DYNAMICAL SYSTEMS ON LATTICES WITH DECAYING INTERACTION II: HYPERBOLIC SETS AND THEIR INVARIANT MANIFOLDS

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Abstract. This is the second part of the work devoted to the study of maps with decay in lattices. Here we apply the general theory developed in [FdlLM10] to the study of hyperbolic sets. In particular, we establish that any close enough perturbation with decay of an uncoupled lattice map with a hyperbolic set has also a hyperbolic set, with dynamics on the hyperbolic set conjugated to the corresponding of the uncoupled map. We also describe how the decay properties of the maps are inherited by the corresponding invariant manifolds.

1. Introduction

The goal of this paper is to study hyperbolic sets in systems of maps with weakly decaying interactions, which have been introduced as a model in several scientific contexts. We refer to the Introduction of [FdlLM10] for more information and motivations for the study of coupled map lattices.

More precisely, in this paper we will study persistence of hyperbolic invariant sets and their invariant manifolds, we will study the regularity of the manifolds and the conjugation as well as their *decay properties*, that is, that the i-th component of an object depends weakly on the dynamics of the map at far away components.

The first main result we present here is the persistence of the hyperbolic set of an uncoupled map under C^1 perturbations with uniformly continuous derivative. Any sufficiently close diffeomorphism must have a hyperbolic set, whose dynamics is conjugated through a homeomorphism to the one of the corresponding hyperbolic set of the original uncoupled map. This result is achieved through a Shadowing Theorem, which also implies the continuity of the conjugation with respect to the diffeomorphism, as well as its uniqueness, provided it is close enough to the identity. Then, by means of the implicit function theorem, we obtain that the conjugation also has decay properties and depends smoothly on the diffeomorphism. We remark that this last approach only provides a semiconjugation and it is the Shadowing Theorem, with its uniqueness statement, which ensures that the semiconjugation actually is a proper conjugation.

The second main result is the existence and decay properties of invariant manifolds of hyperbolic sets when the perturbation has decay properties. It should be noted that the existence of the invariant manifolds, as well as their differentiability, could be inferred without too much work from the standard approach [HP70], see also [CFdlL03], although there are several

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technicalities. Here we also obtain decay properties of these manifolds, which happen to be the same as the ones of the map.

We use the functional analysis framework developed in the companion paper [FdlLM10]. This framework allows us to carry out proofs that follow closely some carefully chosen classical proofs of existence of invariant objects and their stability based on functional analysis. This framework incorporates the decay properties and is such that permits to introduce a differentiable manifold structure on the space of mappings having decay. Of course, we will have to give up some properties such as compactness or separability (the maps, even if uniformly hyperbolic, have uncountably many periodic orbits, except in trivial cases). Hence, we have to pay the price that approximations, extensions are somewhat technical and some topological arguments based on index, closed graph, uniform continuity, etc., have to be completely abandoned. The functional analysis proofs that go through have to be carefully chosen.

One advantage of the framework developed in [FdlLM10] is that the decay properties are preserved by several operations, so that we can formulate our problems as fixed point problems rather than having to use more sophisticated iterative methods.

Similar problems have been considered in other papers in the literature. Notably in [BS88, Jia95, JP98, JdlL00], which present other points of view and consider other problems. Of course, the main goal of [JP98, JdlL00] is invariant measures of coupled map lattices and the geometric properties are only studied as tools. On the other hand, this paper focuses on the geometric properties and we do not discuss invariant measures.

The paper [Jia95] establishes structural stability of hyperbolic sets in some uniform sense. The paper [JP98] uses a different formalism. It maintains the compactness of the space (which is useful for the study of invariant measures they undertake later), but it pays the price that the hyperbolicity is not uniform and that there is no differentiable structure in the space of maps. We also note that in [JP98], since the main goal is the invariant states, the invariant manifolds are studied only through the conjugation, which is defined only on the invariant set and does not give information on their regularity. The methods in the paper [JdlL00] are more closely related to the ones used here, including the use of decay functions and the fact that the conjugation among the systems has also decay. As indicated in [FdlLM10], the formalism that we use now is different from that of [JdlL00] since here we pay attention to the boundary conditions at infinity and we emphasize that the fact that the derivative is determined by the matrix elements is a non-trivial assumption.

In Section 2 we will list the main definitions introduced in [FdlLM10], which we will use extensively here, and state the main theorems of the present work.

Section 3 is devoted to the proof of the first of the main theorems, namely, the structural stability of the maps with decay possessing hyperbolic invariant sets restricted tho their hyperbolic sets if they are close enough to an uncoupled map with a hyperbolic set.

In Section 4 we will prove the second main theorem, concerning the existence and decay properties of the invariant manifolds of the hyperbolic sets of maps with decay close to uncoupled maps.

Finally, four appendices collect technical proofs.

2. Preliminary definitions, setting and main results

We start by recalling some of the definitions and results in [FdlLM10] that we will use here. However, we encourage the reader to go through that paper for the proofs of the different claims and a more detailed information.

2.1. **The lattice manifold.**

We start by introducing the lattice itself, and some related functions.

Let M be a n-dimensional compact Riemannian manifold. The distance in M is

$$d(x,y) = \inf\{\,\text{length}(\gamma) \mid \gamma \text{ is a curve joining } x \text{ and } y\}.$$

Let $\mathcal{F}_M = \{(U_\phi, \phi) \mid \phi : U_\phi \subset M \to \mathbb{R}^n\}$ be a finite atlas of M such that all the transitions maps are C^∞ and, for each r, their r-th derivatives are bounded. Let $2\rho_0$ be the Lebesgue number of the open cover $\cup_{(U_\phi,\phi)\in\mathcal{F}_M} U_\phi$.

We will denote by exp the exponential map in M. Let δ_0 be the injectivity radius of exp.

Let $\rho_\tau = \min\{\rho_0, \delta_0\}$. Let $\tau(x,y) : T_xM \to T_yM$ be the isometry given by the Levi-Civita connection. Let $U_{\rho_\tau} = \{(v,y) \in TM\times M \mid\ d(p(v),y) < \rho_\tau\}$, where $p : TM \to M$ is the tangent bundle projection. We introduce

$$\begin{array}{rccc} \tau : U_{\rho_\tau} \subset & TM\times M & \to & TM \\ & (v,y) & \mapsto & \tau(x,y)v \end{array}$$

where $x = p(v)$.

We will also consider $\mathsf{e} : M \to \mathbb{R}^D$, a C^∞ isometric embedding.

There exists a map C^∞ map $\eta : M\times\mathbb{R}^D \to TM$ such that, for all $x \in M$, $\eta(x)\cdot D\mathsf{e}(x) = \mathrm{Id}_{|T_xM}$.

Given $d \in \mathbb{N}$, we define lattice manifold as the set

$$\mathcal{M} = \prod_{i\in\mathbb{Z}^d} M,$$

which, with the distance

$$d(x,y) = \sup_{i\in\mathbb{Z}^d} d(x_i,y_i), \quad x,y \in \mathcal{M},$$

where the d in the right-hand side is the finite-dimensional one, is a complete metric space. The atlas

$$\mathcal{F}_{\mathcal{M}} = \{(U_\phi,\phi) \mid \phi = (\phi_i)_{i\in\mathbb{Z}^d} \text{ is a sequence with } (U_i,\phi_i) \in \mathcal{F}_M,$$
$$U_\phi = \operatorname{int}\prod_{i\in\mathbb{Z}^d} U_i\}.$$

allows us to model $\mathcal{M}$ as a Banach manifold over $\ell^\infty_{i\in\mathbb{Z}^d}(\mathbb{R}^n) = \ell^\infty(\mathbb{R}^n)$.

Notice that if $g : M \to M$, we can define its *lift* to the lattice $G : \mathcal{M} \to \mathcal{M}$ as $G(x)_i = g(x_i)$, $i \in \mathbb{Z}^d$.

With this differential structure on $\mathcal{M}$, the functions exp, τ, e and η lifted to the lattice $\mathcal{M}$ from the corresponding finite-dimensional ones are C^∞ and their expressions in any chart of $\mathcal{F}_\mathcal{M}$ have derivatives bounded independently of the chart.

2.2. **Linear and k-linear maps with decay.** A *decay function* is a map $\Gamma : \mathbb{Z}^d \to \mathbb{R}^+$ such that

(1) $\sum_{i\in\mathbb{Z}^d} \Gamma(i) \le 1$,
(2) $\sum_{j\in\mathbb{Z}^d} \Gamma(i-j)\Gamma(j-k) \le \Gamma(i-k), \quad i,k \in \mathbb{Z}^d$.

Given $(\mathcal{X}_i)_{i\in\mathbb{Z}^d}$, a family of Banach spaces, we introduce the space $\ell^\infty(\mathcal{X}_i) = \ell^\infty_{i\in\mathbb{Z}^d}(\mathcal{X}_i)$. Given $(\mathcal{X}_i)_{i\in\mathbb{Z}^d}$ and $(\mathcal{Y}_i)_{i\in\mathbb{Z}^d}$, two families of Banach spaces, the space of *linear maps with decay* Γ is

$$L_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i)) = \{A \in L(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i)) \mid \|A\|_\Gamma < \infty\}, \tag{2.1}$$

where L refers to the space of continuous linear maps, with

$$\|A\|_\Gamma = \max\{\|A\|, \gamma(A)\} \tag{2.2}$$

and

$$\gamma(A) = \sup_{i,j\in\mathbb{Z}^d} \sup_{\substack{|u|\le 1\\ \pi_l u=0, l\ne j}} |(Au)_i|\, \Gamma(i-j)^{-1}. \tag{2.3}$$

Using the usual identification between the space of k-linear maps and the space of linear maps on the space of $(k-1)$-linear maps, the space L^k_Γ of k-linear maps with decay is defined recursively by

$$L^k_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i)) = L_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(L^{k-1}_\Gamma(\mathcal{X}_i, \ell^\infty(\mathcal{Y}_i)))),$$

with the norm (2.2).

Linear and k-linear maps are stable under composition and contraction. In particular,

(1) if $A \in L^k_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$ and $v \in \ell^\infty(\mathcal{X}_i)$, $Av \in L^{k-1}_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$ and $\|Av\|_\Gamma \le \|A\|_\Gamma \|v\|$,
(2) If $A \in L^k_\Gamma(\ell^\infty(\mathcal{Y}_i), \ell^\infty(\mathcal{Z}_i))$ and $B_j \in L^{l_j}_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Y}_i))$, for $j = 1, \dots, k$, then the composition $AB_1 \cdots B_k \in L^{l_1+\cdots+l_k}_\Gamma(\ell^\infty(\mathcal{X}_i), \ell^\infty(\mathcal{Z}_i))$ and
$$\|AB_1 \cdots B_k\|_\Gamma \le \|A\|_\Gamma \|B_1\|_\Gamma \cdots \|B_k\|_\Gamma.$$

2.3. **Hölder and C^r functions on $\mathcal{M}$ with decay.** Let $X \subset \mathcal{M}$ be a subset. Given $0 < \alpha \le 1$ and a decay function we define the set of α-Hölder functions with decay as

$$C^\alpha_\Gamma = C^\alpha_\Gamma(X, \mathcal{M}) = \{f : X \to \mathcal{M} \mid f \in C^\alpha,\ \gamma_\alpha(f) < \infty\},$$

where

$$\gamma_\alpha(f) = \sup_{i,j\in\mathbb{Z}^d} \tilde\gamma_{\alpha,j}(f_i)\Gamma(i-j)^{-1} \tag{2.4}$$

with

$$\tilde\gamma_{\alpha,j}(f_i) = \sup_{\substack{x_l=y_l\\ l\ne j}} \sup_{x_j\ne y_j} \frac{d(f_i(x), f_i(y))}{d^\alpha(x_j, y_j)}. \tag{2.5}$$

This set is a complete metric space with the distance defined by

$$d_{C^\alpha_\Gamma}(f,g) = \max(d_{C^\alpha}(f,g), \gamma_\alpha(f,g)). \tag{2.6}$$

where

$$d_{C^\alpha}(f,g) = \max(d_{C^0}(f,g), H_\alpha(f,g)), \tag{2.7}$$

$$H_\alpha(f,g) = \sup_{x\neq y} \frac{|\mathsf{e}(f(x)) - \mathsf{e}(g(x)) - \mathsf{e}(f(y)) + \mathsf{e}(g(y))|}{d^\alpha(x,y)}, \tag{2.8}$$

and, for $f, g \in C^\alpha_\Gamma$,

$$\tilde\gamma_{\alpha,j}(f_i,g_i) = \sup_{\substack{x_l=y_l\\ l\neq j}} \sup_{x_j\neq y_j} \frac{|\mathsf{e}(f_i(x)) - \mathsf{e}(g_i(x)) - \mathsf{e}(f_i(y)) + \mathsf{e}(g_i(y))|}{d^\alpha(x_j,y_j)} \tag{2.9}$$

and

$$\gamma_\alpha(f,g) = \sup_{i,j} \tilde\gamma_{\alpha,j}(f_i,g_i)\Gamma(i-j)^{-1}. \tag{2.10}$$

Let $\mathcal{U} \subset \ell^\infty(\mathcal{X}_i)$ be an open subset. The Banach space of C^r functions with decay is

$$C^r_\Gamma(\mathcal{U}, \ell^\infty(\mathcal{Y}_i)) = \{F \in C^r(\mathcal{U}, \ell^\infty(\mathcal{Y}_i)) \mid D^kF(x) \in L^k_\Gamma, \\ \forall x \in \mathcal{U},\ 1 \le k \le r,\ \|F\|_{C^r_\Gamma} < \infty\}, \tag{2.11}$$

where

$$\|F\|_{C^r_\Gamma} = \max(\|F\|_{C^0}, \max_{1\le k\le r} \sup_x \|D^kF(x)\|_\Gamma). \tag{2.12}$$

Then, given $U \subset \mathcal{M}$, an open set, the Banach space of C^r functions with decay on U is

(2.13)
$$C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n)) = \{G \in C^r(U, \ell^\infty(\mathbb{R}^n)) \mid G\circ\phi^{-1} \in C^r_\Gamma(\phi(U_\phi \cap U), \ell^\infty(\mathbb{R}^n)), \\ \forall (U_\phi, \phi) \in \mathcal{F}_\mathcal{N}, \|G\|_{C^r_\Gamma} < \infty\},$$

with

$$\|G\|_{C^r_\Gamma} = \sup_{(U_\phi,\phi)\in\mathcal{F}_\mathcal{N}} \|G\circ\phi^{-1}\|_{C^r_\Gamma}, \tag{2.14}$$

and

$$C^r_\Gamma(U, \mathcal{M}) = \{G \in C^r(U,\mathcal{M}) \mid \mathsf{e}\circ G \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^D))\}, \tag{2.15}$$

with the distance

$$d_{C^r_\Gamma}(G, \tilde G) = \|\mathsf{e}\circ G - \mathsf{e}\circ \tilde G\|_{C^r_\Gamma}. \tag{2.16}$$

With this distance, $C^r_\Gamma(U,\mathcal{M})$ is a complete metric space.

2.4. **Spaces of sections with decay.** First we introduce

$$S^r_\Gamma(\mathcal{M}) = \{\sigma \in C^r_\Gamma(\mathcal{M},\mathcal{M}) \mid p\circ\sigma = \mathrm{Id}, \|\sigma\|_{C^r_\Gamma} < \infty\} \tag{2.17}$$

the Banach space of C^r_Γ sections on $\mathcal{M}$, where

$$\|\sigma\|_{C^r_\Gamma} = \sup_{(U_\phi,\phi)\in\mathcal{F}_\mathcal{M}} \|\sigma_\phi\|_{C^r_\Gamma},$$

with $\sigma_\phi = \pi_2 \circ T\phi\circ\sigma\circ\phi^{-1}$, the second component of the expression of σ in the coordinate chart $(U_\phi,\phi) \in \mathcal{F}_\mathcal{M}$.

$C^r_\Gamma(\mathcal{M},\mathcal{M})$ can be modeled as a Banach manifold over $S^r_\Gamma(\mathcal{M})$. See Section 5.6 in [FdlLM10], for details.

Given X a topological space (that may be a subset of $\mathcal{M}$) and $u : X \to \mathcal{M}$, we will say that $\nu : X \to T\mathcal{M}$ is a *section covering* u if

$$p \circ \nu(x) = u(x), \tag{2.18}$$

where $p : T\mathcal{M} \to \mathcal{M}$ is the tangent bundle projection.

We define the Banach spaces of bounded and continuous sections by

$$\mathcal{S}_u^b(X) = \mathcal{S}_u^b(X, \mathcal{M}) = \{\nu : X \to T\mathcal{M} \mid p(\nu(x)) = u(x),\ \nu \text{ bounded } \} \tag{2.19}$$

and, for u continuous,

$$\mathcal{S}_u^0(X) = \mathcal{S}_u^0(X, \mathcal{M}) = \{\nu : X \to T\mathcal{M} \mid p(\nu(x)) = u(x),\ \nu \text{ continuous } \}, \tag{2.20}$$

with the norm

$$\|\nu\|_{C^{b,0}} = \sup_{x\in X} \|\nu(x)\| = \sup_{x\in X} \sup_{i\in\mathbb{Z}^d} |\nu(x)_i|_i. \tag{2.21}$$

In [FdlLM10] it is shown that $T_x\mathcal{M} = \ell^\infty(T_{x_i}M)$, for all $x \in \mathcal{M}$. From that it is deduced that $\mathcal{S}_u^b(X, \mathcal{M}) \cong \ell^\infty((\mathcal{S}_u^b(X, M))_i)$.

Finally, assume $X \subset \mathcal{M}$. Given a C_Γ^α function $u : X \to \mathcal{M}$, we define for $0 < \alpha \le 1$,

$$\mathcal{S}_{u,\Gamma}^\alpha(X) = \mathcal{S}_{u,\Gamma}^\alpha(X, \mathcal{M}) = \{\nu \in C^\alpha(X, T\mathcal{M}) \mid p(\nu(x)) = u(x), \|\nu\|_{C_\Gamma^\alpha} < \infty\}, \tag{2.22}$$

where

$$\|\nu\|_{C_\Gamma^\alpha} = \max(\|\nu\|_{C^\alpha}, \gamma_\alpha(\nu)) \tag{2.23}$$

and

$$\gamma_\alpha(\nu) = \sup_{i,j} \tilde\gamma_{\alpha,j}(\nu_i)\Gamma(i-j)^{-1} \tag{2.24}$$

with

$$\tilde\gamma_{\alpha,j}(\nu_i) = \sup_{\substack{x_i=y_i\\ i\neq j}} \sup_{x_j\neq y_j} \frac{|D\mathsf{e}(u_i(y))\nu_i(y) - D\mathsf{e}(u_i(x))\nu_i(x)|}{d^\alpha(x_j, y_j)}. \tag{2.25}$$

With this norm, $\mathcal{S}_{u,\Gamma}^\alpha(X, \mathcal{M})$ is a Banach space.

2.5. **Uncoupled maps and hyperbolic sets.** Let M be a compact n-dimensional manifold and $f : M \to M$ a C^r diffeomorphism. We consider the *uncoupled* lattice map $F : \mathcal{M} \to \mathcal{M}$ defined by f by

$$F_i(x) = f(x_i). \tag{2.26}$$

By Lemma 4.4 in [FdlLM10], F is also a C^r diffeomorphism and that, for any $x \in \mathcal{M}$ and $v \in T_x\mathcal{M}$,

$$D^kF(x)v^{\otimes k} = (D^kf(x_i)v_i^{\otimes k})_{i\in\mathbb{Z}^d}. \tag{2.27}$$

Suppose that $\Lambda = \Lambda_f \subset M$ is an invariant hyperbolic compact set for the map f, that is, there exist a splitting $T_\Lambda M = E^s \oplus E^u$, invariant by Tf, and $\lambda < 1$ such that, for all $x \in \Lambda$,

$$\begin{aligned} |Df(x)_{|E_x^s}| &\le \lambda, \\ |Df^{-1}(x)_{|E_x^u}| &\le \lambda. \end{aligned}$$

As a consequence of the hyperbolicity the splitting is continuous. Moreover, since Λ is compact, the splitting is uniformly continuous. We also have that the projections $\pi^{s,u}(x) : T_xM \to E^{s,u}_x$ are uniformly bounded in Λ. Furthermore, it is well known (see Theorem 19.1.6 in [KH95], for instance) that the splitting is C^{α_f}, for some $\alpha_f > 0$ depending on λ and $\mathrm{Lip}\, f^{-1}$.

The hyperbolic properties of f are naturally lifted to F. The set

$$\Delta = \Delta_F = \prod_{i\in\mathbb{Z}^d} \Lambda = \{x \in \mathcal{M} \mid x_i \in \Lambda\} \tag{2.28}$$

is invariant by F. Moreover, from (2.27) with $k = 1$, we have that the fiber bundles $\mathcal{E}^s$, $\mathcal{E}^u$ defined by

$$\mathcal{E}^{s,u}_x = \{v \in T_x\mathcal{M} \mid v_i \in E^{s,u}_{x_i}\} \tag{2.29}$$

are invariant and

$$\begin{aligned} |DF(x)_{|\mathcal{E}^s_x}| &\leq \lambda, \\ |DF^{-1}(x)_{|\mathcal{E}^u_x}| &\leq \lambda. \end{aligned}$$

Since E^u and E^s are subbundles of $T_\Lambda M$, $\mathcal{E}^s$, $\mathcal{E}^u$ are subbundles of $T_\Delta\mathcal{M}$.

It follows from $T_\Lambda M = E^s \oplus E^u$ that $T_\Delta\mathcal{M} = \mathcal{E}^s \oplus \mathcal{E}^u$. Moreover, the projections $\pi^{s,u}_x : T_x\mathcal{M} \to \mathcal{E}^{s,u}_x$ are defined as $(\pi^{s,u}_x v)_i = \pi^{s,u}_{x_i} v_i$ and are uniformly bounded in x for $x \in \Delta$. Indeed, if $v \in T_\Delta\mathcal{M}$ and $i \in \mathbb{Z}^d$, $v_i = \pi^s_{x_i} v_i + \pi^u_{x_i} v_i$, and $|\pi^{s,u}_{x_i} v_i| \leq C|v_i|$, for some C independent of x_i. Then, the vectors $\pi^{s,u}_x v$ defined by $(\pi^{s,u}_x v)_i = \pi^{s,u}_{x_i} v_i$ belong to $\mathcal{E}^{s,u}_x$ and $v = \pi^s_x v + \pi^u_x v$. Furthermore, since the projections $\pi^{s,u}_{x_i}$ are uniformly bounded, we have that $|(\pi^{s,u}_x v)_i| = |\pi^{s,u}_{x_i} v_i| \leq C|v_i| \leq C|v|$, and, then, $|\pi^{s,u}_x v| \leq C|v|$. In particular, the product norm is equivalent to the original one.

2.6. **Main results.** The first result deals with the structural stability of uncoupled maps with hyperbolic sets in the space of C^r_Γ diffeomorphisms on the lattice manifold $\mathcal{M}$. This result is also true even if no decay properties are assumed on the perturbation of the uncoupled map F, but then, of course, the conjugation does not have decay properties either (see Theorem 3.7, in Section 3.3). If the perturbation has decay properties, these are inherited by the conjugation.

Theorem 2.1. *Assume that the uncoupled map F is C^r, with $r \geq 3$. Let α_f be the Hölder exponent of the splitting of the underlying map f. Then, there exist $0 < \alpha^* \leq \alpha_f$ and a neighborhood $\tilde{\mathcal{V}}$ of F in C^r_Γ such that, for any $\Phi \in \tilde{\mathcal{V}}$ and $0 < \alpha \leq \alpha^*$,*

(1) *there exists a unique close to the identity homeomorphism h_Φ that satisfies*

$$\Phi \circ h_\Phi = h_\Phi \circ F_{|\Delta_F}. \tag{2.30}$$

Moreover, $h_\Phi \in C^\alpha_\Gamma(\Delta_F, \mathcal{M})$.

(2) *The map $\tilde{\mathcal{V}} \to C^\alpha_\Gamma : \Phi \mapsto h_\Phi$ is C^{r-3}.*

We remark that Theorem 2.1 follows in part from a version of the Shadowing Theorem that we prove in the context of lattice manifolds, which may have interest by itself. See Theorem 3.1 in Section 3.2.

In Section 3.6, Proposition 3.11, we will show that the sets $\Delta_\Phi = h_\Phi(\Delta_F)$, $\Phi \in \tilde{\mathcal{V}}$, given by Theorem 2.1 are indeed hyperbolic. Next theorem provides a description of their invariant stable and unstable manifolds, which also have decay properties in several senses.

Theorem 2.2. *Let $F : \mathcal{M} \to \mathcal{M}$ be a C^r uncoupled map, $r \geq 4$, $F = (f)_{i\in\mathbb{Z}^d}$, with $f : M \to M$ of class C^r having a compact hyperbolic invariant set. Let Δ_F be the hyperbolic set of F and $\mathcal{E}^s_F \oplus \mathcal{E}^u_F$ its invariant splitting, as defined in* (2.28)*. Let $\mathcal{V} \subset C^r_\Gamma(\mathcal{M}, \mathcal{M})$ be the neighborhood of F given by Theorem 2.1.*

Given $\Phi \in \mathcal{V}$, let $h_\Phi \in C^\alpha_\Gamma(\Delta_F, \Delta_\Phi)$ be the conjugation given by Theorem 2.1, $\Delta_\Phi = h_\Phi(\Delta_F)$ its hyperbolic set and $\mathcal{E}^s_\Phi \oplus \mathcal{E}^u_\Phi$ its invariant hyperbolic splitting.

Then, the hyperbolic set Δ_Φ possesses stable and unstable invariant manifolds, $W^s_{\Delta_\Phi}$ and $W^u_{\Delta_\Phi}$, tangent to $\mathcal{E}^s_\Phi$ and $\mathcal{E}^u_\Phi$, resp., which are C^α_Γ and are foliated by C^{r-3}_Γ leaves. More concretely, there exists $\rho > 0$ and a map

$$\Upsilon^s : B_\rho = \{v \in \mathcal{E}^s_F \mid |v| < \rho\} \subset \mathcal{E}^s_F \to \mathcal{M},$$

such that

(1) $\mathrm{range}(\Upsilon^s)$ *is invariant by* Φ,
(2) *for all $v \in \mathcal{E}^s_{F,x}$, $d(\Phi^n \circ h_\Phi(x), \Phi^n \circ \Upsilon^s(v)) \to 0$, when $n \to \infty$,*
(3) $\Upsilon^s \in C^\alpha_\Gamma(B_\rho, \mathcal{M})$,
(4) *for each $x \in \Delta_F$, the map $\Upsilon^s_{|\mathcal{E}^s_{F,x}} : B_\rho \cap \mathcal{E}^s_{F,x} \to \mathcal{M}$ is C^{r-3}_Γ, with norm uniformly bounded in x and*
(5) *denoting 0_x the zero vector in $\mathcal{E}^s_{F,x}$, $D(\Upsilon^s_{|\mathcal{E}^s_{F,x}})(0_x)\mathcal{E}^s_{F,x} = \mathcal{E}^s_{\Phi,h_\Phi(x)}$.*

The same claim holds for $W^u_{\Delta_\Phi}$, replacing s by u and n by $-n$.

3. Hyperbolic set

In this section we will obtain the conjugation of perturbations of uncoupled map lattices to the unperturbed system restricted to their respective hyperbolic sets by using a version of the shadowing theorem in lattices. We adapt the proofs of the related results for perturbations of uniformly hyperbolic systems in Shub [Shu78] to this setting. We strongly use that we work with perturbations of uncoupled maps and that Δ, the hyperbolic set of the uncoupled lattice map, is a product of compact sets. Similar results appear in [JP98]. The proof there follows the arguments in [KH95] which need maps of class at least C^2. In the paper [JP98] it is claimed that the invariant manifolds of the points in the perturbed hyperbolic set are the image by the conjugation of the corresponding ones of the unperturbed map. However, the results only provide the conjugation on the hyperbolic set and hence the conjugation may not be defined on the invariant manifolds as would be the case if the hyperbolic set consists of a finite number of points.

3.1. **Extension of the splitting.**

The splitting $T_\Lambda M = E^s \oplus E^u$ can be extended to a continuous splitting in a bounded neighborhood U_Λ of Λ in M. See [HP70] or [HPPS70]. [HP70], p. 148, attributes this to Mather. It is also indicated in [KH95] (see the proofs of Prop. 6.4.4 and 6.4.6 in pp. 264–265). We denote the extension again by the same symbol.

The extended splitting need not be invariant. Let

$$\begin{pmatrix} \tilde{a}_x & \tilde{b}_x \\ \tilde{c}_x & \tilde{d}_x \end{pmatrix}$$

be the matrix of $Df(x)$ represented with respect to the decompositions $T_xM = E^s_x \oplus E^u_x$ and $T_{f(x)}M = E^s_{f(x)} \oplus E^u_{f(x)}$.

By the continuity of Df and the splitting and the compactness of Λ we have that given $\delta_* > 0$, we can reduce the size of the neighborhood U_Λ so that if $x \in U_\Lambda$

$$|\tilde{a}_x|, |\tilde{d}_x^{-1}| < \lambda + \delta_*/2, \qquad |\tilde{b}_x|, |\tilde{c}_x| < \delta_*/2.$$

We recall that δ_0 the injectivity radius of exp, that is, $\exp_x : B(0, \delta_0) \subset T_x\mathcal{M} \to \mathcal{M}$ is a diffeomorphism onto its image, introduced in 4.1 in [FdlLM10]. For $x, z \in M$ such that $d(f(x), z) < \delta_0$ we define the linear map $F_{z,x} : T_xM \to T_zM$ by

$$F_{z,x} = D\exp_z^{-1}(f(x))Df(x).$$

If $x, z \in U_\Lambda$ we can write the matrix of $F_{z,x}$ with respect to the corresponding decompositions of the tangent spaces as

$$\begin{pmatrix} a_{z,x} & b_{z,x} \\ c_{z,x} & d_{z,x} \end{pmatrix}.$$

If $z = f(x)$, $D\exp_z^{-1}(f(x)) = \mathrm{Id}$ and hence we have that if $d(f(x), z)$ is small

$$|a_{z,x}|, |d_{z,x}^{-1}| < \lambda + \delta_*, \qquad |b_{z,x}|, |c_{z,x}| < \delta_*.$$

We choose δ_* such that $\lambda + \delta_* < 1$. Since Λ is compact there is $\rho_* > 0$ such that $\Lambda + \rho_* = \cup_{x\in\Lambda} B(x, \rho_*) \subset U_\Lambda$. Then $\Delta + \rho_* \subset \prod_i(\Lambda + \rho_*) \subset \prod_i U_\Lambda$ and since $\Delta + \rho_*$ is open, it is contained in the interior of $\prod_i U_\Lambda$. We denote $U_\Delta = \Delta + \rho_*$. In U_Δ we have the decomposition $T_{U_\Delta}\mathcal{M} = \mathcal{E}^s \oplus \mathcal{E}^u$, where $\mathcal{E}^s$ and $\mathcal{E}^u$ are defined formally as in (2.29) with the extended splitting in U_Δ.

3.2. **The shadowing theorem.** We denote $\mathrm{Diff}^{1,0}(\mathcal{M})$ the set of C^1 diffeomorphisms such that their derivative is uniformly continuous. The radius ρ_* is the one introduced in Section 3.1 with δ_* small enough.

Theorem 3.1. *Let $F : \mathcal{M} \to \mathcal{M}$ be an uncoupled map $F = (f)_{i\in\mathbb{Z}^d}$, with $f : M \to M$ of class C^1 having a compact hyperbolic invariant set Λ. Let $\Delta = \prod_{i\in\mathbb{Z}^d} \Lambda$. Then, there exist $\varepsilon > 0$, $\delta > 0$, V_Δ neighborhood of Δ in $\mathcal{M}$ and $\mathcal{V}$ neighborhood of F in $\mathrm{Diff}^{1,0}(\mathcal{M})$ such that if X is a topological space, $H : X \to X$ is a homeomorphism, $u : X \to V_\Delta$ is a continuous map and $\Phi \in \mathcal{V}$, such that $d_{C^0}(u \circ H, \Phi \circ u) < \varepsilon$ there exists a unique continuous map $v : X \to \Delta + \rho_*$ such that*

$$v \circ H = \Phi \circ v, \qquad d(u, v) < \delta.$$

Moreover, having fixed u and H, v depends continuously on Φ and there exists $K > 0$ such that

$$d_{C^0}(u, v) < K d_{C^0}(u \circ H, \Phi \circ u). \tag{3.1}$$

Symbolically,

$$\begin{matrix} \Phi \in \mathcal{V} \\ d(u \circ H, \Phi \circ u) < \varepsilon \end{matrix} \qquad \textit{and} \qquad \begin{matrix} & & \Phi & & \\ & \mathcal{M} & \longrightarrow & \mathcal{M} & \\ u & \uparrow & & \uparrow & u \\ & X & \longrightarrow & X & \\ & & H & & \end{matrix}$$

implies

$$\begin{matrix} & & \Phi & & \\ & \mathcal{M} & \rightarrow & \mathcal{M} & \\ v & \uparrow & /// & \uparrow & v \\ & X & \rightarrow & X & \\ & & H & & \end{matrix}$$

Proof. We have to obtain $v \in C^0(X, \mathcal{M})$ near u (satisfying $d(u, v) < \delta$ for some δ) such that

$$v \circ H = \Phi \circ v,$$

or equivalently $v = \tilde{\mathcal{F}}(v)$, where

$$\tilde{\mathcal{F}}(v) = \Phi \circ v \circ H^{-1}.$$

Throughout the proof we will denote d for the distance d_{C^0}. We take $V_\Delta = \Delta + \rho_*/2$. We claim that if $d(u \circ H, \Phi \circ u)$ is sufficiently small $\tilde{\mathcal{F}}$ sends a neighborhood of u in $C^0(X, \mathcal{M})$ to the ball centered at u of radius δ_0, defined in Section 2.1 (see also the beginning of Section 4.1 in [FdlLM10]). More precisely, there exists $0 < \delta_1 < \delta_*/2$ such that if

$$d(u, v) < \delta_1, \quad d(\Phi, F) < \delta_0/4, \quad d(\Phi \circ u, u \circ H) < \delta_0/4$$

then $d(\tilde{\mathcal{F}}(v), u) < \delta_0$. Indeed,

$$d(\tilde{\mathcal{F}}(v), u) \le d(\Phi \circ v \circ H^{-1}, \Phi \circ u \circ H^{-1}) + d(\Phi \circ u \circ H^{-1}, u). \tag{3.2}$$

The first term is bounded by

$$d(\Phi \circ v, \Phi \circ u) \le d(\Phi \circ v, F \circ v) + d(F \circ v, F \circ u) + d(F \circ u, \Phi \circ u)$$

and we use that F is uncoupled and hence

$$d(F \circ v, F \circ u) = \sup_{i \in \mathbb{Z}^d} \sup_{x \in X} d(f(v_i(x)), f(u_i(x))).$$

By the uniform continuity of f on $\overline{U_\Lambda}$, there exists $\delta_1 < \delta_*/2$ such that if $d(u, v) = \sup_i d(u_i, v_i) < \delta_1$ then $d(f(v_i(x)), f(u_i(x))) < \rho_0/4$ for all $x \in X$ and moreover $v(x) \in \Delta + \rho_*$.

The second term in (3.2) can be expressed as

$$d(\Phi \circ u \circ H^{-1}, u \circ H \circ H^{-1}) = d(\Phi \circ u, u \circ H) < \delta_0/4.$$

Then $d(\tilde{\mathcal{F}}(v), u) < \delta_0$.

Following [Mos69], the proof will be reduced to functional analysis in $C^0(X, \mathcal{M})$. Hence, it will be useful to use the chart $\mathcal{A}$ of $C^0(X, \mathcal{M})$, defined in a neighborhood of u, introduced in (5.17) in Section 5.4 of [FdlLM10], i.e.,

$$(\mathcal{A}v)(x) = \exp^{-1}_{u(x)} v(x) = (\exp^{-1}_{u_i(x)} v_i(x))_{i \in \mathbb{Z}^d}.$$

We restrict its domain to the ball $B(u,\delta_1) \subset C^0(X,\mathcal{M})$. We have the diagram

$$\begin{array}{ccccc} & & \tilde{\mathcal{F}} & & \\ & B(u,\delta_1) & \longrightarrow & B(u,\delta_0) & \\ \mathcal{A} & \downarrow & & \downarrow & \mathcal{A} \\ & \mathcal{S}^0_{u,\delta_1}(X,\mathcal{M}) & \longrightarrow & \mathcal{S}^0_u(X,\mathcal{M}) & \\ & & \mathcal{F} & & \end{array}$$

where $\mathcal{S}^0_{u,\delta_1}(X,\mathcal{M})$ is the ball of radius δ_1 in $\mathcal{S}^0_u(X,\mathcal{M})$, the space of continuous sections covering u introduced in Section 2.4 (see also Section 5.3 in [FdlLM10]).

It is clear that $\mathcal{F}$ has a fixed point in $\mathcal{S}^0_{u,\delta_1}(X,\mathcal{M})$ if and only if $\tilde{\mathcal{F}}$ has a fixed point in $B(u,\delta_1)$.

Using this chart the operator $\mathcal{F} = \mathcal{A}\tilde{\mathcal{F}}\mathcal{A}^{-1} : \mathcal{S}^0_{u,\delta_1}(X,\mathcal{M}) \to \mathcal{S}^0_u(X,\mathcal{M})$ has the form

$$(\mathcal{F}\nu)(x) = \exp^{-1}_{u(x)} \Phi(\exp_{u(H^{-1}(x))} \nu(H^{-1}(x))). \tag{3.3}$$

The map $\mathcal{F}$ can be written as the composition $\mathcal{F} = \mathcal{R} \circ \mathcal{C}$ where

$$(\mathcal{C}\nu)(x) = \mathcal{H}(\nu(x)), \tag{3.4}$$

$$\mathcal{H}(\xi) = \exp^{-1}_{u(H(x))} \Phi(\exp_{u(x)} \xi), \quad \text{if } \xi \in T_{u(x)}\mathcal{M}, \tag{3.5}$$

and

$$(\mathcal{R}\nu)(x) = \nu(H^{-1}(x)) \tag{3.6}$$

is the operator defined by (5.27) in [FdlLM10]. Although $\mathcal{C}$ is similar to $\mathcal{L}$ in (5.26) in [FdlLM10], it has not the same structure and does not satisfy the hypotheses of Proposition 5.6 in [FdlLM10], where the regularity of $\mathcal{L}$ was established. However we have

Proposition 3.2. *If Φ is C^1 and $D\Phi$ is uniformly continuous on $\mathcal{M}$, then $\mathcal{C} : \mathcal{S}^0_{u,\delta_1}(X,\mathcal{M}) \to \mathcal{S}^0_{u\circ H}(X,\mathcal{M})$ is C^1 and*

$$(D\mathcal{C}(\nu)\Delta\nu)(x) = D(\mathcal{H}_{|T_{u(x)}\mathcal{M}})(\nu(x))\Delta\nu(x). \tag{3.7}$$

The proof of this proposition is placed in Appendix A.

Combining Proposition 3.2 with (3) of Proposition (5.7) in [FdlLM10], we have that $\mathcal{F} : \mathcal{S}^0_{u,\delta_1}(X,\mathcal{M}) \to \mathcal{S}^0_u(X,\mathcal{M})$ is C^1.

We will apply a fixed point theorem for perturbations of hyperbolic maps which we quote from [Shu78, Proposition 7.7].

Proposition 3.3. *Let E be a Banach space and $T : E \to E$ a hyperbolic linear map. More concretely let $E = E_1 \oplus E_2$ be a decomposition invariant by T, where E_1, E_2 are closed subspaces. Assume that the norm of E is the max norm of the ones of E_1 and E_2. Let $T_j = T_{|E_j}$ and assume that $\|T_1\| \le \lambda$ and $\|T_2^{-1}\| \le \lambda$. Let $f : B(0,r) \subset E \to E$ be such that $\mathrm{Lip}(f - T) < \varepsilon_1$ and $\|f(0)\| \le \varepsilon_2$ with*

$$\lambda + \varepsilon_1 < 1, \qquad \varepsilon_2 < r(1 - \lambda - \varepsilon_1). \tag{3.8}$$

Then f has a unique fixed point p_f in $B(0,r)$ and

$$\|p_f\| < \frac{1}{1 - \lambda - \varepsilon_1}\|f(0)\|. \tag{3.9}$$

Moreover the map $f \mapsto p_f$ *is continuous from* $\{f \in C^1(B(0,r),E) \mid \mathrm{Lip}(f-T) < 1-\lambda\}$ *to* E.

Theorem 3.1 follows directly from the next result.

Proposition 3.4. *Under the hypotheses of Theorem 3.1, the operator* $\mathcal{F}$ *satisfies the hypotheses of the fixed point theorem for perturbations of hyperbolic maps, Proposition 3.3, with* $f = \mathcal{F}$, T *an auxiliary map to be defined below,* $E = \mathcal{S}_u^0(X,\mathcal{M})$, $E_1 = \mathcal{S}_u^0(X,\mathcal{E}^s)$ $E_2 = \mathcal{S}_u^0(X,\mathcal{E}^u)$ *and the radius* r *small enough.*

The proof of Proposition 3.4 is a consequence of the following two lemmas. First we introduce two auxiliary operators $\mathcal{G}$ and $\tilde{\mathcal{G}}$. For $x, z \in \Delta + \rho_*$ such that $d(F(x), z) < \delta_0$ (as in Section 3.1) let

$$A_{z,x} = D\exp_z^{-1}(F(x))DF(x).$$

We write $A_{z,x}$ with respect to the decomposition $T\mathcal{M} = \mathcal{E}^s \oplus \mathcal{E}^u$ as

$$A_{z,x} = \begin{pmatrix} a_{z,x} & b_{z,x} \\ c_{z,x} & d_{z,x} \end{pmatrix}.$$

Since F is uncoupled $A_{z,x}\xi = (D\exp_{z_i}^{-1}(f(x_i))Df(x_i)\xi)_i$ and we have

$$|a_{z,x}|, |d_{z,x}^{-1}| < \lambda + \delta_*, \qquad |b_{z,x}|, |c_{z,x}| < \delta_*.$$

The decomposition $\mathcal{E}_x^s \oplus \mathcal{E}_x^u$ is not invariant by $A_{z,x}$ if some $x_i \notin \Lambda$. However it is invariant by

$$\tilde{A}_{z,x} = \begin{pmatrix} a_{z,x} & 0 \\ 0 & d_{z,x} \end{pmatrix}.$$

We define $\mathcal{G}, \tilde{\mathcal{G}} : \mathcal{S}_u^0(X,\mathcal{M}) \to \mathcal{S}_u^0(X,\mathcal{M})$ by

$$\begin{aligned} \mathcal{G}\nu(x) &= A_{u(x),u(H^{-1}(x))}\nu(H^{-1}(x)), \\ \tilde{\mathcal{G}}\nu(x) &= \tilde{A}_{u(x),u(H^{-1}(x))}\nu(H^{-1}(x)). \end{aligned}$$

Lemma 3.5. *Under the hypotheses of Theorem 3.1,*

(1) $\tilde{\mathcal{G}}$ *is hyperbolic and close to* $\mathcal{G}$ *in the space of linear operators.*
(2) $\mathcal{F}$ *is Lipschitz close to* $\mathcal{G}$.

Proof. (1) First we note that if $\xi \in \mathcal{E}_x^s$, $|\tilde{\mathcal{A}}_{z,x}\xi| < (\lambda+\delta_*)|\xi|$ and if $\eta \in \mathcal{E}_x^u$, $|\tilde{\mathcal{A}}_{z,x}^{-1}\eta| < (\lambda+\delta_*)|\eta|$. Using the max norm in $\mathcal{E}^s \oplus \mathcal{E}^u$, which is equivalent to the original one, we have

$$\begin{aligned} |\mathcal{G}\nu(x) - \tilde{\mathcal{G}}\nu(x)| &= \sup_i |(\mathcal{G}\nu)_i(x) - (\tilde{\mathcal{G}}\nu)_i(x)| \\ &= \sup_i \left| \begin{pmatrix} b_{u(x),u(H^{-1}(x)),i}\nu_i^u(H^{-1}(x)) \\ c_{u(x),u(H^{-1}(x)),i}\nu_i^s(H^{-1}(x)) \end{pmatrix} \right|, \end{aligned}$$

thus $\|\mathcal{G} - \tilde{\mathcal{G}}\| = \sup_{\|\nu\|\le 1} \|\mathcal{G}\nu - \tilde{\mathcal{G}}\nu\| \le \delta_*$.

(2) To estimate the Lipschitz constant of $\mathcal{F}-\mathcal{G}$ we consider the derivative $D(\mathcal{F}-\mathcal{G})(0)$. Since $\mathcal{G}$ is linear

$$\begin{aligned}
&D(\mathcal{F}-\mathcal{G})(0)\nu(x)\\
&\quad=[D\exp^{-1}_{u(x)}(\Phi u H^{-1}(x))D\Phi(u(H^{-1}(x)))\\
&\qquad\quad - D\exp^{-1}_{u(x)}(F u H^{-1}(x))DF(u(H^{-1}(x)))]\nu(H^{-1}(x))\\
&\quad=[D\exp^{-1}_{u(x)}(\Phi u H^{-1}(x)) - D\exp^{-1}_{u(x)}(F u H^{-1}(x))]D\Phi(u(H^{-1}(x)))\nu(H^{-1}(x))\\
&\qquad + D\exp^{-1}_{u(x)}(F u H^{-1}(x))[D\Phi(u(H^{-1}(x))) - DF(u(H^{-1}(x)))]\nu(H^{-1}(x)).
\end{aligned}$$

Then since $\mathcal{F}$ is C^1 there exists a ball in $\mathcal{S}^0_u$ such that in this ball $D\mathcal{F}-D\mathcal{F}(0)$ is close to zero and hence $\mathrm{Lip}(\mathcal{F}-\tilde{\mathcal{G}})$ is close to zero. □

Lemma 3.6. *$\mathcal{F}(0)$ is small.*

Proof. By the expression of $\mathcal{F}$ in (3.3)

$$(\mathcal{F}0)(x) = \exp^{-1}_{u(x)} \Phi(\exp_{u(H^{-1}(x))} 0) = \exp^{-1}_{u(x)}(\Phi u H^{-1}(x)).$$

Since $d(\Phi u H^{-1}, u)$ is small and $\exp^{-1}_x y$ is uniformly continuous in $\{(x,y) \in M \times M \mid x \in \bar{U}_\Lambda,\ d(x,y) \le \delta_0\}$

$$\exp^{-1}_{u(x)}(\Phi u H^{-1}(x)) = \exp^{-1}_{u(x)}(\Phi u H^{-1}(x)) - \exp^{-1}_{u(x)}(u(x)) \tag{3.10}$$

is small. □

The function $v(x) = \exp_{u(x)} \nu(x)$ satisfies the conclusions of Theorem 3.1. In particular (3.1) follows from

$$d(u_i(x), v_i(x)) = d(\exp^{-1}_{u_i(x)} u_i(x), \exp^{-1}_{u_i(x)} v_i(x)) = |0 - \nu_i(x)|,$$

the fact that, by (3.9), $|\nu| \le \frac{1}{1-\lambda-\varepsilon_1}|\mathcal{F}(0)|$ and (3.10).

This ends the proof of Theorem 3.1. □

3.3. **Conjugation.** Now we can prove that, if $\Phi \in \mathrm{Diff}^{1,0}(\mathcal{M})$ is C^1-close to F, it has an invariant set close to the one of F and the dynamics on the invariant sets are topologically conjugated. In Section 3.6 we will show that the obtained invariant set of Φ is hyperbolic.

Theorem 3.7. *Assume that $\Phi \in \mathrm{Diff}^{1,0}(\mathcal{M})$ and $d_{C^1}(\Phi, F)$ is small enough. Then, there exists $\Delta_\Phi \subset \mathcal{M}$ invariant by Φ and such that $\Phi_{|\Delta_\Phi}$ is topologically conjugate to $F_{|\Delta_F}$, that is, there exists a unique close to the identity homeomorphism $h_\Phi : \Delta_F \to \Delta_\Phi$, such that*

$$\Phi \circ h_\Phi = h_\Phi \circ F_{|\Delta_F}.$$

Moreover, $h_F = \mathrm{Id}$ and h_Φ depends continuously on Φ.

Proof. We apply the shadowing theorem to F with different choices of X, H and u. Let ε, δ be as in the statement of Theorem 3.1, $U_\Delta = \Delta + \rho_*/2$, with ρ_* such that $\Delta + \rho_*$ is contained in the neighborhood V_Δ given by Theorem 3.1, and $\mathcal{V}$ the neighborhood by the same theorem.

First we take $X = \Delta$, $H = F$ and u the inclusion i_Δ from Δ into U_Δ. Then, if $\Phi \in \mathcal{V}$ and $d(i_\Delta \circ F, \Phi \circ i_\Delta) \le d(F, \Phi) < \varepsilon$, there exists a unique $v_1 \in C^0(\Delta, \Delta + \rho_*)$ such that $d(i_\Delta, v_1) < \delta$ and

$$v_1 \circ F_{|\Delta} = \Phi \circ v_1. \tag{3.11}$$

By (3.1) we can take $d(\Phi, F)$ so small that $d(i_\Delta, v_1) < \delta/2$. From condition (3.11) we deduce that $\Delta_\Phi := v_1(\Delta) \subset U$ is invariant by Φ.

Next take $X = \Delta_\Phi$, $H = \Phi$ and u the inclusion i_{Δ_Φ} from Δ_Φ into U_Δ. If $d(i_{\Delta_\Phi} \circ \Phi, F \circ i_{\Delta_\Phi}) \le d(\Phi, F) < \varepsilon$, there exists a unique $v_2 \in C^0(\Delta_\Phi, \Delta + \rho_*)$ such that $d(i_{\Delta_\Phi}, v_2) < \delta/2$ and

$$v_2 \circ \Phi_{|\Delta_\Phi} = F \circ v_2.$$

On the other hand $v_2 \circ v_1$ conjugates F to itself. Since $d(v_2 \circ v_1, \mathrm{Id}_\Delta) \le d(v_2 \circ v_1, i_{\Delta_\Phi} \circ v_1) + d(i_{\Delta_\Phi} \circ v_1, \mathrm{Id}_\Delta) < \delta$ and Id_Δ also conjugates F to itself, by the uniqueness conclusion of Theorem 3.1, we must have $v_2 \circ v_1 = \mathrm{Id}_{|\Delta}$. Analogously $v_1 \circ v_2$ is close to the identity and conjugates Φ to itself, thus it must coincide with $\mathrm{Id}_{|\Delta_\Phi}$. This implies that v_1 is a homeomorphism. We take $h_\Phi = v_1$. The continuous dependence of h on Φ follows from the continuous dependence of v on Φ in Theorem 3.1. □

3.4. **Hyperbolicity on spaces of Hölder sections with decay.** Given a section ν covering $\mathsf{i} : \Delta_F \to \mathcal{M}$, the embedding given by $\mathsf{i}(x) = x$, we define its *push forward* by F as the linear operator

$$F_*(\nu)(x) = DF(F^{-1}(x))\nu(F^{-1}(x)), \qquad x \in \Delta_F. \tag{3.12}$$

Since Δ_F is invariant by F, $F_*(\nu)$ is a well defined section covering i.

Proposition 3.8. *There exists $\alpha^* \le \alpha_f$ such that for any $0 < \alpha \le \alpha^*$, the operator $F_* : S^\alpha_{\mathsf{i},\Gamma} \to S^\alpha_{\mathsf{i},\Gamma}$, where $S^\alpha_{\mathsf{i},\Gamma}$ was introduced in (2.22), is continuous and hyperbolic. In particular, $1 \notin \operatorname{spec} F_*$.*

The proof of this proposition is placed in Appendix C.

3.5. **Hölder regularity of the conjugation.** Here we prove Theorem 2.1. That is, that the conjugation h_Φ obtained in Theorem 3.7 is C^α_Γ. First we reformulate the conjugation problem in terms of some suitable sections to be able to apply the implicit function theorem in Banach spaces.

Proof of Theorem 2.1. Let h_Φ be the conjugation given in Theorem 3.7. We have that h_Φ is close to the identity and depends continuously on Φ. We assume that Φ is so close to F that $d(h_\Phi, \mathrm{Id}) < \delta_0$. Then there exists a unique $\nu \in \mathcal{S}^0$ such that

$$h_\Phi(x) = \exp_x \nu(x).$$

Moreover the fact that $\Phi \circ F^{-1}$ is close to the identity means that there exists a unique $\sigma \in \mathcal{S}^r_\Gamma(\mathcal{M})$ (defined in (2.17)) such that

$$\Phi(F^{-1}(x)) = \exp_x \sigma(x).$$

Hence we can write $\Phi(y) = \exp_{F(y)} \sigma(F(y))$. Then the conjugation condition $\Phi \circ h_\Phi = h_\Phi \circ F$ can be rewritten in the form

$$\exp_x^{-1} \exp_{\exp_x v(\nu)(x)} \sigma(\exp_x v(\nu)(x)) - \nu(x) = 0 \tag{3.13}$$

where

$$v(\nu)(x) = \exp_x^{-1} F(\exp_{F^{-1}(x)} \nu(F^{-1}(x))). \tag{3.14}$$

We are let to introduce the operator

$$\mathcal{F} : \mathcal{U} \subset \mathcal{S}^r_\Gamma(\mathcal{M}) \times \mathcal{S}^\alpha_\Gamma(\Delta_F) \to \mathcal{S}^\alpha_\Gamma(\Delta_F)$$

defined by $\mathcal{F}(\sigma,\nu)$ as the left-hand side of (3.13) on a suitable subset $\mathcal{U}$. It is immediate to check that $\mathcal{F}(0,0)=0$. We also have

Lemma 3.9. (1) *$\mathcal{F}$ is C^{r-3} and the linear map $D_\nu\mathcal{F}(0,0)$ is given by*

$$D_\nu\mathcal{F}(0,0)\Delta\nu(x) = DF(F^{-1}(x))\Delta\nu(F^{-1}(x)) - \Delta\nu(x). \tag{3.15}$$

(2) *There exists $0<\alpha^*\le\alpha_f$ such that, for any $0<\alpha\le\alpha^*$, $D_\nu\mathcal{F}(0,0)$ is invertible from $\mathcal{S}^\alpha_\Gamma$ to $\mathcal{S}^\alpha_\Gamma$.*

Proof. Observe that $\mathcal{F}(\sigma,\nu)=\Omega(\sigma,v(\nu))$, being Ω the map introduced in Proposition 5.11 in [FdlLM10], $\hat{\mathcal{H}}$ was defined in Lemma 5.8 in [FdlLM10], with $\mathsf{j}(u)=\exp_x u$, $u\in T_x\mathcal{M}$ and $J(x,w)=\exp_x^{-1}(\exp_y w)$, $x\in\mathcal{M}$, $w\in T_y\mathcal{M}$ and being $v(\nu)$ the right-hand side of (3.14). Then $\mathcal{F}$ is C^{r-3} since, by Proposition 5.11, Ω is C^∞ and, by Propositions 5.6 and 5.7, in [FdlLM10], v is C^{r-3}. Formula (3.15) follows from Propositions 5.6, 5.7, 5.11 and Lemma 5.8 in [FdlLM10].

(2) follows from (3.12) and Proposition 3.8. ◻

As a consequence of Lemma 3.9 we can apply the implicit function theorem to

$$\mathcal{F}(\sigma,\nu)=0 \tag{3.16}$$

and we obtain that there exists a neighborhood $\mathcal{B}$ of 0 in C^r_Γ such that if Φ is close enough to F then $\sigma(x)=\exp_x^{-1}\Phi(F^{-1}(x))$ belongs to $\mathcal{B}$ and there exists a unique $\nu\in\mathcal{S}^\alpha_\Gamma$ close to 0 such that (σ,ν) satisfies (3.16). Then $h(x)=\exp_x\nu(x)$ is the unique conjugation from F to Φ close to the identity. Since $\nu\in\mathcal{S}^\alpha_\Gamma$ we also have $h\in\mathcal{S}^\alpha_\Gamma$. Moreover ν depends C^{r-3} on σ and therefore h depends C^{r-3} on Φ. ◻

Remark 3.10. Notice that we need the Shadowing Theorem in order to prove that h_Φ is a homeomorphism, since the solutions of (3.16) only provide a semiconjugation. If one tries to formulate the equivalent relation for h_Φ^{-1}, the operator is no longer smooth and the implicit function theorem cannot be applied.

3.6. **Perturbation of hyperbolic sets.** We have the following result

Proposition 3.11. *Let F and Δ be as in Theorem 3.1. There exist neighborhoods U_Δ of Δ in $\mathcal{M}$ and $\mathcal{V}$ of F in $\mathrm{Diff}^{1,0}(\mathcal{M})$ such that if $\Phi\in\mathcal{V}$ and Δ_Φ is an invariant set of Φ in U_Δ then Δ_Φ is hyperbolic.*

Remark 3.12. If F is C^r, $r\ge 4$, as a consequence of Theorem 2.2, in Section 4.1, there will exist a neighborhood $\mathcal{V}$ of F in C^r_Γ such that for any $\Phi\in\mathcal{V}$ the invariant splitting associated to its hyperbolic set is C^α_Γ and each fiber can be described, written as a graph, by means of a L_Γ map between appropriate ℓ^∞ spaces.

Proof. Let $U_1=\Delta+\rho_*$ be the neighborhood of Δ and $T_{U_1}\mathcal{M}=\mathcal{E}^s\oplus\mathcal{E}^u$ the decomposition introduced in Section 3.1. Let

$$\begin{pmatrix}\tilde a(x) & \tilde b(x)\\ \tilde c(x) & \tilde d(x)\end{pmatrix}$$

be the matrix representation of $DF(x)$ with respect to this decomposition for $x\in U_2$ such that $U_2\cap F^{-1}(U_2)\subset U_1$.

Given $\delta > 0$, if $\Phi \in \mathcal{V}$ and $\mathcal{V}$ is small enough,

$$D\Phi(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$$

with

$$\|a(x)\| < \lambda + \delta, \qquad \|b(x)\| < \delta, \qquad \|c(x)\| < \delta, \qquad \|d(x)^{-1}\| < \lambda + \delta.$$

If $\Delta_\Phi \subset U_2$ is a closed invariant set for Φ its unstable space is a fixed point of the usual graph transform of a suitable auxiliary function.

Let $\mathcal{D}$ be the bundle over Δ_Φ such that its fiber is $\mathcal{D}_x = L_1(\mathcal{E}^s_x, \mathcal{E}^u_x)$, where $L_1(\mathcal{E}^s_x, \mathcal{E}^u_x)$ is the unit ball in $L(\mathcal{E}^s_x, \mathcal{E}^u_x)$.

Let $u \in \mathcal{S}^* = \{w \in C^0(\Delta_\Phi, \mathcal{D}) \mid w(x) \in \mathcal{D}_x\}$ with the norm $\|w\|_{\mathcal{S}^*} = \sup_{x\in\Delta_\Phi} \|w(x)\|$. Given a section $u \in \mathcal{S}^*$ we want that

$$\begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \begin{pmatrix} u(x) \\ \mathrm{Id} \end{pmatrix}$$

belongs to the graph of $u(\Phi(x))$, that is,

$$a(x)u(x) + b(x) = u(\Phi(x))(c(x)u(x) + d(x)). \tag{3.17}$$

Let B_1 be the ball of radius 1 in $\mathcal{S}^*$. Then we define $\mathcal{T} : B_1 \to \mathcal{S}^*$ by

$$(\mathcal{T}u)(x) = (au+b)(cu+d)^{-1}(\Phi^{-1}(x)) = (au+b)d^{-1}(d^{-1}cu+\mathrm{Id}\,)^{-1}(\Phi^{-1}(x)).$$

Note that this operator is well defined if δ is small enough. The next calculation shows that if δ is small enough $\mathcal{T}(B_1) \subset B_1$.

$$\begin{aligned} \|\mathcal{T}u\| &\le \sup_{x\in\Delta_\Phi} \|(au+b)(\Phi^{-1}(x))\| \, \|(cu+d)^{-1}(\Phi^{-1}(x))\| \\ &\le (\lambda+2\delta)\frac{\lambda+\delta}{1-(\lambda+\delta)\delta}. \end{aligned}$$

The space $\mathcal{S}^*$ is complete. Now we estimate the Lipschitz constant of $\mathcal{T}$:

$$\begin{aligned} &\|\mathcal{T}(u) - \mathcal{T}(v)\| \\ &\quad = \sup_{x\in\Delta_\Phi} \|[(au+b)(cu+d)^{-1} - (av+b)(cv+d)^{-1}](\Phi^{-1}(x))\| \\ &\quad \le \sup_{y\in\Delta_\Phi} \Big(\|(au+b)(cv+d)^{-1}c(v-u)(cu+d)^{-1}(y)\| + \|a(u-v)(cv+d)^{-1}(y)\|\Big) \\ &\quad \le \Big[(\lambda+2\delta)\Big(\frac{\lambda+\delta}{1-(\lambda+\delta)\delta}\Big)^2\delta + (\lambda+\delta)\frac{\lambda+\delta}{1-(\lambda+\delta)\delta}\Big]\|u-v\|. \end{aligned}$$

Taking δ sufficiently small $\mathcal{T}$ is a contraction. The unique fixed point $\hat{u} \in B_1$ of $\mathcal{T}$ gives the unstable subspaces. Indeed, for every $x \in \Delta_\Phi$, $\hat{u}(x)$ verifies

$$\begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \begin{pmatrix} \hat{u}(x) \\ \mathrm{Id} \end{pmatrix} = \begin{pmatrix} \hat{u}(\Phi(x))(c(x)\hat{u}(x)+d(x)) \\ c(x)\hat{u}(x)+d(x) \end{pmatrix} \in \operatorname{graph} \hat{u}(\Phi(x)).$$

Moreover, if $\xi \in \mathcal{E}^u$, $\xi(x) = (\xi_1(x), \xi_2(x)) = (\hat{u}(x)\xi_2(x), \xi_2(x))$ and

$$D\Phi^{-1}(x)\begin{pmatrix} \hat{u}(x)\xi_2(x) \\ \xi_2(x) \end{pmatrix} = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}^{-1} \begin{pmatrix} \hat{u}(x)\xi_2(x) \\ \xi_2(x) \end{pmatrix} = \begin{pmatrix} \hat{u}(\Phi^{-1}(x))\hat{\xi}_2(x) \\ \hat{\xi}_2(x) \end{pmatrix}.$$

From the previous relation we have $\xi_2(x) = (c(x)\hat{u}(\Phi^{-1}(x)) + d(x))\hat{\xi}_2(x)$, thus $\|\hat{\xi}_2(x)\| = \|[c(x)\hat{u}(\Phi^{-1}(x)) + d(x)]^{-1}\xi_2(x)\| \le \frac{\lambda+\delta}{1-(\lambda+\delta)\delta}\|\xi_2(x)\|$. Then, using the max norm in $\mathcal{E}^u \oplus \mathcal{E}^s$,

$$\|D\Phi^{-1}(x)\xi(x)\| \le (\lambda + 2\delta)\|\xi(x)\|$$

with $\lambda + 2\delta < 1$, if δ is small.

To obtain the stable bundle we begin with the invariance condition for subspaces close to $\mathcal{E}^s$: we want that

$$\begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \begin{pmatrix} \mathrm{Id} \\ u(x) \end{pmatrix}$$

be on the graph of $u(\Phi(x))$, that is,

$$u(\Phi(x))(a(x) + b(x)u(x)) = c(x) + d(x)u(x).$$

To have a contraction we rewrite this condition in the form

$$u(x) = d(x)^{-1}[u(\Phi(x))(a(x) + b(x)u(x)) - c(x)].$$

Then, with analogous arguments as before, the stable bundle is obtained as a fixed point.

Then $T_{\Delta_\Phi}\mathcal{M}$ has a hyperbolic splitting and therefore Δ_Φ is a hyperbolic set. □

4. Stable and unstable manifolds of Δ_Φ

4.1. **Introduction.** In the previous section we have dealt with the structural stability, in the C^r_Γ sense, of uncoupled lattice maps of the form (2.26), with an underlying map possessing a hyperbolic set.

In this section, we will go further in the study of the hyperbolic set Δ_Φ of a C^r_Γ perturbations Φ of an uncoupled lattice map F of such type, describing its invariant stable and unstable manifolds. We recall that, in the finite-dimensional case, the stable and unstable invariant manifolds of a compact hyperbolic invariant set of C^r map are in general C^α on the base point in the hyperbolic set. However, they are foliated by C^r invariant manifolds, corresponding to the invariant manifolds of each point in the set. Here, we will prove that the invariant manifolds of Δ_Φ are C^α_Γ as functions on the base point, and that the leaves corresponding to the invariant manifolds of the points in Δ_Φ are C^{r-3}_Γ, in some appropriate sense.

We remark that in the proof of the decay properties of the manifolds it is essential that Φ is close enough to an uncoupled map F, in the C^r_Γ topology. However, the existence of the manifolds is guaranteed simply by the hyperbolicity.

The rest of the section is devoted to the proof of Theorem 2.2. It will be a consequence of Theorem 4.7, concerning the hyperbolicity of a fixed point of a certain operator acting on spaces of sections, whose invariant manifolds will be closely related to those we are looking for. The procedure will be as follows. In Section 4.2, we will introduce the operator $\mathcal{A}_\Phi$ acting on the space of C^α_Γ sections and on the space of bounded sections. By construction, the zero section will be a fixed point of $\mathcal{A}_\Phi$. We will prove that this fixed point is hyperbolic and, hence, has invariant manifolds.

In order to prove that the zero section is indeed a hyperbolic fixed point of the operator $\mathcal{A}_\Phi$, we will need to construct an appropriate splitting of the space of C^α_Γ sections, that we will carry out in Section 4.3. Once we state Theorem 4.7, we will deduce Theorem 2.2 from the latter, in Section 4.5.

4.2. **The action of Φ on sections.** In this section we will use the exponential map in the form $\exp : U_{\delta_0} \subset T\mathcal{M} \to \mathcal{M} \times \mathcal{M}$, with $U_\delta = \{v \in T\mathcal{M} \mid |v| < \delta\}$, and we will write $\exp(v) = (x, \exp_x v)$, for $v \in T_x\mathcal{M}$, i.e., $x = p(v)$.

Let h_Φ be the conjugation given by Theorem 2.1. We define the action of Φ on a section ν covering h_Φ by

$$\mathcal{A}_\Phi(\nu)(x) = \exp^{-1}_{h_\Phi(x)} \circ \Phi \circ \exp_{h_\Phi \circ F^{-1}(x)} \nu(F^{-1}(x)).$$

Note that the above expression is well defined if $\|\nu\|_{C^b} < \delta_1 := \mathrm{Lip}(\Phi)\delta_0$.

For our purposes, it will be more convenient to rewrite $\mathcal{A}_\Phi$, by using the operators $\mathcal{L}_{H_\Phi}$ and $\mathcal{R}_{F^{-1}}$ introduced in (5.26) and (5.27), resp., in [FdlLM10], as

$$\mathcal{A}_\Phi(\nu) = \mathcal{L}_{H_\Phi} \circ \mathcal{R}_{F^{-1}}(\nu) = H_\Phi \circ \nu \circ F^{-1}, \tag{4.1}$$

where $H_\Phi : U_{\delta_1} \to T\mathcal{M}$ is the function

$$H_\Phi = \exp^{-1} \circ (\Phi \circ \pi_1, \Phi \circ \pi_2) \circ \exp, \tag{4.2}$$

and $\pi_i : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, $i = 1, 2$, are the standard projections in the cartesian product.

Remark 4.1. Notice that, by its definition, the value of $\mathcal{A}_\Phi(\nu)$ at a point x only depends on the value of ν at $F^{-1}(x)$, that is, $\mathcal{A}_\Phi(\nu)(x) = \mathcal{A}_\Phi(\tilde{\nu})(x)$, whenever $\nu(F^{-1}(x)) = \tilde{\nu}(F^{-1}(x))$.

Remark 4.2. The idea to obtain invariant manifolds of hyperbolic sets by applying invariant manifolds theories for spaces of sections goes back to [HP70]. Nevertheless, we call attention that our operator is different from the one used in [HP70]. We have to rearrange the scheme so that the dynamics is referred to the dynamics of the uncoupled map.

Let us denote by $\mathbf{0}$ the zero section covering h_Φ, that is, $\mathbf{0}(x) \in T_{h_\Phi(x)}\mathcal{M}$ is the zero vector. Notice that we can identify $\mathbf{0}(x)$ with $h_\Phi(x)$ by means of the exponential, since $\exp_{h_\Phi(x)} \circ \mathbf{0}(x) = h_\Phi(x)$. From the conjugation relation (2.30), we have that $\mathcal{A}_\Phi(\mathbf{0}) = \mathbf{0}$.

In order to prove the hyperbolic character of $\mathbf{0}$ for $\mathcal{A}_\Phi$, we will need to find a suitable splitting of the spaces of sections under consideration. This is accomplished in the next section.

4.3. **Splitting of spaces of sections.** In this section we start with the invariant hyperbolic splitting of $T_{\Delta_F}\mathcal{M} = \mathcal{E}^u \oplus \mathcal{E}^s$ introduced in (2.29) to find a near invariant hyperbolic splitting, under $\mathcal{A}_\Phi$, of the spaces of sections we will work with.

We recall that, by construction, the projections $\pi^u : T_{\Delta_F}\mathcal{M} \to \mathcal{E}^u$ and $\pi^s : T_{\Delta_F}\mathcal{M} \to \mathcal{E}^s$ satisfy

$$(\pi^u(x)v)_j = \pi^u(x_j)v_j, \qquad (\pi^s(x)v)_j = \pi^s(x_j)v_j,$$

for all $x \in \Delta_F$, $v \in T_x\mathcal{M}$, that is, they are uncoupled maps on each fiber, where $\pi^u : T_{\Lambda_f}M \to TM$ and $\pi^s : T_{\Lambda_f}M \to TM$ are the projections on the invariant hyperbolic splitting of the underlying map f, $E^u \oplus E^s$, introduced in Section 2.5. Furthermore, the dependence of the projections on x is C^{α_f}. This fact is quantified in next lemma and, in particular, in inequality (4.3).

Lemma 4.3. *The vector bundles $\mathcal{E}^u$ and $\mathcal{E}^s$ are C^{α_f}, that is, there exists $C > 0$ such that, for all $x, y \in \Delta$, $v \in T_x\mathcal{M}$, $w \in T_y\mathcal{M}$ and for all $i \in \mathbb{Z}^d$,*

$$|(D\mathsf{e}(x)\pi^u(x)v - D\mathsf{e}(y)\pi^u(y)w)_i| \le C(d^{\alpha_f}(x_i, y_i)\max\{|v_i|, |w_i|\} + |D\mathsf{e}(x_i)v_i - D\mathsf{e}(y_i)w_i|), \tag{4.3}$$

and the same inequality holds for π^s, where $\mathsf{e} : \mathcal{M} \to \ell^\infty(\mathbb{R}^D)$ is the embedding introduced in Section 2.1 (see also (4.10)*, in* [FdlLM10]*), and d is the distance in $T\mathcal{M}$ (see also Section 2.1 and* (4.12) *in* [FdlLM10]*).*

Moreover,

$$|D\mathsf{e}(x)\pi^u(x)v - D\mathsf{e}(y)\pi^u(y)w| \le C(d^{\alpha_f}(x, y)\max\{|v|, |w|\} + |D\mathsf{e}(x)v - D\mathsf{e}(y)w|), \tag{4.4}$$

that is, the splitting is α_f-Hölder.

The proof of Lemma 4.3 is placed in Appendix B.

This splitting of $T_{\Delta_F}\mathcal{M}$ induces a splitting of the space of sections over Δ_F, as described below.

Notice that, if ν is a section covering the embedding $\mathrm{i} : \Delta_F \to \mathcal{M}$, $\mathrm{i}(x) = x$, so are the sections $\pi^s \circ \nu$ and $\pi^u \circ \nu$.

In what follows we will use extensively the embedding $\mathsf{e} : \mathcal{M} \to \ell^\infty(\mathbb{R}^D)$, the left inverse of $D\mathsf{e}$, $\eta : \mathcal{M} \times \ell^\infty(\mathbb{R}^D) \to T\mathcal{M}$ and the connector $\tau : \mathcal{M} \times T\mathcal{M} \to T\mathcal{M}$, introduced in Section 2.1 (see also (4.10), (4.13) and (4.9), resp., in Section 4.2 of [FdlLM10], for more details). In particular, all these maps are uncoupled and, when written in charts, have uniformly bounded derivatives.

Lemma 4.4. *Let ν be a section covering* $\mathrm{i} : \Delta_F \to \mathcal{M}$*. Then*

$$\nu = \pi^s \circ \nu \oplus \pi^u \circ \nu.$$

Furthermore, there exists $C > 0$ such that for any $\nu \in \mathcal{S}^\alpha_{\mathrm{i},\Gamma}(\Delta_F)$, with $\alpha \le \alpha_f$, then $\pi^s \circ \nu, \pi^u \circ \nu \in \mathcal{S}^\alpha_{\mathrm{i},\Gamma}(\Delta_F)$ and

$$\|\pi^u \circ \nu\|_{C^\alpha_\Gamma} \le C\|\nu\|_{C^\alpha_\Gamma}, \qquad \|\pi^s \circ \nu\|_{C^\alpha_\Gamma} \le C\|\nu\|_{C^\alpha_\Gamma}. \tag{4.5}$$

Proof. The first claim follows from $v = \pi^u(x)v \oplus \pi^s(x)v$, for all $v \in T_x\mathcal{M}$.

Now we assume that ν is C^α_Γ. To prove (4.5), we take $j \in \mathbb{Z}^d$, and $x, y \in \Delta_F$ such that $x_i = y_i$, $i \ne j$. Then, using that $\alpha \le \alpha_f$ and inequality (4.3), we have that, for any $i \in \mathbb{Z}^d$,

$$\begin{aligned}
&|\big(D\mathsf{e}(x)\pi^u(x)\nu(x) - D\mathsf{e}(x)\pi^u(y)\nu(y)\big)_i| \\
&\quad = |D\mathsf{e}(x_i)\pi^u(x_i)\nu_i(x) - D\mathsf{e}(x_i)\pi^u(y_i)\nu_i(y)| \\
&\quad \le C\big(d^{\alpha_f}(x_i, y_i)\max\{|\nu_i(x)|, |\nu_i(y)|\} + |D\mathsf{e}(x_i)\nu_i(x) - D\mathsf{e}(y_i)\nu_i(y)|\big) \\
&\quad \le (1 + \Gamma(0)^{-1})C\|\nu\|_{C^\alpha_\Gamma}\Gamma(i - j)d^\alpha(x_j, y_j).
\end{aligned}$$

In the same way, for any $x, y \in \Delta_F$, using (4.4), we obtain that

$$|D\mathsf{e}(x)\pi^u(x)\nu(x) - D\mathsf{e}(x)\pi^u(y)\nu(y)| \le 2C\|\nu\|_{C^\alpha_\Gamma}d^\alpha(x, y),$$

and, hence, the Hölder norm of $\pi^u \circ \nu$ is also bounded. □

We define the vector bundles $\mathcal{E}^u_\Phi$ and $\mathcal{E}^s_\Phi$ by

$$\mathcal{E}^u_{\Phi,x} = \tau(x, h_\Phi(x))\mathcal{E}^u_{F,x}, \qquad \mathcal{E}^s_{\Phi,x} = \tau(x, h_\Phi(x))\mathcal{E}^s_{F,x}, \tag{4.6}$$

where $\tau : U_{\rho_\tau} \subset T\mathcal{M}\times\mathcal{M} \to T\mathcal{M}$ is the connector introduced in Section 2.1 (see also (4.9), in [FdlLM10]).

Notice that $\mathcal{E}^u_{\Phi,x}, \mathcal{E}^s_{\Phi,x} \subset T_{h_\Phi(x)}\mathcal{M}$. Moreover, since τ is a linear isometry on each fiber, we have that $T_{h_\Phi(x)}\mathcal{M} = \mathcal{E}^u_{\Phi,x} \oplus \mathcal{E}^s_{\Phi,x}$. We remark that this splitting is not necessarily invariant, although we will see that it is close to invariant.

We will use this splitting to induce a splitting of the space of sections covering h_Φ.

We define the operator τ_{h_Φ} acting on sections covering h_Φ by

$$\tau_{h_\Phi}\nu(x) = \tau(h_\Phi(x), x)\nu(x). \tag{4.7}$$

Since $\tau(h_\Phi(x), x)$ is a map from $T_{h_\Phi(x)}\mathcal{M}$ to $T_x\mathcal{M}$, $\tau_{h_\Phi}\nu$ is a section covering $\mathrm{i} : \Delta_F \to \Delta_F$.

Analogously, the operator τ^{h_Φ} acting on sections covering $\mathrm{i} : \Delta_F \to \Delta_F$ is defined by

$$\tau^{h_\Phi}\nu(x) = \tau(x, h_\Phi(x))\nu(x). \tag{4.8}$$

Then, $\tau^{h_\Phi}\nu$ is a section covering h_Φ.

Lemma 4.5. *Let $\mathcal{S}^\alpha_{h_\Phi,\Gamma}(\Delta_F)$, $\mathcal{S}^\alpha_{\mathrm{i},\Gamma}(\Delta_F)$, $\mathcal{S}^b_{h_\Phi}(\Delta_F)$ and $\mathcal{S}^b_{\mathrm{i}}(\Delta_F)$ be the spaces of C^α_Γ and bounded sections introduced in* (2.22) *and* (2.19) *(see also* (5.13) *and* (5.8)*, in* [FdlLM10]*, resp). The operators τ_{h_Φ} and τ^{h_Φ} defined above satisfy*

(1) *$\tau_{h_\Phi} : \mathcal{S}^\alpha_{h_\Phi,\Gamma}(\Delta_F) \to \mathcal{S}^\alpha_{\mathrm{i},\Gamma}(\Delta_F)$ and $\tau_{h_\Phi} : \mathcal{S}^b_{h_\Phi}(\Delta_F) \to \mathcal{S}^b_{\mathrm{i}}(\Delta_F)$ are linear and bounded,*
(2) *$\tau^{h_\Phi} : \mathcal{S}^\alpha_{\mathrm{i},\Gamma}(\Delta_F) \to \mathcal{S}^\alpha_{h_\Phi,\Gamma}(\Delta_F)$ and $\tau_{h_\Phi} : \mathcal{S}^b_{\mathrm{i}}(\Delta_F) \to \mathcal{S}^b_{h_\Phi}(\Delta_F)$ are linear and bounded.*

Furthermore, $\tau_{h_\Phi} \circ \tau^{h_\Phi} = \mathrm{Id}$.

Proof. We prove (1), (2) being analogous.

We concentrate in proving that $\tau_{h_\Phi} : \mathcal{S}^\alpha_{h_\Phi,\Gamma}(\Delta_F) \to \mathcal{S}^\alpha_{\mathrm{Id},\Gamma}(\Delta_F)$ is bounded. The case $\tau_{h_\Phi} : \mathcal{S}^b_{h_\Phi}(\Delta_F) \to \mathcal{S}^b_{\mathrm{i}}(\Delta_F)$ is straightforward since $\tau(x, y)$ is an isometry on each fiber and depends C^∞ on x and y.

We compute a bound of the C^α_Γ-norm of $\tau_{h_\Phi}\nu$. We take $j \in \mathbb{Z}^d$, $x, y \in \Delta_F$ such that $x_i = y_i$, for $i \neq j$. Let β and β^p be the curves associated to ν and h_Φ given by Lemma A.2, in [FdlLM10]. Let $\tilde\beta : [0, 1] \to \mathcal{M}$ be the curve defined by $\tilde\beta_i(t) = x_i = y_i$, for $i \neq j$ and

$$\tilde\beta_j(t) = \eta(\beta^p_j(t))\big(tv_{y,j} + (1-t)v_{x,j}\big),$$

where

$$v_{x,j} = D\mathsf{e}(h_{\Phi,j}(x)) \exp^{-1}_{h_{\Phi,j}(x)} x_j, \quad v_{y,j} = D\mathsf{e}(h_{\Phi,j}(y)) \exp^{-1}_{h_{\Phi,j}(y)} y_j$$

and $\eta(z)$ is the left inverse of $D\mathsf{e}(z)$ introduced in (4.4), in [FdlLM10]. There exists $C > 0$ depending only on $\mathcal{M}$ and the choice of η, such that the curve $\tilde\beta$ satisfies $d(\tilde\beta(t), \beta^p(t)) \le C d_{C^0}(h_\Phi, \mathrm{Id})$, $|\dot{\tilde\beta}_i(t)| = 0$, for $i \neq j$

and $|\dot{\tilde{\beta}}_j(t)| \le C\big(\gamma_\alpha(h_\Phi)d^\alpha(x_j,y_j) + d_{C^0}(h_\Phi, \mathrm{Id}\,)\big)$. Indeed, if $i \neq j$, since $\beta^p_i(t)$ is the minimal geodesic joining $h_{\Phi,i}(x)$ and $h_{\Phi,i}(y)$ (see Lemma A.2 in [FdlLM10]) and $x_i = y_i$,

$$\begin{aligned} d(\tilde{\beta}_i(t), \beta^p_i(t)) &= d(x_i, \beta^p_i(t)) \\ &\le d(x_i, h_{\Phi,i}(x)) + d(h_{\Phi,i}(x), \beta^p_i(t)) \\ &\le d(x_i, h_{\Phi,i}(x)) + d(h_{\Phi,i}(x), h_{\Phi,i}(y)) \\ &\le 3d(x_i, h_{\Phi,i}(x)) \le 3d_{C^0}(\mathrm{Id}\,, h_\Phi). \end{aligned}$$

For $i = j$, by construction, for some constants C_1 and C_2 depending only on η, e and exp

$$d(\tilde{\beta}_j(t), \beta^p_j(t)) \le C_1(\|v_{x,j}\| + \|v_{y,j}\|) \le C_2 d_{C^0}(\mathrm{Id}\,, h_\Phi).$$

The bound for $\dot{\tilde{\beta}}_j(t)$ follows from the estimates on $\dot{\beta}^p$ given by Lemma A.2 in [FdlLM10].

By Theorem 2.1, we can assume that $Cd_{C^0}(h_\Phi, \mathrm{Id}\,) < \rho_0$, provided that $d_{C^r_\Gamma}(\Phi, F)$ is small enough.

Then, we can write

$$\begin{aligned} (4.9)\quad & |(D\mathsf{e}(x)\tau_{h_\Phi}\nu(x) - D\mathsf{e}(y)\tau_{h_\Phi}\nu(y))_i| \\ &= |D\mathsf{e}(x_i)\tau(h_{\Phi,i}(x), x_i)\nu_i(x) - D\mathsf{e}(y_i)\tau(h_{\Phi,i}(y), y_i)\nu_i(y)| \\ &= \Big|\int_0^1 \frac{d}{dt}\big(D\mathsf{e}(\tilde{\beta}_i(t))\tau(\beta^p_i(t), \tilde{\beta}_i(t))\beta_i(t)\big)\,dt\Big|. \end{aligned}$$

Hence, for a fixed $t \in (0,1)$, let (U_ϕ, ϕ) be a coordinate chart such that $B_{\rho_0}(\beta^p(t)) \subset U_\phi$. Since $d(\tilde{\beta}(t), \beta^p(t)) < \rho_0$, we have that $\tilde{\beta}(t) \in U_\phi$. Let $\beta_\phi = (\beta^p_\phi, \beta^2_\phi)$, $\tilde{\beta}_\phi$, $\tau_\phi = (\tau^1_\phi, \tau^2_\phi)$ and e_ϕ be the expressions of the involved curves and functions in this chart, following the notation introduced in A.1 in [FdlLM10]. By inequalities (A.1), (A.2) and (A.3), in [FdlLM10], we have that the curves β^p_ϕ, β^2_ϕ have decay around the j component, in the sense introduced, in (2.29) in Section 2.10 of [FdlLM10], and, for some constant C independent of ν and the chart ϕ,

$$\|\dot{\beta}^p_\phi\|_{j,\Gamma} \le Cd^\alpha(x_j,y_j), \quad \|\dot{\beta}^2_\phi\|_{j,\Gamma} \le C\|\nu\|_{C^\alpha_\Gamma} d^\alpha(x_j,y_j), \quad \|\beta^2_\phi\| < C\|\nu\|_{C^\alpha_\Gamma},$$

where the norm

$$\|\beta\|_{j,\Gamma} = \sup_{t\in I}\sup_{l\in\mathbb{Z}^d} |\dot{\beta}_l(t)|\Gamma(l-j)^{-1}$$

was defined in (2.29) in [FdlLM10].

Also, since, by construction, $\dot{\tilde{\beta}}_i = 0$, for $i \neq j$, and the definition of $\dot{\tilde{\beta}}_j$, we have that

$$\|\dot{\tilde{\beta}}^p_\phi\|_{j,\Gamma} \le Cd^\alpha(x_j,y_j).$$

Furthermore, since τ_ϕ and e_ϕ are uncoupled, they are C^r_Γ. Their norm only depends on the manifold $\mathcal{M}$ and the choice of the embedding. Hence, by

Lemma 2.18 in [FdlLM10], we have that

$$
\begin{aligned}
&|\frac{d}{dt}\big(D\mathsf{e}(\tilde\beta_i(t))\tau(\beta_i^p(t),\tilde\beta_i(t))\beta_i(t)\big)| \\
&\qquad\qquad = |\frac{d}{dt}\big(D\mathsf{e}_\phi(\tilde\beta_{\phi,i}(t))\tau_\phi(\beta_{\phi,i}^p(t),\tilde\beta_{\phi,i}(t))\beta_{\phi,i}^2(t)\big)| \\
&\qquad\qquad \le C\|\nu\|_{C^\alpha_\Gamma} d^\alpha(x_j,y_j)\Gamma(i-j)^{-1},
\end{aligned}
$$

for some $C > 0$. Claim (1) follows from inserting this last inequality into (4.9).

Last claim follows from the fact that $\tau(x,y)\circ\tau(y,x) = \mathrm{Id}_{|T_x\mathcal{M}}$. □

From the vector bundles $\mathcal{E}^u_\Phi$ and $\mathcal{E}^s_\Phi$ defined in (4.6), we introduce

$$
\mathcal{S}^{\alpha,u}_{h_\Phi,\Gamma}(\Delta_F) = \{\nu \in \mathcal{S}^{\alpha}_{h_\Phi,\Gamma}(\Delta_F) \mid \nu(x)\in\mathcal{E}^u_{\Phi,x},\ \forall x\in\Delta_F\} \tag{4.10}
$$

and

$$
\mathcal{S}^{\alpha,s}_{h_\Phi,\Gamma}(\Delta_F) = \{\nu \in \mathcal{S}^{\alpha}_{h_\Phi,\Gamma}(\Delta_F) \mid \nu(x)\in\mathcal{E}^s_{\Phi,x},\ \forall x\in\Delta_F\}. \tag{4.11}
$$

The spaces $\mathcal{S}^{b,u}_{h_\Phi}(\Delta_F)$ and $\mathcal{S}^{b,s}_{h_\Phi}(\Delta_F)$ of bounded sections are defined analogously.

Then, we have

Lemma 4.6. *(1) $\mathcal{S}^{b}_{h_\Phi}(\Delta_F) = \mathcal{S}^{b,u}_{h_\Phi}(\Delta_F)\oplus\mathcal{S}^{b,s}_{h_\Phi}(\Delta_F)$ and the projections, $\pi^u_{h_\Phi}$ and $\pi^s_{h_\Phi}$, resp., are continuous.*
(2) $\mathcal{S}^{\alpha}_{h_\Phi,\Gamma}(\Delta_F) = \mathcal{S}^{\alpha,u}_{h_\Phi,\Gamma}(\Delta_F)\oplus\mathcal{S}^{\alpha,s}_{h_\Phi,\Gamma}(\Delta_F)$ and the projections, $\pi^u_{h_\Phi}$ and $\pi^s_{h_\Phi}$, resp., are continuous.

Proof. Since $\pi^u_{h_\Phi}$ can be written as

$$
(\pi^u_{h_\Phi}\nu)(x) = \tau(x,h_\Phi(x))\pi^u(x)\tau(h_\Phi(x),x)\nu(x),
$$

the lemma follows from Lemmas 4.4 and 4.5. □

Using the splittings of $\mathcal{S}^{b}_{h_\Phi}(\Delta_F)$ and $\mathcal{S}^{\alpha}_{h_\Phi,\Gamma}(\Delta_F)$, we define a new norm on these spaces by setting

$$
\|\nu\|'_{C^\alpha_\Gamma} = \max\{\|\pi^u_{h_\Phi}\nu\|_{C^\alpha_\Gamma}, \|\pi^u_{h_\Phi}\nu\|_{C^\alpha_\Gamma}\}, \tag{4.12}
$$

$$
\|\nu\|'_{C^b} = \max\{\|\pi^u_{h_\Phi}\nu\|_{C^b}, \|\pi^u_{h_\Phi}\nu\|_{C^b}\}. \tag{4.13}
$$

Lemma 4.6 implies that the prime norms are equivalent to the original norms in $\mathcal{S}^{\alpha}_{h_\Phi,\Gamma}$ and $\mathcal{S}^{b}_{h_\Phi}$. From now on, we will use norms (4.12) and (4.13) in $\mathcal{S}^{\alpha}_{h_\Phi,\Gamma}$ and $\mathcal{S}^{b}_{h_\Phi}$, resp., which we will denote without prime.

4.4. **Hyperbolicity of the operator $\mathcal{A}_\Phi$.** In this section we state Theorem 4.7 on the hyperbolicity of the zero section of $\mathcal{S}^{\alpha}_{h_\Phi,\Gamma}$ and $\mathcal{S}^{b}_{h_\Phi}$, which is a fixed point of the operator $\mathcal{A}_\Phi$, introduced in (4.1). We will also describe the regularity of the invariant manifolds of the zero section. Theorem 4.7 shows that the invariant manifolds produced in the space of sections enjoy properties that allow to project them to geometric objects in the phase space, namely, the invariant manifolds of Theorem 2.2.

Given $\rho > 0$, we will denote by V^α_ρ and V^b_ρ the balls of radius ρ in $\mathcal{S}^{\alpha}_{h_\Phi,\Gamma}(\Delta_F)$ and $\mathcal{S}^{b}_{h_\Phi}(\Delta_F)$, resp., $V^{\alpha,s}_\rho = V^\alpha_\rho\cap\mathcal{S}^{\alpha,s}_{h_\Phi,\Gamma}(\Delta_F)$ and, analogously, $V^{b,s}_\rho$.

Using the splittings of $\mathcal{S}^{\alpha}_{h_\Phi,\Gamma}(\Delta_F)$ and $\mathcal{S}^{b}_{h_\Phi}(\Delta_F)$ given by Lemma 4.6, we will denote $\nu = (\nu^s, \nu^u)$, where $\nu^u = \pi^u_{h_\Phi}\nu$ and $\nu^s = \pi^s_{h_\Phi}\nu$.

Theorem 4.7. *Assume that Φ is C^r_Γ close to F, $r \geq 4$. Then, there exists $\rho > 0$ such that the following holds true.*
(1) The map $\mathcal{A}_\Phi : V^\alpha_\rho \to \mathcal{S}^{\alpha}_{h_\Phi,\Gamma}(\Delta_F)$ is well defined and C^{r-3}. The zero section in $\mathcal{S}^{\alpha}_{h_\Phi,\Gamma}(\Delta_F)$ is a hyperbolic fixed point of $\mathcal{A}_\Phi$. Let $W^{\alpha,s}_{loc}$ denote its local stable invariant manifold. There exists a C^{r-3} function $\Psi^{\alpha,s} : V^{\alpha,s}_\rho \to \mathcal{S}^{\alpha,u}_{h_\Phi,\Gamma}(\Delta_F)$ such that

$$W^{\alpha,s}_{loc} \cap V^\alpha_\rho = \{(\nu^s, \Psi^s_\alpha(\nu^s)) \mid \nu^s \in V^{\alpha,s}_\rho\}.$$

Moreover, if $\nu^s, \tilde{\nu}^s \in V^{\alpha,s}_\rho$ satisfy $\nu^s(x) = \tilde{\nu}^s(x)$, for some $x \in \Delta_F$, then $\Psi^{\alpha,s}(\nu^s)(x) = \Psi^{\alpha,s}(\tilde{\nu}^s)(x)$.
(2) The map $\mathcal{A}_\Phi : V^b_\rho \to \mathcal{S}^{b}_{h_\Phi}(\Delta_F)$ is well defined and C^{r-2}_Γ. The zero section in $\mathcal{S}^{b}_{h_\Phi}(\Delta_F)$ is a hyperbolic fixed point of $\mathcal{A}_\Phi$. Let $W^{b,s}_{loc}$ denote its local stable invariant manifold. There exists a C^{r-3}_Γ function $\Psi^{b,s} : V^{b,s}_\rho \to \mathcal{S}^{b,u}_{h_\Phi}(\Delta_F)$ such that

$$W^{b,s}_{loc} \cap V^b_\rho = \{(\nu^s, \Psi^{b,s}(\nu^s)) \mid \nu^s \in V^{b,s}_\rho\}.$$

Moreover, if $\nu^s, \tilde{\nu}^s \in V^{b,s}_\rho$ satisfy $\nu^s(x) = \tilde{\nu}^s(x)$, for some $x \in \Delta_F$, then $\Psi^{b,s}(\nu^s)(x) = \Psi^{b,s}(\tilde{\nu}^s)(x)$.
(3) $\Psi^{\alpha,s} = \Psi^{b,s}{}_{|V^{\alpha,s}_\rho}$.
(4) For any $x \in \Delta_F$ and $v \in \mathcal{E}^s_{\Phi,x}$ with $\|v\| < \rho$, the map, induced by $D\Psi^{b,s}$,

$$A_\Psi(v) : \mathcal{E}^s_{\Phi,x} \to \mathcal{E}^u_{\Phi,x} : w \mapsto D\Psi^{b,s}(\nu)\hat{\nu}(x),$$

where $\nu^s \in V^{b,s}_\rho$ and $\hat{\nu}^s \in \mathcal{S}^{b,s}_{h_\Phi}(\Delta_F)$ are such that $\nu^s(x) = v$ and $\hat{\nu}^s(x) = w$, is well defined.
(5) The map $A_\Psi(v)$ is Γ-linear and its Γ-norm is uniformly bounded in $x \in \Delta_F$.

4.5. **Deduction of Theorem 2.2 from Theorem 4.7.** We start by introducing two auxiliary lemmas we will use in the deduction of Theorem 2.2 from Theorem 4.7.

First we claim that we can construct (uncoupled) C^α sections covering h_Φ with a prescribed value at a given point, and that this construction is linear (hence, regular) on each fiber of $T_{\Delta_F}\mathcal{M}$. Actually, the map $\Omega^s_{h_\Phi}$ introduced in the next lemma assigns to every $v \in \mathcal{E}^s_{F,x}$ a C^α section ν such that $\nu(x)$ is the transport of v to $h_\Phi(x)$ by the connector τ.

Lemma 4.8. *There exists a map $\Omega^s_{h_\Phi} : \mathcal{E}^s_F \to \mathcal{S}^{\alpha,s}_{h_\Phi,\Gamma}(\mathcal{M})$ with the following properties:*

1. *it is Lipschitz,*
2. *$\Omega^s_{h_\Phi}(v)(x) = \tau(x, h_\Phi(x))v$, for all $v \in \mathcal{E}^s_{F,x} \subset T_x\mathcal{M}$ and*
3. *it is linear on each fiber.*

In particular, $\Omega^s_{h_\Phi}$ is C^∞ on each fiber.

Furthermore, $\Omega^s_{h_\Phi}$ is uncoupled in the following sense: given $j \in \mathbb{Z}^d$, if $x, \tilde{x} \in \Delta_F$ satisfy $x_i = \tilde{x}_i$ whenever $i \neq j$, and $v \in T_x\mathcal{M}$, $\tilde{v} \in T_{\tilde{x}}\mathcal{M}$ satisfy $v_i = \tilde{v}_i$, for $i \neq j$ (this comparison makes sense since $T_x\mathcal{M} = \ell^\infty(T_{x_i}M)$), then $((\Omega^s_{h_\Phi}(v) - \Omega^s_{h_\Phi}(\tilde{v}))(z))_i = 0$, for $i \neq j$, $z \in \Delta_F$.

The proof of this lemma is deferred to Appendix E.

Next we check the regularity of the evaluation operator.

Lemma 4.9. *(1) The map* ev : $\mathcal{S}^{\alpha}_{h_\Phi,\Gamma}(\Delta_F) \times \Delta_F \to T_{\Delta_\Phi}\mathcal{M}$ *defined by* $\mathrm{ev}(\nu, x) = \nu(x)$ *is continuous. Moreover, it is linear with respect its first variable – which implies it is* C^∞ *with respect to its first variable – and* C^α_Γ *with respect to* x *and*

$$\gamma_\alpha(\mathrm{ev}(\nu,\cdot)) \le \max\{\gamma_\alpha(h_\Phi), \|\nu\|_{C^\alpha_\Gamma}\}. \tag{4.14}$$

In particular, ev *is* α*-Hölder.*

(2) The map ev : $\mathcal{S}^{b}_{h_\Phi}(\Delta_F) \times \Delta_F \to T_{\Delta_\Phi}\mathcal{M}$ *defined by* $\mathrm{ev}(\nu, x) = \nu(x)$ *is linear with respect its first variable and* $\|\mathrm{ev}(\cdot, x)\| \le 1$*. It is uncoupled with respect to the identification* $\mathcal{S}^{b}_{h_\Phi}(\Delta_F) = \ell^\infty(\mathcal{S}^{b}_{h_\Phi}(\Delta_F))$*, that is,* $\mathrm{ev}(\nu, x)_i = \nu_i(x_i)$*.*

Proof. We recall that the distance in $T\mathcal{M}$ was defined as

$$d(v, \tilde v) = \max\{d(x, \tilde x), \|D\mathsf{e}(x)v - D\mathsf{e}(\tilde x)\tilde v\|\},$$

for $v \in T_x\mathcal{M}$, $\tilde v \in T_{\tilde x}\mathcal{M}$.

In order to prove (1), let $(\nu, x), (\tilde\nu, \tilde x) \in \mathcal{S}^{\alpha}_{h_\Phi,\Gamma}(\Delta_F) \times \Delta_F$. Then

$$\begin{aligned} d(\mathrm{ev}(\nu, x), \mathrm{ev}(\tilde\nu, \tilde x)) \\ = \max\{d(h_\Phi(x), h_\Phi(\tilde x)), \|D\mathsf{e}(h_\Phi(x))\nu(x) - D\mathsf{e}(h_\Phi(\tilde x))\tilde\nu(\tilde x)\|\}. \end{aligned} \tag{4.15}$$

Since h_Φ is C^α, we have that $d(h_\Phi(x), h_\Phi(\tilde x)) \le H(h_\Phi) d^\alpha(x, \tilde x)$. Moreover

$$\begin{aligned} &\|D\mathsf{e}(h_\Phi(x))\nu(x) - D\mathsf{e}(h_\Phi(\tilde x))\tilde\nu(\tilde x)\| \\ &\quad\le \|D\mathsf{e}(h_\Phi(x))\nu(x) - D\mathsf{e}(h_\Phi(\tilde x))\nu(\tilde x)\| + \|D\mathsf{e}(h_\Phi(\tilde x))(\nu(\tilde x) - \tilde\nu(\tilde x))\| \\ &\quad\le \|\nu\|_{C^\alpha} d^\alpha(x, \tilde x) + C\|\nu - \tilde\nu\|_{C^b}, \end{aligned}$$

which yields the continuity of ev. By definition, it is linear with respect to ν.

To finish the proof of (1), it only remains to see that ev is C^α_Γ with respect to x. To this end, we take $\nu \in \mathcal{S}^{\alpha}_{h_\Phi,\Gamma}(\Delta_F)$ and $j \in \mathbb{Z}^d$ and we let $x, \tilde x \in \Delta_F$ such that $x_k = \tilde x_k$, for $k \ne j$. By the C^α_Γ continuity of h_Φ, we have that $d(h_{\Phi,i}(x), h_{\Phi,i}(\tilde x)) \le \gamma_\alpha(h_\Phi)\Gamma(i-j) d^\alpha(x_j, \tilde x_j)$. Moreover,

$$\begin{aligned} &\|(D\mathsf{e}(h_\Phi(x))\nu(x) - D\mathsf{e}(h_\Phi(\tilde x))\nu(\tilde x))_i\| \\ &\qquad\le \|D\mathsf{e}(h_{\Phi,i}(x))\nu_i(x) - D\mathsf{e}(h_{\Phi,i}(\tilde x))\nu_i(\tilde x)\| \\ &\qquad\le \|\nu\|_{C^\alpha_\Gamma}\Gamma(i-j) d^\alpha(x_j, \tilde x_j). \end{aligned}$$

Hence, in view of (4.15),

$$d(\mathrm{ev}(\nu, x)_i, \mathrm{ev}(\nu, \tilde x)_i) \le \max\{\gamma_\alpha(h_\Phi), \|\nu\|_{C^\alpha_\Gamma}\}\Gamma(i-j) d^\alpha(x_j, \tilde x_j).$$

To prove (2), note that the linearity is obvious and , for any $\nu \in \mathcal{S}^{b}_{h_\Phi}(\Delta_F)$,

$$\|\nu(x)\| \le \|\nu\|, \qquad \text{for all } x \in \Delta_F.$$

By definition, the map is uncoupled. □

Deduction of Theorem 2.2 from Theorem 4.7. Let $\Omega^s_{h_\Phi}$ be the map given in Lemma 4.8, and $\Psi^{\alpha,s}$ be the map such that its graph is invariant by $\mathcal{A}_\Phi$, given in Theorem 4.7. We define, for $x \in \Delta_F$, $v \in \mathcal{E}^s_{F,x} \subset T_x\mathcal{M}$,

$$\Upsilon^s(v) = \pi_2 \circ \exp \circ\, \mathrm{ev} \circ ((\mathrm{Id}\,, \Psi^{\alpha,s}) \circ \Omega^s_{h_\Phi}(v), x). \tag{4.16}$$

To see (1) note that, by construction, since the graph of $\Psi^{\alpha,s}$ is invariant by $\mathcal{A}_\Phi$, the range of Υ^s is invariant by Φ. More concretely, we claim that, if $v \in \mathcal{E}^s_{F,x}$, $\|v\| \leq \rho$,

$$\Phi(\Upsilon^s(v)) = \Upsilon^s(w),$$

where

$$w = \pi^s_{F(x)} \circ \tau(h_\Phi \circ F(x), F(x)) \circ \exp^{-1}_{h_\Phi \circ F(x)} \Phi(\Upsilon^s(v))$$

and $\pi^s_{F(x)} : T_{F(x)}\mathcal{M} \to \mathcal{E}^s_{F,F(x)}$ is the natural projection. Note that $w \in \mathcal{E}^s_{F,F(x)}$. Indeed, for $\nu_v = (\Omega^s_{h_\Phi}(v), \Psi^{\alpha,s} \circ \Omega^s_{h_\Phi}(v))$ we have that

$$\mathcal{A}_\Phi(\nu_v)(F(x)) = \exp^{-1}_{h_\Phi \circ F(x)} \Phi(\Upsilon^s(v)).$$

Since ν_v lies on the stable manifold of $\mathbf{0}$ for $\mathcal{A}_\Phi$, the claim follows.

We remark that the definition of Υ^s in (4.16) is independent of the choice of the function $\Omega^s_{h_\Phi}$, as soon as satisfies the properties listed in Lemma 4.8.

Let us denote $(\mathrm{Id}\,, \Psi^{\alpha,s})$ by Ψ during the rest of the proof.

Since $\Psi^{\alpha,s}$ parameterizes the local stable manifold of the zero section in $\mathcal{S}^\alpha_{h_\Phi,\Gamma}$, for all $x \in \Delta_F$ and $v \in \mathcal{E}^s_{F,x}$ with $\|v\| \leq \rho$, $d(\Phi^n \circ h_\Phi(x), \Phi^n \circ \Upsilon^s(v)) \to 0$, when $n \to \infty$. This proves (2) of Theorem 2.2.

Now we check (3) of Theorem 2.2. First we note that, by Lemmas 4.8 and 4.9, since $\Psi^{\alpha,s}$ is C^{r-3}, we have that Υ^s is α-Hölder. Next, we take $j \in \mathbb{Z}^d$, $x, \tilde{x} \in \Delta_F$ such that $x_i = \tilde{x}_i$ for $i \neq j$, and $v \in T_x\mathcal{M}$, $\tilde{v} \in T_{\tilde{x}}\mathcal{M}$ such that $v_i = \tilde{v}_i$, for $i \neq j$. We first observe that, since $\Omega^s_{h_\Phi}$ is uncoupled,

$$\Omega^s_{h_\Phi}(v)_i(z) = \Omega^s_{h_\Phi}(\tilde{v})_i(z), \quad \text{for all } z \in \Delta_F \text{ and } i \neq j. \tag{4.17}$$

On the other hand, since exp is uncoupled and uniformly C^∞, there exists $C > 0$, depending only on M, such that

$$\begin{aligned} d(\Upsilon^s(v)_i, \Upsilon^s(\tilde{v})_i) \leq & C d(\mathrm{ev}_i \circ (\Psi \circ \Omega^s_{h_\Phi}(v), x), \mathrm{ev}_i \circ (\Psi \circ \Omega^s_{h_\Phi}(\tilde{v}), \tilde{x})) \\ \leq & C\big(d(\mathrm{ev}_i \circ (\Psi \circ \Omega^s_{h_\Phi}(v), x), \mathrm{ev}_i \circ (\Psi \circ \Omega^s_{h_\Phi}(v), \tilde{x})) \\ & + d(\mathrm{ev}_i \circ (\Psi \circ \Omega^s_{h_\Phi}(v), \tilde{x}), \mathrm{ev}_i \circ (\Psi \circ \Omega^s_{h_\Phi}(\tilde{v}), \tilde{x}))\big). \end{aligned} \tag{4.18}$$

By Lemma 4.9, the map ev is C^α_Γ with respect to its second variable, and, by (4.14),

$$\begin{aligned} & d(\mathrm{ev}_i \circ (\Psi \circ \Omega^s_{h_\Phi}(v), x), \mathrm{ev}_i \circ (\Psi \circ \Omega^s_{h_\Phi}(v), \tilde{x})) \\ & \qquad \leq \max\{\gamma_\alpha(h_\Phi), \|\Psi \circ \Omega^s_{h_\Phi}(v)\|_{C^\alpha_\Gamma}\}\Gamma(i-j) d^\alpha(x_j, \tilde{x}_j). \end{aligned} \tag{4.19}$$

On the other hand, we recall that

$$\begin{aligned} & d(\mathrm{ev}_i \circ (\Psi \circ \Omega^s_{h_\Phi}(v), \tilde{x}), \mathrm{ev}_i \circ (\Psi \circ \Omega^s_{h_\Phi}(\tilde{v}), \tilde{x}) \\ & \qquad = \max\{d(x_i, \tilde{x}_i), \|(\Psi \circ \Omega^s_{h_\Phi}(v)(\tilde{x}) - \Psi \circ \Omega^s_{h_\Phi}(\tilde{v})(\tilde{x}))_i\|)\} \end{aligned}$$

and, denoting $\nu = \Omega^s_{h_\Phi}(v)$ and $\tilde{\nu} = \Omega^s_{h_\Phi}(\tilde{v})$,

$$\|(\Psi \circ \nu(\tilde{x}) - \Psi \circ \tilde{\nu}(\tilde{x}))_i\| = \|\Big(\int_0^1 D\Psi(\tilde{\nu} + t(\nu - \tilde{\nu}))(\nu - \tilde{\nu})(\tilde{x})\, dt\Big)_i\|. \tag{4.20}$$

By Theorem 4.7, $D\Psi(\hat{\nu})$ induces a L_Γ map at each point z, with $\|D\Psi(\hat{\nu})(z)\|_\Gamma \leq C$, for some positive constant C. Hence, since, by (4.17), $(\nu(\tilde{x}) - \tilde{\nu}(\tilde{x}))_k = 0$

for $k \neq j$, applying (1) in Lemma 4.8, we have that, for some constants $C, \tilde{C} > 0$,

$$\begin{aligned}\|(\Psi \circ \nu(\tilde{x}) - \Psi \circ \tilde{\nu}(\tilde{x}))_i\| &\leq C\|\Omega^s_{h_\Phi}(v)_i(\tilde{x}) - \Omega^s_{h_\Phi}(\tilde{v})_i(\tilde{x})\|\Gamma(i-j)\\ &\leq \tilde{C}\Gamma(i-j)\|v_j - \tilde{v}_j\|.\end{aligned}$$

Inserting this last inequality into (4.20), the claim is proven.

(4) follows from the regularity of each map in the definition of Υ^s in (4.16) given by Theorem 4.7, Lemmas 4.8 and 4.9.

(5) follows from the invariance of range(Υ^s) and the fact that is C^{r-2}_Γ on each fiber. □

4.6. **Proof of Theorem 4.7.** Let $\mathbf{0}$ denote the zero section in $\mathcal{S}^\alpha_{h_\Phi,\Gamma} \cap \mathcal{S}^b_{h_\Phi}$. By construction, $\mathcal{A}_\Phi(\mathbf{0}) = \mathbf{0}$. The rest of the claim follows from proving that $\mathbf{0}$ is a hyperbolic fixed point for $\mathcal{A}_\Phi$ in $\mathcal{S}^\alpha_{h_\Phi,\Gamma}$ if $d_{C^r_\Gamma}(F,\Phi)$ is small enough. To do so, we will see that $\mathcal{A}_\Phi$ is close, in the appropriate topology, to a map for which $\mathbf{0}$ is a hyperbolic fixed point.

To begin with, we consider the map $\mathcal{A}_F(\nu) = H_F \circ \nu \circ F^{-1}$ (see formulas (4.1) and (4.2)). By Propositions 5.6 and 5.7 in [FdlLM10], $\mathcal{A}_F$ is C^{r-3} when considered acting on $\mathcal{S}^\alpha_{\mathrm{Id},\Gamma}$ sections and C^{r-2}_Γ when acting on $\mathcal{S}^b_{\mathrm{Id}}$. Furthermore, by (5.28) and Proposition 5.7 in [FdlLM10],

$$(D\mathcal{A}_F(\mathbf{0})\nu)(x) = DF(F^{-1}(x))\nu(F^{-1}(x)). \tag{4.21}$$

By Lemma 4.4, $\mathcal{S}^\alpha_{\mathrm{Id},\Gamma} = \mathcal{S}^{\alpha,s}_{\mathrm{Id},\Gamma} \oplus \mathcal{S}^{\alpha,u}_{\mathrm{Id},\Gamma}$. Using this decomposition, since the splitting $\mathcal{E}^u_F \oplus \mathcal{E}^s_F$ is invariant by F, we can write

$$D\mathcal{A}_F(\mathbf{0}) = \begin{pmatrix} A_{F,s,s} & 0 \\ 0 & A_{F,u,u}\end{pmatrix}, \tag{4.22}$$

where $A_{F,s,s} = \pi_{\mathcal{S}^{\alpha,s}_{\mathrm{Id},\Gamma}} \circ D\mathcal{A}_F(\mathbf{0}) \circ \pi_{\mathcal{S}^{\alpha,s}_{\mathrm{Id},\Gamma}}$, and $A_{F,u,u}$ is defined analogously.

Lemma 4.10. *Under the standing hypotheses on F, for any $0 < \alpha \leq \alpha_f$, $A_{F,u,u} : \mathcal{S}^{\alpha,u}_{\mathrm{Id},\Gamma} \to \mathcal{S}^{\alpha,u}_{\mathrm{Id},\Gamma}$, $A_{F,s,s} : \mathcal{S}^{\alpha,s}_{\mathrm{Id},\Gamma} \to \mathcal{S}^{\alpha,s}_{\mathrm{Id},\Gamma}$ are bounded linear maps and $A_{F,u,u}$ is invertible.*

Moreover, there exist $0 < \alpha^* < \alpha_f$ such that, for any $0 < \alpha < \alpha^*$ and any $0 < \tilde{\lambda} < 1$, there exists $N > 0$ such that

$$\|A^{-1}_{F^N,u,u}\| \leq \tilde{\lambda}, \quad \|A_{F^N,s,s}\| \leq \tilde{\lambda}. \tag{4.23}$$

The proof of this lemma is deferred to Appendix C.

Note that $\mathcal{A}_{F^N} = \mathcal{A}^N_F$.

We would like to compare $\mathcal{A}_{F^N}$ with $\mathcal{A}_{\Phi^N}$. However, a direct comparison is impossible since the operators act on different spaces. Because of this reason we introduce

$$\tilde{\mathcal{A}}_{F^N} = \tau_{h_\Phi} \circ \mathcal{A}_{F^N} \circ \tau^{h_\Phi}, \tag{4.24}$$

with the operators τ_{h_Φ} and τ^{h_Φ} defined in (4.7) and (4.8), resp. Since, by Lemma 4.5, these operators are linear and bounded, $D\tilde{\mathcal{A}}_{F^N}(0) = \tau_{h_\Phi} \circ D\mathcal{A}_{F^N}(0) \circ \tau^{h_\Phi}$. Using the decomposition $\mathcal{S}^\alpha_{h_\Phi,\Gamma} = \mathcal{S}^{\alpha,s}_{h_\Phi,\Gamma} \oplus \mathcal{S}^{\alpha,u}_{h_\Phi,\Gamma}$ and the fact that, by construction, $\tau^{h_\Phi}(\mathcal{S}^{\alpha,u,s}_{h_\Phi,\Gamma}) = \mathcal{S}^{\alpha,u,s}_{\mathrm{Id},\Gamma}$, we have that

$$D\tilde{\mathcal{A}}_{F^N}(\mathbf{0}) = \begin{pmatrix} \tilde{A}_{F^N,s,s} & 0 \\ 0 & \tilde{A}_{F^N,u,u}\end{pmatrix}. \tag{4.25}$$

Lemma 4.10 implies that

$$\|\tilde{A}^{-1}_{F^N,u,u}\|, \|\tilde{A}_{F^N,s,s}\| \le \|\tau_{h_\Phi}\|\|\tau^{h_\Phi}\|\tilde{\lambda}. \tag{4.26}$$

We choose $\tilde{\lambda}$ in Lemma 4.10 such that $0 < \|\tau_{h_\Phi}\|\|\tau^{h_\Phi}\|\tilde{\lambda} < 1$.

Lemma 4.11. *Assume $r \ge 4$. There exist $C > 0$ and $\varepsilon_0 > 0$ such that, if $d_{C^r_\Gamma}(\Phi, F) < \varepsilon_0$,*

(1) $\|D\mathcal{A}_\Phi(0) - D\tilde{\mathcal{A}}_F(0)\|_{L(\mathcal{S}^\alpha_{h_\Phi,\Gamma}, \mathcal{S}^\alpha_{h_\Phi,\Gamma})} < C d_{C^r_\Gamma}(\Phi, F)$,

(2) $\|D\mathcal{A}_\Phi(0) - D\tilde{\mathcal{A}}_F(0)\|_{L_\Gamma(\mathcal{S}^b_{h_\Phi}, \mathcal{S}^b_{h_\Phi})} < C d_{C^r_\Gamma}(\Phi, F)$.

The proof of Lemma 4.11 is rather cumbersome and is placed in Appendix D.

Now, after Lemma 4.11, Theorem 4.7 follows almost immediately.

Indeed, from (1) in Propositions 5.6 and 5.7 in [FdlLM10], we have that $\mathcal{A}_\Phi$ is a C^{r-3} map from some ball around $\mathbf{0}$ in $\mathcal{S}^\alpha_{h_\Phi,\Gamma}$ to $\mathcal{S}^\alpha_{h_\Phi,\Gamma}$. Moreover, by (1) in Lemma 4.11, we have that, if $d_{C^r_\Gamma}(\Phi, F)$ is small enough, the zero section $\mathbf{0} \in \mathcal{S}^\alpha_{h_\Phi,\Gamma}$ is a hyperbolic fixed point for $\mathcal{A}_\Phi$. Hence, it possesses a C^{r-3} stable invariant manifold. More concretely, there exists a C^{r-3} map $\Psi^{\alpha,s} : V^\alpha_\rho \subset \mathcal{S}^{\alpha,s}_{h_\Phi,\Gamma} \to \mathcal{S}^{\alpha,u}_{h_\Phi,\Gamma}$ whose graph is invariant by $\mathcal{A}_\Phi$. Furthermore, it is well known that this function $\Psi^{\alpha,s}$ is an attracting fixed point of the graph transform operator in the space of C^{r-4} maps, $\Psi \mapsto \mathcal{G}(\Psi)$, with

$$\begin{aligned}\mathcal{G}(\Psi)(\nu^s) = A^{-1}_{u,u}\big(\Psi(A_{s,s}\nu^s + A_{s,u}\Psi(\nu^s) + R_s(\nu^s, \Psi(\nu^s)) \\ - A_{u,s}\nu^s - R_u(\nu^s, \Psi(\nu^s))\big),\end{aligned} \tag{4.27}$$

where $\mathcal{A}_\Phi = D\mathcal{A}_\Phi(0) + \mathcal{R}$, $D\mathcal{A}_\Phi(0) = \begin{pmatrix} A_{s,s} & A_{s,u} \\ A_{u,s} & A_{u,u} \end{pmatrix}$ and $\mathcal{R} = (R_s, R_u)$. Now, since $\mathcal{A}_\Phi(\nu)(x) = \mathcal{A}_\Phi(\tilde{\nu})(x)$, whenever $\nu(F^{-1}(x)) = \tilde{\nu}(F^{-1}(x))$ (see Remark 4.1) and in view of (4.27), we have that, if Ψ satisfies also $\Psi(\nu^s)(x) = \Psi(\tilde{\nu}^s)(x)$, whenever $\nu^s(x) = \tilde{\nu}^s(x)$ (for instance, if $\Psi = \mathbf{0}$), then so does $\Psi^k = \mathcal{G}^k(\Psi)$. Hence, $\Psi^{\alpha,s} = \lim_{k\to\infty} \Psi^k$ satisfies the same property. This concludes the proof of (1) in Theorem 4.7.

Now we prove (2) in Theorem 4.7. From (2) in Propositions 5.6 and 5.7 in [FdlLM10], we have that $\mathcal{A}_\Phi$ is a C^{r-2}_Γ map from some ball around $\mathbf{0}$ in $\mathcal{S}^b_{h_\Phi}$ to $\mathcal{S}^b_{h_\Phi}$. Hence, by (2) in Lemma 4.11 and Theorem 3.1 in [FdlLM10], the invariant manifolds of $\mathbf{0} \in \mathcal{S}^b_{h_\Phi}$ are C^{r-3}_Γ, that is, there exists a C^{r-3}_Γ map $\Psi^{b,s} : V^b_\rho \subset \mathcal{S}^{b,s}_{h_\Phi} \to \mathcal{S}^{b,u}_{h_\Phi}$ whose graph is invariant by $\mathcal{A}_\Phi$. This map coincides with $\Psi^{\alpha,s}$ in the intersection of their domains, since both are attracting fixed points of the same operator $\mathcal{G}$, $\mathcal{S}^\alpha_{h_\Phi,\Gamma} \subset \mathcal{S}^0_{h_\Phi}$ and the inclusion is continuous. With the same argument as before, we have that $\Psi^{b,s}(\nu^s)(x) = \Psi^{b,s}(\tilde{\nu}^s)(x)$, whenever $\nu^s(x) = \tilde{\nu}^s(x)$.

(3) follows from the fact that $\mathcal{S}^\alpha_\Gamma \subset \mathcal{S}^b$ and the inclusion is continuous.

(4) and (5) simply follow from the fact that $\Psi^{\alpha,s}(\nu)(x)$ only depends on the the value of ν at x and the fact that $\Psi^{b,s}$ is a C^{r-2}_Γ map. □

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Appendix A. Proof of Proposition 3.2

Instead of working with the explicit expression of $\mathcal{H}$, we highlight the set of its properties that permit to prove C^1 regularity. We have that $\mathcal{H} : T_{u(X)}\mathcal{M} \to T_{u(X)}\mathcal{M}$ satisfies

(1) $\mathcal{H}$ is continuous,
(2) for all $x \in X$, $\mathcal{H}(T_{u(x)}\mathcal{M}) \subset T_{u(h(x))}\mathcal{M}$,
(3) for all $x \in X$, $\mathcal{H}_{|T_{u(x)}\mathcal{M}}$ is C^1 and
(4) $D(\mathcal{H}_{|T_{u(x)}\mathcal{M}})$ is uniformly continuous and its modulus of continuity is bounded independently of $x \in X$.

The first three properties are quite obvious. The fourth one follows from the fact that exp and $\exp^{-1}$ are uncoupled, M is compact and Φ is C^1 with $D\Phi$ uniformly continuous.

First we check that $\mathcal{C}$ is differentiable. Let $\nu \in \mathcal{S}^0_{u,\delta_1}$ and $\Delta\nu \in \mathcal{S}^0_u$ with $\|\Delta\nu\|$ small.

We have that

$$
\begin{aligned}
(\mathcal{C}(\nu + \Delta\nu) - \mathcal{C}(\nu) - D\mathcal{C}(\nu)\Delta\nu)(x) \\
&= \mathcal{H}(\nu(x) + \Delta\nu(x)) - \mathcal{H}(\nu(x)) - D(\mathcal{H}_{|T_{u(x)}\mathcal{M}})(\nu(x))\Delta\nu(x) \\
&\quad + R(\nu, \Delta\nu)(x),
\end{aligned}
$$

where

$$
R(\nu, \Delta\nu)(x) = \int_0^1 \big(D(\mathcal{H}_{|T_{u(x)}\mathcal{M}})(\nu(x)+t\Delta\nu(x)) - D(\mathcal{H}_{|T_{u(x)}\mathcal{M}})(\nu(x))\big)\Delta\nu(x)\, dt.
$$

Property (4) implies that $|R(\nu, \Delta\nu)(x)| \le \varepsilon\|\Delta\nu\|$ if $\|\Delta\nu\|$ is small enough, then $\mathcal{C}$ is differentiable at ν.

The continuity of $D\mathcal{C}$ follows from

$$
\begin{aligned}
&\|D\mathcal{C}(\nu') - D\mathcal{C}(\nu)\| \\
&\qquad = \sup_{\|\Delta\nu\|\le 1} \sup_{x\in X} \big|\big(D(\mathcal{H}_{|T_{u(x)}\mathcal{M}})(\nu'(x)) - D(\mathcal{H}_{|T_{u(x)}\mathcal{M}})(\nu(x))\big)\Delta\nu(x)\big| \\
&\qquad \le \varepsilon\|\Delta\nu\| \le \varepsilon
\end{aligned}
$$

if $\|\nu' - \nu\|$ small enough, by property (4).

Appendix B. Proof of Lemma 4.3

The proof follows from the same claim for the finite-dimensional case.

Lemma B.1. *Let $T_{\Lambda_f}M = E^u \oplus E^s$ be the hyperbolic splitting of the hyperbolic set Λ, introduced in Section 2.5. There exists $C > 0$ such that the projections $\pi^{s,u} : T_\Lambda M \to T_\Lambda M$ satisfy, for all $x, y \in \Lambda$ and $v \in T_xM$, $w \in T_yM$,*

$$
\begin{aligned}
\text{(B.1)}\quad & |D\mathsf{e}(x)\pi^{s,u}(x)v - D\mathsf{e}(y)\pi^{s,u}(y)w| \\
&\qquad \le C(d^{\alpha_f}(x, y)\max\{|v|, |w|\} + |D\mathsf{e}(x)v - D\mathsf{e}(y)w|),
\end{aligned}
$$

where $\mathsf{e} : M \to \mathbb{R}^D$ *is the embedding introduced in Section 2.1 (see also Section 4.1 in* [FdlLM10]*).*

Proof. Let $(TU_\phi, T\phi)$ be a chart of TM. The expression of the projections π^u and π^s in this chart are

$$\begin{array}{ccc} T_\Lambda M \cap TU_\phi \subset TU_\phi & \xrightarrow{\pi^u,\pi^s} & TM \\ T\phi \downarrow & & \downarrow T\phi \\ \mathbb{R}^n \times \mathbb{R}^n & \xrightarrow{\pi^u_\phi,\pi^s_\phi} & \mathbb{R}^n \times \mathbb{R}^n \end{array},$$

π^u_ϕ and π^s_ϕ, satisfy $\pi^{s,u}_\phi(x,v) = (x, A^{s,u}_\phi(x)v)$, where for some constant C depending only on M,

$$|A^{s,u}_\phi(x)v - A^{s,u}_\phi(y)w| \le C(d^{\alpha_f}(x,y)\min\{|v|,|w|\} + |v-w|). \tag{B.2}$$

On the other hand, we remark that there exists a constant C such that, if $x, y \in U_\phi$ (the domain of the chart ϕ) and $v \in T_xM$, $w \in T_yM$, then

$$|D\phi(x)v - D\phi(y)w| \le C(|D\mathsf{e}(x)v - D\mathsf{e}(y)w| + d(x,y)\min\{|v|,|w|\}). \tag{B.3}$$

Indeed, using $\eta(x) = D\mathsf{e}^{-1}(x)$, the left inverse of $D\mathsf{e}^{-1}(x)$, we have that, for some $C > 0$,

$$\begin{aligned} |D\phi(x)v - D\phi(y)w| =& |\eta_\phi(\phi^{-1}(x))D\mathsf{e}_\phi(\phi^{-1}(x))D\phi(x)v \\ & - \eta_\phi(\phi^{-1}(y))D\mathsf{e}_\phi(\phi^{-1}(y))D\phi(y)w| \\ \le& C(|D\mathsf{e}(x)v - D\mathsf{e}(y)w| + d(x,y)\min\{|v|,|w|\}). \end{aligned}$$

Let $2\rho_0 < 1$ be the Lebesgue number of the open covering $\{U_\Phi\}$ of M. Assume first that $d(x,y) < \rho_0$. Hence, there exists U_ϕ such that $x, y \in U_\phi$. Then, from (B.2) and (B.3), for any $v \in T_xM$, $w \in T_yM$ and assuming $|v| = \min\{|v|,|w|\}$,

$$\begin{aligned} |D\mathsf{e}(x)\pi^u(x)v - D\mathsf{e}(y)\pi^u(y)w| &= |D\mathsf{e}_\phi(x)\pi^u_\phi(x)D\phi(x)v - D\mathsf{e}_\phi(y)\pi^u_\phi(y)D\phi(y)w| \\ &\le |(D\mathsf{e}_\phi(x) - D\mathsf{e}_\phi(y))\pi^u_\phi(x)D\phi(x)v| \\ &\quad + |D\mathsf{e}_\phi(y)(\pi^u_\phi(x)D\phi(x)v - \pi^u_\phi(y)D\phi(y)w)| \\ &\le Cd(x,y)|v| + C(d^{\alpha_f}(x,y)\min(|D\phi(x)v|, |D\phi(y)w|) \\ &\quad + |D\phi(x)v - D\phi(y)w|) \\ &\le C(d^{\alpha_f}(x,y)\min\{|v|,|w|\} + |D\mathsf{e}(x)v - D\mathsf{e}(y)w|). \end{aligned}$$

If $d(x,y) > \rho$,

$$\begin{aligned} |D\mathsf{e}(x)\pi^u(x)v - D\mathsf{e}(y)\pi^u(y)w| &\le |D\mathsf{e}(x)\pi^u(x)v| + |D\mathsf{e}(y)\pi^u(y)w| \\ &\le C\max\{|v|,|w|\} \\ &\le C\rho^{-\alpha_f}(d^{\alpha_f}(x,y)\max\{|v|,|w|\} \\ &\quad + |D\mathsf{e}(x)v - D\mathsf{e}(y)w|). \end{aligned}$$

□

Then, since the projections $\pi^{u,s} : T_{\Delta_F}\mathcal{M} \to \mathcal{E}^{u,s}$ are uncoupled, that is, $(\pi^{u,s}(x)v)_i = \pi^{u,s}(x_i)v_i$, inequality (4.3) follows immediately from (B.1). Finally, inequality (4.4) is obtained by taking supremum with respect to i in (4.3).

Appendix C. Proof of Proposition 3.8 and Lemma 4.10

Proposition 3.8 follows directly from Lemma 4.10.

Now we proceed with the proof of Lemma 4.10.

The maps $A_{F,u,u}$ and $A_{F,s,s}$ are, by definition, linear and, by Lemma 4.4, $\|A_{F,s,s}\| \leq \|\pi_{\mathcal{S}^{\alpha,s}_{\mathrm{Id},\Gamma}}\|^2 \|D\mathcal{A}_F(0)\|$.

It follows from (4.21) that

$$(D\mathcal{A}_F(\mathbf{0}))^{-1}\nu(x) = DF^{-1}(F(x))\nu(F(x)) = D\mathcal{A}_{F^{-1}}(\mathbf{0})\nu(x).$$

As a consequence, $A_{F,u,u}$ is invertible and, for any $\nu^u \in \mathcal{S}^{\alpha,u}_{\mathrm{Id},\Gamma}$,

$$A_{F,u,u}\nu^u(x) = DF^{-1}(F(x))\nu^u(F(x)). \tag{C.1}$$

In particular, $\|A^{-1}_{F,u,u}\| \leq \|\pi_{\mathcal{S}^{\alpha,u}_{\mathrm{Id},\Gamma}}\|^2 \|D\mathcal{A}_{F^{-1}}(0)\|$.

We proceed to prove (4.23) for $A_{F^N,s,s}$, the other inequality being analogous. To simplify notation, we will denote $A_{F^N,s,s}$ by A_N.

It follows directly from the definition of $A_{F^N,s,s}$ that, for any $\nu \in \mathcal{S}^{\alpha,s}_{\mathrm{Id},\Gamma}$,

$$\begin{aligned} \|A_N\nu\|_{C^0} &= \sup_{x\in\Delta_F} \|\pi^s(x)D(F^N)(F^{-N}(x))\pi^s(F^{-N}(x))\nu(F^{-N}(x))\| \\ &< \sup_{x\in\Delta_F} \|\pi^s(x)\|\lambda^N\|\nu\|_{C^0}. \end{aligned} \tag{C.2}$$

Now we compute $\gamma_\alpha(A_N\nu)$. Let $\rho > 0$. We claim that there exists $C \geq 1$ such that, for any $i, j \in \mathbb{Z}^d$ and any $x, y \in \Delta_F$ satisfying $x_k = y_k$, $k \neq j$, we have that

(I) if $d(x, y) \geq \rho$,

$$\begin{aligned} &d((A_N\nu)(x)_i, (A_N\nu)(y)_i) \\ &\qquad \leq \begin{cases} C\lambda^N\rho^{-\alpha}\|\nu\|_{C^\alpha_\Gamma}\Gamma(0)d^\alpha(x_j, y_j), & i = j \\ C\lambda^N(\operatorname{Lip} f^{-1})^{\alpha N}\|\nu\|_{C^\alpha_\Gamma}\Gamma(i-j)d^\alpha(x_j, y_j), & i \neq j, \end{cases} \end{aligned} \tag{C.3}$$

(II) if $d(x, y) < \rho$,

$$\begin{aligned} &d((A_N\nu)(x)_i, (A_N\nu)(y)_i) \\ &\qquad \leq C\big(\lambda^N + N\|Df\|^{N-1}\rho^{1-\alpha} + \|Df\|^N(\operatorname{Lip} f^{-1})^N\rho^{\alpha_f-\alpha} \\ &\qquad\quad + \lambda^N(\operatorname{Lip} f^{-1})^{\alpha N}\big)\|\nu\|_{C^\alpha_\Gamma}\Gamma(i-j)d^\alpha(x_j, y_j). \end{aligned} \tag{C.4}$$

Assume for the moment that the claims for (I) and (II) hold. Let $0 < \tilde\lambda < 1$. If we take $\rho = (2\lambda/\tilde\lambda)^{\frac{1}{\alpha}}\lambda^{\frac{N-1}{\alpha}}$, then, substituting in (C.3) and (C.4), we have that

$$\begin{aligned} &d((A_N\nu)(x)_i, (A_N\nu)(y)_i)\Gamma(i-j)^{-1}d^{-\alpha}(x_j, y_j) \\ &\qquad \leq \max\{\tilde\lambda, \tilde C(\lambda^N + N(\lambda^{\frac{1-\alpha}{\alpha}}\|Df\|)^N + (\|Df\| \operatorname{Lip} f^{-1}\lambda^{\frac{\alpha_f-\alpha}{\alpha}})^N \\ &\qquad\quad + (\lambda(\operatorname{Lip} f^{-1})^\alpha)^N)\}\|\nu\|_{C^\alpha_\Gamma}, \end{aligned} \tag{C.5}$$

where $\tilde C = \min\{C, (C/\tilde\lambda)^{\frac{\alpha_f-\alpha}{\alpha}}\|Df\|^{-1}\}$. Since $0 < \lambda < 1$, we can choose $0 < \alpha* < \alpha_f$ such that for any $0 < \alpha \leq \alpha^*$,

$$\|Df\| \operatorname{Lip} f^{-1}\lambda^{\frac{\alpha_f-\alpha}{\alpha}} < 1,$$

and, then, N large enough such that the right hand side of (C.5) is smaller than $\tilde{\lambda}$.

It remains to prove the claims for (I) and (II).

We start by proving (I). If $i = j$, then, using that $d(x,y)\rho^{-1} \geq 1$,

$$d((A_N\nu)(x)_i, (A_N\nu)(y)_i) \leq 2 \sup_{x\in\Delta_F} \|(A_N\nu)(x)_i\| \\ \leq 2K^2\lambda^N\|\nu\|_{C^0} \leq C\lambda^N\rho^{-\alpha}\|\nu\|_{C^\alpha_\Gamma}\Gamma(0)d^\alpha(x_j, y_j),$$

where $K = \sup_{x\in\Delta_F}\|\pi^s(x)\|$ and $C = 2K^2\Gamma(0)^{-1}$. This proves the first part of the claim.

If $i \neq j$, since F and π^s are uncoupled, we have that

$$\begin{aligned} &d((A_N\nu)(x)_i, (A_N\nu)(y)_i) \\ &\quad= \|\pi^s_i(x_i)Df^N(f^{-N}(x))\pi_i(f^{-N}(x_i))(\nu_i(F^{-N}(x)) - \nu_i(F^{-N}(y)))\| \\ &\quad\leq K^2\lambda^N d(\nu_i(F^{-N}(x)), \nu_i(F^{-N}(y))) \\ &\quad\leq K^2\lambda^N(\operatorname{Lip} f^{-1})^{\alpha N}\|\nu\|_{C^\alpha_\Gamma}\Gamma(i-j)d^\alpha(x_j, y_j). \end{aligned}$$

Finally we prove (II). Using the distance on TM induced by the embedding e and the map η introduced in Section 2.1 (see also (4.4) in [FdlLM10]), we have that

$$\begin{aligned} &d((A_N\nu)(x)_i, (A_N\nu)(y)_i) \\ &\qquad=|D\mathsf{e}(x_i)\pi(x_i)Df^N(f^{-N}(x_i))\pi(f^{-N}(x_i))\nu_i(F^{-N}(x)) \\ &\qquad\quad - D\mathsf{e}(y_i)\pi(y_i)Df^N(f^{-N}(y_i))\pi(f^{-N}(y_i))\nu_i(F^{-N}(y))| \\ &\qquad\leq \mathrm{a} + \mathrm{b} + \mathrm{c} + \mathrm{d}, \end{aligned}$$

where

$$\begin{aligned} a =&|(D\mathsf{e}(x_i)\pi(x_i)\eta(x_i) - D\mathsf{e}(y_i)\pi(y_i)\eta(y_i)) \\ &\times D\mathsf{e}(x_i)Df^N(f^{-N}(x_i))\pi(f^{-N}(x_i))\nu_i(F^{-N}(x))|, \\ b =&|D\mathsf{e}(y_i)\pi(y_i)\eta(y_i) \\ &\times (D\mathsf{e}(x_i)Df^N(f^{-N}(x_i))\eta(f^{-N}(x_i)) - D\mathsf{e}(y_i)Df^N(f^{-N}(y_i))\eta(f^{-N}(y_i))) \\ &\times D\mathsf{e}(f^{-N}(x_i))\pi(f^{-N}(x_i))\nu_i(F^{-N}(x))|, \\ c =&|D\mathsf{e}(y_i)\pi(y_i)Df^N(f^{-N}(y_i))\eta(f^{-N}(y_i)) \\ &\times (D\mathsf{e}(f^{-N}(x_i))\pi(f^{-N}(x_i))\eta(f^{-N}(x_i)) - D\mathsf{e}(f^{-N}(y_i))\pi(f^{-N}(y_i))\eta(f^{-N}(y_i))) \\ &\times D\mathsf{e}(f^{-N}(x_i))\nu_i(F^{-N}(x))|, \\ d =&|D\mathsf{e}(y_i)\pi(y_i)Df^N(f^{-N}(y_i))\pi(f^{-N}(y_i))\eta(f^{-N}(y_i)) \\ &\times (D\mathsf{e}(f^{-N}(x_i))\nu_i(F^{-N}(x)) - D\mathsf{e}(f^{-N}(y_i))\nu_i(F^{-N}(y)))|. \end{aligned}$$

The proof of (II) can be obtained by observing that, for some constant $C > 0$, depending only on M, the choice of e and η, the value of $\Gamma(0)$ and f, and using that $\alpha < \alpha_f$ we have the following simple bounds (C.6) to (C.9) for a, b, c and d.

$$a \leq \begin{cases} 0 & \text{if } i \neq j, \\ C\lambda^N \|\nu\|_{C^\alpha_\Gamma} \Gamma(0) d^\alpha(x_j, y_j) & \text{if } i = j. \end{cases} \tag{C.6}$$

Since, for any chart, $\|D^2(f^N)\| \leq KN\|Df\|^{N-1}$,

$$b \leq \begin{cases} 0 & \text{if } i \neq j, \\ CN\|Df\|^{N-1} \|\nu\|_{C^\alpha_\Gamma} \Gamma(0) d^\alpha(x_j, y_j)\rho^{1-\alpha} & \text{if } i = j. \end{cases} \tag{C.7}$$

By (B.1),

$$c \leq \begin{cases} 0 & \text{if } i \neq j, \\ C\|Df\|^N (\operatorname{Lip} f^{-1})^N \|\nu\|_{C^\alpha_\Gamma} \Gamma(0) d^\alpha(x_j, y_j)\rho^{\alpha_f - \alpha} & \text{if } i = j. \end{cases} \tag{C.8}$$

And

$$d \leq C\lambda^N (\operatorname{Lip} f^{-1})^{\alpha N} \|\nu\|_{C^\alpha_\Gamma} \Gamma(i-j) d^\alpha(x_j, y_j). \tag{C.9}$$

Inequalities (C.6) to (C.9) imply (II). □

Appendix D. Proof of Lemma 4.11

We start by proving (1). To begin with, we notice that, since $d_{C^0}(h_\Phi, \mathrm{Id})$ is small, there exists $C > 0$ such that for all $x \in \Delta_F$, the linear maps $\tau(h_\Phi(x), x)$ and $\tau(x, h_\Phi(x))$ are well defined and

$$\|(D\mathsf{e}(x)\tau(h_\Phi(x), x) - D\mathsf{e}(h_\Phi(x)))v\| \leq C d_{C^0}(h_\Phi, \mathrm{Id})|v|,$$

for all $v \in T_{h_\Phi(x)}\mathcal{M}$ and

$$\|(D\mathsf{e}(h_\Phi(x))\tau(x, h_\Phi(x)) - D\mathsf{e}(x))v\| \leq C d_{C^0}(h_\Phi, \mathrm{Id})|v|,$$

for all $v \in T_x\mathcal{M}$. Hence, by the definition of $\mathcal{A}_\Phi$ and $\tilde{\mathcal{A}}_F$ in (4.1) and (4.24), resp., we have that, for some constant $C > 0$,

$$\|(D\mathcal{A}_\Phi(0) - D\tilde{\mathcal{A}}_F(0))\nu\|_{C^0} \leq C(d_{C^0}(h_\Phi, \mathrm{Id}) + d_{C^1}(\Phi, F))\|\nu\|_{C^0}.$$

Now we proceed to compute $\gamma_\alpha((D\mathcal{A}_\Phi - D\tilde{\mathcal{A}}_F)(0)\nu)$. To do so, we will use some auxiliary functions provided by the following two lemmas.

Lemma D.1. *Let $X \subset \mathcal{M}$. There exist $C > 0$ and $\tilde{C} > 0$ such that for any $h \in C^\alpha_\Gamma(X, \mathcal{M})$ with $d_{C^\alpha_\Gamma}(h, \mathrm{Id}) < \tilde{C}\rho_0$, $j \in \mathbb{Z}^d$ and $x, y \in X$ such that $x_k \neq y_k$, for $k \neq j$, there exists a C^∞ map $B : [0,1] \times [0,1] \to \mathcal{M}$ such that, for any $i \in \mathbb{Z}^d$, and any chart (U_ϕ, ϕ) with $B(s,t) \in U_\phi$,*

1. *$B(0,0) = x$, $B(0,1) = y$, $B(1,0) = h(x)$ and $B(1,1) = h(y)$,*
2. *$|\frac{d}{dt}B(s,t)_i| \leq C(1 + \gamma_\alpha(h))\Gamma(i-j)d^\alpha(x_j, y_j)$,*
3. *$|\frac{d}{ds}B(s,t)_i| \leq C d_{C^0}(h, \mathrm{Id})$,*
4. *$|\frac{d^2}{dtds}(\phi \circ B)(s,t)_i| \leq C d_{C^\alpha_\Gamma}(h, \mathrm{Id})\Gamma(i-j)d^\alpha(x_j, y_j)$.*

Proof. Let $h \in C^\alpha_\Gamma(X, \mathcal{M})$, $j \in \mathbb{Z}^d$ and $x, y \in X$ such that $x_i = y_i$, for $i \neq j$. By Lemma A.2 in [FdlLM10], there exists a C^∞ curve $\beta^p : [0,1] \to \mathcal{M}$ such that $\beta^p(0) = h(x)$, $\beta^p(1) = h(y)$ and $|\dot{\beta}^p_i(t)| \leq C_1\gamma_\alpha(h)\Gamma(i-j)d^\alpha(x_j, y_j)$, for some constant C_1 independent of the choice of h, j and x, y.

As a first step, we claim that we can choose two coordinate charts, (U_ϕ, ϕ) and (U_ψ, ψ), such that $x, h(x) \in U_\phi$, $y, h(y) \in U_\psi$, $U_\phi = \operatorname{int} \prod_i U_{\phi_i}$, $U_\psi = \operatorname{int} \prod_i U_{\psi_i}$, $U_{\phi_i} = U_{\psi_i}$, for $i \neq j$ and $\beta^p(t)_i \in U_{\phi_i}$ for $i \neq j$, $t \in [0,1]$.

If, furthermore, $d(x_j, y_j) < \rho_0$, where $2\rho_0$ is the Lebesgue number of the family $(U_\phi)_{(U_\phi,\phi)\in\mathcal{F}_M}$, first introduced in Section 4.1 of [FdlLM10], then it is possible to assume that $U_\phi = U_\psi$ and $\beta^p(t)_j \in U_{\phi_j}$, for $t \in [0,1]$.

Indeed, associated to $x = (x_i)$, let $(U_{\phi_i}, \phi_i) \in \mathcal{F}_M$ be a collection of charts in the chosen atlas on M (see Section 4.1 in [FdlLM10]) such that $B_{2\rho_0}(x_i) \subset U_{\phi_i}$. Let $U_\phi = \operatorname{int} \prod_i U_{\phi_i}$ and $\phi = (\phi_i)$. We have that $(U_\phi, \phi) \in \mathcal{F}_\mathcal{M}$, and $x \in U_\phi$ (since $B_{2\rho_0}(x) \subset \prod_i B_{2\rho_0}(x_i) \subset \prod_i U_{\phi_i}$, we have that $B_{2\rho_0}(x) = \operatorname{int} B_{2\rho_0}(x) \subset \operatorname{int} \prod_i B_{2\rho_0}(x_i) \subset \operatorname{int} \prod_i U_{\phi_i} = U_\phi$). Moreover, taking $0 < \tilde{C} < 1$, since $d_{C^\alpha_\Gamma}(\mathrm{Id}\,, h) < \tilde{C}\rho_0 < \rho_0$, we have that $h(x) \in U_\phi$. Notice that, since $x_i = y_i$, for $i \neq j$, we have that $h(y)_i \in B_{\rho_0}(h(y)_i) \subset B_{2\rho_0}(x_i) \subset U_{\phi_i}$, for $i \neq j$. Let (U_{ψ_j}, ψ_j) be such that $B_{2\rho_0}(y_j) \subset U_{\psi_j}$, and $(U_{\psi_i}, \psi_i) = (U_{\phi_i}, \phi_i)$, for $i \neq j$. Then, taking $U_\psi = \operatorname{int} \prod_i U_{\psi_i}$ and $\psi = (\psi_i)$, $(U_\psi, \psi) \in \mathcal{F}_\mathcal{M}$ and $y, h(y) \in U_\psi$. Also, since $d(x_i, \beta^p_i(t)) \le d(x_i, h(x)_i) + d(h(x)_i, \beta^p_i(t))$ and, for some constant $\hat{C} > 0$ depending only on M and the embedding e, for $i \neq j$,

$$
\begin{aligned}
d(h(x)_i, \beta^p_i(t)) &\le d(h(x)_i, h(y)_i) \\
&\le \hat{C} \| \mathsf{e} \circ h(x)_i - \mathsf{e} \circ h(y)_i \| \\
&= \hat{C} \| \mathsf{e} \circ h(x)_i - \mathsf{e}(x_i) - (\mathsf{e} \circ h(y)_i - \mathsf{e}(y_i)) \| \\
&\le \hat{C} d_{C^\alpha_\Gamma}(\mathrm{Id}\,, h) \Gamma(i-j) d^\alpha(x_j, y_j),
\end{aligned}
$$

we have that $d(x_i, \beta^p_i(t)) < \rho_0$, if $\tilde{C} < (1+\hat{C} \sup_i \Gamma(i-j) \sup_{x_j, y_j \in M} d^\alpha(x_j, y_j))^{-1}$. Finally, if $d(x_j, y_j) < \rho_0$, we simply take U_{ϕ_j} such that $B_{2\rho_0}(x_j) \in U_{\phi_j}$, and, since we are dealing with a single coordinate, the claim on the existence of the two charts follows.

Now we proceed to define the map B. We start by assuming $d(x_j, y_j) \ge \rho_0$. Let $\beta^p_\phi = \phi \circ \beta^p$ and $\beta^p_\psi = \psi \circ \beta^p$ be the expressions of β^p in these charts. By the construction of the charts, $\beta^p_{\phi,i} = \beta^p_{\psi,i}$, for $i \neq j$. Let $0 < t_1 < 1/2$ such that $\beta^p_j(t) \in U_{\phi_j}$ for all $t \in [0, t_1]$ and $\beta^p_j(t) \in U_{\psi_j}$ for all $t \in [1 - t_1, 1]$. Let $\chi : [0,1] \to \mathbb{R}$ be a C^∞ function such that $0 \le \chi(t) \le 1$ for all t, $\chi(0) = 1$, $\chi(1) = 1$ and $\chi(t) = 0$ for $t \in [t_1, 1 - t_1]$. Let $\tilde{\beta}^p_{\phi,j} : [0, t_1] \to \phi_j(U_{\phi,j})$ and $\tilde{\beta}^p_{\psi,j} : [1 - t_1, 1] \to \psi_j(U_{\psi,j})$ be the curves defined by

$$
\begin{aligned}
\tilde{\beta}^p_{\phi,j}(t) &= (\phi_j(x_j) - \phi_j(h_j(x)))\chi(t) + \beta^p_{\phi,j}(t) \\
\tilde{\beta}^p_{\psi,j}(t) &= (\psi_j(y_j) - \psi_j(h_j(y)))\chi(t) + \beta^p_{\psi,j}(t),
\end{aligned}
$$

resp. We define, for $0 \le s \le 1$,

$$
B(s,t)_j = \begin{cases} \phi_j^{-1}((1-s)\tilde{\beta}^p_{\phi,j}(t) + s\beta^p_{\phi,j}(t)), & t \in [0, t_1), \\ \beta^p_j(t), & t \in [t_1, 1 - t_1], \\ \psi_j^{-1}((1-s)\tilde{\beta}^p_{\psi,j}(t) + s\beta^p_{\psi,j}(t)), & t \in (1 - t_1, 1], \end{cases}
$$

and, for $i \neq j$,

$$
B(s,t)_i = \phi_i^{-1}((1-s)\phi_i(x_i) + s\beta^p_{\phi,i}(t)).
$$

Now we check that the function B just defined satisfies (1), (2), (3) and (4). By construction, B is C^∞. Moreover, for $i \neq j$, we have that

$$|\frac{d}{dt}B(s,t)_i| = |s||\frac{d}{dt}\beta_i^p(t)| \leq C_1\gamma_\alpha(h)\Gamma(k-j)d^\alpha(x_j,y_j).$$

Also, for some $C > 1$,

$$\begin{aligned}|\frac{d}{ds}B(s,t)_i| &= |\beta_{\phi,i}^p(t) - \phi_i(x_i)| \\ &\leq |\beta_{\phi,i}^p(t) - \phi_i \circ h_i(x)| + |\phi_i \circ h_i(x) - \phi_i(x_i)| \\ &\leq C(|\phi_i \circ h_i(y) - \phi_i \circ h_i(x)| + d_{C^0}(\mathrm{Id}\,, h)) \\ &\leq C(|\phi_i \circ h_i(y) - \phi_i(y_i)| + |\phi_i(x_i) - \phi_i \circ h_i(x)|) + d_{C^0}(\mathrm{Id}\,, h) \\ &\leq 3Cd_{C^0}(\mathrm{Id}\,, h).\end{aligned}$$

Finally, for $i \neq j$, using the same argument,

$$|\frac{d^2}{dtds}(\phi \circ B)(s,t)_i| \leq C|\frac{d}{dt}\beta_i^p(t)| \leq d_{C^\alpha_\Gamma}(h, \mathrm{Id}\,)\Gamma(i-j)d^\alpha(x_j,y_j).$$

Notice that in these bounds we have not used that $d(x_j,y_j) \geq \rho_0$, assumption that only plays a role when $i = j$.

For $i = j$ we have that, for $t \in [0,t_1)$, since $\rho_0^{-\alpha}d^\alpha(x_j,y_j) > 1$,

$$\begin{aligned}|\frac{d}{dt}B(s,t)_j| &= |(1-s)(\phi_j(x_j) - \phi_j(h_j(x)))\dot\chi(t) + \dot\beta_{\phi,j}^p(t)| \\ &\leq \sup_t|\dot\chi(t)|d_{C^0}(\mathrm{Id}\,, h) + \gamma_\alpha(h)\Gamma(0)d^\alpha(x_j,y_j) \\ &\leq (\tilde C\rho_0^{1-\alpha}\sup_t|\dot\chi(t)|\Gamma(0)^{-1} + \gamma_\alpha(h))\Gamma(0)d^\alpha(x_j,y_j).\end{aligned}$$

Also, for some $C > 0$ depending only on M,

$$|\frac{d}{ds}B(s,t)_j| = |(\phi_j(x_j) - \phi_j(h_j(x))| \leq Cd_{C^\alpha_\Gamma}(h, \mathrm{Id}\,)$$

and

$$\begin{aligned}|\frac{d^2}{dtds}(\phi \circ B)(s,t)_j| &= |(\phi_j(x_j) - \phi_j(h_j(x)))\dot\chi(t)| \\ &\leq Cd_{C^0}(h, \mathrm{Id}\,) \\ &\leq C\rho_0^{-\alpha}d_{C^\alpha_\Gamma}(h, \mathrm{Id}\,)\Gamma(0)^{-1}\Gamma(0)d^\alpha(x_j,y_j).\end{aligned}$$

The bounds for $t \in [t_1, 1-t_1]$ and $t \in (1-t_1, 1]$ are obtained in the same way.

Now we assume $d(x_j,y_j) < \rho_0$. For $i \neq j$, we define $B(s,t)_i$ as in the case $d(x_j,y_j) \geq \rho_0$. The bounds of their derivatives are already computed. For $i = j$, since $x_j, y_j, h_j(x), h_j(y), \beta_j^p(t) \in U_{\phi_j}$, we define

$$B(s,t)_j = \phi_j^{-1}((1-s)[(1-t)(\phi_j(x_j) - \phi_j(h_j(x))) + t(\phi_j(y_j) - \phi_j(h_j(y)))] + s\beta_{\phi,j}^p(t)).$$

Then we have that

$$|\frac{d}{dt}B(s,t)_j| \leq 2\hat C d_{C^\alpha_\Gamma}(\mathrm{Id}\,, h) + |\dot\beta_j^p(t)| \leq (2\hat C\rho_0^{1-\alpha}\Gamma(0)^{-1} + \gamma_\alpha(h))\Gamma(0)d^\alpha(x_j,y_j).$$

Also,

$$|\frac{d}{ds}B(s,t)_j| \leq 2\hat C d_{C^\alpha_\Gamma}(\mathrm{Id}\,, h)$$

and, finally

$$|\frac{d^2}{dtds}(\phi \circ B)(s,t)_j| \le 2\hat{C} d_{C^\alpha_\Gamma}(\mathrm{Id}\,, h).$$

□

Lemma D.2. *Let $F : \mathcal{M} \to \mathcal{M}$ be an uncoupled C^r diffeomorphism. There exist $C > 0$ and $\varepsilon_0 > 0$ such that for any $\Phi : \mathcal{M} \to \mathcal{M}$, C^r_Γ diffeomorphism such that $d_{C^r_\Gamma}(F, \Phi) < \varepsilon_0$ there exists a map $G_{F,\Phi} : [0, 1] \times \mathcal{M} \to \mathcal{M}$ such that $G_{F,\Phi}(0, x) = F(x)$, $G_{F,\Phi}(1, x) = \Phi(x)$, and, for any $(s, x) \in [0, 1] \times \mathcal{M}$ and any (U_ϕ, ϕ) such that $G_{F,\Phi}(s, x) \in U_\phi$,*

$$\|\frac{\partial}{\partial s}\phi \circ G_{F,\Phi}(s, x)\| \le C d_{C^r_\Gamma}(F, \Phi), \tag{D.1}$$

$$\|\frac{\partial}{\partial x}\phi \circ G_{F,\Phi}(s, x)\|_{L_\Gamma} \le C\|F\|_{C^1_\Gamma}, \tag{D.2}$$

$$\|\frac{\partial^2}{\partial s \partial x}\phi \circ G_{F,\Phi}(s, x)\|_{L_\Gamma} \le C d_{C^r_\Gamma}(F, \Phi), \tag{D.3}$$

Proof. We recall the definitions of $\exp : T\mathcal{M} \to \mathcal{M} \times \mathcal{M}$ and $\exp^{-1} : \mathcal{M} \times \mathcal{M} \to T\mathcal{M}$. We define

$$G(s, x) = \pi_2 \circ \exp(s \exp^{-1}(F(x), \Phi(x))). \tag{D.4}$$

The proof consists of checking that G satisfies all the properties claimed by the Lemma, which follow from the chain rule, the algebra properties of C^r_Γ functions and the fact that $d_{C^r_\Gamma}(F, \Phi)$ is small. □

We will also need the following technical result.

Lemma D.3. *Let $F : \mathcal{M} \to \mathcal{M}$ a C^r uncoupled diffeomorphism. There exists $C > 0$ and ε_0 such that if $\Phi : \mathcal{M} \to \mathcal{M}$ is a C^r_Γ uncoupled diffeomorphism with $d_{C^r_\Gamma}(F, \Phi) < \varepsilon_0$, then Φ^{-1} is also C^r_Γ and $d_{C^r_\Gamma}(F^{-1}, \Phi^{-1}) < C d_{C^r_\Gamma}(F, \Phi)$.*

Proof. Since F is uncoupled, so is F^{-1}. In particular, since F is C^r, both F and F^{-1} are C^r_Γ. When expressed using the charts $\mathcal{A}_F$ and $\mathcal{A}_{\mathrm{Id}}$, where

$$\mathcal{A}_F(\Phi)(x) = \exp^{-1}(x, \Phi \circ F^{-1}(x))$$

(see Section 5.6 in [FdlLM10] for more details on the subject), the map $\Phi \mapsto \Phi \circ F^{-1}$ becomes $\nu \to \nu \circ F^{-1}$, which is linear and by Lemma 2.17 in [FdlLM10], is continuous from the space of C^r_Γ sections to itself, with bounded inverse. As a consequence, we can assume that Φ is C^r_Γ close to the identity. By this reason, we only need to deal with the decay properties of Φ^{-1}.

Since, by hypothesis, Φ is a diffeomorphism, we only need to check that Φ^{-1} is a C^r_Γ function. But, for any chart ϕ, $D\Phi^{-1}_\phi = (D\Phi_\phi)^{-1} \circ \Phi^{-1}_\phi$ and, since $D\Phi_\phi = \mathrm{Id} + \nu$, with $\|\nu\|_{L_\Gamma} < \varepsilon_0$, we have that, if $\varepsilon_0 < 1$, the series $D\Phi^{-1}_\phi = \sum_{k \ge 0}(-1)^k \nu^k$ converges to an element of L_Γ with norm bounded by $\Gamma(0)^{-1} - \varepsilon_0(1 - \varepsilon_0)^{-1}$. Then, the rest of the derivatives of Φ^{-1} are obtained directly by applying the chain rule. The last statement follows from $D\Phi^{-1} - \mathrm{Id} = \sum_{k \ge 1}(-1)^k \nu^k$. □

Now we are in a position to compute $\gamma_\alpha((D\mathcal{A}_\Phi - D\tilde{\mathcal{A}}_F)(0)\nu)$. Given $j \in \mathbb{Z}^d$, $x, y \in \Delta_F$ with $x_i = y_i$ for $i \neq j$, let B be the function given by Lemma D.1. We also introduce

$$\mathcal{G}(s,t) = \frac{\partial G_{F,\Phi}}{\partial x}(s, G_{F^{-1},\Phi^{-1}}(s, B(s,t))),$$

which, by Lemmas D.2 and D.3 is well defined. Denoting $\Delta\mathcal{A}_{F,\Phi} = (D\mathcal{A}_\Phi - D\tilde{\mathcal{A}}_F)(0)$, we have that

$$\begin{aligned}
&\|D\mathsf{e}(h_\Phi(y))\Delta\mathcal{A}_{F,\Phi}\nu(y) - D\mathsf{e}(h_\Phi(x))\Delta\mathcal{A}_{F,\Phi}\nu(x)\| \\
&= \int_0^1 \frac{d}{dt}[D\mathsf{e}(B(1,t)) \int_0^1 \frac{d}{ds}(\tau(B(s,t), B(1,t))\mathcal{G}(s,t) \\
&\qquad\qquad \tau(\Phi^{-1} \circ B(1,t), G_{F^{-1},\Phi^{-1}}(s, B(s,t)))\, ds]\, dt
\end{aligned}$$

Taking charts and derivatives above, and applying the bounds in Lemmas D.1, D.2 and D.3, the bound for $\gamma_\alpha((D\mathcal{A}_\Phi - D\tilde{\mathcal{A}}_F)(0)\nu)$ follows.

Appendix E. Construction of sections

This Appendix is devoted to prove Lemma 4.8. We start with the case of a finite-dimensional manifold and we will continue by lifting the construction to the lattice $\mathcal{M}$. The proof is divided in several technical lemmas.

We define the space of C^α sections covering the identity in M by

$$\mathcal{S}^\alpha_{\mathrm{Id}}(M) = \{\nu \in C^\alpha(M, TM) \mid p \circ \nu = \mathrm{Id}_M,\ \|\nu\|_{C^\alpha} < \infty\}, \tag{E.1}$$

where $p : TM \to M$ is the projection and

$$\|\nu\|_{C^\alpha} = \max\{\|\nu\|_{C^0}, \sup_{x,y\in M, x\neq y} \|D\mathsf{e}(x)\nu(x) - D\mathsf{e}(y)\nu(y)\| d^{-\alpha}(x,y)\}. \tag{E.2}$$

With this norm, $S^\alpha_{\mathrm{Id}}(M, TM)$ is a Banach space.

Given an open set $U \subset \mathbb{R}^n$, a map $\xi \in C^r(U \times M, TM)$ such that $p \circ \xi(u,x) = x$, and $0 \le k \le r$, we introduce

$$\|\xi\|_{C^k} = \sup_{(U_\phi,\phi)\in\mathcal{F}_M} \|\xi_\phi\|_{C^k}, \tag{E.3}$$

where $\xi_\phi(u,x) = T\phi \circ \xi(u, \phi^{-1}(x))$ is the expression of ξ in the chart ϕ.

Lemma E.1. *Let $U \subset \mathbb{R}^n$ be an open convex set and $\xi \in C^2(U \times M, TM)$ a map such that $p \circ \xi(u,x) = x$. Assume that $\|\xi\|_{C^2} < \infty$. Then, the map $\bar{\xi} : U \to \mathcal{S}^\alpha_{\mathrm{Id}}(M)$ defined by $\bar{\xi}(u)(x) = \xi(u,x)$ is well defined. There exists $C > 0$ such that*

$$\|\xi(u,\cdot)\|_{C^\alpha} \le C\|\xi\|_{C^1} \tag{E.4}$$

and

$$\mathrm{Lip}(\bar{\xi}) \le C\|D_u\xi\|_{C^1}. \tag{E.5}$$

Furthermore, if $\xi \in C^3$ and $\|\xi\|_{C^3} < \infty$, $\bar{\xi}$ is C^1 and $\|\bar{\xi}\|_{C^1} \le C\|D_u\xi\|_{C^1}$, and, if $\xi \in C^\infty$, $\bar{\xi}$ is C^∞.

Proof. Let $x, \tilde{x} \in M$, and let $\beta : [0,1] \to M$ be a minimizing geodesic joining them, that is, $\beta(0) = \tilde{x}$, $\beta(1) = x$, $|\dot{\beta}(t)| = d(x,\tilde{x})$. Then, since

$$|D\mathsf{e}(x)\xi(u,x) - D\mathsf{e}(\tilde{x})\xi(u,\tilde{x})| = \Big| \int_0^1 \frac{d}{dt}\Big(D\mathsf{e}(\beta(t))\xi(u,\beta(t))\Big)\, dt \Big|, \tag{E.6}$$

for any fixed $t \in (0,1)$, we consider a chart $(U_\phi, \phi) \in \mathcal{F}_M$ such that $\beta(t) \in U_\phi$ and, following the notation introduced in A.1 in [FdlLM10], using the expressions e_ϕ, β_ϕ and ξ_ϕ of e, β and ξ in this chart and the corresponding one of TM, we can compute

$$\begin{aligned}
&|\frac{d}{dt}(D\mathsf{e}(\beta(t))\xi(u,\beta(t)))| \\
&=|\frac{d}{dt}(D\mathsf{e}_\phi(\beta_\phi(t))\xi_\phi(u,\beta_\phi(t)))| \\
&\le|D^2\mathsf{e}_\phi(\beta_\phi(t))\dot{\beta}_\phi(t)\xi_\phi(u,\beta_\phi(t))| + |D\mathsf{e}_\phi(\beta_\phi(t))D_2\xi_\phi(u,\beta_\phi(t))\dot{\beta}_\phi(t)| \\
&\le(\|\mathsf{e}_\phi\|_{C^2}\|\xi_\phi\|_{C^0} + \|\mathsf{e}_\phi\|_{C^1}\|\xi_\phi\|_{C^1})d(x,y) \\
&\le C\|\xi\|_{C^1} d^\alpha(x,y),
\end{aligned} \tag{E.7}$$

where $C = \sup_\phi \|\mathsf{e}_\phi\|_{C^1} \sup_{x,y\in M} d^{1-\alpha}(x,y)$. Inequality (E.4) follows from inserting inequality (E.7) into (E.6).

Now we prove inequality (E.5). Let $u, \tilde{u} \in U$. We have that

$$\xi(u,x) - \xi(\tilde{u},x) = \eta(u,\tilde{u},x)(u-\tilde{u}),$$

where

$$\eta(u,\tilde{u},x) = \int_0^1 D_u\xi(\tilde{u} + t(u-\tilde{u}),x)\, dt. \tag{E.8}$$

Hence, applying inequality (E.4) to η we obtain

$$\begin{aligned}
\|\bar{\xi}(u) - \bar{\xi}(\tilde{u})\|_{C^\alpha} &\le \|\eta(u,\tilde{u},\cdot)\|_{C^\alpha}|u-\tilde{u}| \\
&\le \|D_u\xi\|_{C^1}|u-\tilde{u}|.
\end{aligned}$$

Now we assume that $\xi \in C^3$. By Taylor's formula, we have that

$$\bar{\xi}(u+\tilde{u})(x) = \xi(u,x) + D_u\xi(u,x)\tilde{u} + R(u,\tilde{u},x)\tilde{u},$$

where

$$R(u,\tilde{u},x) = \int_0^1 (D_u\xi(u+t\tilde{u},x) - D_u\xi(u,x))\, dt.$$

By (E.5), the maps $u \mapsto D_u\xi(u,\cdot)$ and $(u,\tilde{u}) \mapsto R(u,\tilde{u},\cdot)$ to $L(\mathcal{S}^\alpha_{\mathrm{Id}}(M), \mathcal{S}^\alpha_{\mathrm{Id}}(M))$ are continuous. The Converse Taylor's Theorem yields the claim.

For the C^∞ case, we use the same argument, developing by Taylor up to order k, for any k. □

Lemma E.2. *There exists a map $\omega : TM \to \mathcal{S}^\alpha_{\mathrm{Id}}(M)$ such that*

$$\omega(v)(x) = v, \qquad \text{for all } v \in T_xM,$$

Moreover, ω is C^∞ and linear on each fiber.

Proof. Let $\{\nu_\phi\}_{\{(U_\phi,\phi)\in\mathcal{F}_M\}}$ be a C^∞ partition of unity associated to the atlas $\mathcal{F}_M$ of M. We recall that $\mathcal{F}_M$ was introduced in Section 2.1 (see also Section 4.1 in [FdlLM10]). We define $\omega : TM \to S^\alpha_{\mathrm{Id}}(M, TM)$ by

$$\omega(v)(\tilde{x}) = \sum_{(U_\phi,\phi)\in\mathcal{F}_M} \nu_\phi^{1/2}(\tilde{x})\nu_\phi^{1/2}(x) D\phi(\tilde{x})^{-1} D\phi(x)v, \quad v \in T_xM. \tag{E.9}$$

It is immediate from the definition that, for all $x \in M$ and for all $v \in T_xM$, $\omega(v)$ is, in fact, a C^∞ section covering the identity, it is linear on each fiber and $\omega(v)(x) = v$, for all $v \in T_xM$. To check the differentiability of the map ω, we choose a coordinate chart $(TU_\psi, T\psi)$ of TM. The expression of ω in this chart is

$$\omega_\psi(y,w)(\tilde{x}) = \sum_{(U_\phi,\phi)\in\mathcal{F}_M} \nu_\phi^{1/2}(\tilde{x})\nu_\phi^{1/2}\circ\psi^{-1}(y) D\phi(\tilde{x})^{-1} D(\phi\circ\psi^{-1})(y)w,$$

for $(y,w) \in T\psi(TU_\psi)$ and $\tilde{x} \in M$, and depends C^∞ on $(y,w,\tilde{x})$. Applying the last part of Lemma E.1 to ω_ψ, the claim follows. □

Now we lift the above result in the finite-dimensional manifold M to the lattice $\mathcal{M}$.

Lemma E.3. *Let $\omega : TM \to \mathcal{S}^\alpha_{\mathrm{Id}}(M)$ be the map given by Lemma E.2. Then the map $\Omega : T\mathcal{M} \to \ell^\infty(\mathcal{S}^\alpha_{\mathrm{Id}}(M))$ defined by*

$$\Omega(v)_i(\tilde{x}) = \omega(v_i)(\tilde{x}_i), \quad v \in T_x\mathcal{M},\ \tilde{x} \in \mathcal{M}$$

is C^∞ and linear on each fiber.

Proof. Given any chart $T\psi$ of $T\mathcal{M}$, we have that for any $(x,v) \in T\psi(TU_\psi)$, $\tilde{x} \in \mathcal{M}$,

$$\Omega_{\psi,i}(x,v)(\tilde{x}) = \omega(x_i, v_i)(\tilde{x}_i).$$

We can apply Corollary 2.3 in [FdlLM10], and the result follows. □

Lemma E.4. *The map $\imath : \ell^\infty(\mathcal{S}^\alpha_{\mathrm{Id}}(M)) \to \mathcal{S}^\alpha_{\Gamma,\mathrm{Id}}(\mathcal{M})$ defined by*

$$\imath(\nu)(x)_i = \nu_i(x_i)$$

is linear and bounded.

Proof. Linearity follows immediately from the definition. Since $\imath(\nu)$ is uncoupled, we have that

$$\|\imath(\nu)\|_{S^\alpha_\Gamma} \le \Gamma(0)^{-1} \sup_{i\in\mathbb{Z}^d} \|\nu_i\|_{C^\alpha}.$$

□

Proof of Lemma 4.8. Using the notations and results given by Lemmas 4.4, 4.5, E.3 and E.4, the map

$$\Omega^s_{h_\Phi} = \pi^s \circ \tau^{h_\Phi} \circ \imath \circ \Omega_{|\mathcal{E}^s_F}$$

satisfies all the claimed properties. □

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