2D CONTINUED FRACTIONS AND POSITIVE MATRICES

R. F. WILLIAMS

1. Abstract

The driving force of this paper is a local symmetry in lattices. The goal is two theorems: a partial converse to the Perron-Frobenius theorem in dimension 3 and a characterization of conjugacy in $Sl^3(\mathbb{Z})$. In the process we develop a geometric approach to higher dimension continued fractions, HDCF. HDCF is an active area with a long history: see for example Lagarias, [L],[Br].

The algorithm: Let Z_r be the set of all lattice points within $r > 0$ of a ray $L = \{mP \in \mathbb{R}^n : m > 0\}$ Let z_1 denote the point in Z_r closest to the origin. Having defined $z_1, .., z_i, 1 \leq i < n$, let z_{i+1} be the point of Z_r closest point into the origin, which is independent of $z_1, ...z_i$.

Conjecture 1. *$z_1, ..., z_n$ is a basis of $\mathbb{Z}^n$.*

The proof of this conjecture in dimension $n = 3$ occupies the bulk of the paper. Here $P \in \mathbb{R}^n$ is taken to have all positive components as usual, and we use a metric in $L^\perp$. For further discussion and the relation of the conjecture to the Minkowski sequential minima theorem, see the section "Remarks on the algorithm" below.

We have no such arithmetic theorems in dimension $n > 3$. However, in the first place, the symmetry theorem, its relation to arithmetic, and part of the bifurcation theorem are proved in all dimensions. Secondly, lots of computer studies in the next three dimensions have been done and we find no obstruction to this conjecture; in particular, for the partial converse mentioned above, we easily find bases of $\mathbb{Z}^n$, relative to which the matrix becomes positive, for $n = 3, 4, 5, 6$. A few examples are given below.

Supported in part by a grant from the National Science Foundation.

We wish to thank The Institute for Advanced Study and the Mathematics Department of Montana State University for their hospitality.

lattices, symmetry, Perron-Frobenius, $Gl^3(\mathbb{Z})$, higher dimensional continued fractions.

2. Introduction

We will need to prove that there really are n such lattice points which are (just) linearly independent; this is done in terms of the $n-1$ dimensional cohomology of the n dimensional torus.

Given this setting and $r > 0$, we determine a certain $n-1$-dimensional cell complex K_r. This complex is homotopy equivalent to the $n-1$-skeleton of the n-dimensional torus, and is a smooth (in fact flat) branched manifold [W3]. The $n-1$-cells $E(z)$ of K_r, are labeled with lattice points $z \in \mathbb{Z}^n$; the geometry of these cells is intimately tied up with the arithmetic of $\mathbb{Z}^n$; and there is a recipe (equivalent to the algorithm) for choosing n of these cells, $E(z_1), ..., E(z_n)$, where $z_1, ...z_n$, is the sequence in $\mathbb{Z}^n$ chosen by our algorithm.

There are infinitely many bifurcations of K_r, and thus a sequence of bases $\{a_i, b_i, c_i\}$ of $\mathbb{Z}^3$. These bases limit down on the ray $\rho = \{xP : x \in \mathbb{R}, x > 0\}$ and the 'update' matrix relating the ith basis to the $i+1$st is among a certain small set of matrices. Recall that the complete (ordinary) continued fraction is determined by the corresponding transition matrices—there are only two, to wit the 2 'elementary' 2×2 matrices: each a_i in the continued fraction expansion, is just the number of one of these that occur before the other one shows up. Thus, we have a geometric theory of 2 dimensional (2D) continued fractions See [St] for a geometric presentation of ordinary continued fractions.

We use a computer to find these bases; the programs are elementary, are potentially polynomial in the dimension n, and proceed by seeking out **best approximants** directly. (see definitions, below) Thus much of our theory can be regarded as just finding out 'what's there' near a line in a 3-dimensional symmetric lattice.

Theorem 1. *Given a Pisot matrix A (3×3, integral entries determinant 1,having one eigenvalue bigger than 1, and the other 2 less than 1 in modulus), there exists $B \in GL(3, \mathbb{Z})$ such that no entry of BAB^{-1} is negative.*

We claimed to have proved this theorem before, in particular in [W1], where the argument is incomplete.

The corresponding converse is known in dimension 2 [ATW][W4]; it also follows from the fact that the transition matrices in ordinary continued fractions have only non-negative entries. Unfortunately, in the next dimension these matrices have occasional negatives, especially when the ray ρ is too close to a rational subspace of dimension 2. Finally, in the (eventually) periodic case, the complete machinery (the bifurcations, and the complex K_r is easily shown to be (eventually) periodic.

Another application is solution to the conjugacy problem for matrices $A, B \in SL^3(\mathbb{Z})$. (See [ATW] for this result in dimension2.) That is, there is a finite process for deciding whether A and B are conjugate over $\mathbb{Z}$. First, if one eigen value $\lambda > 1$ and the other two complex, and the inner product gotten by declaring the complex eigen vectors to be normal and of the same length, our whole structure is eventually periodic. For the totally real case, periodicity is obtained by changing the metric continually, to offset the different rates of contractions of the two small eigenvalues. Of course, either a given $A \in Gl^3(\mathbb{Z})$ or its inverse is one of these two types. We understand that there is a solution of this problem, by very different methods, in [BS], p 128.

This work has taken many years; among those who have helped are Rafael de la Llave, Marcy Barge, Harold Stark, Jeff Vaaler and John Tate.

The paper is organized as follows. The complexes K_r and Q_r are described and we prove the symmetry result, this far for all dimensions. Then the bifurcations are described and shown to the only ones for $n = 3$. Next, we find the simplest examples of the complexes Q_r and K_r and show that they are models in all cases ($n = 3$ here and below).

Almost half the paper is devoted to proving the main proposition, that our choice gives a basis for $\mathbb{Z}^3$. First we show that the cochains dual to the cells $E(z_i)$ of Q_r or K_r which are *convex* span the 3-dimensional cohomology of K_r : it follows that the corresponding $z_i's$ span $\mathbb{Z}^3$. Hence there are always at least 3 convex cells; 4, 5, and 6 can occur. We next show that any cell is a *positive* linear linear combination of the convex cells. Here we abuse notation by using the same symbol for a cell E the label $z \in \mathbb{Z}^n$ such that $E = E(z)$. In dimension 3 we show that there is a subset of at most 4 convex cells in terms of which any other cell as can be written as a positive linear combination.

This leads us to the special cases where four convex cells, say $E(z), z = a, b, c, d$ are actually required: when the ray ρ is near a rational subspace, generated by $a, b \in \mathbb{Z}^3$. In such cases a relation like $a + d = X = b + c$ is shown to hold, where $E(X)$ is a cell of K_r. Of course this yields a basis, say a, b, c, which suffices for our version of 2 dimensional continued fractions, but is not good enough for our positivity results. Hence we must work a bit harder in such cases,

The last 2 sections concern the periodic case and conjugating Pisot matrices to ones with no negative entries.

3. Contents

4. Remarks on the algorithm

First, a more detailed account of the algorithm: Let $\pi : \mathbb{R}^n \to P^\perp$ be the projection along the vector P. For $x \in \mathbb{R}^n$, let the *weight* of $x = wt(x) = P \cdot x$, be the distance from x to the hyper-plane $P^\perp$, measured along the vector P. Given a real number $r > 0$, let

$$Z_r = \{z \in \mathbb{Z}^n : |\pi(z)| < 2r \text{ and } wt(z) > 0\}.$$

Say $z_1 \in Z_r$ minimizes the wt in Z_r. Assume that $z_1, z_2, ..., z_m, m < n$ have been chosen; then there exists a $z \in Z_r$ such that:
(1) z is independent of $z_1, ..., z_m$.

Remark 1. *One shows below that this algorithm is unchanged if one adds the condition (2) if $z' \in Z_r$ has less weight than z, then $N_r(\pi(z)) \cap N_r(\pi(z')) \cap N_r(O) = \emptyset$, where $O = \pi(L)$ is the origin.*

Among all such z, let z_{m+1} be the one of least weight.

The lattice point z_1 is a *best approximant* for r provided it minimizes the weight for lattice points in Z_r.

Remark 2. *This is vaguely analogous to the choice in the Minkowski sequential minima* [C] *theorem but is essentially different: there is no inequality here; Minkowski enlarges his set in all dimensions and takes the first lattice point it hits which is independent of the earlier choices; whereas, by the definition of Z_r, we are only enlarging in one dimension. And as is well known, the Minkowski process does not always get a basis of the full lattice.*

Remark 3. *We have to prove the existence of the full sequence, $z_1, ..., z_n$, which is done in terms of the cohomology of the complex K_r. That this forms a basis of $\mathbb{Z}^n$ is easy for $n = 2$, (essentially classical) and is proved below for $n = 3$.*

Remark 4. *The cells satisfying 1) and 2) above are convex in Q_r; all other cells have concave boundaries. This is also true in the complex K_r, where this makes sense as it is endowed with a flat structure.*

Remark 5. *Here we used the Euclidean inner product—however, other metrics work and others have been used.*

5. The complexes K_r, Q_r and Q_r^R

Recall that P is a vector in $\mathbb{R}^n$ with all positive coordinates and $P^\perp$ is the plane through the origin, O, perpendicular to P. Let Q_r be the open, round disk in $P^\perp$, of radius r centered at O. For each $z \in \mathbb{Z}^n$, let $Q_r(z) = Q_r + z$, the translation of Q_r to the lattice point z. Our first definition is in terms of decomposition spaces, which allows us to use the Vietoris-Beagle- mapping theorem. We give an equivalent definition in the next paragraph, avoiding quotient spaces. Let $\mathcal{L}$ be the family of all lines in $\mathbb{R}^n$ parallel to the vector P and let $\mathcal{C}_{rt}$ be the family of all connected components of

$$L \cap \{\mathbb{R}^n - \cup_{z\in\mathbb{Z}^n} Q_r(z)\} \text{ for } L \in \mathcal{L}.$$

In the quotient space, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, let Q_{rt} and $\mathcal{C}_{rt}$ be the images of Q_r and $\mathcal{C}_r$ respectively. Then $\mathbb{T}^n - Q_{rt}$ is foliated by the set $\mathcal{C}_t$ of all the connected components C of $L \cap (\mathbb{T}^n - Q_{rt})$, all $L \in \mathcal{C}_t$.

Standing hypothesis: each $C \in \mathcal{C}_t$ is a line interval.

This is of course true if the coordinates of the vector P are rationally independent—but, as is customary, this weaker assumption allow for some rational dependence in our lines, as long as the radius r is not too small. For example, if $n = 3$ and $P = (1, 1, 1)$, then for $\sqrt{2}/3 < r < \sqrt{3}/2$, this all works. The upper bound is to keep the coordinates of the various $z_i \geq 0$. See also the section on rational subspaces, especially for $n = 3$.

Finally, let $K_r = (\mathbb{T}^n - Q_{rt})/\mathcal{C}_{rt}$. That is, K_r is the quotient space in which all the intervals of $\mathcal{C}_{rt}$ are collapsed to points, and let $q : \mathbb{T}^n - Q_t \to K_r$ be the quotient map. Then K_r has the homotopy type of the $(n-1)$-skeleton of $\mathbb{T}^n$, (which is the union of n, $(n-1)$-dimensional tori.) Furthermore, K_r is an orientable, $(n-1)$-dimensional branched manifold. This follows from general results in [W3], or can easily be seen directly. Let $\mathcal{C}_0$ be the family of all line intervals $(t, b) \in \mathcal{C}$, having their bottom end, b in $Q_r = Q_r(O)$. Note that

$$\cup_{C\in\mathcal{C}_0)} C \cup Q_r$$

is a fundamental domain for $\mathbb{T}^n$. Finally, For each $z \in \mathbb{Z}^n$ such that some $(b, t) \in \mathcal{C}_0$ has its top $t \in Q_r(z)$, and bottom end in $Q_r = Q_r(O)$, define

$$E(z) = \{b \in Q_r : \text{ for some } t \in Q_r(z), z \in \mathbb{Z}^n, (b, t) \in \mathcal{C}_0\}.$$

This complex is finite, that is $S = \{z|E(z) \neq \emptyset\}$ is finite because the lines parallel to L are densely and uniformly winding in the torus. We can avoid the quotient space terminology, with the following equivalent definition:

$$E(z) = [\pi(Q(z)) - \cup_{x\in S, wt(x)<wt(z)}\pi(Q(x)] \cap Q_r.$$

Here, π is the projection along the line L; in this sum, any set larger than S, for example $\mathbb{Z}^n$ itself, will give the same answer. The set S is determined as $\{z \in \mathbb{Z}^n : E(z) \neq \emptyset\}$.

We distinguish Q_r (actually, Q_r union its boundary) as a separate cell complex and there is the identification map $Q_r \to K_r$. This identification can be described as follows: each interior $n-1$-cell of the complex Q_r, is identified to its (unique for $n > 2$) translate in the boundary of Q_r. In all our figures, it is Q_r that is drawn.

More generally, if $R \subset \mathbb{R}^n$, is a rational **subspace**, define

$$E(z) = [\pi(Q(z)) - \cup_{x\in S\cap R, wt(x)<wt(z)}\pi(Q(x)] \cap Q_r.$$

Remark 6. *These cells $E_R(z)$ may not cover the open disk Q_r but do form a a complex, Q_r^R, useful below where it is used as a 'background'.*

It is striking that each codimension 1 cell in K_r bounds 3 and only 3 cells of dimension $n-1$. (For $n = 2$ this is fails at bifurcation points, but is always true for higher dimensions.)

6. Symmetry

This is the key to our approach and ties the geometry to the arithmetic.

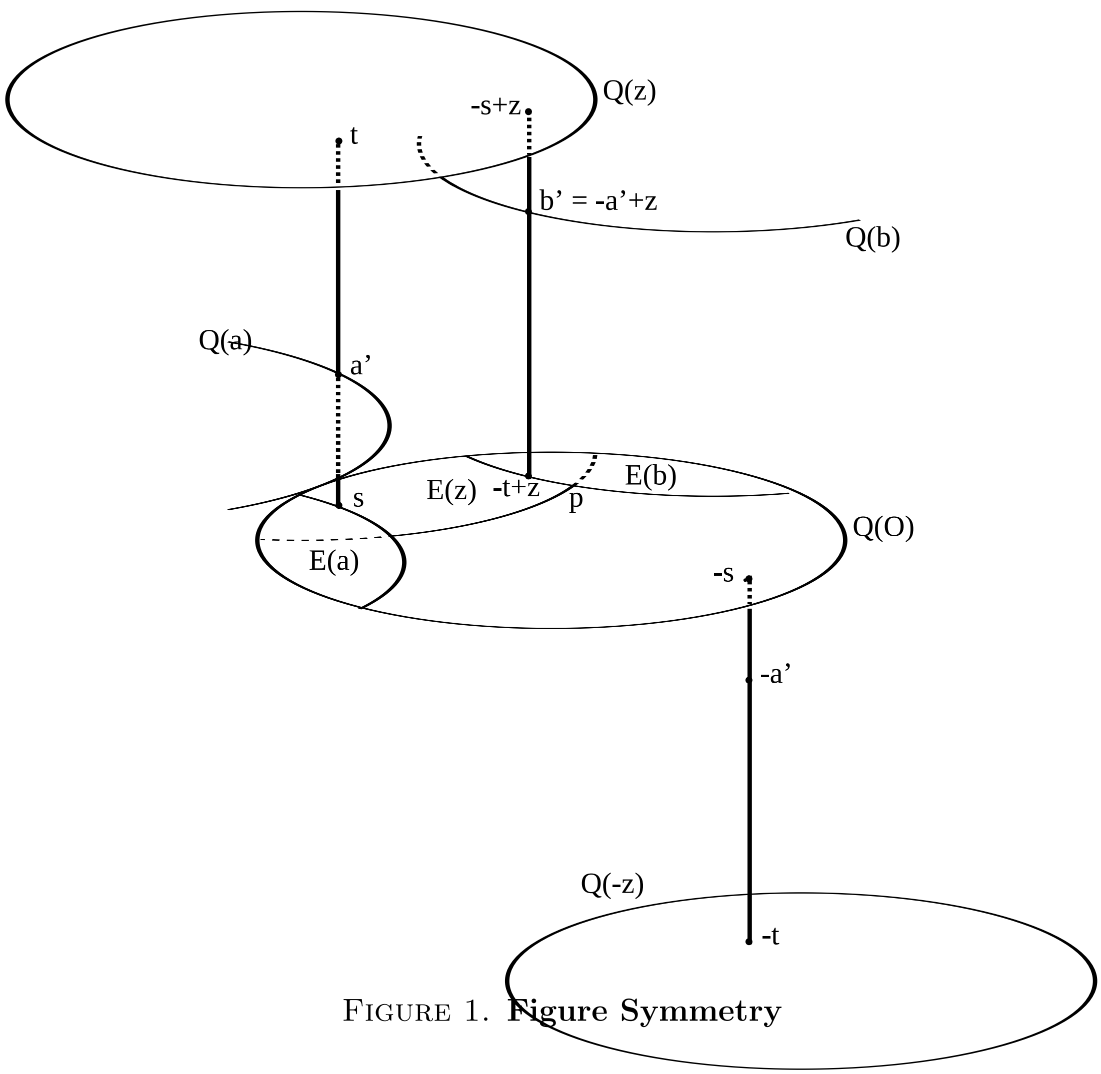

FIGURE 1. **Figure Symmetry**

Theorem 2. *Each cell of Q_r is symmetric under the reflection about its center. If a cell $E(z)$ has τ as a concave boundary then the symmetric boundary τ' is also concave; if $E(a), E(b)$ are the other cells in Q_r having τ, τ' as (convex) boundaries then $z = a + b$. Furthermore, the symmetry reverses the order of the weights of all cells bounding $E(z)$.*

Notation We say that the cells a and b are *dual* with respect to the cell z.

Remark 7. *For R a rational subspace, the complex Q_r^R satisfies this because the symmetry map, defined below, leaves any subspace invariant;and thus satisfies many of the consequences listed below as well.*

Proof of the theorem. Let $R : \mathbb{R}^n \to \mathbb{R}^n$ be given by $x \mapsto -x$. Then R respects all of the structure that enters into the definition of K_r : the disks $Q_r(z)$ go into $Q_r(-z)$, the lines of our foliation go into other lines, etc. The family $\mathcal{C}_0$ is mapped into a 'dual' family, $\mathcal{C}_0'$ which covers another fundamental domain of $\mathbb{T}^n$.

Recall that $\pi : Q_r(z) \to P^\perp$, is the projection along our foliation $\mathcal{L}$; let $\bar{Q}_r(z) = \pi(Q_r(z))$. Let $T = T_z$ be the translation $x \mapsto x + z$,

Then R maps $P^\perp$ to itself by reflection about O and $(\pi \circ T)|P^\perp$, is the translation of the hyperplane $P^\perp$ sending $Q_r(0)$ to $\bar{Q}_r(z)$, the projection of $Q_r(z)$, and in particular, sending the origin to $\bar{z} = \pi(z)$. Then $\pi \circ T \circ R$ sends x to $\bar{z} - x$, which has fixed point $\bar{z}/2$. Then this composition sends

$$\bar{z}/2 + x \to \bar{z} - \bar{z}/2 - x = \bar{z}/2 - x;$$

this is the reflection R' of the plane $P^\perp$ about the point $\bar{z}/2$ midway between the centers of these 2 disks; the center of $B = \bar{Q}_r(z) \cap Q_r$, is $\bar{z}/2$. See the figure.

Now suppose $s \in E(z)$. Then based at s there is an interval $(s,t) \in \mathcal{C}_0$ and $T \circ R$ takes this interval into another interval in $\mathcal{C}_0$. As $-s \in Q_r$, $-s + z \in Q_r(z)$ and $-t + z \in Q_r$ so that $R'(s) = -t + z \in E(z)$. That is, R' is a reflection symmetry of $E(z)$, as claimed.

Now if s is in a concave boundary of $E(z)$, we see that the interval (s,t) intersects the boundary of one of the disks, say at $a' \in \partial Q(a)$. Since both the maps R, T, send these disks into others, we see that $TR(a')$ is in the boundary one, say $Q(b)$. Hence, (see the figure), $R'(s)$ is likewise in a concave boundary of $E(z)$, and $TR(a) = -a + z = b$, or $z = a + b$ as claimed.

The fact about reversing the order of neighboring cells, is really just the fact that R reverses the direction of intervals of $\mathcal{C}_r$ and is detailed in the

Lemma 1. *The symmetry R' satisfies the equation $wt(R'(E)) = wt(E(z)) - wt(E)$, for each cell E touching the boundary of $E(z)$.*

Corollary 1. *Each cell which has part of its boundary on the boundary of $Q_r(0)$ has a convex inner edge; moreover R' interchanges the inner and outer boundaries.*

Proof For the reflection sends the outer boundary of the carrying lenticular region B into the interior of $Q_r(O)$.

Corollary 2. *Each cell has an even number of faces and these faces occur in dual pairs which the symmetry interchanges. For s and s' dual pairs, the tangent plane to a point $x \in s$ is parallel to the point $x' \in s'$, dual to x' under the symmetry; unless two faces are dual, they cannot have parallel tangents.*

Proof First, duality is defined by the symmetry, which is an involution. Since the symmetry is differentiable the result follows.

Corollary 3. *Suppose D is an open disk in the boundary of a cell E, away from edges of smaller dimension, is diametrical to its dual D', in the sense that when D and D' are translated to the boundary of Q, are exactly diametrical; D' is unique in that no other disk in the boundary of E is diametrical to D.*

Proof. This is just the fact that when two spherical disks intersect they cut off diametrical segments of each other. This implies the corollary in case the cell is convex. And if a "bite" is taken out of a cell by another cell, this is principle applies to show that the disk removed is diametrical to that which replaces it on the boundary. The corollary follows by induction in the number of "bites" that have been taken out of the cell.

If a cell has dual concave sides, then as r is decreased they approach, become tangent and finally overlap, thus disconnecting the cell. However,

Corollary 4. *For $n = 3$ no cell has more than 2 components.*

(This is seems true in higher dimensions but is more involved and not needed below.)

Proof First, only concave sides can intersect at one of their interior points, and they first meet at a point A, of tangency. Hence the meeting sides are dual. At this radius the geometry at A is unique in the cell and hence fixed under its symmetry and therefore is the center of the cell. For larger values of the radius r, the cell has two connected components, which the symmetry interchanges. Subsequently, dual sides are in different connected components, and therefore cannot become tangent or intersect intersect. A cell with two connected components can be capped, if it has a convex side, and thus gain a new pair of dual concave sides—however these are in deferent components, and thus cannot lead to a further cutting of the cell into more components. This uses the uniqueness of dual edges.

Corollary 5 (The same cell lemma)**.** *Each cusp in Q_r has a dual cusp and they belong o the same cell.*

Proof. For the cell containing a portion of one cusp, has a symmetry and only two cusps can be interchanged by a symmetry. And the only cusp parallel to a given one, is that at the other end of the surviving arc of the given cusp.

7. That the selection $z_1, z_2, ...z_n$ exists

There is a commutative diagram [W1][W2] linking the first homology of $\mathbb{T}^n$ to the $n-1$th cohomology of K_r. Since the lattice points of $\mathbb{Z}^n$ correspond directly to the first homology of $\mathbb{T}^n$, this shows that the $n-1$th cohomology is free abelian of rank n. Finally, as each cell of K_r can be written in terms of its convex cells, this shows that the z_i's can be so chosen.

There is a different way of seeing this: we need to see that the arithmetic in Corollary 5.3 agrees with that in the $n-1$st cohomology: let $E(a), E(b)$ be dual with respect to z and let F be a concave face between $E(a)$ and $E(z)$ (in Q_r.) Then reflecting F about the center of $E(z)$ and reflecting it again about $E(b)$, we get a face F' of $E(b)$ on the boundary of Q_r. But the composition of two reflections is a translation, so that $F' = F$ in K_r. Let C^{n-2} be the cochain dual to F; then the coboundary of C^{n-2} is $E(z) - E(a) - E(b)$. This proves that the arithmetic in K_r agrees with that in its cohomology.

8. Some terminology

: **approximant**: a lattice point z is a best approximant to L provided that no other lattice point is closer to both the origin and the line L.

: **cell** z: at times we say the "cell" z meaning the cell $E(z)$.

: **hex cell**: a special cell (see the 'simplest cases' section) with 6 sides, 2 convex and 4 concave.

: **shell**: a cell, say $E(z)$ which contains the two extreme points of $\pi(Q_r(z)) \cap Q_r$. Examples: convex cells, cells that have just been capped, such as $E(z_2)$ for $r > r_0$, in the capping bifurcation below.See figure 4.

: **span cell**: a cell c with inner boundary joining two shells a, b which are dual with respect to c. For example the cell z_2 in both the breakout and upstaging bifurcation. Figures 2 and 3.

: **end points of a shell**: the two extreme points in the definition of 'shell'.

: **stack**: See the section on Rational subspaces.

: **capped cell**: a cell in Q_r whose intersection with the boundary of Q_r is disconnected, See the section on rational subspaces.

: **small cell**: cells grow faster when they are 'small', providing thus for the breakout bifurcation. (If 2 unit circles overlap by Δ, measured along their radii, then the width of the arc they cut off is

$$w = \sqrt{4\Delta - \Delta^2}.)$$

Also, cells may become small and then vanish—in itself a bifurcation.

: **duality**: The dual to a boundary point of a cell is its image under the symmetry of the cell; similarly faces of Q_r interchanged by the symmetry are dual, as are the cells bounding those faces. Note that when dual arcs are translated to the boundary circle, they are exactly antipodal.

: **chasing duals** In the complex Q_r, proceeding from one edge to another by duality; in particular, a double dual is a translation, allowing one to compute boundary identifications forming K_r from Q_r. to-f

: **parallel cells**: $n-2$-cells in the complex Q a said to be parallel if they are translates of sets which are (exactly) antipodal on the $n-2$-sphere. For example, the two sides of a convex cell are parallel, as are the dual sides of any cell.

: **π-span**: 2 cells in capping, or 3 cells in upstaging which take up π radians of the boundary of Q_r.

: **background** The background of a cell $E(z), z \in \mathbb{Z}^3$, is $Q(O) \cap \pi(Q(z))$, where π is the projection along the the line L.

: **background of a stack** For a stack based on a and b and the rational subspace, R that they span, for each z for which $E(z)$ is a stack cell, define

$$E^R(z) = [\pi(Q(z)) - \cup_{x \in S \cap R, wt(x) < wt(z)} \pi(Q(x)] \cap Q_r.$$

Here S is the set of all x such that $E(x)$ is a stack cell. Then Q_r^R is the complex consisting of all these $E^R(x)$.

9. The Bifurcations

We use the radius r of the disk Q_r as a bifurcation parameter and distinguish 3 bifurcations which happen at $r = r_0$, say; these each occur when 2 cells collide and they all involve 3 cells $E(z_i), i = 1, 2, 3$. See figures 2, 3. 4. There are two types of collisions: when end points (on the boundary of Q_r) of two shells touch and when two cells touch in the interior of Q_r. The first happens in two ways: first, 'upstage' when a right end point of one shell touches the left end point of another. Say the right endpoint of z_1 touches the left endpoint of z_2 where

$wt(z_1) < wt(z_2)$. Then for $r > r_0, z_2$ will have a concave edge on its left, and hence it must have a concave edge on its right, made by, say z_3. Hence z_1 and z_3 are dual relative to z_2 and $z_1 + z_3 = z_2$. Of course left and right could be switched here. See figure 2.

Secondly, *breakout* occurs when the shells touch at their left end points, say z_1 and z_2, where $wt(z_1) < wt(z_2)$; see the figure. Note that just before $r = r_0$, both endpoints of z_1 were between z_2's endpoints on the boundary of Q_r and the cell z_1 is growing faster than z_2 as it is 'younger'. Just after $r = r_0, z_2$ will have a concave side on its right and thus there is another cell, z_3 with $wt(z_3) < wt(z_2)$, making a concave side on z_2's right. Hence z_1 andz_3 are dual with respect to z_2.

As above, we can determine a π-span by following parallel tangents over several—this time 4—points:

(1) the exterior tangent at z_1's right end point;
(2) the interior tangent at z_1's left end point;
(3) the interior tangent to z_2's right end point; and
(4) the exterior tangent to z_3's right end point.

Finally, the cells z_1, z_2 could touch in an interior point of Q_r; as the radius is increasing the circular arcs in Q_r are moving in the direction of the outward normal and when two sides touch they are tangent, say at a point a. Then the cell of less weight, say z_1, will proceed to "top" the other, say z_2. Then at the point $\bar{a}$, dual to a in z_2, a new convex cell will be 'born'. Note that $\bar{a}$ is on the boundary of Q_r, and that Q_r is actually tangent to the disk $\pi(E(z_3))$. As r increases, these two disks will intersect and thus $E(z_3)$ becomes a cell. At this point, z_1 and z_3 are dual with respect to z_2 and $z_1 + z_3 = z_2$. Note that at the point of bifurcation the tangent line to the boundary of z_1 and z_2 is parallel to the tangents at both the dual to a in z_2 and the dual to a in z_1, There are two cases here: first, if there is a spanning cell between z_1 and z_2, we see that these three cells occupy an arc of more than π on the boundary of Q_r. This is very useful in *crowding* arguments below. Secondly, if there is no spanning cell, then neither of the two half circles is picked out.

Upstaging.

(1) The cells $E(z_i)$ are shells and are mutually disjoint for $r < r_0$.
(2) For $r = r_0$ the three shells just touch; the extreme points of $E(z_2)$ are (respectively) the left extreme point and right extreme point of $E(z_1)$ and $E(z_3)$.
(3) At $r = r_0$ the other extreme points of $E(z_1), E(z_3)$ are antipodal on the $n - 1$ sphere $\partial Q_r(O)$.

(4) For $r > r_0$, the middle cell has lost its shell character and is also a span cell.
(5) $z_1 + z_3 = z_2$.
(6) A totally concave cell is destroyed and another is formed. [The central cell is destroyed and replaced.]

Breakout.

(1) The cells $E(z_i)$ are shells, $E(z_2)$ separates $E(z_1)$ from the rest of the complex Q_r and $E(z_1)$ and $E(z_3)$ are duals with respect to $E(z_2)$;
(2) For $r = r_0$ the extreme points of $E(z_2)$ are the left (respectively, right) extreme points of $E(z_1)$ and $E(z_3)$.
(3) At $r = r_0, E(z_1) \cup E(z_2) \cup E(z_3)$ spans $\pi+$ the angle of $E(z_1)$.
(4) For $r > r_0$, the middle cell has lost its shell character.
(5) $z_1 + z_3 = z_2$.
(6) The span cell $E(z_2 + z_3)$ (only) is destroyed; it has 2 convex sides.

Capping.

(1) The cells $E(z_1), E(z_2)$ have nearby interior convex edges e_1, e_2, for $r < r_0$.
(2) At $r = r_0$. the convex edges of part (1) are tangent at a point a.
(3) The weights satisfy $wt(z_1) < wt(z_2)$.
(4) For $r > r_0$ there is a new convex cell, $E(z_1 - z_2)$ which appears on the spot a' of the boundary of $E(z_3)$ dual to the point a.
(5) $E(z_2)$ is not convex; in fact, $\partial E(z_2) \cap \partial Q_r$ is not connected, thus $E(z_2)$ is a 'capped' cell.
(6) The points a' and a'' are antipodal on ∂Q_r, where a'' is dual to a in $E(z_1)$. In particular, if there is a cell spanning $E(z_1)$ and $E(z_2)$ then these three cells contain a π-span.

Proposition 1. *These bifurcations occur in all dimensions, $n \geq 3$, except that part 6 of upstaging and part 6 of breakout are only for $n = 3$. (See "small cells" below.) For $n = 3$ these are the only bifurcations that occur for a generic path, as r changes.*

Proof. We next go over some details about these bifurcations, say at $r = r_0$. For the 'upstaging' bifurcation we have already verified everything except conditions (3) and part of (6). For $r > r_0$, using the symmetries of cells, $E(z_3), E(z_2), E(z_1)$ in that order, we see (The Spanning Pie figure) that the arc from the outer extreme points of $E(z_1), E(z_3)$, is $\pi+$ the arc CC'. Thus at $r = r_0$, the angle is π, which proves part (3).

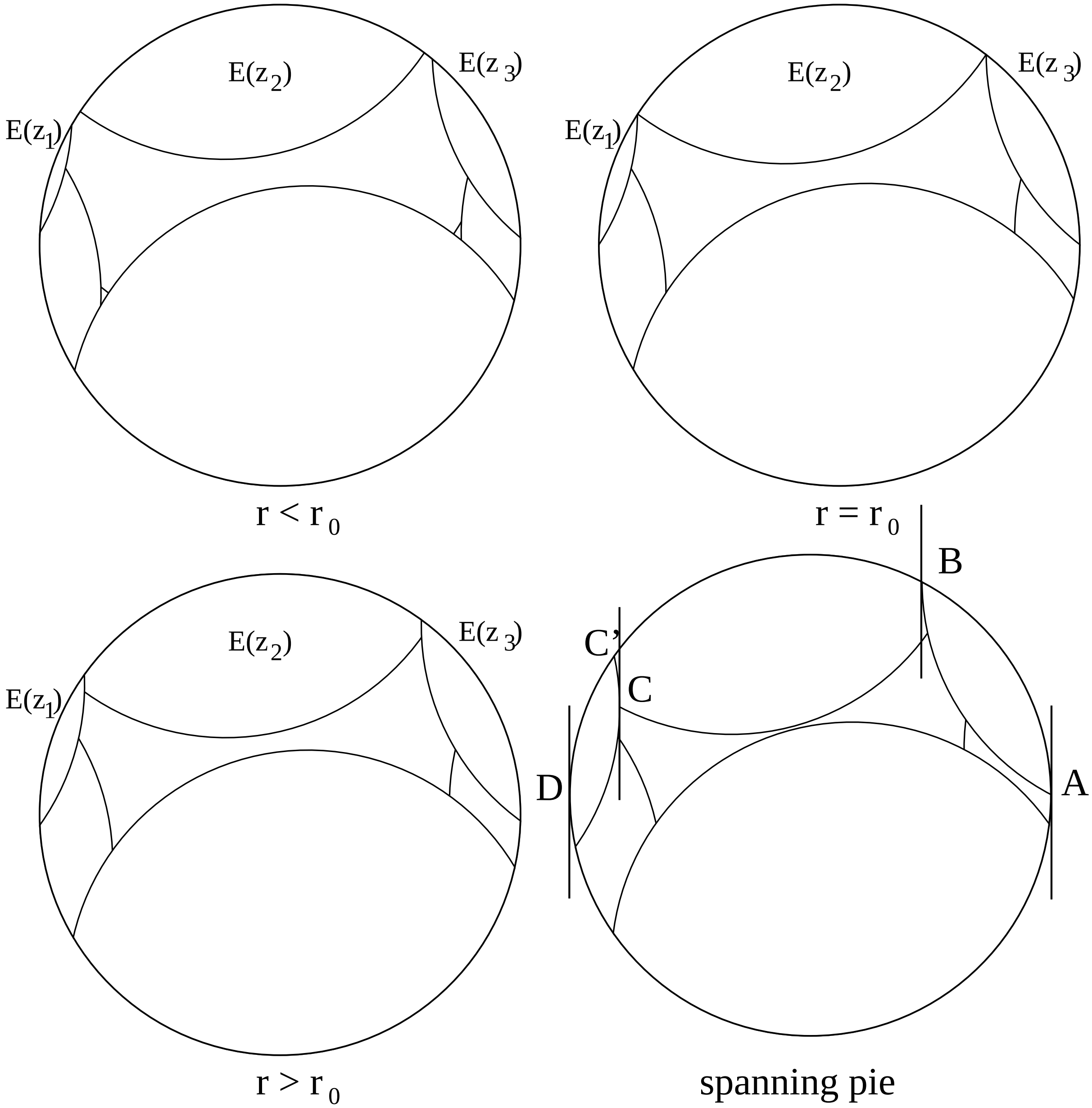

Figure 2. **The Upstaging Bifurcation**

For r just above r_0, we use the segment CC' above: applying the general duality proposition, we see (Spanning pie) that $z_2 - z_1 = z_3$, Finally, since at the bifurcation, only 2 intervals in ∂Q_r are lost and these are on the boundary of Q_r,

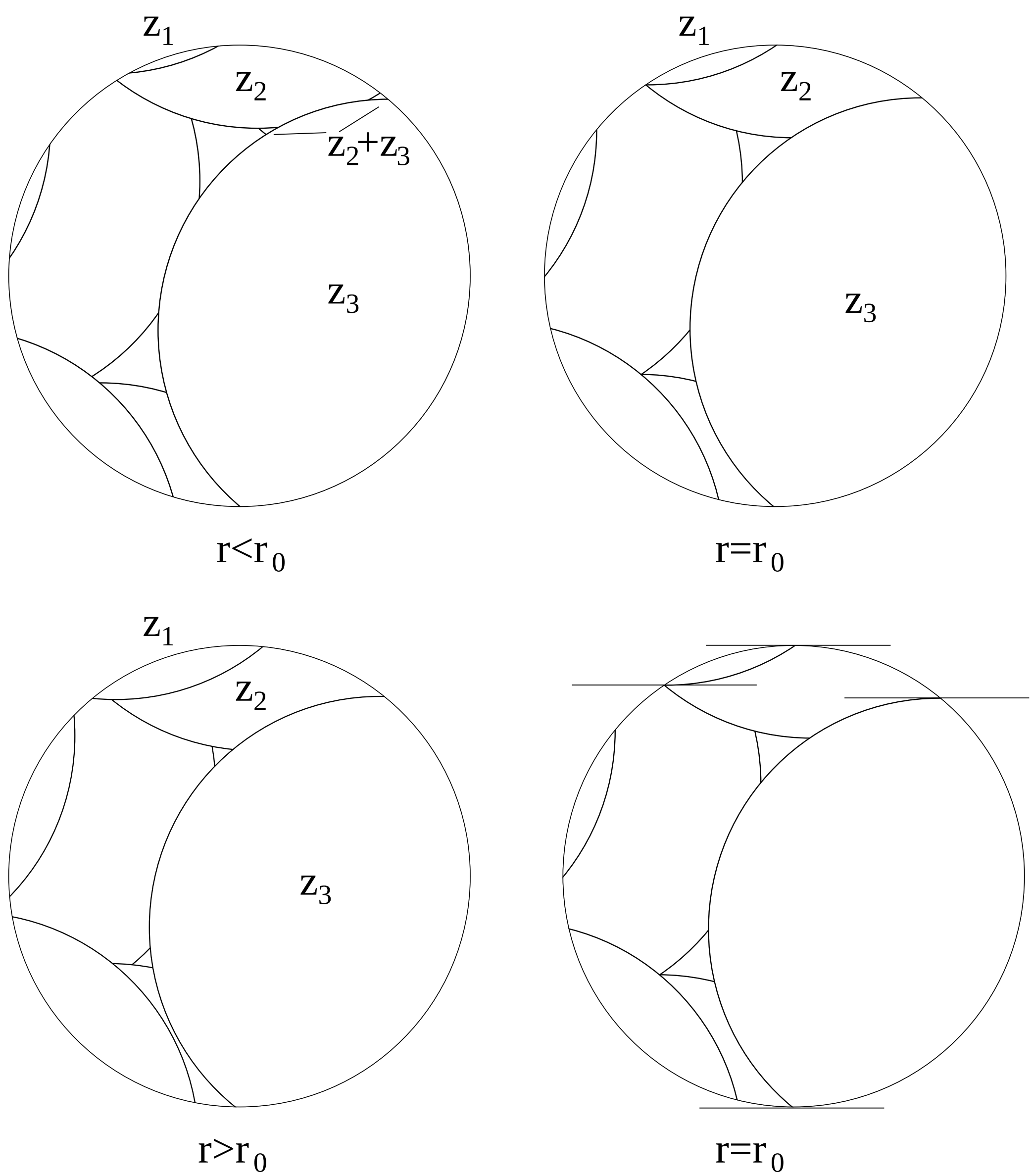

FIGURE 3. **The Breakout Bifurcation**

Now cell E'' containing these two intervals and their duals for $r < r_0$, can have no convex sides, for $r > r_0$ as is seen as follows: first, it cannot have boundary points of Q on both sides of the π-span $E(z_1)E(z_2)E(Z_3)$ and hence by symmetry it cannot have any boundary points at all. Thus it is totally concave sided.

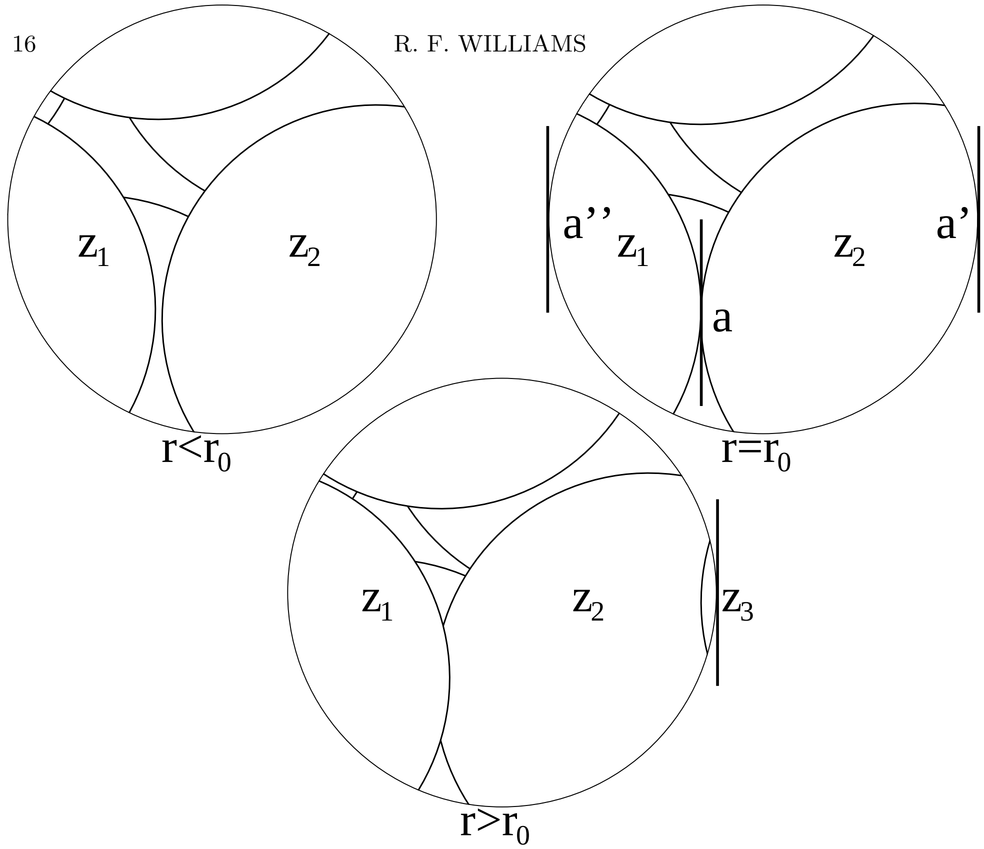

FIGURE 4. **The Capping Bifurcation**

Secondly, the duals to these two intervals, relative to E" are in the interior of Q_r for $r < r_0$, and completely disappear for $r \geq r_0$. Thus they must have been part of the boundary of a disconnected cell consisting of 2 small triangles. The fact that there is only one totally concave cell is proved below. With this exception, the proof of the 'upstaging' bifurcation is complete.

The real difference between the breakout and upstaging bifurcation is in parts (3), (6) and the second part of (5). To see part (3), use the symmetries of $E(z_1), E(z_2), E(z_3)$ in that order, (see the 4th picture in figure 3) in that order to see that the lines drawn are parallel, which proves (3). The second part of (5): At $r = r_0$, the inner convex edge of $E(z_2)$ connects the inner edge of $E(z_3)$ to the left end point of $E(z_1)$. Thus when $E(z_2)$ gets larger is spans the other cells as required.

Finally, just before the bifurcation, there is the cell $E(z_2+z_3)$, having its boundary convex segment dual (in $E(z_2)$) to the gap between the leftmost points of $E(z_1)$ and $E(z_2)$. This boundary segment is in a "triangle" with one convex segment, and the segment is wiped out at $r = r_0$. Thus, so is the triangle and therefore the whole cell, which proves (6) and finishes the proof in the case of the breakout bifurcation .

We have covered enough of the proof of the capping bifurcation, above.

So far, we have shown that if one cell collides with or obstructs another at $r = r_0$, then 1 of the 3 bifurcations does occur.

For the rest of the paper we will be working with the case $n = 3$.

There is one other type of bifurcation that could *a priori* happen here: a cell E or one of its sides, becomes very small, then at the bifurcation disappears all together. This happens at each the of three bifurcations, as we have seen above. We show below in the section on "Small cells," that this happens *only* at these bifurcations.

10. The simplest cases

Here we consider the case that each cell of Q_r has at most one pair of convex edges—equivalently, the intersection of each cell with ∂Q_r, is connected. We show this happens in just two ways; these turn out to be quite general models, as every Q_r is a variation of one of these two types, fugue 5.

Proposition 2. *If all cells of Q_r have connected intersection with the boundary of Q_r, then there are 7 and only 7 cells; furthermore, this occurs in two and only two topological types: one with 3 span cells and one with 2 span cells and a hex cell.*

Proof. First note that all cells except any possible totally concave cell(s) have just 2 convex edges, one on ∂Q_r and one interior; in particular, all shells are convex.

Case 1. Each adjacent pair of convex cells has a span cell.

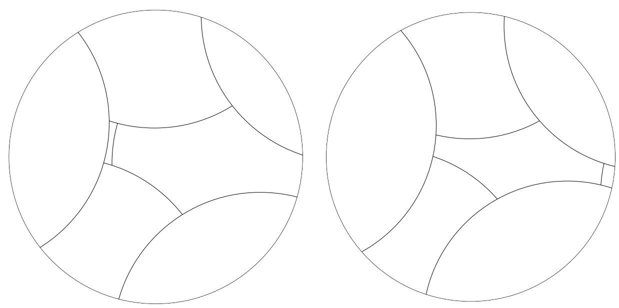

FIGURE 5. **The simplest cases**

Then there can be only 3 convexes and labeling them a, b, c, the span cells are the sums of these in pairs. These 6 use up the exterior boundary of Q_r : hence there is one other cell, which is six-sided and totally concave. Then there are 3 ways to compute remaining cell, but they all come out $a + b + c$. Thus, there is only one concave cell and the complex is determined.

Lemma 2. *If a, b, c are three adjacent convex cells then there is a span cell between either a and b or between b and c.*

Proof. Label three of the convex cells in the order a, b, c around the boundary of Q_r. Suppose that there is no span cell between a and b; let α be arc on the boundary of Q_r joining extreme points of a and b and let β that between b and c. Let A be the the cell having the portion of α nearest b as its exterior side. Then A must have a concave side, say γ, with one end on the inner boundary of b. Any other edge having an end point on ∂b would also have to curl towards β, because adjacent edges are on the same cell and were there a pair of adjacent edges curling toward each other, then they would bound a cell having 2 interior convex edges. But this would mean that there are two exterior convex edges, contrary to our assumption that the exterior boundary of each cell is connected. Let β' be the closest of these edges to β, measured along the boundary of b. It follows that β' is dual to β relative to B—the cell with β as exterior edge —because B has only two convex edges. Thus B is a span cell which proves the lemma.

Let d be the next convex cell, after a in order along the boundary of Q_r. We claim that $d = c$. Assume otherwise.for purpose of contradiction. Now apply the lemma to the cells d, a, b : as no cell spans a and b, there must be one spanning d and a. But this makes 2 disjoint π-spans, a contradiction. This proves our claim that $d = c$. Now apply the lemma to c, a, b : thus there is a cell spanning c and a. At this point we have 3 convex cells a, b, c and cells spanning b, c and c, a.

By "chasing duals" it is easy to see that the dual arcs to our 5 known interior arcs span everything on the circle except α. Thus the edge α' dual to α in A, is exactly the last missing piece. At this point we know that A has the convex sides α, α' and (at least) one pair of consecutive concave sides: the inner boundaries of b and the cell spanning b and c. This leaves 3 sides available to be sides or parts of sides of A : the inner boundaries of c and a and the span cell between them. As the number of sides must be even, is must be 6; occurring in the order convex, concave, concave, convex, concave, concave. Thus α' must connect the two span cells; then the last cell is totally concave, having boundaries α' and parts of the inner boundaries of c and the two span cells. See figure 5.

11. Small cells and the central cell

Recall that we are now working only in the case $n = 3$.

Lemma 3. *If upon changing r a cell E of Q_r or one of its sides decreases to (diameter) zero at $r = r_0$, then this is accounted for by one of the three bifurcations.*

First in case E is convex, then it can only disappear at a capping bifurcation and as r decreases.

Thus assume that a cell E has at least one pair of concave sides and at least one side that gets small and disappears at $r = r_0$.

First we claim this cannot happen as r decreases.

Proof. The totally concave case will be dealt with in the section. Thus we assume that E has convex sides also. Thus $S = E \cap \partial Q_r \neq \emptyset$, and if not connected, E cannot have small diameter until S becomes connected. At this point, there is a (reverse) capping, which leaves S connected. Then E has a convex side in the interior of Q_r and thus not small diameter while r is decreasing. Then more reverse cappings or a reverse upstaging can reduce the concave sides, still without danger of E having a side go to zero length. Thus E becomes convex before disappearing and we are in the previous case.

Thus except for the totally concave case, the remaining case is that an edge of E goes to zero as r is increasing. Now as concave sides increase in length as r increases, it follows that E has a pair of concave sides that have become tangent and then crossed; in other words, a cell a has capped another, say b yielding c, where $a + c = b$. Then $E = a + b = 2a + c$ is disconnected and as r increases, the two components become further apart.

Assume as case 1 that E has two pairs of convex sides. It follows that these meet at a vertex, A. If (for our purposes the interesting case) the line L is not actually *on* the subspace generated by a and c, then b may break out of E; this turns E into a span cell, by removing a pair of its convex sides. In this case no cell disappears.

If L *is* on the subspace generated by a and c, then by symmetry of the situation, there will be no breakout and no side of this cell going to zero diameter.

As case 2) assume that $E = E(x)$ has only one pair of convex sides; then it has a side on ∂Q_r by symmetry.

Now we know that E consists of two 3-sided pieces and each piece has one convex and two concave sides. It follows that this cell disappears with a breakout bifurcation. Here the cell a plays the role of z_3 in the breakout bifurcation (figure 3) and b the role of z_2. Then E corresponds to the cell $z_2 + z_3$, which disappears in a breakout bifurcation, as claimed.

This completes the proof of the proposition except for the case in which E is totally concave; this case will be covered in the next section where we prove at the same time that there is always one and only one totally concave cell. In addition a structure theorem will proved be at the same time. We complete this section with some terminology and a fact about central cells.

Finally,

Lemma 4. *The central cell can have at most 6 sides.*

Proof. We will see below that at birth with r increasing the central cell has either 4 or 6 sides. Thus it suffices to show that they cannot gain sides as r increases.

Suppose two of the sides have their centers at A and A'. To prove the lemma, take a coordinate system (see figure 6) with the two centers at $(\pm a, 0)$. We are concerned with the vertex, $(0, v)$, which is the

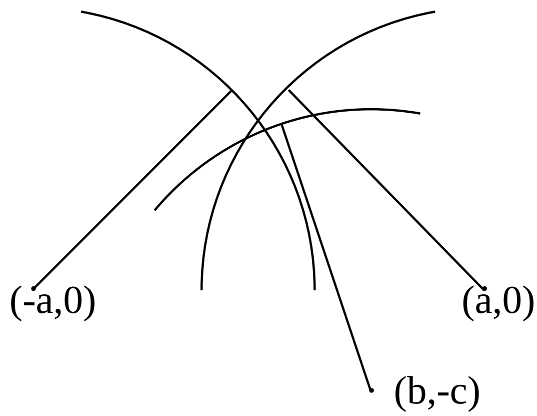

FIGURE 6. Side loss for the central cell

intersection of the two sides; and whether another arc can approach it as r increases.

Were there such an arc, say α, to approach the vertex, it would have to do this from below the vertex. The coordinates of its center $(b, -c)$, must satisfy $-a < b < a$ and $c > 0$, for this crossing to happen. Let $(0, d)$ be where α crosses the y-axis. This crossing is $D = v - d$ units below the vertex, where

$$D = \sqrt{r^2 - a^2} - \sqrt{r^2 - b^2} + c.$$

One computes that as r increases, D decreases monotonely. Hence if α *does* hit the vertex, it approaches it from *above*. If this happens (and it does if c sufficiently small) then the cell looses a side—rather than gaining one. This completes the proof of the lemma.

12. Simple cases as models

In this section we prove that each Q_r has one and only one cell with all sides concave. Existence is easy: consider E' the last open cell added that has a boundary point, say $p \in \partial Q_r$; the dual p' to p relative to E' is an interior point of Q_r and is certainly not in E'. Nor could it be in a previous chosen cell, as then E' would not have been chosen. Then p' is in some cell, say E. Now no face of E can be convex as then its dual relative to E would be in ∂Q_r. Hence E is totally concave.

We know of no simple proof of uniqueness. This proceeds by induction, taking a generic path through the space of complexes, using the bifurcations deduced from symmetry. At the same time we prove a structure theorem for the Q_r and K_r.

Definition 1. *We say Q_r is of 3-span type provided it has three specified shells and a span cell for each pair of these shells. Furthermore, a unique cell C with all sides concave; C has 6 sides, and the dual cells with respect to C are a shell x_i and span cell $s_i, i = 1, 2, 3$.*

On the other hand, Q_r is of 2-span hex type (hex for short) provided it has three specified shells, a, b, c, a span cell of two pairs of these three shells, a unique (central) cell C of all concave sides, and a hex cell H. Furthermore, C has 4 sides and with respect to C, the dual pairs are a, H and b', c' where b' and c' are the span cells. Finally, with respect to H the two dual pairs are b, b' and c, c'; H has 2 convex sides.

The definitions of 3-span and 2-span-hex include the two simple cases (in which there are no other cells) but much more, as we allow other shells and capping. Of course capping introduces new cells and new sides of old cells.

Theorem 3. *Each Q_r is either 3-span or 2-span-hex.*

Proof. This is proved by induction along a generic path by showing that each bifurcation retains this property by changing it in the correct way. Thus the result follows from the following:

Lemma 5. (1) *The cell types: span, hex, and central are not changed, nor are any new cells of these types created by the capping bifurcation.*

(2) *Just before a breakout bifurcation, Q_r must be 3-span. At this bifurcation, a six sided central cell looses a pair of sides and after this bifurcation, Q_r is of hex type.*

(3) *Just before an upstaging, Q_r must be of hex type. At each upstaging the old central cell is destroyed and another created, having either 4 or 6 sides.*

(4) *If 6, the resulting Q_r will be 3-span: 2-span-hex if 4.*

Proof. The first part is covered in the section describing this bifurcations.

For the second part, we use the notation set above in describing the breakout bifurcation: assume z_1 breaks out of z_3, at $r = r_0$, and note that $z_2 + z_3$ is the span cell lost at the bifurcation. As the inner boundary of this span cell lies in the boundary of the central cell, the central cell looses a pair of sides.

We need to show that there are three span cells, (just before the breakout.) Assume for purpose of contradiction, that this is false. Let f denote the inner convex edge of $z_2 + z_3$. Then by the induction hypothesis, there must be hex cell, say t with convex edge e, spanning f and another span cell, say s. Then f cuts off the central cell, C (and a hex cell.) We see than s spans either z_3 and another shell or z_2 and another shell, because there cannot be two disjoint π-spans. Then the dual e' of e, with respect to C, lies on either ∂z_3 or ∂z_2. In either case, e and e' both have end points on f and thus coincide at, say $r = r_1 \leq r_0$. But then e and e' become tangent at $r = r_2$ and r_2 must be $< r_1$, because the parallel points of dual sides are closer together than their end points. This makes $r = r_2$ a point of bifurcation, to wit a capping. Thus a capping bifurcation occurs before the breakout, contrary to our assumption.

Hence just before the breakout Q_r is of 3-span type. Say z_1 breaks out of z_2 where z_3 is dual to z_1 relative to z_2. Then $z_1 + z_3$ spans z_2 and z_3. Since there can be no π-span disjoint from this one, it must be part of the 3-span setup: thus there is another shell w, a cell A spanning z_2 and w, and a cell B spanning z_3 and w. By induction, the inner boundaries of the span cells are in the boundary of the central cell. Thus it has these 3 sides, and their duals with respect to the central cell, and thus 6 sides. When z_1 breaks out, the span cell $z_2 + z_3$ is lost, decreasing the central cell to 4 sides. In addition, as z_1 extends into A, z_2 becomes a span cell of z_1 and z_3, and its inner boundary adds another concave side to A; this and its dual make 6 and A becomes a hex cell. This proves part 2.

For the third part, assume for purpose of contradiction that just before an upstaging, Q_r is of 3-span type. Say $x_i$are the shells, s_i the spans, where s_i is dual to x_i relative to the central cell, $i = 1, 2, 3$. Then no two of the x_i can be involved in the upstaging, as they are further apart than π.. Furthermore, the full upstaging cannot be “cut off” by a single cell, as the outer boundary of single cells span less that π. Therefore there must be a breakout before the upstaging. This is a contradiction, and thus before an upstaging, Q_r must be of hex type.

We must have a capping, as no 3 of these shells can make an upstaging. We use the notation above. As the upstaging is to occur next, without breakout, the three shells in the upstaging must include b and

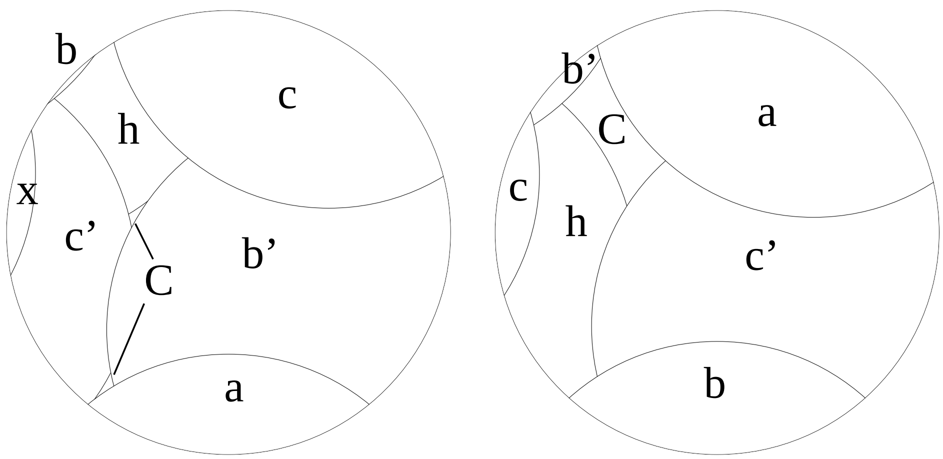

FIGURE 7. **Before and after upstaging, case b**

c. Hence the capping must supply a shell either between them or adjacent to one of them. Using the notation above, we see that there are three and only three possibilities:

a) a caps the hex cell $h = a + b + c$ creating $x = b + c$ and cutting the central cell into two 'triangles'. Then b and c upstage x. Note that the capped hex cell has two short exterior edges—between b and x and x and c— that disappear at the bifurcation. Their duals are edges of C, and hence the cell C itself disappears. After the upstaging, x spans b and c, and thus Q_r is 3-span. The old hex cell becomes the new central cell and has 6 sides: the interior convex sides of the 3 span cells, together with their duals. This is quite simple compared to the next case. Refer to figure 7.

b) b' caps c' (as these are dual with respect to C) creating $x = a + c - (a + b) = c - b$. Then x and b upstage c. Here the capped cell c', has two exterior edges, only one of which—that between c' and x—disappears with the bifurcation. This removes two of the convex edges c' got from capping, leaving two; it also got a pair of concave edges, and thus the resulting cell, h, has six sides which is right for a hex cell.

On the other hand, the old hex cell looses its convex edge, being between b and c, and now has 4 concave edges and is the new central cell. The duals of the two exterior edges that disappear, also disappear: as these duals are edges of the old central cell, it disappears. The new cell x becomes one of the shells, c in our standard notation. Other changes: $a \to b; b \to b'; c \to a; b' \to c'; c' \to h$. It is straightforward to check the duality requirements, for example, b' and b dual with respect to h translates to c' and b' are dual with respect to C.

c) c caps c' (or the symmetric case, b caps b'.) This creates the cell x where $c + x = c' = a + b$. There is no possible upstaging here: the only triple not obviously too far apart are c, x, b. But these don't add up. This completes the proof of the lemma and the proposition. The following diagram summarizes the results of this section.

13. Rational Subspaces

An important special case (of any theory of continued fractions) occurs when the vector P is close to a rational subspace of $\mathbb{R}^3$

Suppose R is a 2 dimensional rational subspace of $\mathbb{R}^3$ with normal N, where the coordinates are integral and relative prime. We have

Remark 8. *For the rational subspace R, the critical distance is $1/|N|$: the distance from R to the closest lattice points not on R is $1/|N|$.*

Proof. For $z \in \mathbb{Z}^3$ we have $k = |z \cdot N| = |N||z|cos\theta$, where k is an integer. If z is not on R, the minimum value for k is 1. The minimum is attained at any lattice point (a, b, c) chosen using the relative prime assumption, to give $a \cdot N_1 + b \cdot N_2 + c \cdot N_3 = 1$. As the distance from z to the subspace R is $|z \cdot cos\theta|$ the result follows.

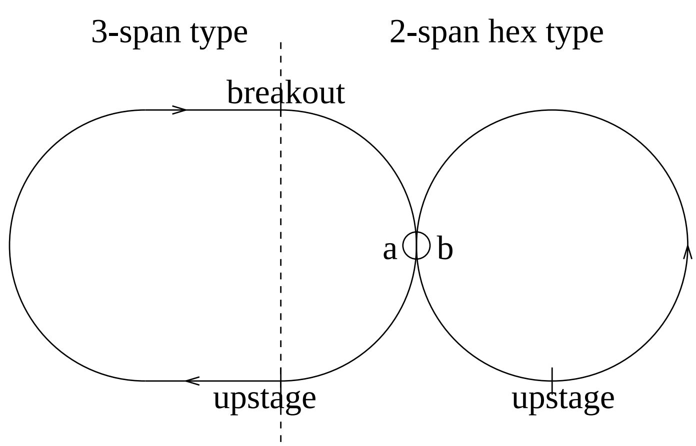

FIGURE 8. **How Q_r varies between 3-span and 2-span hex as r increases. a if a caps hex: b if b' caps c'.**

Corollary 6. *If Q_r is a stack then it is so relative to a rational subspace R, with, say, normal N in lowest terms. If the radius is $\leq$ one half the critical distance $d_{crit} = 1/|N|$, there is a positive basis.*

13.1. **For P on a rational plane.** First, if P is actually *on* R^2, and the radius is less than r_{crit} we get a 2D interpretation of a 1D continued fraction. That is, varying P on this subspace, is like varying a vector in the plane, so that the complexes Q_r that we get are parameterized by a full set of ordinary continued fractions. Furthermore, since we are in 3 dimensions, we can 'see' *all* of a terminal section of the continued fraction in each picture of figure 12. Thus the convex cells a and b will be on the right and left sides, say, and 'behind' these are the remaining, partially hidden and trailing off to infinity (see the appendix). In particular, for our theory we have to limit the radius to be $> r_{crit}$, a restriction like the one for the case when the vector P is rational itself. Here, this concerns us mainly in that, when *near* a rational plane, can have an arbitrarily large number of cells, by continuity.

We begin with a lemma needed in the stack proposition.

Lemma 6. *Every cell in Q_r is one of the following:*

1. *the central cell;*
2. *a capped cell, that is a cell with exterior boundary disconnected;*
3. *a span cell;*
4. *a hex cell; or*
5. *a convex cell.*

Remark 9. *These are not exclusive: for example, both hex and span cells can be capped.*

Proof.Assume X is a cell and not one of the first 4. Then $\partial X \cap \partial Q_r$ is (1) nonempty, (2) connected, and has at least one pair of convex sides. Since X is not a span cell, it must have the extreme points of its background $= \pi(E(z)) \cap Q_r$. Thus it is a shell and since it is not capped, it is convex.

13.2. **Stack Proposition.** The purpose of the section is to get a handle on stacks.

Proposition 3. *If there is a capped cell in Q_r, then then there are cells a, b, X, such that*

1. *a and b contain diametrically opposite points on ∂Q_r;*
2. *a and b are dual with respect to X;*
3. *$a \cup X \cup b$ separates Q_r into two sets, one above and one below the diameter in part 1;*
4. *b is convex.*

Proof. As there is a capped cell, let X be one with the least weight. Thus X has at least one, say k pairs of dual concave sides. Each such pair leads to a pair a, b of cells satisfying the first 2 conditions. We claim that $k = 1, 2$, or 3. To see this, assume for purpose of contradiction that $k \geq 4$. Now label the first k a's $a_0, ...a_{k-1}$ and their duals $a_i, a_{i+k}, mod 2k$. Then as r increases a little, which would result in an upstaging, say a_i and a_{i+2} upstage a_{i+1}, or a capping. But then these span π radians, even though they lie entirely on one side of the diameter between points of a_{i+3} and a_{i+3+k}. $(i + 3 + k \neq i.)$ A capping would lead to a similar contradiction. This proves our claim. Note that the outer boundary of X consists of $k + 1$ intervals.

Let a, b be one such pair; were part 3 to fail, by lemma 6, one of a, b, X would be disconnected. Suppose first that X is disconnected. Then X has a pair of dual concave edges which overlap; let Y, Z denote the corresponding cells, dual with respect to X. Then one of Y, Z caps the other. This contradicts our choice of X, as both Y and Z have less weight than X. Next, assume that either a or b is disconnected. Then reasoning the same way, we arrive at a (or b) capping cell that has less weight than a (or b.) But then this cell has less weight than X, again yielding a contradiction. This proves part 3 holds in each of the k cases.

Note b 1) is not capped, by our choice of X; 2) is not central as it has a convex edge; 3)could not be a hex cell as the outer boundary of a hex cells is bounded by shells, leaving no room for X's disconnected sides; 3) could not be upstaged, as this would leave no room for X to have its disconnected boundary. Thus b is convex.

Similarly, a is neither capped nor central; to see that it not hex, recall that if x caps y then they are dual, relative to a cell between them where the dual edges overlap and x has less weight than y. But a hex cell has move weight than than any other cell, except the central cell and thus cannot cap another cell. So far we know that a is either a convex cell or a span cell, by the lemma above.

To go further we must proceed by cases: for $k = 2$, and a_1, b_1, a_2, b_2 the two pairs, a_1 cannot be a span cell because the resulting π-span would have to be on one side of the diameter determined by a_2, b_2. Similarly for a_2. Hence the a_i are convex.

For $k = 3$ (and similar notation) a_2 might span a_1 and a_3, but then the other two could not be span cells and thus are convex. For $k = 1, a$ might well span two (necessarily convex) cells, and often does.

Corollary 7. *Under the conditions of the previous proposition, a may fail to be convex only in the cases $k = 1$ or 3. Furthermore, in the exceptional case where $k = 1$ and a is not convex, a is a span cell and there is a positive basis.*

Proof. The first statement has been proved. Next, if a is not convex, it spans two cells c, d, resulting in a π-span. Just as for a, c and d satisfy 1-4 above and thus are convex, as neither can be span cells. We claim b, c, d forms a positive basis. Now assume for purpose of contradiction that there is a convex cell e not in their positive span—say e is between b and c. Let r increase until e touches. First, suppose e touches c; this cannot be a capping as it would be above the diameter determined by the dual cells a and b. Second, could c and another cell x upstage e? Only if $x = b$, again because of the dual cells a and b. But this contradicts the assumption that e is not in the positive span of b, c, e.

Finally, could e touch b but not c? Yes, with e and d upstaging b; but b would have to break out first, and this is impossible as a is dual to b relative to X, and a is spanned and thus not a shell.

Corollary 8. *In these three cases where :*

- $k = 1$**:** *and a is convex, we have a stack with convexes a, b and cells $a + b$ and possibly others of the form $Aa + Bb, A, B \in \mathbb{Z}^+$. There may or may not be a positive basis.*
- $k = 2$**:** *we have $a_1 + b_1 = X = a_2 + b_2$ and we have a basis, but not a positive one.*
- $k = 3$**:** *Q_r has 5 or 6 convexes and Q_r takes the form given in figure 10; there is a positive basis.*

Proof.In the first case, there is nothing to prove. However, note that this case includes the 'finite stack' as described below. In particular, there may be many more cells of the form specified. The second case is clear.

The third case is the special case of 5 or 6 (the maximum number possible) convexes mentioned above. Choosing notation a_0, a_1, a_2 and their duals, a_3, a_4, a_5, these occur in order around ∂Q_r, and a_i and a_{i+3}, mod 6, are dual with respect to X. At an upstaging among these, say $a_0 + a_2 = a_1$, we can deduce, $a_2 + a_4 = a_3$ and $a_4 + a_0 = a_5$. Thus a_0, a_2, a_4 forms a positive basis.

14. Stacks

Lemma 7. *If a stack disconnects Q_r, then each cell is either on one side of the stack (say above or below it) or in the stack.*

Proof. Assume for purpose of contradiction, that there is a cell neither in the stack, above nor below the stack. Such a cell is the union of two connected pieces one above and one below the stack. Among all such, let $E(x)$ be one whose upper piece is not lower than any one adjacent.

Then we know that the two pieces of $E(x)$ are cut off by a side of $E(x)$, say the right side, passing over its dual, left side. Note that the edge just to the right of the upper and lower pieces, bound the same cell, say $E(b)$. This is true because these two arcs lie on the same circle, and no two cells of Q_r have arcs of the same circle as convex edges. Similarly, the cell $E(a)$ contains the pieces on the left sides of the top and bottom parts of $E(x)$.

Hence, by our assumption, each of these two cells must be in the stack by our choice of $E(x)$. But then, $a + b = x$, which means that $E(x)$ is also in the stack. This contradiction proves the lemma.

14.1. **Stack background.** If there is a stack, say determined by the rational subspace R, then there is background complex Q_r^R defined above. Choose a coordinate system with the line of centers of $\pi(a)$ and $\pi(b)$ horizontal. Then $E(a)$ intersects the top of ∂Q_r and is concave to the left—let $E(x)$ be the rightmost cell of Q_r^R having these properties. Similarly, let $E(y)$ be the leftmost cell of this complex, concave to the right and intersecting the top. Let A be the top end point of $E(x)$ and D be the top end point of (y). Note: at a bifurcation (breakout) the cell $E(x + y)$ will be empty, and then the point $B = C$.

Lemma 8. *The boundary β of the complex Q_r^R consists of portions of the boundaries of 3 cells $E(x), E(y), E(x+y) \in Q_r^R$. It proceeds from A along $\partial E(x)$ to the vertex B of the cell $E(x+y)$, then along $\partial E(x+y)$, to the vertex C of the cell $E(y)$, then along $\partial E(y)$ back to D.*

See the example, figure 9.

Proof. We have indicated why the cells $E(x)$ and $E(y)$ exist. To see that $E(x + y)$ exists, (except at a breakout) let K be the line segment joining the centers of $\pi(x)$ and $\pi(y)$ and C_K its midpoint, and $K^\perp$ its perpendicular bisector. Then C_K is the center of symmetry of the putative cell $E(x + y)$, and thus $x + y$ contains a neighborhood of the lower intersection $\partial Q_r \cap K^\perp$.

Let A' be the dual of A, with respect to $E(x), B', C'$ the duals with respect to $E(x + y)$ and D' the dual of D with respect to $E(y)$. Note that $A' = C'$: certainly C' is on the boundary of $E(x)$ which proves this. Similarly, $B' = D'$. Elementary properties of convex sets show that the convex sets $\pi(E(v))$, for $v = x, y, x + y$ together, contain all of Q_r below β thus finishing the proof.

Proposition 4. *The background of a stack intersects a non-stack cell if and only if there is a cell on the bottom of the stack.*

Proof. First, note that the symmetry of each stack cell interchanges top and bottom. Thus if a non stack cell intersects the background on

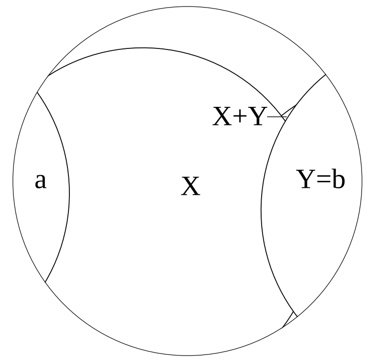

FIGURE 9. **The background of a stack**

either one of top or bottom, there is a dual non-stack cell intersecting the other.

15. Bounding the number of convex cells

Proposition 5. *There are never more than 6 convex cells.*

Assume on the contrary that there is a radius for which there are 7 (or more) convex cells. Now we allow r to increase and concentrate on the bifurcations. Consider the following:

(1) There is a capping producing a new cell v which then breaks out.
(2) Cells a_1, c_1 upstage another, b_1.
(3) A capping producing v which does not break out.

But since this last does not decrease the number of convex cells, we can ignore it. Call such a bifurcation (1 or 2) a *touch*. Note that each touch produces a π-span with the two ends intact; they are much the same, except that the first uses at most two of our 6 convex cells, the other (at most) three.

Case 1: there is no touch in which one end is a shell containing more than one of our 7 convex cells.

The first makes a π-span and uses at most 3 convex cells, leaving at least 4 untouched, and in the complement. Thus there must be another touch and the π-span it makes cannot be disjoint from the first, and thus overlaps it;

Case 1u: x, one of the remaining 4 convex cells, together with an end cell of the first π-span, upstages y, another one of the four remaining

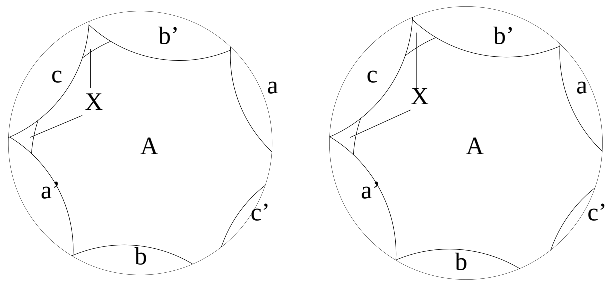

FIGURE 10. A case of 6 convex cells

convexes. The overlap with the first π-span is just one cell, and uses at most two of our convexes; there are at least 2 convex cells remaining. Thus there must be a third touch; first, suppose it is

Case 1uu: another upstaging. The π-span made by another touch, which could overlap the previous two by at most one cell, and therefore is disjoint from one of the first two. As this is impossible, this contradicts our assumption and finishes the proof in this case.

Note that were there only 6 convex cells, then the last of our 7 cells could be upstaged by the two cells on the ends of the previous (overlapping) π-spans.

This is what happens in a certain case: metric induced by projecting onto the plane with normal = (0.57605, 0.522974, 0.719093), $P = (0.625822, 0.547594, 0.555417)$ and radius 0.46-0.50.} as in figure 10. The reader may wish to check that from the first figure one can compute $a' = b + c, b' = a + c$ and $c' = a + b$.

Case 1c: the second touch is a capping. Here we need some notation: say the first π-span is made by cells a, c, with a spanning cell b. Let A

denote the end point of the π-span at a and let $-A$ denote the other end, antipodal to a. Now the new π-span cannot be disjoint from the first; say x one of the four remaining convexes, caps a. This creates another convex cell , d, at first cut off by a. Let B denote the point at which d breaks out of a, and $-B$ the point antipodal to B; this is near an end point of x. not that the two π-span 's take up all of ∂Q_r, except the shorter arc between $-A$ and $-B$. At this point there are at least 3 of our 7 convexes left and there must be another touch. It must overlap with one of the first two π-span 's, but it cannot overlap both, as there are other convexes in the way. Thus we have two disjoint π-span 's which is impossible. This completes the proof of case 1.

As Case 2, we assume that there is a stack, that is case $k = 1$ of the stack proposition, with a and b convex.

As Case 2A, we assume that there is a shell x on one side of the stack which cuts off i convex cell in Q_r, where $i \geq 2$. Say these are $a, b, c, ...,$ and are on the top of the stack. Since the cell x has i concave edges on top, it must have i on the bottom; as these edges must be convex downward, they must belong to stack cells. But there are only three such stack cells, and thus $i \leq 3$. Say that there are j other convex cells on top of the stack. Each of these has a consequent convex cell on the bottom of the stack making a total of $2 + 2j + i$ so that $j \geq 1$. Let r increase and note that none of the i convexes can touch another, while cut off by the cell x.

Case 2Aa. A convex cell a breaks out of x say on the left. Then the dual to a with respect to x is a stack cell, say it is X and is on the right and each of the j convexes on top are to the left of a. Also, $x = X + a$. As in case 1 above, the next touching among our convexes cannot be disjoint from the π-span a, x, X.

Case 2Aaa: a touches.

Case 2Aaaj: a touches one of the j convex cells, say d, together with a upstages another. This is impossible as it makes a π-span lying entirely above the stack, which has less than π radians.

Case 2Aaas: one of the j convexes is upstaged by a and a stack cell Y, which is thus on the left. Since there can be no cell between a and d, we have $j = 1$ and $i = 3$. There is still a convex cell on the bottom, and it must expand and touch something. It must be either X or Y; it cannot be upstaged by X and Y as the resulting π-span would contain all of Q_r except the cells above the stack—which has less than π radians. Hence it is disjoint from one of the first 2 π-spans, a contradiction.

Case 2Aaac: a caps a cell. This must be one of the $j \geq 0$ cells on top, as these cells are to the left of a and between a and the stack cells on the left. This is impossible as the top is of less than π radians.

Case 2Aaac': a is capped by one of the j cells to the left of a. This is impossible as both of these cells are on top, which is of less than π radians. Thus a does not touch.

Case 2AaX: X touches a cell on bottom. It cannot be another stack cell Y as Y would be a convex leftward cell, and together they would cut off a diameter of Q_r. Thus X and a convex cell u on bottom must thus must upstage another, v on the bottom. At this stage we still have two convex cells on top corresponding to the bottom cells u and v, and as we may suppose a does not touch, we are again trapped with a contradiction.

At this point we know that under the special assumption about the cell x, that there cannot be a breakout. But then one of the the i cells cut off by x must touch. But this is impossible, as it would make a π-span cut off by x though cells never span π radians. Thus the special assumption about x itself is impossible.

Hence, in the case of a stack, the case 1 argument, made under the special assumption that there was no cell such as x, proves the proposition.

Corollary 9. *There cannot be 4 convex cell on a side, in case there is a stack.*

Proof. For at least one of four would have to a stack cell and thus determine at least one on the other side. These 5 along with the generators a, b of the stack make 7. □.

Case 3: $k = 2$ in the stack proposition. But this is a special case of a stack—in two different ways, and thus already covered. Alternately, a very easy proof is possible in this case, as the two pairs $a_i, b_i, i = 1, 2.$ mean that each touch would have to involve both a_i or both b_i. Similarly, the $k = 2$ case is already covered.

16. Four convex cells suffice: there is always a basis

Proposition 6. *Every cell label of K_r is a positive linear combination of the convex cell labels. More simply, every cell in K_r is a sum of its convex cells.*

Proof. If a cell z has a concave edge bounding a cell a, then the dual to this edge is also concave, and bounds a cell, b. Then $z = a + b$. Note that both a and b have less weight than z; if either a or b has a concave side, it is the sum of cells that have less weight. Thus continuing this

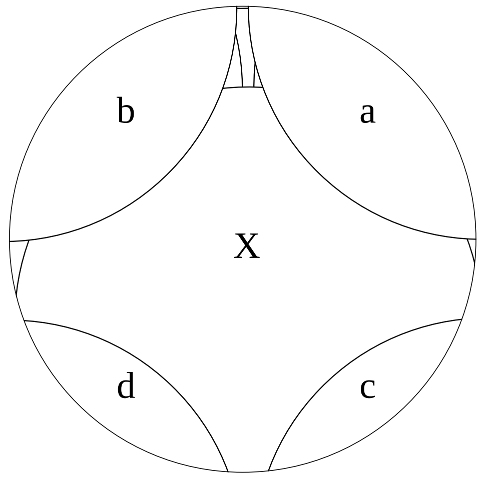

FIGURE 11. **Four cells are required**

this process must end with cells that have no concave sides, that is are convex. This proves the proposition by induction. [Note also that this proof is valid in all dimensions.]

In fact we can do better; the following proposition is crucial in our proof.

Proposition 7. *Four convex cells (or perhaps just three) suffice for positive labels of the other cells.*

The example in figure 11 shows that we sometimes *need* four: note that $a+d = X = b+c$; in particular, any three of $\{a, b, c, d\}$ is a basis of $\mathbb{Z}^3$, though *not* a positive basis, for example, $d = a+b-c$.

First of all, if there is no capping, then there are just 3 convex cells and they suffice.

Thus suppose we have a stack, say based on the convex cells a and b.; then the stack divides Q_r into two pieces, each spanning less than π radians. Now, among all convex cells on top of the stack, let c denote the one with least weight.

Lemma 9. *if $x \neq c$ is a cell on the top of the stack, then $x = c + \Sigma X_i$ where the X_i are stack cells.*

Proof. First, if $x \neq c$ is convex, then it is too close to c to cap or be capped by it. But a touch will eventually happen and can only be that one is upstaged by the other and a stack cell. But it could not be $c = x + X, X$ a stack cell, as then c would have more weight than x. Thus. $x = c + X$. So suppose x is top cell which is not convex. Then x has at least two concave sides, can be written $x = X_1 + u_1$. where X_1 has a a convex downward side and u_1 has the dual side, with respect to x. It follows that X_1 is a stack cell, since the side dual the convex downward side with respect to X is on the bottom boundary of Q_r. This uses lemma (14.7) that any cell containing part above and part below the stack, is a stack cell. If u_1 is convex, we are done. Otherwise we apply the first step again and get $u_1 = X_2 + u_2$, and thus $x = X_1 + u_1 = X_1 + X_2 + u_2$, where X_2 is a stack cell and u_2 a top cell, which clearly has less weight than u_1. Thus by induction we have $x = \Sigma X_i + y$, where the X_i are stack cells and y is a convex top cell. Again, y is within π of c and thus $y = c$, or $y = c + X', X'$ a stack cell. in either case the lemma is proved. Similarly,

Lemma 10. *if $x \neq d$ is a cell on the bottom of the stack, then $x = d + \Sigma X_i$ where the X_i are stack cells.*

Proof of the proposition. If there are no convex cells on the bottom, then a, b, c suffice to write any cell, and hence is a (positive) basis. Otherwise, we note that any cell in the stack can be written as a positive combination in the end cells, a and b. Thus, any cell in Q_r is a positive sum of the four cells a, b, c, d.

Corollary 10. *The lattice points a, b together with either c or d forms a basis of $\mathbb{Z}^3$.*

Proof. So suppose there is a cell on bottom. Then the convex cell d exists. Some cell, say y must hit a stack cell X"; let x denote the dual cell (with respect to X"); x is first, a top cell, and secondly satisfies X" $= y + x$. Using the lemma above we have $x + y = X$" $= d + c + \Sigma X_i$, and hence each of c and d can be written as a linear combination of the other together with a and b. This leads us to the corollary.

Corollary 11. *Our algorithm which chooses three of a, b, c, d does form a basis of $\mathbb{Z}^3$.*

Proof. Recall that the first two chosen are the two convexes of least weight. (The second is automatically not in the span of the first one.) The third choice is that convex cell that 1) is not in the span of the first two, and 2) is of least weight among those satisfying part 1. Adjust the notation so that $wt(a) \leq wt(b)$. and $wt(c) \leq wt(d)$ and take as

Case A that $wt(d) > wt(a)$. Then the choice would be a, b, c, which is a basis as d can be written as a linear combination of c, a, b. Thus we may assume

Case B, $wt(c), wt(d) \leq wt(a), wt(b)$.

Therefore equation $d + c + \Sigma X_i = X''$ means that $wt(X") \leq wt(d + c) \leq wt(a + b) \leq w(x + y)$. But as $X'' = x + y$, these are all equalities. In particular, as $x = c + \Sigma X_i$ and $y = d + \Sigma X'_i$, we have $x = c$ and $y = d$.

As $X"$ is a positive linear combination of a, b, we have one of the following: $X" = ma, m \in \mathbb{Z}, X" = mb, m \in \mathbb{Z}$ or $X'' = a + b$.

Case B1: $X" = a + b$. Then $a + b = x + y = c + d$, and any three of $a, b.c.d$ form a basis.

Remark 10. *Note that case B1 is the case $k = 2$ of the stack proposition.*

Case B2: $X" = ma$. Were $m > 1$, the cell X'' would be behind a and its center further out form the center of Q_r, and thus would not be (the name of) a cell at all. Thus $m = 1$ and $X'' = a = x + y = d + c$. Then $wt(a) > wt(c), wt(d)$. Hence c and d are chosen before a and thus a is not chosen. Hence the three chosen are b, c, d which is a basis.

Case B3. $X'' = mb$. As in case 2, $m = 1$ and thus $X'' = b = c + d$. Then b would not be chosen before c and d and thus would not be chosen. Then a, c, d is the chosen basis.

17. That decreasing r yields a positive basis

Theorem 4. *Given r and K_r, there exists a radius $r' \leq r$ for which there is a positive basis of K_r.*

Proof. This is clear if Q_r is one of the two simple cases, that is if there is no capping.

Next, the case $k = 3$ of the stack proposition has already been covered—there is a positive basis without perturbing r. The case $k = 2$ is a special case of the $k = 1$ case, in that is a stack.

This brings us the case $k = 1$, and the part of this case in which a and b are convex and are the ends of a stack. As above, we let c denote the convex cell on top of the stack which has least weight and d convex, of least weight on the bottom. We proved above that any cell can be written as a *positive* linear combination of the four convexes, a, b, c, d.

Now begin to decrease r; there may be various changes that are not important, for example, one of a, b, c, d may be replaced with others but as long as there is a stack with top and bottom cells in Q_r, the property just mentioned holds. Thus there is a value of r, say $r = t$, at which something fails. This uses corollary 6 of the section on rational subspaces.

Case 1. There is still a stack, but there are no cells on bottom. Then we claim that a, b, c, forms a positive basis. Because any top element can be written as a positive linear combination of c and the stack elements, and thus as positive linear combinations of this putative basis. And of course, the elements of the stack can be written as positive linear combination of a and b alone.

Case 2. There is no longer a convex cell to play the role of a. Since we still have a capping the stack proposition still applies. If $k = 2$ then this is still a stack and this is case 1 again. Thus we must be in the special case of $k = 2$ in which a is not convex. However, in this case we proved above that there is a positive basis.

Finally, we could no longer have a capping. Thus we are in one of the 2 special cases and there is a positive basis.

18. Convex bases

Definition. We say that a basis is *convex* provided it has the eigenvector P in its (positive) cone. That is, a basis a, b, c of $\mathbb{Z}^3$ is convex provided the eigenvector V has the form $V = Aa + Bb + Cc$, where $A, B, C > 0$.

Lemma 11. *Every positive basis is convex.*

Proof. There are cells that span a triangle with the center in the interior. For x, y, z such cells, we can write the eigenvector as positive linear combination of them, and turn, we can write these as a linear combination of the positive basis. Note that this also implies that the centers of the last cells form a triangle with the origin in the interior.

19. The periodic case

As has been noticed before for other 2D continued fractions, the sequence of transition matrices is periodic if there is a 3×3 'Pisot' matrix A with small eigenvalues complex. Here, in addition, the spaces K_r themselves are periodic. There is also uniqueness here, provided that the we are using a metric or inner product on the plane $P^{\perp}$.

Theorem 5. *The sequence of spaces K_{r_i} is eventually periodic if*

(1) *the vector P is the eigen vector of the eigen value λ bigger than 1.*
(2) *the other 2 eigen values of A are complex, say $\alpha \pm i\beta$.*
(3) *the bifurcation parameter is the radius r of the round disk Q_r as given by an inner product on the plane $P^{\perp}$,*

(4) *the inner product is that induced on $P^\perp$ by the metric on the eigen plane of the complex eigenvalues by taking the vectors u, v in the formal eigen vector $u+iv$ to be of equal size and normal.*

Furthermore, if the update matrices $\{A_i\}$ determined are eventually periodic, and part 3 holds, then the spaces are eventually periodic and parts 1 and 2 are true.

Conjecture 2. *We can add part 4 to the last part of this theorem.*

Proof. We assume that A is such a matrix and that we are using the indicated metric. In particular we may [H],[**?**]and do assume that the entries of the P are all positive. We find a radius r, small enough so that a, b, c each have only non-negative entries and so that r is *not* a bifurcation parameter. Let r_0 be the bifurcation value just before this. Note that applying the matrix preserves everything in our setup: the positive ray of the the eigenvector P, the plane of u, v, the integral lattice $\mathbb{Z}^n$, and finally multiplies distances in the $u - v$ plane by the factor $\sqrt{\lambda}$. note that in addition, the complex Q_{r_0} is exactly the same as $Q_{r_{m-1}}$ except rotated by the angle $\theta = arccos\alpha$ and with all distances multiplied by the factor $\sqrt{\lambda}$.

Hence $\sqrt{\lambda}r_0$ is also a bifurcation value, and letting m be the number of bifurcations so far, $r_{m-1} = \sqrt{\lambda}r_0$. Hence the product of the transition matrices up to this point is A. The further bifurcation values are of the form $r_i(\sqrt{\lambda})^n, 0 \le i \le m-1, n = 1, 2, 3, ...$ Periodicity follows. This proves the first part.

Next, assume we have an eventually periodic sequence $\{A_i\}$ of transition matrices, say periodic after the radius r_0. Also take r_0 to be *not* a bifurcation value. Let A be the product of a full period of the transition matrices. Choose r_0 small enough so that the further matrices are periodic and so that the entries of the three basis vectors are all positive. At this point we know a bit about the spectrum of $A :$ it has one eigen value bigger than unity and the other two less than unity in absolute value. The eigen vector for the big eigenvalue could not be different from P, as the lattice points of the special bases converge to P. Let $\{a, b, c\}$ denote the first basis set.

Now assume for purposes of contradiction that the small eigen values are real, say μ, ν, and let the corresponding eigen vectors be u, v. We introduce another bifurcation parameter $t, 0 < t \le 1$ and form another sequence of spaces, K'_t. This will differ from all the other sequences of spaces, in that we will not be using a single metric throughout the sequence. For $t = 1$ we have a round disk Q'_1 of radius r_0, and for $t = 1/2$, the disk is an ellipse, given by $A(Q'_1)$. Fill in the values of $Q'_t, 1/2 < t \le 1$, say linearly. Now extend to $1/4 \le t \le 1/2$ by

$Q'_t = A(Q'_{2t})$, and so on. Similarly we have corresponding ellipses at all the lattice points z defined by $Q'_t + z$.. This yields a theory with corresponding bifurcations, because the basic symmetry result follows, then the bifurcations, etc. Next note that in this theory we also get a periodic sequence of transition matrices say B_i with the product over a full period equal to A. The period may be different.

We thus get two periodic sequences, K_i, K'_i, with perhaps quite different transition matrices as well as different periods, but agreeing on their chosen bases, at the end of each period. to wit $\{A^n(a), A^n(b), A^n(c)\}$. The first set has round disks decreasing in radius, the second set ellipses decreasing in size and with growing eccentricities.

The diameter of a round disk Q_{r_0} in the direction v is covered by a certain number , say k cells in K_0 (and K'_0 as they are exactly the same.) Then these disks form a 'chain' $D_i, i = 1, ..k$. with $D_i \cap D_{i+1} \neq \emptyset, i = 1, k-1$. However, at a later stage, either the elliptical disks are much longer in the u direction than the round disks or much shorter in the v direction. As these two possibilities are similar, we suppose that the ellipses are much shorter in the v direction. Now there is a chain of k round disks, with centers say $\pi(z_1), ...\pi(z_k)$ that make it across the round disk and thus *with the same centers* there are k disks forming a continuous *chain* made up of short elliptical disks. But the elliptical disks are too short to make up such a chain. This contradiction finishes the proof of the theorem.

We close his section with an application analogous to the use of continued fractions to solve the Pell equation.

Corollary 12. *This gives a finite process of deciding whether two matrices in $SL^3(\mathbb{Z})$ are conjugate over the integers, provided not all their eigen values are on the unit circle.*

Proof. Two A, B such matrices, are conjugate iff their inverses are. Hence we may assume A, B, have an eigen value > 1 and the other 2 are inside the unit circle. If the small eigen values are complex, the the metric mentioned above to give a periodic sequence of transition matrices for each matrix. Then the matrices are conjugate in $GL^3(\mathbb{Z})$ if and only if their sequence of transition matrices from a full period of one is just a cyclic permutation of the the other. This follows from the fact that a conjugacy R preserves the integral lattice and must send eigen vectors of one into the eigen vectors of the other. Hence the basis $\mathcal{B}'_i$ chosen by the second sequence, must be $R(\mathcal{B}_i)$, the image under R of the basis chosen by the first sequence. Thus, grouping the transition matrices, we have $A = ST$ and $B = TS$. Then S is a conjugacy from A to B.

If the small eigenvalues are real, then the device introduced above, where we change the metric continuously, yields periodic sequences of update matrices, and the proof continues as in the complex case.

We close this section with a few computations of the invariants attached to matrices (with characteristic polynomial $t^3 - 4t^2 - 1$.)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix} : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{bmatrix} ; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 4 \end{bmatrix} : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} : \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

20. Positive matrices

In [W1] we gave an argument for the 2×2 case, though number theorists knew this as we mentioned above. In [W1, W2] we indicated how the argument used would extend to higher dimensions provided one could find a radius such that there were only three convex cells. Though we still do not know this, our slightly weaker result: there is a radius for which there are three convex cells which suffice to write all the other cells as positive sums. So suppose our complex K_r has such a basis, a, b, c. then the $n - 1 = 2$ cochains dual to these, say A, B, C, are a basis of the 2-dimensional cohomology, and using this basis, an orientation preserving map of K_r to itself determines a matrix C which we claim to be positive. For though there may be other convex cells, d, e, their corresponding cochains can be written as positive sums of our basis, A, B, C. With this minor modification the same proof goes through.

In an earlier paper [W1] and in several talks, the author naively conjectured that this would be true in all dimensions. However, there are Pisot numbers with negative trace, for example in dimension 38, as had been pointed out me by Mike Mossinghoff [M]. However we have looked at many cases of Pisot matrices in dimensions 4, 5, and 6, and have had no trouble so far in finding positive bases, in terms of which the matrix has no negative terms. For example

$$\begin{array}{cccc} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}.$$

In the table below, we give several examples of good bases, and just after each,the resulting positive matrix

$$\begin{array}{cccc||cccc} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 13 & 4 & 9 & 2 & 1 \\ 1 & 1 & 5 & 20 & 2 & 3 & 4 & 1 \end{array}$$

$$\begin{array}{cccc||cccc} 1 & 4 & 17 & 73 & 1 & 0 & 1 & 0 \\ 2 & 8 & 35 & 150 & 1 & 0 & 0 & 1 \\ 3 & 13 & 56 & 240 & 3 & 0 & 1 & 1 \\ 7 & 31 & 133 & 571 & 6 & 1 & 3 & 2 \end{array}$$

$$\begin{array}{cccc||cccc} 296 & 1270 & 5449 & 23379 & 0 & 0 & 1 & 0 \\ 905 & 3883 & 16660 & 71480 & 1 & 0 & 0 & 1 \\ 1270 & 5449 & 23379 & 100308 & 2 & 0 & 1 & 1 \\ 3587 & 15390 & 66031 & 283307 & 4 & 1 & 2 & 3 \end{array}$$

Not always: There are Pisot numbers with negative trace—so in such a case this positive representation cannot occur; this yields a counterexample to the naive conjecture of the author (talks and [W2]).

References

[ATW] Adler, R., Tresser, C, and Worfolk, P. Topological conjugacy of linear automorphisms of the 2-torus, Trans. Act. Amer. Math. Soc., 349(1997) no 4, 1633-1652.

[BW] Barge, M and Williams R. F. Classification of Denjoy continua, Topology and its Applications, 106, 2000, no. 1, 77-89.

[Be] Berend, Dani Private communication

[BS] Borevitch and Schafarevich. Number Theory.

[Br] Brentjes, A. J. A 2-dimensional continued fraction algorithm for best approximations with an application in cubic number fields, J. Reine Angewante Math (326) 1981 18-55.

[C] Cassels, J.W.S. An introduction to the geometry of numbers, Springer, New York, 1971.

[H] Handelman, D. Positive matrices and dimension groups affiliated with C* algebras and topological Markov Chains, J Operator Theory 6 (1981), no. 1, 55-74.

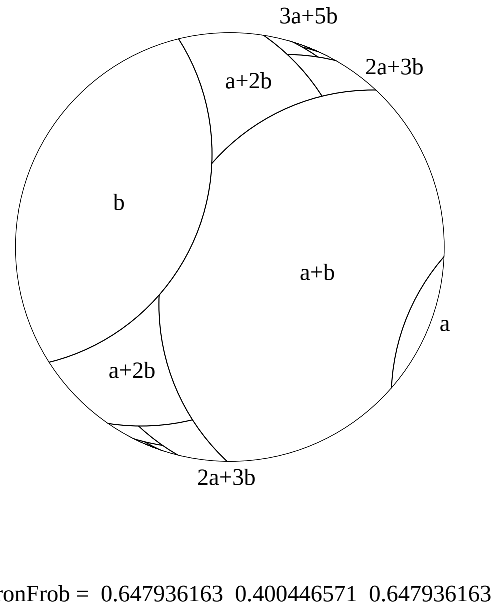

FIGURE 12. Q_r **near a rational subspace**

[L] Lagarias, J., The Quality of Diophantine approximation found by the Jacobi-Perron and related algorithms, Monatshefte für Math. 115 (1993) 299-328

[M] Michael Mossinghoff, private communication.

[St] Stark, Harold M. An introduction to number theory, 1970, Markham Publishing Company.

[W1] Williams, R. F. Pisot-Vijayarghavan numbers and positive matrices, Proceedings of the international Conference on Dynamical Systems and Chaos, Tokyo, 1994.

[W2] Spaces that won't say no, International Conference on dynamical systems, Montevideo 1995, Ledrappier, Lewowicz, and Newhouse, editors. Pitman research notes in Mathematics number 363, Longman (1996) 236-46

[W3] Williams, R.F.Expanding attractors, Institute des Hautes Études Scientifique Publ. Math. no. 43(1973)169-203.

[W4] Williams, R.F. The "DA" maps of Smale and structural stability, Global Analysis, vol. XIV of Amer. Math. Soc. Proc. of Symposia in Pure and Appl. Math., 14(1970) 239-334.

R.F. Williams
Department of Mathematics
The University of Texas at Austin
Austin, TX 78712 U.S.A.

APPENDIX A. APPENDIX: STACKS VIA 1D CF DATA

Given a line on the rational subspace Π, with normal N, and 1 dimensional continued fraction data, $\{n_i\}, i = 0, \infty$, determined by this line in Π, we describe the resulting complex Q_r for $r < r_{crit}$. First, there exist lattice points a_0, b_0, determining with the origin the subspace Π, and infinite sequences $\{a_i\}, \{b_j\}$, such that

(1) each of $E(a_i), E(b_j)$ is a cell of $Q_r, i, j = 1, ...\infty$;
(2) $b_{j+1} = a_0 + b_j, j = 0, ...n_0 - 1$;
(3) $a_{i+1} = b_{n_0} + a_i, i = 0, ...n_1 - 1$;
(4) $b_{j+1} = a_{n_1} + b_j, j = n_0, ..n_0 + n_2 - 1$;
(5) $a_{i+1} = b_{n_2} + a_i, i = n_1, ...n_1 + n_3 - 1$;
(6) $b_{j+1} = a_{n_1+n_3} + b_j, j = n_0 + n_2, ..n_0 + n_2 + n_4 - 1$;

etcetera. The a's are on the left, say, with their boundary edges covering the open left half of the circle, and the b's on the right with their boundary edges covering the open right half of the circle. See the illustration above, where $n_i = 1, i = 0, 1, ...$

Remark 11. *This does satisfy our definition of a stack—except that there infinitely many cells.*

Now if the line is moved off the subspace Π, by adding a small multiple of the given normal N, the resulting complex Q_r will be a finite approximation to this. (With r still less than half the critical distance.) There now will be at least 3 convex regions and by the remark above, there will be a positive basis consisting of a_0, b_0 and a third, say c, at the top. The top will be a four sided region (explained above), filled with cells $E(x), x$ not on the subspace Π, and the bottom will contain no such cells.