOF

Remark 2-1. Equations (2-1) specify an infinite family of ordinary differential equations with a continuous vector parameter whose solutions are the time dependent Fourier transforms of $\vec{U}(t,\vec{x}), \vec{F}(t,\vec{x}), \vec{U}_0(\vec{x}), P(t,\vec{x})$.

Definition 2-1. $L_{\vec{\omega}} \hat{\vec{F}}(t,\vec{\omega}) = \vec{\omega} \cdot \hat{\vec{F}}(t,\vec{\omega}), \vec{\omega} \in \Omega^3, t \geq 0 \qquad (2\text{-}8)$

Remark 2-2. For any $\vec{\omega}, L_{\vec{\omega}} : \vec{C}^\infty[0,\infty) \to C^\infty[0,\infty)$. The linear operator $L_{\vec{\omega}}$ is a map from vectors of infinitely continuous differentiable functions on $[0,\infty)$ to a scalar continuous function on $[0,\infty)$.

Proposition 2-1. Any derivative of finite order of the velocity $\frac{d^k \hat{\vec{U}}}{dt^k}, k = 0,1,2,...$ is in $\eta(L_{\vec{\omega}})$.

PROOF

By (2-1) the assertion holds for the Fourier transform of the equation of continuity $k = 0$. Take the derivatives of order $k = 1,2,3,...$ with respect to t of both sides of

$$\hat{\vec{U}}(t,\vec{\omega}) \cdot \vec{\omega} = 0, \vec{\omega} \in \Omega^3, t \geq 0 \qquad (2\text{-}9)$$

to complete the proof.

END PROOF

Since differentiation of a function with respect to the space variables corresponds to frequency multiplication of its transform, the following Banach space is the most economical one needed to establish that solutions of the Navier-Stokes equations are smooth in $(t,\vec{x}) \in [0,\infty) \times R^3$.

Definition 2-2. The Schwartz space $\hat{\vec{S}}$ is the set of all three component vectors of functions which are Schwartz in the frequency variable on $\vec{\omega} \in \Omega^3$. The solution space for the Navier-Stokes equations consists of functions which are smooth and bounded for all forward time, uniformly in $\vec{\omega} \in \Omega^3$ and Schwartz in the frequency for all $t \geq 0$. Any finite order mixed partial derivative with respect to frequency is bounded in the L^p norm for any $p = 1,2,...,\infty$ with respect to monomial frequency weights of any finite order on the space $\hat{\vec{S}}(\Omega^3)$ **.**

The following lemma shows that the family of differential equations simplifies to a linear ordinary vector differential equation at $\vec{\omega} = \vec{0}$.

Lemma 2-2. If $\vec{\omega} = \vec{0}$ the Navier-Stokes ordinary equations for the Fourier transform of the velocity reduce to

$$\frac{d\hat{\vec{U}}}{dt}(t,\vec{0}) = \hat{\vec{F}}(t,\vec{0}), \hat{\vec{U}}(0,\vec{0}) = \hat{\vec{U}}_0(\vec{0}) . \tag{2-10}$$

PROOF

Formula (2-10) follows immediately from equation (2-1).

END PROOF

The Fourier transform function is an auxiliary function for a family of equations which involves it provided it does not appear in an equivalent form. The following lemma establishes that the Fourier transform of the pressure function is an auxiliary function for the family of (ordinary differential) equations (2-1). A closed formula for the Fourier transform of the pressure is supplied.

Lemma 2-3. Equations (2-1) can be placed into the following equivalent form

$$\begin{aligned}
&\hat{\vec{U}}_t = -\eta \,|\, \vec{\omega} \,|^2\, \hat{\vec{U}} + \left[\frac{\vec{\omega}\vec{\omega}^t}{|\,\vec{\omega}\,|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] [i(\hat{\vec{U}}\vec{\omega}^t) * \hat{\vec{U}} - \hat{\vec{F}}], t \geq 0, \vec{\omega} \in \Omega^3, \vec{\omega} \neq \vec{0} \\
&\hat{\vec{U}}(0,\vec{\omega}) = \hat{\vec{U}}_0(\vec{\omega}), \vec{\omega} \in \Omega^3 \\
&\hat{\vec{U}} \in \hat{\vec{S}}([0,T) \times \Omega^3), T > 0.
\end{aligned} \tag{2-11}$$

The pressure coefficients satisfy the following equations

$$\begin{aligned}
&\hat{P}(t,\vec{\omega}) = \frac{1}{|\vec{\omega}|^2} \{-\vec{\omega} \cdot i(\hat{\vec{U}}(t,\vec{\omega})\vec{\omega}^t) * \hat{\vec{U}} + \vec{\omega} \cdot \hat{\vec{F}}(t,\vec{\omega})\}, t \geq 0, \vec{\omega} \in \Omega^3, \vec{\omega} \neq \vec{0} \\
&\hat{P}(0^+,\vec{\omega}) = \frac{1}{|\vec{\omega}|^2} \{-\vec{\omega} \cdot i(\hat{\vec{U}}_0(\vec{\omega})\vec{\omega}^t) * \hat{\vec{U}}_0(\vec{\omega}) + \vec{\omega} \cdot \hat{\vec{F}}(0^+,\vec{\omega}\}, \vec{\omega} \in \Omega^3, \vec{\omega} \neq \vec{0}.
\end{aligned} \tag{2-12}$$

PROOF

Apply the linear operator $L_{\vec{\omega}}$ of definition 2-1 to each side of equation (2-1) – where, by 2-5, $(\hat{\hat{U}}\vec{\omega}^t) * \hat{\hat{U}} = \int_{\Phi^3} (\hat{\hat{U}}(t,\vec{\omega}-\vec{\phi})(\vec{\omega}-\vec{\phi})^t \hat{\hat{U}}(t,\vec{\phi})d\vec{\phi}$ -by forming the dot product of each term with the vector $\vec{\omega} \in \Omega^3$ to obtain

$$\vec{\omega} \cdot \frac{d\hat{\hat{U}}(t,\vec{\omega})}{dt} = -\eta\vec{\omega} \cdot |\vec{\omega}|^2 \hat{\hat{U}}(t,\vec{\omega})$$
$$-\vec{\omega} \cdot \int_{\Phi^3} i\hat{\hat{U}}(t,\vec{\omega}-\vec{\phi})(\vec{\omega}-\vec{\phi})^t \hat{\hat{U}}(t,\vec{\phi}d\vec{\phi} - \vec{\omega}\cdot\vec{\omega}i\hat{P}(t,\vec{\omega}) + \vec{\omega}\cdot\hat{\hat{F}}(t,\vec{\omega}), \vec{\omega} \in \Omega^3, t \geq 0. \quad (2\text{-}13)$$

By proposition 2-1, $\hat{\hat{U}}(t,\vec{\omega}), \hat{\hat{U}}_t(t,\vec{\omega}) \in \eta(L_{\vec{\omega}}), \vec{\omega} \in \Omega^3$ hence (2-13) reduces to

$$0 = -\vec{\omega} \cdot \int_{\Phi^3} i\hat{\hat{U}}(t,\vec{\omega}-\vec{\phi})(\vec{\omega}-\vec{\phi})^t \hat{\hat{U}}(t,\vec{\phi})d\vec{\phi} - \vec{\omega}\cdot\vec{\omega}i\hat{P}(t,\vec{\omega}) + \vec{\omega}\cdot\hat{\hat{F}}(t,\vec{\omega}), \vec{\omega} \in \Omega^3, t \geq 0. \quad (2\text{-}14)$$

Solve (2-14) for $\hat{P}(t,\vec{\omega})$ to obtain the first line of the pressure coefficient formulas. Insert the first line of the pressure formula into (2-1) to obtain the first line of the velocity coefficient formulas. The second line of the pressure transform formulas are obtained from the first line from the initial transform $\hat{\hat{U}}(t,\vec{\omega})$.

The only remaining question is to show that $\hat{P}(t,\vec{\omega})$ is well defined. Equivalently, is it bounded for $\vec{\omega} \neq 0$?

By the Schwartz property in the frequency parameters,

$$\exists M > 0 : \max|\hat{\hat{F}}(t,\vec{\omega}\}|, |\hat{\hat{F}}\vec{\omega}^t| \leq M. \quad (2\text{-}15)$$

Thus

$$\sup_{\vec{\omega}\in\Omega^3} \left|\frac{\vec{\omega}\cdot\hat{\hat{F}}(t,\vec{\omega}\}}{|\vec{\omega}|^2}\right| = \sup_{\vec{\omega}\in\Omega^3} \frac{|\hat{\hat{F}}(t,\vec{\omega}\}|}{|\vec{\omega}|} \leq$$
$$\max\{\sup_{\vec{\omega}\in\Omega^3}|\hat{\hat{F}}|, \sup_{\vec{\omega}\in\Omega^3}\frac{|\hat{\hat{F}}\vec{\omega}^t|}{|\vec{\omega}|}\} = \sup_{\vec{\omega}\in\Omega^3}|\hat{\hat{F}}|, t \geq 0. \quad (2\text{-}16)$$

Since the convolution in the numerator is of quadratic order in $\vec{\omega}$,

$$\frac{\vec{\omega}}{|\vec{\omega}|^2} \cdot \int_{\Phi^3} \hat{\hat{U}}(t,\vec{\omega}-\vec{\phi})(\vec{\omega}-\vec{\phi})^t \hat{\hat{U}}(t,\vec{\phi})d\vec{\phi} < \infty, t \geq 0 \quad . \quad (2\text{-}17)$$

Thus, the definition of $\hat{P}(t,\vec{\omega}), \vec{\omega} \neq 0$ can be smoothly extended to $\hat{P}(t,\vec{\omega}), \vec{\omega} = 0, t \geq 0$ by calculating

$$\lim_{\vec{\omega}\to\vec{0}} \hat{P}(t,\vec{\omega}), t \geq 0.. \tag{2-18}$$

END PROOF

Remark 2-3. From higher order derivatives of (2-1) and the projection of the k[th] order equation of continuity, one can calculate formulas for $\hat{P}^{(k)}(t,\vec{\omega})$.

The equation that results from calculating any higher order derivative of (2-13a) is

$$\hat{\vec{U}}^{(k+1)} = -\eta\,|\,\vec{\omega}\,|^2\,\hat{\vec{U}}^{(k)} + \left[\frac{\vec{\omega}\vec{\omega}^t}{|\,\vec{\omega}\,|^2} - \begin{pmatrix}1\,0\,0\\0\,1\,0\\0\,0\,1\end{pmatrix}\right][(i(\hat{\vec{U}}\vec{\omega}^t) * \hat{\vec{U}})^{(k)} - \hat{\vec{F}}^{(k)}],$$

$$t \geq 0, \vec{\omega} \in \Omega^3, k \in N \tag{2-19a}$$

$$\hat{\vec{U}}^{(k)}(0,\vec{\omega}) = \vec{0}, \vec{\omega} \in \Omega^3, k \in N$$

$$\hat{\vec{U}}^{(k)}(t,\vec{\omega}) \cdot \vec{\omega} = 0, t \geq 0, \vec{\omega} \in \Omega^3, k \in N.$$

Remark 2-4. For coefficients in discrete Schwartz spaces the integer weights are unrestricted. Hence any sum of squares can be exceeded by a product which is a single square. The result is that the Schwartz norm has two equivalent formulations

$$\sup_{\vec{p}\in W^3} |\,\omega_1^{p(1)}\omega_2^{p(2)}\omega_3^{p(3)}\,||\,\hat{\vec{U}}_0\,| = \sup_{p\in W} |\,\vec{\omega}\,|^p |\,\hat{\vec{U}}_0\,|. \tag{2-19b}$$

The first theorem establishes the existence of unique solutions for sufficiently short forward time starting at time $t = 0^+$. The following upper bound on the matrix operator of (2-19) is useful in the proof of the first theorem

$$\left[\frac{\vec{\omega}\vec{\omega}^t}{|\,\vec{\omega}\,|^2} - \begin{pmatrix}1\,0\,0\\0\,1\,0\\0\,0\,1\end{pmatrix}\right] \leq$$

$$\frac{1}{|\,\vec{\omega}\,|^2}\left|\begin{pmatrix}\omega_1^2 & \omega_1\omega_2 & \omega_1\omega_3\\ \omega_1\omega_2 & \omega_2^2 & \omega_2\omega_3\\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2\end{pmatrix} - \begin{pmatrix}1\,0\,0\\0\,1\,0\\0\,0\,1\end{pmatrix}\right| < \left|\begin{pmatrix}\frac{\omega_1^2 - |\,\vec{\omega}\,|^2}{|\,\vec{\omega}\,|^2} & 0 & 0\\ 0 & \frac{\omega_2^2 - |\,\vec{\omega}\,|^2}{|\,\vec{\omega}\,|^2} & 0\\ 0 & 0 & \frac{\omega_3^2 - |\,\vec{\omega}\,|^2}{|\,\vec{\omega}\,|^2}\end{pmatrix}\right| < \frac{3\,|\,\vec{\omega}\,|^2 - |\,\vec{\omega}\,|^2}{|\,\vec{\omega}\,|^2} = 2. \tag{2-20}$$

Theorem 2-1. Suppose

$$\hat{\vec{F}}, \hat{\vec{U}} \in \{\vec{C}^{\infty}([0,T]), \vec{x} \in R^3\} \cap \{\hat{\vec{S}}(\Omega^3), t \geq 0\}$$
$$\hat{\vec{U}}_0 \in \hat{\vec{S}}(\Omega^3). \tag{2-21}$$

There exists $T > 0$ such that $\hat{\vec{U}} \in \vec{C}^{\infty}([0,T)) \cap \hat{\vec{S}}(\Omega^3)$, satisfies (2-11) and $\hat{P} \in C^{\infty}([0,T)) \cap \hat{\vec{S}}(\Omega^3)$.

PROOF

The goal is to establish

a. Continuity of $\hat{\vec{U}}^{(k)}$ in time

$$\exists T > 0 : |\hat{\vec{U}}^{(k)}(t_2) - \hat{\vec{U}}^{(k)}(t_1)| < M(k,\vec{\omega})|t_2 - t_1| < \infty, \forall t_1, t_2 \in [0,T], M$$
$$= \sup_{0 \leq t \leq T} \hat{\vec{U}}^{(k+1)}(t). \tag{2-22}$$

b. The continuity of $\hat{\vec{U}}^{(k)}$ in time is uniform in $k \in W$

$$\exists T > 0 : |\hat{\vec{U}}^{(k)}(t_2) - \hat{\vec{U}}^{(k)}(t_1)| < M(\vec{\omega})|t_2 - t_1| < \infty,$$
$$\forall t_1, t_2 \in [0,T], \widetilde{M} = \sup_{k \in W} \sup_{0 \leq t \leq T} \hat{\vec{U}}^{(k+1)}(t). \tag{2-23}$$

c. The continuity of $\hat{\vec{U}}^{(k)}$ is uniform in $\vec{\omega} \in \Omega^3$

$$\exists T > 0 : |\hat{\vec{U}}^{(k)}(t_2) - \hat{\vec{U}}^{(k)}(t_1)| < M|t_2 - t_1| < \infty, \forall t_1, t_2 \in [0,T], \tilde{\widetilde{M}} =$$
$$\sup_{\vec{\omega} \in \Omega^3} \sup_{k \in W} \sup_{0 \leq t \leq T} \hat{\vec{U}}^{(k+1)}(t). \tag{2-24}$$

Use the variation of constants formula to solve for the general Fourier transform of the momentum

$$\hat{\vec{U}}(t,\vec{\omega}) =$$
$$e^{-\eta|\vec{\omega}|^2 t}\hat{\vec{U}}_0 + \int_0^t e^{-\eta|\vec{\omega}|^2(t-s)} \left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] [(i(\hat{\vec{U}}\vec{\omega}^t) * \hat{\vec{U}}) - \hat{\vec{F}}]ds, t \geq 0, \vec{\omega} \in \Omega^3. \tag{2-25}$$

Form the difference to establish continuity of the $\hat{\vec{U}}$ in time

$$\hat{\hat{U}}(t_2,\vec{\omega})-\hat{\hat{U}}(t_1,\vec{\omega})=e^{-\eta|\vec{\omega}|^2 t(2)}\hat{\hat{U}}_0-e^{-\eta|\vec{\omega}|^2 t(1)}\hat{\hat{U}}_0+$$
$$\int_0^{t(2)} e^{-\eta|\vec{\omega}|^2(t(2)-s)}\left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2}-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right][(i(\hat{\hat{U}}\vec{\omega}^t)*\hat{\hat{U}})-\hat{\hat{F}}]ds- \tag{2-26}$$
$$\int_0^{t(1)} e^{-\eta|\vec{\omega}|^2(t(1)-s)}\left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2}-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right][(i(\hat{\hat{U}}\vec{\omega}^t)*\hat{\hat{U}})-\hat{\hat{F}}]ds$$
$$t_2>t_1\ge 0,\vec{\omega}\in\Omega^3$$

Simplify the expression on the right side of (2-26)

$$\hat{\hat{U}}(t_2,\vec{\omega})-\hat{\hat{U}}(t_1,\vec{\omega})=$$
$$\{e^{-\eta|\vec{\omega}|^2 t(2)}-e^{-\eta|\vec{\omega}|^2 t(1)}\}\hat{\hat{U}}_0+$$
$$\int_{t(1)}^{t(2)} e^{-\eta|\vec{\omega}|^2(t(2)-s)}\left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2}-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right][(i(\hat{\hat{U}}\vec{\omega}^t)*\hat{\hat{U}})-\hat{\hat{F}}]ds- \tag{2-27}$$
$$\int_0^{t(1)} [e^{-\eta|\vec{\omega}|^2(t(2)-s)}-e^{-\eta|\vec{\omega}|^2(t(1)-s)}]\left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2}-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right][(i(\hat{\hat{U}}\vec{\omega}^t)*\hat{\hat{U}})-\hat{\hat{F}}]ds$$
$$t_2>t_1\ge 0,\vec{\omega}\in\Omega^3.$$

Examine continuity near time 0 by setting $t_2=t,t_1=0$ in (2-27)

$$\hat{\hat{U}}(t,\vec{\omega})-\hat{\hat{U}}(0,\vec{\omega})=\{e^{-\eta|\vec{\omega}|^2 t}-1\}\hat{\hat{U}}_0+$$
$$\int_0^{t} e^{-\eta|\vec{\omega}|^2(t(2)-s)}\left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2}-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right][((\hat{\hat{U}}\vec{\omega}^t)*\hat{\hat{U}})-\hat{\hat{F}}]ds- \tag{2-28}$$
$$t\ge 0,\vec{\omega}\in\Omega^3.$$

By (2-20)

$$\int_0^t e^{-\eta|\vec{\omega}|^2(t-s)} \left| \left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] \right| |((\hat{\vec{U}}\vec{\omega}^t) * \hat{\vec{U}})| \, ds \le$$

$$2 \sup_{\vec{\omega}\in\Omega^3} \sup_{s\ge 0} |((\hat{\vec{U}}\vec{\omega}^t) * \hat{\vec{U}})\vec{\omega}| \left| \int_0^t e^{-\eta|\vec{\omega}|^2(t-s)} ds \right| = \tag{2-29}$$

$$2 \sup_{\vec{\omega}\in\Omega^3} \sup_{s\ge 0} |(\hat{\vec{U}} * \hat{\vec{U}})| \, |\vec{\omega}|^2 \, \frac{1}{\eta |\vec{\omega}|^2} \le \frac{2M_1}{\eta}, t \ge 0$$

Also

$$\sup_{\vec{\omega}\in\Omega^3} \int_0^t e^{-\eta|\vec{\omega}|^2(t-s)} \left| \left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] \right| |\hat{\vec{F}}| \, ds$$

$$2 \sup_{\vec{\omega}\in\Omega^3} \int_0^t e^{-\eta|\vec{\omega}|^2(t-s)} \sup_{s\ge 0} |\hat{\vec{F}}| |\vec{\omega}|^2 ds \le \sup_{\vec{\omega}\in\Omega^3} \sup_{s\ge 0} |\hat{\vec{F}}| |\vec{\omega}|^2 \left| \int_0^t e^{-\eta|\vec{\omega}|^2(t-s)} ds \right| \tag{2-30}$$

$$= 2 \sup_{\vec{\omega}\in\Omega^3} \sup_{s\ge 0} |\hat{\vec{F}}| |\vec{\omega}|^2 \, \frac{1}{\eta |\vec{\omega}|^2} \le \frac{1}{\eta} \sup_{\vec{\omega}\in\Omega^3} \sup_{s\ge 0} |\hat{\vec{F}}| < 2\frac{M_2}{\eta}, t \ge 0.$$

It follows that

$$\sup_{\vec{\omega}\in\Omega^3} |\hat{\vec{U}}(t,\vec{\omega})| \le Lt + 2[\frac{M_1}{\eta} + \frac{M_2}{\eta}], t \ge 0, \eta > 0. \tag{2-31}$$

For derivatives of the momentum coefficients of any finite order $k = 1,2,3,...$ the variation of constants formula yields

$$\hat{\vec{U}}^{(k)}(t,\vec{\omega}) = (-\eta)^k \, |\vec{\omega}|^{2k} \, e^{-\eta|\vec{\omega}|^2 t} \hat{\vec{U}}_0$$
$$+ \int_0^t e^{-\eta|\vec{\omega}|^2(t-s)} \left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] [i(\hat{\vec{U}}\vec{\omega}^t) * \hat{\vec{U}})^{(k)} - \hat{\vec{F}}^{(k)}] ds, \tag{2-32}$$
$$t \ge 0, \vec{\omega} \in \Omega^3, k = 1,2,3,...$$

Form the difference to prove continuity of the kth derivative of the momentum coefficient in time,

$$\hat{\vec{U}}^{(k)}(t_2,\vec{\omega}) - \hat{\vec{U}}^{(k)}(t_1,\vec{\omega}) = (-\eta)^k \mid \vec{\omega} \mid^{2k} \; e^{-\eta|\vec{\omega}|^2 t(2)} \hat{\vec{U}}_0 - (-\eta)^k \mid \vec{\omega} \mid^{2k} \; e^{-\eta|\vec{\omega}|^2 t(1)} \hat{\vec{U}}_0$$
$$+ \int_0^{t(2)} e^{-\eta|\vec{\omega}|^2 (t(2)-s)} \left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] [(i(\hat{\vec{U}}\vec{\omega}^t) * \hat{\vec{U}})^{(k)} - \hat{\vec{F}}^{(k)}] ds -$$
$$+ \int_0^{t(1)} e^{-\eta|\vec{\omega}|^2 (t(1)-s)} \left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] [(i(\hat{\vec{U}}\vec{\omega}^t) * \hat{\vec{U}})^{(k)} - \hat{\vec{F}}^{(k)}] ds \qquad (2\text{-}33)$$
$$t_2 > t_1 \geq 0, \vec{\omega} \in \Omega^3, k = 1,2,3,..$$

Simplify the difference of differentiated Fourier transforms at two distinct times

$$\hat{\vec{U}}^{(k)}(t_2,\vec{\omega}) - \hat{\vec{U}}^{(k)}(t_1,\vec{\omega}) = (-\eta)^k \mid \vec{\omega} \mid^{2k} \; [e^{-\eta|\vec{\omega}|^2 t(2)} - e^{-\eta|\vec{\omega}|^2 t(1)}] \hat{\vec{U}}_0$$
$$+ \int_{t(1)}^{t(2)} e^{-\eta|\vec{\omega}|^2 (t(2)-s)} \left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] [(i(\hat{\vec{U}}\vec{\omega}^t) * \hat{\vec{U}})^{(k)} - \hat{\vec{F}}^{(k)}] ds -$$
$$\int_0^{t(1)} [e^{-\eta|\vec{\omega}|^2 (t(2)-s)} - e^{-\eta|\vec{\omega}|^2 (t(1)-s)}] \left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] [(i(\hat{\vec{U}}\vec{\omega}^t) * \hat{\vec{U}})^{(k)} - \hat{\vec{F}}^{(k)}] ds \qquad (2\text{-}34)$$
$$t_2 > t_1 \geq 0, \vec{\omega} \in \Omega^3, k = 1,2,3,...$$

To investigate continuity of $\hat{\vec{U}}^{(k)}(t,\vec{\omega})$ near time 0, evaluate (2-34) at $t_1 = 0, t_2 = t$

$$\hat{\vec{U}}^{(k)}(t,\vec{\omega}) = (-\eta)^k \mid \vec{\omega} \mid^{2k} \; [e^{-\eta|\vec{\omega}|^2 t} - 1] \hat{\vec{U}}_0$$
$$+ \int_0^{t} e^{-\eta|\vec{\omega}|^2 (t-s)} \left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] [(i(\hat{\vec{U}}\vec{\omega}^t) * \hat{\vec{U}})^{(k)} - \hat{\vec{F}}^{(k)}] ds \qquad (2\text{-}35)$$
$$t \geq 0, \vec{\omega} \in \Omega^3, k = 1,2,3,...$$

The upper bound on the momentum transform is

$$|\hat{\vec{U}}^{(k)}(t,\vec{r})| \le |\eta|^k |\vec{\omega}|^{2k} [1-e^{-\eta|\vec{\omega}|^2 t}] |\hat{\vec{U}}_0|$$
$$+\int_0^t e^{-\eta|\vec{\omega}|^2(t-s)} \left| \left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] \right| [|(i(\hat{\vec{U}}\vec{\omega}^t) * \hat{\vec{U}})^{(k)}| + |\hat{\vec{F}}^{(k)}|] ds \tag{2-36}$$
$$t \ge 0, \vec{\omega} \in \Omega^3, k = 1,2,3,...$$

The upper bound of (2-36) can be simplified as follows.

$$|\hat{\vec{U}}^{(k)}(t,\vec{\omega})| \le |\eta|^k |\vec{\omega}|^{2k}\ \eta |\vec{\omega}|^2 |\hat{\vec{U}}_0| t$$
$$+\int_0^t e^{-\eta|\vec{\omega}|^2(t-s)} \left| \left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] \right| [|(i(\hat{\vec{U}}\vec{\omega}^t) * \hat{\vec{U}})^{(k)}| + |\hat{\vec{F}}^{(k)}|] ds \tag{2-37}$$
$$t \ge 0, \vec{\omega} \in \Omega^3, k = 1,2,3,...$$

By the hypothesis on the transforms of the data functions

$$\sup_{t\ge 0} |\hat{\vec{U}}_0(\vec{\omega})| < L \tag{2-38a}$$
$$\sup_{s\ge 0} |\eta|^k |\vec{\omega}|^{2k} |\hat{\vec{F}}^{(k)}(s,\vec{\omega})| < |\vec{\omega}|^{5k} |\hat{\vec{F}}^{(k)}(s,\vec{\omega})| < M_2(\vec{r}) < M_2, |\vec{\omega} > \eta| \tag{2-38b}$$

Since the variation of constants operator is defined on the smooth/ Schwartz transforms, $\hat{\vec{U}}(s,\vec{\omega})\vec{\omega}^t$ is Schwartz and the convolution of Schwartz functions is a Schwartz function with no effect on the smoothness in time,

$$\sup_{s\ge 0} |([(\hat{\vec{U}}(s,\vec{\omega})\vec{\omega}^t) * \hat{\vec{U}}(s,\vec{\omega})]^{(k)}| < M_1(\vec{\omega}) < M_1, k = 1,2,3,... \tag{2-39}$$

By (2-20),

$$\int_0^t e^{-\eta|\vec{\omega}|^2(t-s)} \left| \left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2} - \begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix} \right] \right| |((\hat{\vec{U}}\vec{\omega}^t) * \hat{\vec{U}})^{(k)}| ds \le 2\frac{M_1(k)}{\eta}, k = 1,2,3,... \tag{2-40}$$

Also

$$\sup_{\vec{\omega}\in\Omega^3}\int_0^t e^{-\eta|\vec{\omega}|^2(t-s)}\left|\left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2}-\begin{pmatrix}1\,0\,0\\0\,1\,0\\0\,0\,1\end{pmatrix}\right]\right|\,|\hat{\vec{F}}^{(k)}|\,ds$$

$$\sup_{\vec{\omega}\in\Omega^3}\int_0^t e^{-\eta|\vec{\omega}|^2(t-s)}\sup_{s\ge 0}|\hat{\vec{F}}^{(k)}|\,|\vec{\omega}|^2 ds \le \sup_{\vec{\omega}\in\Omega^3}\sup_{s\ge 0}|\hat{\vec{F}}^{(k)}|\,|\vec{\omega}|^2\left|\int_0^t e^{-\eta|\vec{\omega}|^2(t-s)}ds\right| \tag{2-41}$$

$$=\sup_{\vec{\omega}\in\Omega^3}\sup_{s\ge 0}|\hat{\vec{F}}^{(k)}|\,|\vec{\omega}|^2\frac{1}{\eta|\vec{\omega}|^2}\le\frac{1}{\eta}\sup_{\vec{\omega}\in\Omega^3}\sup_{s\ge 0}|\hat{\vec{F}}^{(k)}|<\frac{M_2(k)}{\eta},t\ge 0,k=1,2,3,...$$

It follows that

$$\sup_{\vec{\omega}\in\Omega^3}|\hat{\vec{U}}^{(k)}(t,\vec{\omega})|\le Lt+2\frac{M_1(k)}{\eta}+\frac{M_2(k)}{\eta},t\ge 0,\eta>0,k=1,2,3,... \tag{2-42}$$

Not only are the $|\hat{\vec{U}}^{(k)}(t,\vec{\omega})|$ bounded for sufficiently short time but for all finite forward time. Since $\hat{\vec{U}}^{(k)},\hat{\vec{F}}^{(k)}$ are Schwartz, $\hat{P}^{(k)}$ is Schwartz in Ω^3 for all forward time follow automatically from the given conditions on the boundary data.

END PROOF

The following proposition is used as a lemma for the next theorem.

Proposition 2-2. The inner product of $\hat{\vec{U}}^{(k)}(t,\vec{\omega})$ with the transformed Euler (convolution) term satisfies

$$\hat{\vec{U}}^{(k)}(t,\vec{\omega})[\hat{\vec{U}}\vec{\omega}^t*\hat{\vec{U}}(t,\vec{\omega})]^{(k)}=0,k=0,1,2,3,...,t\ge 0,\vec{\omega}\in\Omega^3.$$

PROOF

The Fourier transform of the equation of continuity is

$$\hat{\vec{U}}^{(k)}(t,\vec{\omega})\cdot\vec{\omega}=0,t\ge 0. \tag{2-43}$$

By the Liebnitz rule, the k[th] order Euler term can be written

$$\hat{\vec{U}}^{(k)}(t,\vec{\omega})\cdot[\vec{U}(t,\vec{\omega})\vec{\omega}^t * \vec{U}(t,\vec{\omega})]^{(k)} =$$
$$\hat{\vec{U}}^{(k)}(t,\vec{\omega})\cdot\sum_{l=0}^{k}\binom{k}{l}[\vec{U}(t,\vec{\omega})\vec{\omega}^t]^{(l)} * \vec{U}^{(k-l)}(t,\vec{\omega}) =$$
$$\hat{\vec{U}}^{(k)}(t,\vec{\omega})\cdot\int_{\Omega^3}\sum_{l=0}^{k}\binom{k}{l}[\vec{U}(t,\vec{\omega}-\vec{\sigma})]^{(l)}(\vec{\omega}-\vec{\sigma})^t\vec{U}^{(k-l)}(t,\vec{\sigma})d\vec{\sigma} \qquad (2\text{-}44)$$
$$= \int_{\Omega^3}\sum_{l=0}^{k}\binom{k}{l}[\vec{U}(t,\vec{\omega}-\vec{\sigma})]^{(l)}\cdot\hat{\vec{U}}^{(k)}(t,\vec{\omega})(\vec{\omega}-\vec{\sigma})\cdot\vec{U}^{(k-l)}(t,\vec{\sigma})d\vec{\sigma}.$$

The previous quantity is bounded above by

$$\le \int_{\Omega^3}\sum_{l=0}^{k}\binom{k}{l}|\vec{U}(t,\vec{\omega}-\vec{\sigma})]^{(l)}|^2|\hat{\vec{U}}^{(k)}(t,\vec{\omega})|^2|(\vec{\omega}-\vec{\sigma})|^2|\vec{U}^{(k-l)}(t,\vec{\sigma})|^2\ d\vec{\sigma}$$
$$\le \int_{\Omega^3}\sum_{l=0}^{k}\binom{k}{l}|\hat{\vec{U}}^{(k)}(t,\vec{\omega})|^2|\hat{\vec{U}}^{(k-l)}(t,\vec{\sigma})|^2|\hat{\vec{U}}(t,\vec{\omega}-\vec{\sigma})]^{(l)}|^2|(\vec{\omega}-\vec{\sigma})|^2\ d\vec{\sigma} \qquad (2\text{-}45)$$
$$= \int_{\Omega^3}\sum_{l=0}^{k}\binom{k}{l}|\hat{\vec{U}}^{(k)}(t,\vec{\omega})|^2|\hat{\vec{U}}^{(k-l)}(t,\vec{\sigma})|^2|\hat{\vec{U}}(t,\vec{\omega}-\vec{\sigma})]^{(l)}\cdot(\vec{\omega}-\vec{\sigma})|^2\ d\sigma = 0.$$

The first equality follows by the Liebnitz rule and the fact that the time derivative distributes over the convolution, the second by the definition of the convolution. The third line follows by matrix vector multiplication, the fourth by the Schwartz inequality applied continuously to the both families of vector dot products in R^3 parameterized by $\vec{\omega}\in\Omega^3$, the fifth by the definition of the inner product of a vector with itself $\vec{x}\cdot\vec{x} =|\vec{x}|^2$. The final inequality follows by applying the higher order equation of continuity to each term $(\hat{\vec{U}}^{(l)}(t,\vec{\omega})\cdot\vec{\omega} = 0, l = 0,1,2,...,k, t\ge 0, \vec{\omega}\in\Omega^3$.
Similarly,

$$\hat{\vec{U}}^{(k)}(t,\vec{\omega})\cdot[\vec{U}(t,\vec{\omega})\vec{\omega}^t * \vec{U}(t,\vec{\omega})]^{(k)} \ge$$
$$-\int_{\Omega^3}\sum_{l=0}^{k}\binom{k}{l}|\hat{\vec{U}}^{(k)}(t,\vec{\omega})|^2|\hat{\vec{U}}^{(k-l)}(t,\vec{\sigma})|^2|\hat{\vec{U}}(t,\vec{\omega}-\vec{\sigma})]^{(l)}\cdot(\vec{\omega}-\vec{\sigma})|^2\ d\sigma = 0, k = 0,1,2,... \qquad (2\text{-}46)$$

Hence

$$\hat{\vec{U}}^{(k)}(t,\vec{\omega})\cdot[\vec{U}(t,\vec{\omega})\vec{\omega}^t * \vec{U}(t,\vec{\omega})]^{(k)} = 0, t\ge 0, \vec{\omega}\in\Omega^3, k = 0,1,2,.... \qquad (2\text{-}47)$$

END PROOF

The next theorem extends the domain of definition of solutions of (2-11) by showing that solutions are bounded for all forward time. A frequency domain formula for the total mechanical energy of the average velocity and any time derivative of it also appears. This is the frequency domain analog of the extension of Leray's energy law. It is equivalent to formula (1-4).

Theorem 2-2. Suppose $\neg\{\vec{U}_0^{(k)} \equiv 0\}, k = 0,1,2,...$ and

$$\frac{1}{(2\pi)^3}\int_0^t \int_{\Omega^3} \hat{\vec{U}}^{(k)}(s,\vec{\omega}) \cdot \hat{\vec{F}}^{(k)}(s,\vec{\omega}) d\vec{\omega}ds < \eta \frac{1}{(2\pi)^3}\int_0^t \int_{\Omega^3} |\vec{\omega}|^2 |\hat{\vec{U}}^{(k)}|^2 \, d\vec{\omega}ds, \tag{2-47}$$
$$t \geq 0, \eta > 0, k = 0,1,2,...$$

a. Then the following formulas are well defined

$$\frac{1}{(2\pi)^3}\int_{\Omega^3} |\hat{\vec{U}}^{(k)}|^2 (t,\vec{\omega}) d\vec{\omega} - \frac{1}{(2\pi)^3}\int_{\Omega^3} |\hat{\vec{U}}_0^{(k)}|^2 (\vec{\omega}) d\vec{\omega} =$$
$$-\eta \frac{1}{(2\pi)^3}\int_0^t \int_{\Omega^3} |\vec{\omega}|^2 |\hat{\vec{U}}^{(k)}(s,\vec{\omega})|^2 d\vec{\omega}ds + \frac{1}{(2\pi)^3}\int_0^t \int_{\Omega^3} \hat{\vec{U}}^{(k)}(s,\vec{\omega}) \cdot \hat{\vec{F}}^{(k)}(s,\vec{\omega}) d\vec{\omega}ds, \tag{2-48}$$
$$k = 0,1,2,...$$

b. The solution of (2-12) and every finite time derivative $\hat{\vec{U}}^{(k)}(t,\vec{\omega}), k = 0,1,2,..$ of it has a unique extension (with respect to time) which is bounded and continuous in *t* for all forward time, continuous (and asymptotically vanishing in Ω^3) and jointly continuous and bounded almost everywhere on $[0,\infty) \times \Omega^3$ such that $\hat{\vec{U}} \in (\vec{L}^1 \cap \vec{L}^2)(\Omega^3), t \geq 0$.

PROOF

First note that

$$\neg\{\vec{U}_0^{(k)} \equiv 0\}, k = 0,1,2,... \Rightarrow \int_{R^3} |\hat{\vec{U}}_0^{(k)}|^2 (\vec{x}) d\vec{x} > 0, k = 0,1,2,... \tag{2-49}$$

Thus

$$\frac{1}{(2\pi)^3}\int_{\Omega^3} |\hat{\vec{U}}_0^{(k)}|^2 (\vec{\omega}) d\vec{\omega} > 0, k = 0,1,2,... \tag{2-50}$$

By Plancherel's theorem

$$\frac{1}{(2\pi)^3}\int_{\Omega^3} |\hat{\vec{U}}_0^{(k)}|^2 (\vec{\omega}) d\vec{\omega} < \infty \Leftrightarrow \int_{R^3} |\hat{\vec{U}}_0^{(k)}|^2 (\vec{x}) d\vec{x} < \infty, k = 0,1,2,... \tag{2-51}$$

Next construct the formulas for the average energy of $\hat{\vec{U}}^{(k)}, k = 0,1,2,...$ Then eliminate the modified Euler terms to simplify these formulas. Form the dot (Hermitian) product of each equation (2-19) with the complex vector $\hat{\vec{U}}^{*(k)}(t,\vec{\omega}), k = 0,1,2,...$, integrate over $\vec{\omega} \in \Omega^3$ and integrate from 0 to *t* to obtain

$$\frac{1}{(2\pi)^3}\int_{\Omega^3} |\hat{\vec{U}}^{(k)}(t,\vec{\omega})|^2 d\vec{\omega} - \frac{1}{(2\pi)^3}\int_{\Omega^3} |\hat{\vec{U}}^{(k)}(0,\vec{\omega})|^2 d\vec{\omega} = -\eta \frac{1}{(2\pi)^3}\int_0^t \int_{\Omega^3} |\vec{\omega}|^2 |\hat{\vec{U}}^{(k)}|^2 d\vec{\omega} ds$$
$$+ i\frac{1}{(2\pi)^3}\int_0^t \int_{\Omega^3}\int_{\Phi^3} (\hat{\vec{U}}^{(k)}(s,\vec{\omega}-\vec{\phi}) \cdot (\vec{\omega}-\vec{\phi}))\hat{\vec{U}}^{(k)}(s,\vec{\phi})\hat{\vec{U}}^{*}(s,\vec{\omega}) d\vec{\phi} d\vec{\omega} \qquad (2\text{-}52)$$
$$+ \frac{1}{(2\pi)^3}\int_0^t \int_{\Omega^3} \hat{\vec{U}}^{*(k)} \cdot \hat{\vec{F}}^{(k)} d\vec{\omega} ds.$$

Prior to integration from 0 to *t*, the transformed Euler term which would appear in (2-48) is

$$\int_{\Omega^3}\int_{\Phi^3} (\hat{\vec{U}}(s,\vec{\omega}-\vec{\phi}) \cdot (\vec{\omega}-\vec{\phi}))\hat{\vec{U}}(s,\vec{\phi})\hat{\vec{U}}^{*}(s,\vec{\omega}) d\vec{\phi} d\vec{\omega}. \qquad (2\text{-}53)$$

The higher order time derivatives of the transformed Euler term

$$i\int_{\Omega^3}\int_{\Phi^3} \{(\hat{\vec{U}}(s,\vec{\omega}-\vec{\phi}) \cdot (\vec{\omega}-\vec{\phi}))\hat{\vec{U}}(s,\vec{\phi})\}^{(k)} \hat{\vec{U}}^{(k)*}(s,\vec{\omega}) d\vec{\phi} d\vec{\omega}, k = 0,1,2,... \qquad (2\text{-}54)$$

vanish by proposition 2-2.

It follows immediately by (2-47), that

$$\int_{\Omega^3} |\hat{\vec{U}}^{(k}(t_2,\vec{\omega})|^2 d\vec{\omega} - \int_{\Omega^3} |\hat{\vec{U}}^{(k)}(t_1,\vec{\omega})|^2 d\vec{\omega} < 0, 0 \le t_1 < t_2, k = 0,1,2,.... \qquad (2\text{-}55)$$

In particular

$$\frac{1}{(2\pi)^3}\int_{\Omega^3} |\hat{\vec{U}}^{(k)}|^2 (t,\vec{\omega}) d\vec{\omega} \le \frac{1}{(2\pi)^3}\int_{\Omega^3} |\hat{\vec{U}}_0^{(k)}|^2 (0,\vec{\omega}) d\vec{\omega} = M_k < \infty, t \ge 0, k = 0,1,2,... \qquad (2\text{-}56)$$

By (2-56) and the strict inequality of (2-47) , it follows that the time average of the potential energy is bounded for all forward time. Thus all terms appearing in formulas (2-48) are well defined for all forward time.

Since $\|\hat{\vec{U}}\|_{L_2} < \infty$ it follows that $\hat{\vec{U}}^{(k)}(t,\vec{\omega}), k = 0,1,2,....$ is bounded in *t* for all forward time uniformly bounded with respect to $\vec{\omega} \in \Omega^3$. By the fundamental (extension) theory of ordinary differential equations $\hat{\vec{U}}^{(k)}(t,\vec{\omega}), k = 0,1,2,...$ is continuous for all forward time uniformly with respect to $\vec{\omega} \in \Omega^3$.

By the corresponding inequalities (1-5) it follows that $\vec{U}^{(k)}(t,\vec{x}), k = 0,1,2,...$ is in $(L^1 \cap L^2)(R^3)$ for all forward time, hence $\hat{\vec{U}}^{(k)}(t,\vec{\omega}), k = 0,1,2,..., t \geq 0$ is continuous and asymptotically vanishing as a function of $\vec{\omega}$. Thus $\hat{\vec{U}}^{(k)} \in \vec{C}_0(\Omega^3)$. By the inequalities (2-47), $\hat{\vec{U}}^{(k)}(\vec{L}^1 \cap \vec{L}^2)(\Omega^3), k = 0,1,2,..., t \geq 0$..

END PROOF

Lemma 2-4. If $\vec{U}^{(k)} \cdot \vec{F}^{(k)}, \nabla\vec{U}^{(k)} \in \hat{\vec{S}}(R^3) \subset L^2(R^3), k = 0,1,2,..., t \geq 0$ such that $\vec{U}^{(k)}$ satisfies the Navier-Stokes partial differential equation (1-1) and its finite time derivatives satisfy

$$\int_0^t \int_{R^3} \vec{U}^{(k)} \cdot \vec{F}^{(k)}(s,\vec{x}) d\vec{x} ds < \eta \int_0^t \int_{R^3} |\nabla \vec{U}^{(k)}(s,\vec{x})|^2 d\vec{x} ds, t \geq 0, k = 0,1,2,... \qquad \text{(2-57a)}$$

for all forward time if and only if $\hat{\vec{U}}^{(k)} \cdot \hat{\vec{F}}^{(k)}, |\vec{\omega}|^2 \hat{\vec{U}}^{(k)} \in L^2(\Omega^3), k = 0,1,2,... t \geq 0$ and the solutions of the Navier-Stokes ordinary differential equations (2-1) and its finite time derivatives (2-20) satisfy

$$\int_0^t \int_{\Omega^3} \hat{\vec{U}}^{(k)}(s,\vec{\omega}) \cdot \hat{\vec{F}}^{(k)}(s,\vec{\omega}) d\vec{\omega} ds < \eta \int_0^t \int_{\Omega^3} |\vec{\omega}|^2 |\hat{\vec{U}}^{(k)}(s,\vec{\omega})|^2 ds, t \geq 0, k = 0,1,2,... \qquad \text{(2-57b)}$$

for all forward time.

PROOF

By the strict inequality, the integral on the left of (2-57a) over R^3 must be finite. In order for the Fourier transforms which appear in (2-57b) to be well defined
$\vec{F}^{(k)} \cdot \vec{U}^{(k)}, \nabla\vec{U}^{(k)} \in L^2(R^3), k = 0,1,2,..., t \geq 0$ if and only if
$\hat{\vec{U}}^{(k)} \cdot \hat{\vec{F}}^{(k)}, |\vec{\omega}|^2 \hat{\vec{U}}^{(k)} \in L^2(\Omega^3), k = 0,1,2,..., t \geq 0$
Note that, since $\vec{F}^{(k)}, k = 0,1,2,...$ is Schwartz in $\vec{x} \in R^3$, $\vec{U}^{(k)} \cdot \vec{F}^{(k)}$ is integrable by *Hölder's* inequality if only $\vec{U}^{(k)} \in \vec{L}^1(R^3)$ without assuming (2-57a).

Now suppose

$$\int_0^t \int_{\Omega^3} \hat{\vec{U}}^{(k)}(s,\vec{\omega}) \cdot \hat{\vec{F}}^{(k)}(s,\vec{\omega}) d\vec{\omega} ds < \eta \int_0^t \int_{\Omega^3} |\vec{\omega}|^2 |\hat{\vec{U}}^{(k)}(s,\vec{\omega})|^2 \, d\vec{\omega} ds, t \geq 0, k = 0,1,2,... \tag{2-58}$$

By the definition of the Fourier transform, the previous inequalities hold if and only if

$$\int_0^t \int_{\Omega^3} \int_{R^3} \vec{U}^{(k)}(s,\vec{x}) \cdot \vec{F}^{(k)}(s,\vec{x}) |e^{-i\vec{\omega}\cdot\vec{x}}|^2 d\vec{x} d\vec{\omega} ds < \eta \int_0^t \int_{\Omega^3} \int_{R^3} |\nabla \vec{U}^{(k)}(s,\vec{x})|^2 |e^{-i\vec{\omega}\cdot\vec{x}}|^2 d\vec{x} d\vec{\omega} ds, \tag{2-59}$$
$k = 0,1,2,..., t \geq 0.$

But these inequalities hold if and only if

$$\int_0^t \int_{R^3} \vec{U}^{(k)}(s,\vec{x}) \cdot \vec{F}^{(k)}(s,\vec{x}) d\vec{x} ds < \eta \int_0^t \int_{R^3} |\nabla \vec{U}^{(k)}(s,\vec{x})|^2 d\vec{x} ds, t \geq 0, k = 0,1,2,... \tag{2-60}$$

because $|e^{-i\vec{\omega}\cdot\vec{x}}|^2 = 1, \vec{\omega} \in \Omega^3, \vec{x} \in R^3$.

END PROOF

Theorem 2-3. If $\hat{\vec{F}}(t,\vec{\omega})$, and all time derivatives are Schwartz in $\vec{\omega} \in \Omega^3$ for all forward time and $\hat{\vec{U}}_0(\vec{\omega})$ is Schwartz in $\vec{\omega} \in \Omega^3$ then any finite time derivative of $\hat{\vec{U}}(t,\vec{\omega})$ (the solution of the equation of lemma 2-2) is Schwartz in $\vec{\omega} \in \Omega^3$.

PROOF

a. For any fixed t, k the kth derivative is Schwartz in $\vec{\omega} \in \Omega^3$

$$\sup_{\vec{\omega} \in \Omega^3} |\vec{\omega}|^p |\hat{\vec{U}}^{(k)}| < \infty, \forall p \in W. \tag{2-61}$$

b. The weighted upper bound is uniform in k

$$\sup_{k \in W} \sup_{\vec{\omega} \in \Omega^3} |\vec{\omega}|^p |\hat{\vec{U}}^{(k)}| < \infty, \forall p \in W. \tag{2-62}$$

c. The weighted upper bound is uniform for all $t \geq 0$.

$$\sup_{t \geq 0} \sup_{k \in W} \sup_{\vec{\omega} \in \Omega^3} |\vec{\omega}|^p |\hat{\vec{U}}^{(k)}| < \infty, \forall p \in W. \tag{2-63}$$

By the variation of constants formula, it suffices to show that the convolution integral is discrete Schwartz since $\hat{\hat{\Phi}}(t,\vec{\omega}) = e^{-\eta|\vec{\omega}|^2 t}\hat{\hat{U}}_0$ are Schwartz in $\vec{\omega} \in \Omega^3$ by inspection given the hypotheses on the initial and boundary conditions.

Multiply the time convolution by any monomial formed by the product of any finite powers of the frequency components

$$\sup_{\vec{\omega}\in\Omega^3} |\omega_1^{p(1)}\omega_2^{p(2)}\omega_3^{p(3)}||\hat{\hat{U}}| \le \sup_{\vec{\omega}\in\Omega^3} e^{-\eta|\vec{\omega}|^2 t}|\hat{\hat{U}}_0||\omega_1^{p(1)}\omega_2^{p(2)}\omega_3^{p(3)}| +$$

$$\sup_{\vec{\omega}\in\Omega^3} |\omega_1^{p(1)}\omega_2^{p(2)}\omega_3^{p(3)}||\int_0^t e^{-\eta|\vec{\omega}|^2(t-s)}\left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2} - \begin{pmatrix}1\,0\,0\\0\,1\,0\\0\,0\,1\end{pmatrix}\right][i((\hat{\hat{U}}\vec{\omega}^t)*\hat{\hat{U}}) - \hat{\hat{F}}]ds| \tag{2-64}$$

$\eta > 0, \vec{p} \in W^3, t \ge 0, k = 1,2,3,$

Simplify the upper bound on the Schwartz weighted Fourier transform of the momentum of (2-64)

$$\sup_{\vec{\omega}\in\Omega^3} |\omega_1^{p(1)}\omega_2^{p(2)}\omega_3^{p(3)}||\hat{\hat{U}}^{(k)}(t,\vec{r})| \le$$

$$\sup_{\vec{\omega}\in\Omega^3} |\omega_1^{p(1)}\omega_2^{p(2)}\omega_3^{p(3)}||(-4\eta)^k|\pi^{2k}|\vec{r}|^{2k} e^{-\eta|\vec{\omega}|^2 t}|\hat{\hat{U}}_0|$$

$$+\sup_{\vec{\omega}\in\Omega^3} |\omega_1^{p(1)}\omega_2^{p(2)}\omega_3^{p(3)}||\int_0^t e^{-\eta|\vec{\omega}|^2 t(t-s)}\left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2} - \begin{pmatrix}1\,0\,0\\0\,1\,0\\0\,0\,1\end{pmatrix}\right][i((\hat{\hat{U}}\vec{\omega}^t)*\hat{\hat{U}})^{(k)} - \hat{\hat{F}}^{(k)}]ds| \tag{2-65}$$

$\eta > 0, \vec{p} \in W, t \ge 0, \vec{\omega} \in \Omega^3, k = 1,2,3,$

Since $\hat{\hat{U}}_0 \in \hat{\hat{S}}(\Omega^3)$

$\forall t \ge 0, \vec{p} \in W^3$

$$\sup_{\vec{\omega}\in\Omega^3} e^{-\eta|\vec{\omega}|^2 t}|\hat{\hat{U}}_0||\omega_1^{p(1)}\omega_2^{p(2)}\omega_3^{p(3)}| \le \sup_{\vec{\omega}\in\Omega^3} |\omega_1^{p(1)}\omega_2^{p(2)}\omega_3^{p(3)}||\hat{\hat{U}}_0| \le M_2 \tag{2-66}$$

Since

$$\hat{\hat{U}} \in \hat{\hat{S}}(\Omega^3) \Rightarrow (\hat{\hat{U}}\vec{\omega}^t)*\hat{\hat{U}} \in \hat{\hat{S}}(\Omega^3), \hat{\hat{F}} \in \hat{\hat{S}}(\Omega^3) \tag{2-67}$$

it follows by (2-20) (the norm on the matrix operator) that

$$\sup_{\vec{\omega}\in\Omega^3} |\omega_1^{p(1)}\omega_2^{p(2)}\omega_3^{p(3)}||\hat{\vec{U}}^{(k)}| \le$$

$$\sup_{\vec{\omega}\in\Omega^3} |\omega_1^{p(1)}\omega_2^{p(2)}\omega_3^{p(3)}||\int_0^t e^{-\eta|\vec{\omega}|^2(t-s)}\left[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2}-\begin{pmatrix}1\,0\,0\\0\,1\,0\\0\,0\,1\end{pmatrix}\right][i((\hat{\vec{U}}\vec{r}^{\,t})*\hat{\vec{U}})-\hat{\vec{F}}]ds| \tag{2-68}$$

$$\le 2\int_0^t e^{-\eta|\vec{\omega}|^2 t(t-s)}\sup_{\vec{r}\in N^3}\sup_{s\ge 0}|\omega_1^{p(1)}\omega_2^{p(2)}\omega_3^{p(3)}|((\hat{\vec{U}}\vec{\omega}^t)*\hat{\vec{U}})ds|$$

$$+2\int_0^t e^{-\eta|\vec{\omega}|^2(t-s)}\sup_{s\ge 0}|\omega_1^{p(1)}\omega_2^{p(2)}\omega_3^{p(3)}||\hat{\vec{F}}|ds|$$

Now switch to the vector norm Schwartz condition.

It suffices to consider the integral.

$$\sup_{\vec{\omega}\in\Omega^3}|\vec{\omega}|^{p+2}|\int_0^t e^{-\eta|\vec{\omega}|^2(t-s)}[\frac{\vec{\omega}\vec{\omega}^t}{|\vec{\omega}|^2}-\begin{pmatrix}1\,0\,0\\0\,1\,0\\0\,0\,1\end{pmatrix}][(\hat{\vec{U}}\vec{\omega}^t*\hat{\vec{U}})^{(k)}-\vec{F}^{(k)}]ds| \le \tag{2-69}$$

$$2(\sup_{\vec{\omega}\in\Omega^3}|\vec{\omega}|^2|\int_0^t e^{-\eta|\vec{\omega}|^2(t-s)}ds|(\sup_{\vec{\omega}\in\Omega^3}\sup_{t\ge 0}|\vec{\omega}|^p[|(\hat{\vec{U}}\vec{\omega}^t*\hat{\vec{U}})^{(k)}|+|\vec{F}^{(k)}|])$$

The previous inequality follows by the product inequality, the inequality for the product of suprema, the inequality relating suprema with respect to one vector parameter vs. one vector parameter and one variable and the fact that $|\vec{\omega}|^p|.|\le \sup_{\vec{\omega}\in\Omega^3}|\vec{\omega}|^p|.|$.

$$2(\sup_{\vec{\omega}\in\Omega^3}|\vec{\omega}|^2|\int_0^t e^{-\eta|\omega|^2(t-s)}ds|)\times(\sup_{\vec{\omega}\in\Omega^3}\sup_{t\ge 0}|\vec{\omega}|^p[|(\hat{\vec{U}}\vec{\omega}^t*\hat{\vec{U}})^{(k)}|+|\vec{F}^{(k)}|])$$
$$\le \frac{2}{\eta}\{M_1(k)+M_2(k)\}. \tag{2-70}$$

The inequality above follows by integration for the first factor and the Schwartz bounds for each term of the second factor.

The homogeneous term has a uniform upper bound

$$\forall t\ge 0, \vec{p}\in W^3$$
$$\sup_{\vec{\omega}\in\Omega^3} e^{-\eta|\vec{\omega}|^2 t}|\omega_1^{p(1)}\omega_2^{p(2)}\omega_3^{p(3)}||\hat{\vec{U}}_0|\le \sup_{\vec{\omega}\in\Omega^3}|\omega_1^{p(1)}\omega_2^{p(2)}\omega_3^{p(3)}||\hat{\vec{U}}_0|\le M_2 \tag{2-71}$$

The pressure function

$$\hat{P}(t,\vec{r})=\frac{1}{|\omega|^2}\{\vec{r}\cdot[\hat{\vec{U}}\vec{\omega}^t*\hat{\vec{U}}-\hat{\vec{F}}(t,\vec{\omega})]\}, t\ge 0, \vec{\omega}\in\Omega^3-\vec{0} \tag{2-72}$$

is Schwartz in $\vec{\omega}$ since $\hat{\vec{U}},\hat{\vec{U}}\vec{\omega}^t$ are Schwartz by hypothesis, the convolution of discrete Schwartz functions is Schwartz and the difference of Schwartz functions $[\hat{\vec{U}}\vec{\omega}^t * \hat{\vec{U}} - \hat{\vec{F}}(t,\vec{\omega})]$ is Schwartz.

By the same reasoning any finite order time derivative of the pressure transform $\hat{P}(t,\vec{\omega})$ is Schwartz.

END PROOF

The following theorem is the main result of this paper.

Theorem 2-4. Suppose $\vec{U}^{(k)} \cdot \vec{F}^{(k)}, \nabla\vec{U}^{(k)} \in \vec{S}(R^3), k = 0,1,2,..., t \geq 0$ where the $\vec{U}^{(k)}, k = 0,1,2,...$ satisfy the Navier-Stokes partial differential equations and its finite order time derivatives such that

$$\int_0^t\int_D \vec{U}^{(k)} \cdot \vec{F}^{(k)} d\vec{x}ds < \eta\int_0^t\int_D \| \vec{U}^{(k)} \cdot \nabla \|^2 d\vec{x}ds, t \geq 0, \eta > 0, k = 0,1,2,... \tag{2-73}$$

where $\vec{F}^{(k)}$ is jointly smooth in $(t,\vec{x})$, Schwartz in $\vec{x} \in R^3$ and bounded in $t \in [0,\infty)$. Then every finite time derivative of the solution of the Navier-Stokes momentum equation is bounded, continuous and uniquely determined in *t*. It is also smooth in $\vec{x} \in R^3$ for all forward time.

PROOF

The conclusion follows directly by the properties of the inverse Fourier transform representation $\hat{\hat{\vec{U}}}$ of the velocity function, theorem 2-2 and theorem 2-3. In particular,

$$\begin{aligned} &\hat{\vec{U}} \in \hat{\vec{S}}(\Omega^3), t \geq 0 \Rightarrow \hat{\hat{\vec{U}}} = \vec{U} \in \vec{C}^\infty(R^3), t \geq 0 \\ &\hat{\vec{U}} \in \vec{C}^\infty([0,\infty)), \vec{\omega} \in \Omega^3 \Rightarrow \hat{\hat{\vec{U}}} = \vec{U} \in \vec{C}^\infty([0,\infty)), \vec{x} \in R^3. \end{aligned} \tag{2-74}$$

By the formula in lemma 2-2, each component of the pressure gradient satisfies the same properties as each component of the momentum vector (marginal smoothness in $t,\vec{x}$, uniqueness, and boundedness for all forward time).

END PROOF

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