A class of analytic solutions for force-free electromagnetic fields

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Abstract. A method for producing a large class of force-free electromagnetic field solutions in curved spacetime is presented. Analytic examples in flat spacetime are given. All the solutions start with a null electromagnetic free-field solution with special properties. Then, a transformation of the null free-field solution produces a solution to the force-free problem. Examples based on the Hopf-Rañada electromagnetic knot are presented. All of the solutions considered have the property that both invariants of the Faraday tensor, $F^{\mu\nu}F_{\mu\nu}$ and $F^{\mu\nu}(*F)_{\mu\nu}$, are zero. This requires that in a local Lorentz frame the electric and magnetic fields are perpendicular and equal in magnitude. Thus these degenerate null fields are quite idealized cases, and they may only be relevant to real physical systems in certain limits. We provide a software program which can be used to check the validity of at least one of the solutions found.

Keywords: force-free, plasma, magnetohydrodynamics, Hopf-Rañada, electromagnetic knot

PACS numbers: 03.50.De, 52.65, 52.30, 52.27

AMS classification scheme numbers: 53Z05, 55P99, 76W05, 85-08

1. Introduction

Force-free electromagnetic theory has found numerous applications in astrophysics, solar physics, and laboratory plasma physics [1, 2]. Since there are a number of applications of force-free electrodynamics in astrophysics where the spacetime is curved, the presentation here shall work with the pseudo-Riemannian geometry of general relativity. We use the formalism of differential forms. Our metric signature is (-,+,+,+). Units are such that c=1. Our goal is to present a new class of solutions to the force-free electromagnetic problem. Let the vector potential 1-form for the electromagnetic field be denoted by **A** and the Faraday 2-form by

$$\mathbf{F} = d\mathbf{A} \tag{1}$$

where d denotes exterior derivative. The tensors for **A** and **F** are related to their form representations by the following

$$\mathbf{A} = A_\alpha dx^\alpha \tag{2}$$

$$\mathbf{F} = \sum_{\alpha,\beta,\alpha<\beta} F_{\alpha\beta} dx^\alpha \wedge dx^\beta = \frac{1}{2!}\sum_{\alpha,\beta} F_{\alpha\beta} dx^\alpha \wedge dx^\beta \tag{3}$$

Maxwell's equations are written

$$d\mathbf{F} = 0,\ d * \mathbf{F} = 4\pi * \mathbf{J} \tag{4}$$

or in tensor notation

$$\nabla_\alpha F^{\alpha\beta} = \frac{1}{\sqrt{|g|}}\partial_\alpha\left(\sqrt{|g|}F^{\alpha\beta}\right) = -4\pi J^\beta \tag{5}$$

$$\nabla_{[\alpha}F_{\beta\gamma]} = \partial_{[\alpha}F_{\beta\gamma]} = 0 \tag{6}$$

where square brackets denote the usual antisymmetrization operation,∇ is the covariant derivative, **J** is the current 1-form, and * is the Hodge dual operator which in tensor notation takes the form

$$(*\mathbf{F})^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} \tag{7}$$

$$(*\mathbf{J})^{\alpha\beta\gamma} = \epsilon^{\alpha\beta\gamma\delta}J_\delta \tag{8}$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the Levi-Civita totally antisymmetric tensor which satisfies

$$\epsilon_{\alpha\beta\gamma\delta} = |g|^{1/2}\,\widetilde{\epsilon}_{\alpha\beta\gamma\delta}, \quad \epsilon^{\alpha\beta\gamma\delta} = -\,|g|^{-1}\,\epsilon_{\alpha\beta\gamma\delta} \quad g = \det\left(g_{\alpha\beta}\right) \tag{9}$$

where the $\widetilde{\epsilon}$ represents the Levi-Civita symbol. The force-free condition is simply

$$F_{\alpha\beta}J^\beta = 0 \tag{10}$$

It follows from this equation alone that

$$det(F_{\alpha\beta}) = 0 \tag{11}$$

and consequently that (in spacetime dimension four)

$$\mathbf{F} \wedge \mathbf{F} = 0,\ (*\mathbf{F})_{\alpha\beta} F^{\beta\gamma} = 0 \tag{12}$$

The helicity 4-current $\mathbf{K}$ or Chern-Simons current is defined by the 1-Form

$$\mathbf{K} \equiv *(\mathbf{A} \wedge \mathbf{F}) \tag{13}$$

and it is conserved because of the force-free condition

$$\nabla_\mu K^\mu = 0,\ d * \mathbf{K} = 0; \tag{14}$$

2. Euler potentials

Uchida [2] has derived the following local representation for the Faraday tensor under very broad conditions for a force-free field

$$F_{\alpha\beta} = \partial_\alpha s_1 \partial_\beta s_2 - \partial_\alpha s_2 \partial_\beta s_1 \tag{15}$$

Actually, Uchida's proof does not apply to the case where both invariants of the Faraday tensor vanish, but we shall find solutions with an Euler potential even for that case here. The two scalar functions $s_1$and s_2 are the Euler potentials. This is equivalent to the Farady 2-form being written as

$$\mathbf{F} = ds_1 \wedge ds_2 \tag{16}$$

The vector potential is not uniquely determined by these potentials. One may write in general

$$\mathbf{A} = s_1 ds_2 + d\psi \tag{17}$$

where ψ is a gauge potential. The Euler potentials are not unique either, as discussed in [2].
If the Faraday tensor is nonzero inside a domain D, then the one forms $ds_1$and ds_2 must be linearly independent inside D.

3. A class of analytic solutions

We look for solutions which satisfy the following ansatz for the Hodge-dual current 3-form.

$$*\mathbf{J} = \frac{1}{4\pi} dh \wedge d\phi_1 \wedge d\phi_2 \tag{18}$$

for scalar functions h, ϕ_1, ϕ_2. The rationale for considering solutions which have this property is simply that they exist and can be studied rather easily. The Maxwell equations become

$$d\mathbf{F} = 0,\ d * \mathbf{F} = dh \wedge d\phi_1 \wedge d\phi_2 \tag{19}$$

Consider a candidate solution of the following form

$$*\mathbf{F} = h d\phi_1 \wedge d\phi_2 \tag{20}$$

The inhomogeneous equation is satisfied automatically. Taking the dual of this we have

$$\mathbf{F} = -h * (d\phi_1 \wedge d\phi_2) \tag{21}$$

and so to have a solution we must require

$$d\mathbf{F} = d\left(-h * (d\phi_1 \wedge d\phi_2)\right) = 0 \tag{22}$$

In order to satisfy this equation, we note the following. Suppose that the 2-form $\mathbf{R} = (d\phi_1 \wedge d\phi_2)$ is a Hopf-Rañada electromagnetic knot solution to Maxwell's equation [3–5]. For such a solution we have

$$\mathbf{R} = \frac{1}{2\pi i} \frac{\partial_\mu \bar{\varphi} \partial_\upsilon \varphi - \partial_\upsilon \bar{\varphi} \partial_\mu \varphi}{\left(1 + \bar{\varphi}\varphi\right)^2} \tag{23}$$

and the dual satisfies

$$*\mathbf{R} = -\frac{1}{2\pi i} \frac{\partial_\mu \bar{\theta} \partial_\upsilon \theta - \partial_\upsilon \bar{\theta} \partial_\mu \theta}{\left(1 + \bar{\theta}\theta\right)^2} \tag{24}$$

for complex scalar functions φ and θ are scalars, and where $\bar{\varphi}$ and $\bar{\theta}$ denote complex conjugates. These forms are both closed so that $d\mathbf{R} = 0$ and $d * \mathbf{R} = 0$, and they are also both exact. As Rañada shows, it is possible to write them in terms of real-valued Euler (or Clebsch) potentials as

$$\mathbf{R} = d\phi_1 \wedge d\phi_2 \tag{25}$$

$$*\mathbf{R} = du_1 \wedge du_2 \tag{26}$$

by making the following transformation with $\varphi = S_\varphi exp(i2\pi\gamma)$ and $\theta = S_\theta exp(i2\pi\rho)$and then using

$$\phi_1 = \frac{1}{1 + S_\varphi^2},\ \phi_2 = \gamma,\ u_1 = \frac{1}{1 + S_\theta^2},\ u_2 = \rho \tag{27}$$

We can satisfy Maxwell's equations if we make the following choice

$$h = u_1 \tag{28}$$

so that

$$*\mathbf{F} = u_1 \mathbf{R} \tag{29}$$

and

$$\mathbf{F} = -u_1 * \mathbf{R} = -u_1 du_1 \wedge du_2 = d\frac{-\left(u_1\right)^2}{2} \wedge du_2 \tag{30}$$

so that $\mathbf{F}$ has the Euler form (15) and Maxwell's equations are satisfied. We next find a condition that ensures that this solution is force-free. The Lorentz force on a current element is proportional to

$$f_\mu = F_{\mu\nu} J^\nu = F_{\mu\nu} \frac{1}{3!} \varepsilon^{\nu\alpha\beta\gamma} \left(*J\right)_{\alpha\beta\gamma} \tag{31}$$

$$f_\mu = (-u_1 du_1 \wedge du_2)_{\mu\nu} \frac{1}{3!} \varepsilon^{\nu\alpha\beta\gamma} \left(\frac{1}{4\pi} du_1 \wedge d\phi_1 \wedge d\phi_2 \right)_{\alpha\beta\gamma} \tag{32}$$

$$f_\mu = -u_1 \partial_\mu u_1 \partial_\nu u_2 \frac{1}{3!} \varepsilon^{\nu\alpha\beta\gamma} \left(\frac{1}{4\pi} du_1 \wedge d\phi_1 \wedge d\phi_2 \right)_{\alpha\beta\gamma} \tag{33}$$

$$f_\mu = -u_1 \partial_\mu u_1 \frac{1}{3!} \varepsilon^{\nu\alpha\beta\gamma} \left(\frac{1}{4\pi} \partial_\nu u_2 \partial_\alpha u_1 \partial_\beta \phi_1 \partial_\gamma \phi_2 \right)_{\alpha\beta\gamma} \tag{34}$$

$$f_\mu = \frac{1}{12\pi u_1} \partial_\mu u_1 F_{\alpha\beta} F^{\alpha\beta} \tag{35}$$

Now we note that this vanishes provided that

$$F_{\alpha\beta} F^{\alpha\beta} = 0 \tag{36}$$

But this requires

$$R_{\alpha\beta} R^{\alpha\beta} = 0 \tag{37}$$

The force-free condition is also equivalent to the following equation

$$\mathbf{J} \wedge *\mathbf{F} = 0; \tag{38}$$

The choice (28) is not unique. We could more generally have used (for any smooth function f)

$$h = f(u_1, u_2) \tag{39}$$

and still obtain a force-free solution.

And so that we must restrict the electromagnetic knot solutions even further by requiring (37). Not all of the electromagnetic knot solutions satisfy (37), however some do. For example, the solution based on the Hopf fibration satisfies it [3, 6–8]. It follows further that in this case we have

$$J^\mu J_\mu = 0 \tag{40}$$

4. Examples

4.1. Hopf-Rañada electromagnetic knot solution

This electromagnetic free-field solution is based on the Hopf mapping from $S^3 \to S^2$as described in [3, 8]. We use the notation of [3]. One finds for this case (where the spacetime is flat Minkowski space and the coordinates have been made dimensionless by dividing by a constant length parameter) with

$$\varphi(\mathbf{r}, t) = \frac{(Ax - tz) + i(Ay + t(A-1))}{(Az + tx) + i(A(A-1) - ty)} \tag{41}$$

$$\theta(\mathbf{r}, t) = \frac{(Ay + t(A-1)) + i(Az + tx)}{(Ax - tz) + i(A(A-1) - ty)} \tag{42}$$

$$A = \frac{x^2 + y^2 + z^2 - t^2 + 1}{2} \tag{43}$$

A set of Euler potentials may be obtained using (27). We obtain

$$\phi_1 = \frac{1}{1 + \bar{\varphi}\varphi}, \ \phi_2 = \frac{1}{4\pi i} ln(\frac{\varphi}{\bar{\varphi}}) \tag{44}$$

$$u_1 = \frac{1}{1 + \theta\bar{\theta}}, \ u_2 = \frac{1}{4\pi i} ln(\frac{\theta}{\bar{\theta}}) \tag{45}$$

Notice that ϕ_2 and $u_2$are potentially multivalued because of the branch point in the logarithm at the origin in the complex plane. Since we want the Faraday tensor to be single valued, we can ensure this by choosing $h = u_1$ so that the force free electromagnetic field in this case is given by the Faraday 2-form

$$\mathbf{F} = -\frac{*\mathbf{R}}{1 + \theta\bar{\theta}} \tag{46}$$

where R is the Faraday 2-form for the free-field electromagnetic knot solution for the Hopf fibration given by (41) and (42) obtained by applying (24). In the appendix we verify that this is indeed a force-free solution with a non vanishing electromagnetic current vector, but vanishing Lorentz force vector.

As discussed in [3], one can generate other electromagnetic knot solutions based on higher homotopy classes of the Hopf fibration using the formulas

$$\varphi^{(n)}(\mathbf{r}, t) = \left(\frac{(Ax - tz) + i(Ay + t(A-1))}{(Az + tx) + i(A(A-1) - ty)} \right)^n \tag{47}$$

$$\theta^{(n)}(\mathbf{r}, t) = \left(\frac{(Ay + t(A-1)) + i(Az + tx)}{(Ax - tz) + i(A(A-1) - ty)} \right)^n \tag{48}$$

and once again using (46) we obtain force-free field solutions with nonzero charge.

In all these cases the fields satisfy $R_{\mu\nu}R^{\mu\nu} = R_{\mu\nu}(*R)^{\mu\nu} = F_{\mu\nu}F^{\mu\nu} = F_{\mu\nu}(*F)^{\mu\nu} = 0$.

In appendix A we present software code which can be run under the REDUCE computer algebra system [9] and which uses EXCALC, a package for Cartan algebra and differential forms [10]. This software validates that the above solutions are valid force-free solutions.

Bialynicki-Birula and others have analyzed a generalization of Robinson-Trautman fields which is achieved by multiplying the Riemann-Silberstein vector by a scalar prefactor [6, 8]. Provided a Euler representation for these fields can be found, then they would provide a large class of models on which to base force-free plasma solutions of the type presented here.

5. Shocks in the force-free solutions

It is well known that shocks may occur in force-free solutions to relativistic plasmas. The electromagnetic knot solutions of Rañada satisfy the free-field Maxwell equations, and therefore they can't generate nonlinear shocks, although there can be initial conditions which might focus energy to a point, curve, or surface in space-time, and for these cases one could still have a singular electromagnetic field. In optics, this type of singularity is called a caustic. But this is a different effect from the nonlinear shocks that occur in fluids and plasmas. Caustics are relatively easy to predict and study, whereas nonlinear shocks are much more complex. In the force-free solutions presented here, the only singularities of the fields would be singularities in the Rañada free-field solution used to construct the force-free solution, and therefore they would be relatively easy to identify and study.

6. Conclusion

We have presented a new class of solutions to the charged force-free plasma problem. The current 4-vector is light-like for all of them. They are topologically interesting as they are derived by a transformation of electromagnetic knot solutions. Of course, as for all force-free plasmas, the Chern Simons current is preserved, leading to magnetic helicity conservation. It is hoped that they will prove interesting from a theoretical point of view, and may serve as approximations to some real experimental or astrophysical systems. Perhaps they may suggest a way to generate force-free solutions which are not null degenerate, but only degenerate. An interesting question arises as to whether the field lines are still closed and linked in the force-free case as they are for the free-field electromagnetic knot solutions.

Appendix A. Computer confirmation that the proposed solution is a force-free field

The following program is written in the REDUCE computer algebra system which is open-source. It uses the open-source Cartan algebra package EXCALC. After the code runs, the user can examine the contents of variables to check that the Maxwell equations are satisfied and to check that the electric current is non-zero, but the electromagnetic force is zero. The program with N=1 (Hopf index 1) takes about 19 minutes on a Pentium 4, 2 GHz processor. For N=2 it did not finish after several days of calculation, and so the run time for that case is unknown, but the program is written to handle any value of N.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This is a program to test the force-free solutions in this paper based on the Hopf fibration
off output; % Leave off until calculation is finished to save time and avoid problems.
load_package EXCALC$; %This package is needed for exterior algebra
% Setup for Minkowski space
spacedim 4;
coframe o(t) = d t,
o(x) = d x,
o(y) = d y,
o(z) = d z
with signature (-1,1,1,1);
frame e;
% Declare some variables
```

```
pform F=2,HF=2,FF=2,HFF=2,phi=0,theta=0,phiconj=0,thetaconj=0,h=0;
fdomain A=A(t,x,y,z), phi=phi(t,x,y,z), phiconj=phiconj(t,x,y,z);
fdomain theta=theta(t,x,y,z), thetaconj=thetaconj(t,x,y,z), F=F(t,x,y,z), HF=HF(t,x,y,z);
A:=(x*x+y*y+z*z-t*t+1)/2;
% N**2 is the is the Hopf index
N:=1;
phi:=(((A*x-t*z)+I*(A*y+t*(A-1)))/((A*z+t*x)+I*(A*(A-1)-t*y)))**N;
phiconj:=(((A*x-t*z)-I*(A*y+t*(A-1)))/((A*z+t*x)-I*(A*(A-1)-t*y)))**N;
theta:=(((A*y+t*(A-1))+I*(A*z+t*x))/((A*x-t*z)+I*(A*(A-1)-t*y)))**N;
thetaconj:=(((A*y+t*(A-1))-I*(A*z+t*x))/((A*x-t*z)-I*(A*(A-1)-t*y)))**N;
% The Faraday 2-form F and it's Hodge dual HF for the Rañada electromagnetic map based on
Hopf fibration
% This is a charge-free field.
F:=(1/(2*PI*I))*((d(phiconj)) ^(d(phi)))/((1+phiconj*phi)**2);
HF:=(1/(2*PI*I))*((d(theta))^(d(thetaconj)))/((1+thetaconj*theta)**2);
% Calculate the currents for these. They should be zero
HJmag := d F;
HJelec := d HF;
Jmag := #HJmag;
Jelec := #HJelec;
% Calculate the force-free Faraday 2-form FF and its Hodge dual HFF using the approach described
in the paper.
h := 1/(1+theta*thetaconj);
HFF := h*F; % The Hodge dual of the Faraday 2-form for the force free field
FF := -#HFF; % The Faraday 2-form for the force free field
% Calculate the electric and magnetic currents for these. The magnetic current should be zero.
% The electric current should be non-zero
HelecFFJ := d HFF;
% The magnetic current should be zero
HmagFFJ := d FF;
% calculate the force condition as a 3-form
HForce := (#HelecFFJ)^HFF; % This should be exactly zero.
% Now we calculate the Lorentz force directly
pform J(a)=0, JF=1; tvector JV; JF := #HelecFFJ;
J(t) := e(-t) _| JF; J(x) := e(-x) _| JF; J(y) := e(-y) _| JF; J(z) := e(-z) _| JF;
JV:=J(t)*e(t)+J(x)*e(x)+J(y)*e(y)+J(z)*e(z);
Jsquared := JV _| JF; % This should be zero
LorentzForce := JV _| FF; % This should be zero
on output; %Turn on output so that we can interactively examine variables
end;
```

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