LOWER BOUNDS ON THE EIGENVALUE SUMS OF THE SCHRÖDINGER OPERATOR AND THE SPECTRAL CONSERVATION LAW

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1. Statement of the main results

In this paper we consider the Schrödinger operator

$$-\Delta - V(x), \quad V > 0,$$

acting in the space $L^2(\mathbb{R}^d)$. We study the relation between the behavior of V at the infinity and the properties of the negative spectrum of H. According to the Cwikel-Lieb-Rozenblum estimate [3], [23],[26] the number of negative eigenvalues of H satisfies the relation

$$N \le C \int V^{d/2} dx, \qquad d \ge 3,$$

where C is independent of V. Similar estimates hold in dimensions $d = 1$ and $d = 2$. Therefore, if V decays at the infinity fast enough, then N is finite. The question arise if finiteness of N implies a qualified decay of $V > 0$ at the infinity?

One can try to formulate theorems whose assumptions contain as less as possible information about V. But then it is not clear how to define H. To keep our arguments simple, we shall assume that

$$V \in L^\infty(\mathbb{R}^d).$$

The first result which is related to the two dimensional case is proven in [12].

Theorem 1.1. (Grigoryan-Netrusov-Yau [12] **)** *Let $d = 2$ and let $V > 0$ be a bounded function on $\mathbb{R}^2$. Assume that the negative spectrum of H consists of N eigenvalues. Then $V \in L^1(\mathbb{R}^2)$ and*

$$\int_{\mathbb{R}^2} V dx \le c_0 N. \tag{1.1}$$

where the constant c_0 does not depend on V.

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Instead of proving (1.1), we will establish the estimate

$$\int_{\mathbb{R}^2} V dx \le (6^4 + 12 + 4N)N. \tag{1.2}$$

While the estimate (1.2) does not give any new information compared to (1.1), we think that comprehension of the arguments related to this estimate will help to understand the idea of Theorem 1.3.

One is tempted to say that a straitforward generalization of this result to the case $d \ge 3$ should establish finiteness of the integral $\int V^{d/2} dx$ under the condition that $N < \infty$. However, this generalization has not been obtained. Instead, one can easily show that

$$N < \infty \implies \int_{|x|>1} \frac{V(x)}{|x|^{d-2+\varepsilon}} dx < \infty, \qquad \forall \varepsilon > 0.$$

Indeed, without loss of generality, we can assume that $H \ge 0$ on the complement of a large ball B. It means that

$$\int V(x)|\phi(x)|^2 dx \le \int |\nabla \phi(x)|^2 dx, \qquad \forall \phi \in C_0^\infty(\mathbb{R}^d \setminus B).$$

It remains to take $\phi(x) = |x|^{-d/2+1-\varepsilon/2}$ for $|x|$ large enough.

Theorem 1.1 deals with the number of negative eigenvalues λ_n, whereas one can study similar questions related to the sums

$$\sum_n |\lambda_n|^p, \qquad p > 0.$$

Theorem 1.2. **(Damanik-Remling** [9]**)** *Let $V > 0$ be a bounded function on $\mathbb{R}^d$, $d \ge 1$. Assume that the negative spectrum of $H = -\Delta - V$ consists of discrete eigenvalues λ_n. Then*

$$\int_{|x|>1} |V(x)|^{1/2+p} |x|^{1-d} dx \le C\Big(1 + \sum_n |\lambda_n|^p\Big), \qquad 0 < p \le 1/2, \tag{1.3}$$

where the constant C depends on d and p but does not depend on V. Moreover,

$$\sum_{n=1}^\infty \Big(\int_{n<|x|\le n+1} V(x)|x|^{1-d} dx\Big)^{2p} \le C_{\lambda_0}\Big(1 + \sum_n |\lambda_n|^p\Big), \quad p \ge 1/2, \tag{1.4}$$

where C_{λ_0} depends on d, p and the lowest eigenvalue λ_0 of H.

The proof of this theorem will be given in Section 3, which contains a description of a simple trick reducing the case $d \ge 2$ to the case $d = 1$. In order to complete the proof, it remains to to note that the case $d = 1$ was considered by Damanik and Remling [9].

We hope that our paper will provoke different mathematicians to study similar questions, so we decided to pose a problem to work on.

Conjecture. Let $d \geq 3$ and let $V > 0$. Suppose that $V \in L^{d/2}_{loc}(\mathbb{R}^d)$ and H is semi-bounded in the sense of quadratic forms. Assume that the negative spectrum of H consists of N eigenvalues. Then $V \in L^{d/2}(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} V^{d/2} dx \leq C \exp(c_0 N), \tag{1.5}$$

where the constants c_0 and C do not depend on V.

Let us discuss the case when V changes its sign:

$$V = V_+ - V_-, \qquad 2V_\pm = |V| \pm V.$$

Clearly, it is insufficient to consider only one Schrödinger operator in this case. One has to treat V and $-V$ symmetrically and study the spectra of two operators $H_+ = -\Delta + V$ and $H_- = -\Delta - V$.

Theorem 1.3. (see [30]) *Let $V \in L^\infty(\mathbb{R}^d)$ be a real function. Let the essential spectrum of both operators $-\Delta+V$ and $-\Delta-V$ be either positive or empty. Assume that the negative eigenvalues of the operators $-\Delta + V$ and $-\Delta - V$, denoted by $\lambda_n(V)$ and $\lambda_n(-V)$, satisfy the condition*

$$\sum_n \sqrt{|\lambda_n(V)|} + \sum_n \sqrt{|\lambda_n(-V)|} < \infty.$$

Then

$$V = V_0 + \mathrm{div}(A) + |A|^2$$

where V_0 and $\mathrm{div}A$ are locally bounded, A is continuous and has locally square integrable derivatives,

$$\int (|V_0| + |A|^2)|x|^{1-d} dx < \infty.$$

Thus, if $\lambda_n(\pm V)$ are in $\ell^{1/2}$, then V either oscillates or decays at the infinity. As a consequence, we obtain the following statement about the absolutely continuous spectrum of $H_\pm$.

Theorem 1.4. *Let $V \in L^\infty(\mathbb{R}^d)$ be a real function. Assume that the negative spectra of the operators $H_+ = -\Delta+V$ and $H_- = -\Delta-V$ consist only of eigenvalues, denoted by $\lambda_n(V)$ and $\lambda_n(-V)$, which satisfy the condition*

$$\sum_n \sqrt{|\lambda_n(V)|} + \sum_n \sqrt{|\lambda_n(-V)|} < \infty.$$

Then the absolutely continuous spectra of both operators are essentially supported by $[0, +\infty)$, i.e. the spectral projection $E_{H_\pm}(\delta)$ associated to any subset $\delta \subset \mathbb{R}_+$ of positive Lebesgue measure is different from zero.

While the proof of this result is already given in [30], we would like to give it again in order to fill in the gaps and make it more clear. Note that this theorem is proven in $d = 1$ by Damanik and Remling [9]. One of the missing parts of the proof in $d \geq 2$ was the so called trace formula obtained in [21].

Below, we discuss examples of oscillating potentials for which the corresponding eigenvalue sums are convergent.

Example (see [28]). Let $d \geq 3$ and $V \in L^{d+1}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be a real potential whose Fourier transform is square integrable near the origin. Then

$$\sum_n \sqrt{|\lambda_n(V)|} + \sum_n \sqrt{|\lambda_n(-V)|} < \infty.$$

In particular, the a.c. spectrum of $-\Delta + V$ is essentially supported by the interval $[0, \infty)$.

Example (see [32]). Let $d \geq 3$ and let V_ω be a real potential of the form

$$V_\omega(x) = \sum_{n \in \mathbb{Z}^d} \omega_n v_n \chi(x - n),$$

where v_n are fixed real numbers, χ is the characteristic function of the unit cube $[0, 1)^d$ and ω_n are bounded independent identically distributed random variables with the zero expectation $\mathbb{E}(\omega_n) = 0$. If $V_\omega \in L^{d+2\gamma}(\mathbb{R}^d)$ with $\gamma \geq 0$ for all $\omega = \{\omega_n\}$, then

$$\sum_n |\lambda_n(V_\omega)|^\gamma < \infty, \qquad \text{almost surely}.$$

In particular, if $V_\omega \in L^{d+1}(\mathbb{R}^d)$ for all $\omega = \{\omega_n\}$, then the a.c. spectrum of $-\Delta + V_\omega$ is essentially supported by the interval $[0, \infty)$.

Note, that one can not omit the condition on the spectrum of one of the operators $H_\pm$ in Theorem 1.4, because the property of being essentially supported by $\mathbb{R}_+$ is not shared by the spectra of all positive Schrödinger operators. One can conclude very little about the spectrum from the fact that $-\Delta + V \geq 0$, even if V is bounded. For instance, the theory of random operators has examples of Schrödinger operators with positive V and pure point spectra. Therefore it is natural to combine the information given for V and $-V$. This idea was used in [6] in dimension $d = 1$ to prove the following striking result:

Theorem 1.5. **(Damanik-Killip** [6]**)** *Let $V \in L^\infty(\mathbb{R})$. If the negative spectra of the operators $-\frac{d^2}{dx^2} + V$ and $-\frac{d^2}{dx^2} - V$ on the real line are finite, then the positive spectra of these operators are purely absolutely continuous.*

It becomes clear from the nature of the obtained results that there is a conservation law hiding behind them. Appearance of negative eigenvalues happens on the expense of the absolutely continuous spectrum which becomes "thinner". Therefore the thickness of the positive spectrum can be estimated by the number of the negative energy levels. If the number of these levels is "small", then the positive spectrum is absolutely continuous. Put it differently, one can give a quantitative formulation of Theorem 1.4. In order to do that we have to introduce the spectral measure of the operator H_+ corresponding to an element f:

$$\big((H_+ - z)^{-1} f, f\big) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{t - z}, \qquad f \in L^2, \quad \forall z \in \mathbb{C} \setminus \mathbb{R}.$$

The integration in the right hands side is carried out with respect to μ which is the spectral measure of H_+.

Theorem 1.6. *Let $V \in L^\infty(\mathbb{R}^d)$ be a real function. Assume that the negative spectra of the operators $H_+ = -\Delta + V$ and $H_- = -\Delta - V$ consist only of eigenvalues, denoted by $\lambda_n(V)$ and $\lambda_n(-V)$, which satisfy the condition*

$$\sum_n \sqrt{|\lambda_n(V)|} + \sum_n \sqrt{|\lambda_n(-V)|} < \infty.$$

Then there exists an element $f \in L^2$ with $||f|| = 1$ such that for any continuous compactly supported function $\phi \geq 0$ on the positive half-line $(0, \infty)$,

$$\int_0^\infty \log\Big(\frac{\mu'(\lambda)}{\phi(\lambda)}\Big)\phi(\lambda)d\lambda \geq -C\big(\sum_n \sqrt{|\lambda_n(V)|} + \sum_n \sqrt{|\lambda_n(-V)|} + \sqrt{||V||_\infty} + 1\big), \tag{1.6}$$

where $||V||_\infty$ denotes the L^∞-norm of the function V and $C > 0$ depends on ϕ.

Our section "References" suggests a list of papers containing the material on both topics: eigenvalue estimates and absolutely continuous spectrum of Schrödinger operators. All papers [1]-[33] are highly recommended.

2. Proof of the estimate 1.2

Our arguments can be compared with the constructions of the paper [9] where similar questions were studied for the case $d = 1$. The following statement is obvious

Lemma 2.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with a piecewise smooth boundary. Let ϕ be a real valued bounded function with bounded derivatives. Suppose that $-\Delta\psi - V\psi = \lambda\psi$ and the product $\phi\psi$ vanishes on the boundary of Ω. Then*

$$\int_\Omega \Big(|\nabla(\phi\psi)|^2 - V|\phi\psi|^2\Big)dx = \int_\Omega \Big(|\nabla\phi|^2\psi^2 + \lambda|\phi\psi|^2\Big)dx.$$

Let us introduce the unit square $Q = (-1,1)^2$ in the plane $\mathbb{R}^2$.

Lemma 2.2. *Let Ω be an open domain with a piecewise smooth boundary. Assume that the lowest eigenvalue $-\gamma^2$ of H on Ω is negative. Then there exists a square D,*

$$D = x_0 + 6\gamma^{-1}Q, \tag{2.1}$$

such that H restricted onto $\Omega \cap D$ has an eigenvalue not bigger than $-\gamma^2/2$

Proof. Let ψ be the eigenfunction corresponding to the eigenvalue $-\gamma^2$ for the problem on the domain Ω with the Dirichlet boundary conditions. Put $L = 2\gamma^{-1}$ and pick a point x_0 which gives the maximum to the functional $\int_{x_0+LQ} |\psi|^2 dx$. The latter integral is a continuous function of x_0, tending to zero as $|x_0| \to \infty$, so it does have a maximum. Suppose that the coordinates of x_0 are the numbers a and b, i.e. $x_0 = (a,b)$. Define

$$\phi(x) = \min\{\phi_0(\xi - a), \phi_0(\eta - b)\}, \qquad x = (\xi, \eta),$$

where

$$\phi_0(t) = \begin{cases} 1, & \text{if} \quad |t| < L, \\ 0, & \text{if} \quad |t| \geq 3L, \\ 3/2 - |t|/(2L), & \text{otherwise.} \end{cases} \tag{2.2}$$

It is clear that the support of ϕ is the square D given by (2.1). Now the interesting fact is that

$$\int_\Omega |\nabla\phi|^2\psi^2 dx \leq \frac{\gamma^2}{2}\int_\Omega |\phi\psi|^2 dx$$

which is guaranteed by the choice of x_0. Indeed, $|\nabla\phi|$ vanishes everywhere except for eight rectangles with the side length $2L$, where $|\nabla\phi|$ equals $1/(2L)$. Consequently,

$$\int_\Omega |\nabla\phi|^2\psi^2 dx \leq 2L^{-2}\int_{x_0+LQ} |\psi|^2 dx.$$

Therefore by Lemma 2.1

$$\int_{\Omega\cap D} \Big(|\nabla(\phi\psi)|^2 + V|\phi\psi|^2\Big)dx \leq -\frac{\gamma^2}{2}\int_{\Omega\cap D} |\phi\psi|^2 dx.$$

That proves the statement. $\square$

We will also need the following result

Proposition 2.1. *Let $H \geq -\gamma^2$ on a domain Ω. Then*

$$\int_\Omega V(x)|\phi|^2 dx \leq \Big(\gamma^2 \int_\Omega |\phi|^2 dx + \int_\Omega |\nabla \phi|^2 dx\Big) \tag{2.3}$$

for any $\phi \in C_0^\infty(\Omega)$. The inequality (2.3) *can be extended to functions from the Sobolev space $W_2^{1,0}(\Omega)$.*

The proof of the following statement is a rather obvious consequence of Lemma 2.2

Lemma 2.3. *Suppose that the operator H on the whole plane $\mathbb{R}^2$ has N negative eigenvalues. There is a collection of not more than N rectangles $\Omega_n = x_n + L_n Q$ and not more than N numbers $\epsilon_n > 0$, such that $L_n = 6\epsilon_n^{-1/2}$ and*

$$H \geq -\epsilon_n, \qquad \text{on} \qquad \mathbb{R}^2 \setminus \cup_{j<n} \Omega_j \tag{2.4}$$

The collection of sets fulfills the condition, that $H \geq 0$ on the set $\mathbb{R}^d \setminus \cup_n \Omega_n$. Moreover

$$H \geq -\epsilon_n/2, \quad \text{on } \Omega_n \setminus \cup_{j<n} \Omega_j.$$

We continue our reasoning as follows. Let ϕ_0 be the function of one real variable defined in (2.2) with $L = 1$. We introduce the functions ψ_n on $\mathbb{R}^2$ by setting

$$\psi_n(x) = \min\{\phi_0(L_n^{-1}(\xi - a_n)), \phi_0(L_n^{-1}(\eta - b_n))\}, \qquad \text{where} \;\; x = (\xi, \eta), \;\; x_n = (a_n, b_n).$$

Note, that the functions ψ_n are equal to 1 on the cubes Ω_n, but nevertheless they are compactly supported.

Now we introduce another collection of functions ζ_n equal to 1 on the union $\cup_{j \leq n} \Omega_j$. Set $\zeta_1(x) = \psi_1(x)$. Given ζ_j for $j < n$, we define the functions ζ_n and $\tilde{\psi}_n$ as

$$\zeta_n(x) = \max\{\zeta_{n-1}, \psi_n(x)\},$$

$$\tilde{\psi}_n(x) = \min\{(1 - \zeta_{n-1}(x)), \psi_n((x - 2x_n)/3)\}. \tag{2.5}$$

It is easy to see that $\zeta_n = 1$ on $\cup_{j \leq n} \Omega_j$ and the support of $\tilde{\psi}_n$ is contained in $\mathbb{R}^2 \setminus \cup_{j<n} \Omega_j$. Since the support of $\tilde{\psi}_n$ is contained in $\mathbb{R}^2 \setminus \cup_{j<n} \Omega_j$ and the operator H on this set is larger than $-\epsilon_n$, we obtain that

$$\int |\nabla \tilde{\psi}_n|^2 dx - \int V(x) |\tilde{\psi}_n|^2 dx \geq -\epsilon_n \int |\tilde{\psi}_n|^2 dx$$

due to (2.4). Using (2.5) we derive the estimate

$$\int V(x)|\tilde{\psi}_n|^2dx \le \epsilon_n(6L_n)^2 + \int |\nabla\zeta_{n-1}|^2dx + \int |\nabla\psi_n|^2dx, \tag{2.6}$$

where the constant C does not depend on V. Since the functions ζ_{n-1} and ψ_n are not larger than 1, it is easy to see that

$$\int |\nabla\zeta_n|^2dx \le \int |\nabla\zeta_{n-1}|^2dx + \int |\nabla\psi_n|^2dx,$$

which leads to the estimate

$$\int |\nabla\zeta_n|^2dx \le 8n, \qquad \forall n \le N. \tag{2.7}$$

Combining (2.6) and (2.7), we obtain that

$$\int_{\text{supp}\psi_n\setminus\text{supp}\zeta_{n-1}} V(x)dx \le \int V(x)|\tilde{\psi}_n|^2dx \le 6^4 + 8n,$$

On the other hand, it is obvious that

$$\text{supp}\zeta_n = \text{supp}\psi_n \cup \text{supp}\zeta_{n-1}, \qquad \forall n \le N,$$

which leads to the estimate

$$\int_{\text{supp}\zeta_N} V(x)dx \le \sum_n \int_{\text{supp}\psi_n\setminus\text{supp}\zeta_{n-1}} V(x)dx \le (6^4 + 4(N+1))N, \tag{2.8}$$

Define now the function ψ_{N+1} by

$$\psi_{N+1} = \min\{\phi_0(\xi/R), \phi_0(\eta/R)\}, \qquad x = (\xi, \eta)$$

for R large enough. Then for $\tilde{\psi} = \min\{(1-\zeta_N), \psi_N\}$ we have the inequality

$$\int |\nabla\tilde{\psi}|^2dx - \int V(x)|\tilde{\psi}|^2dx \ge 0,$$

which implies that

$$\int_{\mathbb{R}^d\setminus\text{supp}\zeta_N} V(x)dx \le \lim_{R\to\infty} \int |\nabla\tilde{\psi}|^2dx \le 8(1+N). \tag{2.9}$$

Actually, one can even get read of 1 in the right hand side of (2.9) by considering a slightly different family of functions ψ_{N+1}. Combining (2.8) with (2.9) we obtain (1.2).

3. Proof of Theorems 1.2

Here we prove the estimate (1.3) and (1.4). Let $N(\lambda, A)$ be the number of eigenvalues of an operator $A = A^*$ situated to the left the point $\lambda \in \mathbb{R}$. Then according to the variational principle,

$$
\begin{aligned}
&N(\lambda, A) = \max \dim F \\
&(Au, u) \leq \lambda ||u||^2, \qquad u \in F.
\end{aligned} \tag{3.1}
$$

It follows immediately from (3.1), that if P_0 denotes the orthogonal projection onto the space of spherically symmetric functions, then

$$N(\lambda, P_0 H P_0) \leq N(\lambda, H), \qquad \lambda \leq 0,$$

because, we diminish the number of considered subspaces by saying that F in (3.1) consists only of spherically symmetric functions.

On the other side, $P_0 H P_0$ is unitary equivalent to the orthogonal sum of the zero operator and the one-dimensional Schrödinger operator

$$-\frac{d^2}{dr^2} + \frac{\alpha_d}{r^2} + \bar{V}(r),$$

where

$$\bar{V}(r) = \frac{1}{c_d r^{d-1}} \int_{|x|=r} V(x) dS$$

and c_d is the area of the unit sphere. It remains to note that Theorem (1.2) is proven for $d = 1$ in [9].

4. Proof of Theorem 1.3

Following the main idea of [9], we find the regions where the eigenfunctions of $H_\pm$ live. Such regions have the property that after introducing the Dirichlet condition on their boundary, the corresponding negative eigenvalue does not get too close to zero. The next statement is obvious

Lemma 4.1. *Let ϕ be a real valued bounded function with bounded derivatives. Suppose that $-\Delta\psi \pm V\psi = \lambda\psi$ and the product $\phi\psi$ vanishes on the boundary of the domain $\{a < |x| < b\}$. Then*

$$\int_{a<|x|<b} \Big(|\nabla(\phi\psi)|^2 \pm V|\phi\psi|^2\Big) dx = \int_{a<|x|<b} \Big(|\nabla\phi|^2\psi^2 + \lambda|\phi\psi|^2\Big) dx$$

Before stating a very important lemma we introduce the notion of the internal size (width) of a spherical layer $\{a < |x| < b\}$, which is $b - a$. We will also say that one set $X \subset \mathbb{R}^d$ is situated to the "left" of another $Y \subset \mathbb{R}^d$ (correspondingly, Y is situated to the "right" of X) if $|x| < |y|$ for all $x \in X$ and $y \in Y$.

Lemma 4.2. *Assume that the lowest eigenvalue $-\gamma^2$ of $H_\pm$ on the domain $\{a < |x| < b\}$ is negative. If $b-a \geq 6\gamma^{-1}$, then there exist a spherical layer $\Omega \subset \{a < |x| < b\}$ whose width is $d(\Omega) = 6\gamma^{-1}$ such that $H_\pm$ restricted onto Ω has an eigenvalue not bigger than $-\gamma^2/2$.*

Proof. Let ψ be the eigenfunction corresponding to the eigenvalue $-\gamma^2$ for the problem on the domain $\{a < |x| < b\}$ with the Dirichlet boundary conditions. Put $L = \gamma^{-1}$ and pick a number $c > 0$ which gives the maximum to the functional $\int_{c-L<|x|<c+L} |\psi|^2 dx$. The latter functional is a continuous function of c, tending to zero as $c \to \infty$, so it does have a maximum. Define

$$\phi(x) = \begin{cases} 1, & \text{if} \quad ||x| - c| < L, \\ 0, & \text{if} \quad ||x| - c| \geq 3L, \\ 3/2 - ||x| - c|/(2L), & \qquad \text{otherwise}. \end{cases} \tag{4.1}$$

Let Ω be the intersection of the support of ϕ with $\{a < |x| < b\}$. Without loss of generality we can assume that $d(\Omega) = 6\gamma^{-1}$. Otherwise, we can make it larger, so that the bottom of the spectrum will not increase. Now the interesting fact is that

$$\int_{a<|x|<b} |\nabla\phi|^2 \psi^2 dx \leq \frac{\gamma^2}{2} \int_{a<|x|<b} |\phi\psi|^2 dx$$

which is guaranteed by the choice of c. Indeed, $|\nabla\phi|$ vanishes everywhere except for two spherical layers of width $2L$, where it equals $\gamma/2$. Consequently,

$$\int_{a<|x|<b} |\nabla\phi|^2 \psi^2 dx \leq \frac{\gamma^2}{2} \int_{c-L<|x|<c+L} |\psi|^2 dx.$$

Therefore by Lemma 4.1

$$\int_{a<|x|<b} \Big(|\nabla(\phi\psi)|^2 \pm V|\phi\psi|^2 \Big) dx \leq -\frac{\gamma^2}{2} \int_{a<|x|<b} |\phi\psi|^2 dx.$$

That proves the result. $\square$

We also need the following elementary statement:

Lemma 4.3. *Let $H_\pm \geq -\gamma^2$ on a bounded spherical layer $\Omega = \{a < |x| < b\}$, $a > 0$, for both indexes $\pm$. Then $V + \gamma^2 = \mathrm{div} A + |A|^2$ on Ω, where A satisfies the estimate*

$$\int_{a<|x|<b} |\phi|^2 |A(x)|^2 dx \leq C\Big(\gamma^2 \int_{a<|x|<b} |\phi|^2 dx + \int_{a<|x|<b} |\nabla\phi|^2 dx \Big) \tag{4.2}$$

for any $\phi \in C_0^\infty(\Omega)$ with a constant C independent of γ, Ω and ϕ.

Proof. The representation $V = -\gamma^2 + \mathrm{div}A + |A|^2$ on Ω follows from the results of [29]. The idea is to define the vector potential by $A = u^{-1}\nabla u$, where u is a positive solution of the equation $-\Delta u + Vu = \gamma^2 u$. Now

$$\int_{a<|x|<b} \Big(|\nabla\phi|^2 - V|\phi|^2\Big) dx \geq -\gamma^2 \int_{a<|x|<b} |\phi|^2 dx$$

which leads to the inequality (4.2) due to the estimate

$$\int_{a<|x|<b} \mathrm{div}A|\phi|^2 dx \leq \epsilon \int_{a<|x|<b} |A|^2|\phi|^2 dx + \epsilon^{-1} \int_{a<|x|<b} |\nabla\phi|^2 dx.$$

The proof is completed. $\square$

The main ingredients of our proof are in the following technical lemmas, which can be compared with the corresponding set of statements from [9]. Our proof is shorter because instead of functions with symmetric graphs we will use functions whose graphs will have three different slopes. This will influence the choice of sets Ω_n.

Lemma 4.4. *Let $V(x) = 0$ for $|x| < 2$. There is a sequence of spherical layers Ω_n and a sequence of numbers $\epsilon_n > 0$, such that $\sum_n \epsilon_n^{1/2} < \infty$ and the width of Ω_n is bounded by $C\epsilon_n^{-1/2}$ with some C independent of n. The sequence of sets fulfills the condition, that $H_\pm \geq 0$ on the set $\mathbb{R}^d \setminus \cup_n \Omega_n$. Moreover*

$$H_\pm \geq -\epsilon_{j(n)}, \quad \text{on } \Omega_n$$

where $j(n)$ is the lowest number j such that $\Omega_j \cap \Omega_n \neq \emptyset$. If $\Omega_j \cap \Omega_n \neq \emptyset$ and the width of $\Omega_j \cap \Omega_n$ is bounded from below by $6(1 - 20^{-1})\epsilon_k^{-1/2}$:

$$d\Big(\Omega_j \cap \Omega_n\Big) \geq 6(1 - 20^{-1})\epsilon_k^{-1/2},$$

where $k = \min\{j, n\}$. The choice of the sequences can be so made that for each n the number of the sets Ω_j intersecting Ω_n is not bigger than 2 and

$$\mathrm{dist}\Big(\Omega_n, \cup_{m<j(n)}\Omega_m\Big) \geq \frac{3}{10\epsilon_{j(n)}^{1/2}}.$$

Proof. In the proof, we also need to construct some sequence of sets $\omega_n \subset \Omega_n$. Put $\Omega_0 = B_2$ and $\omega_0 = B_1$ where B_r denotes the ball of radius $r > 0$ about the origin; ϵ_0 can be any sufficiently large number, having the property that $-\epsilon_0$ lies below the bottom of the spectra of operators $H_\pm$.

Given $\omega_n \subset \Omega_n$ and ϵ_n for $n < N$ we consider the set

$$S = \mathbb{R}^d \setminus \cup_{n<N}\Omega_n$$

and define $-\epsilon_N$ as the lowest eigenvalue of both operators $H_\pm$ on S. Note that it follows immediately that

$$\epsilon_j \geq \epsilon_{j+1}.$$

Let $\omega_N \subset S$ be the largest spherical layer where one of the operators $H_\pm$ has spectrum below $-\epsilon_N/2$, i.e.

$$\inf \sigma(H_\pm) \leq -\epsilon_N/2,$$

and the width of ω_N is not bigger than $L = 6\epsilon_N^{-1/2}$. The existence of this set is proven in Lemma 2.2. We would like to stress that there are two options:

1) either the boundary of ω_N contains at least one interior point of S or
2) the boundary of ω_N is contained in the boundary of S.

In the first case, the width of ω_N is equal to $L = 6\epsilon_N^{-1/2}$.

Denote by S_+ and S_- the connected component of $S \setminus \omega_N$ situated to the right and to the left of ω_N correspondingly. In the second of the above cases, both sets $S_\pm$ are empty. Let $\Omega_j =: \Omega_-$ and $\Omega_k =: \Omega_+$, $j, k < N$, be the two sets which have common boundary with S_- and S_+ correspondingly. Denote $\omega_- = \omega_j$ and $\omega_+ = \omega_k$. Our construction (or induction assumptions) allow us to assume that

$$\text{dist}\{\omega_\pm, S_\pm\} \geq L_\pm, \text{ where } L_- = 6\epsilon_j^{-1/2} \text{ and } L_+ = 6\epsilon_k^{-1/2}$$

If the width of $S_\pm$ is not bigger than $3L$ we include $\Omega_\pm \setminus \{x : \text{ dist}(x, \omega_\pm) \leq L_\pm/20\}$ into Ω_N by definition by demanding that $\{x : \text{ dist}(x, \omega_\pm) \leq L_\pm/20\}$ and Ω_N has a non- empty piece of common boundary and $\{x : \text{dist}(x, \omega_\pm) \leq L_\pm/20\} \cap \Omega_N = \emptyset$. Otherwise,

$$S_\pm \setminus \{x : \text{ dist}(x, \omega_N) < L\} = S_\pm \setminus \Omega_N.$$

Obviously Ω_N is contained in $\mathbb{R}^d \setminus \cup_{i<j(N)} \Omega_i$ where $j(N)$ is the lowest number i such that $\Omega_i \cap \Omega_N \neq \emptyset$. Therefore,

$$H_\pm \geq -\epsilon_{j(N)} \quad \text{on} \quad \Omega_N.$$

Observe that the distance from the boundary of Ω_N to ω_N is not less than the smallest of the numbers $19L_\pm/20$, which equals $57/(10\sqrt{\epsilon_{j(N)}})$. Obviously, for any $\gamma > 0$ there exist a number N such that the infinum of the spectrum of both operators $H_\pm$ on the domain

$$\mathbb{R}^d \setminus \cup_{n<N} \Omega_n$$

is higher than $-\gamma$. Assume the opposite. Then for any N one of the operators $H_\pm$ on the domain

$$\mathbb{R}^d \setminus \cup_{n<N} \Omega_n$$

has an eigenvalue which is not bigger than $-\gamma$. Then there is an eigenvalue of one of the the operators on ω_N which is not bigger than $-\gamma/2$. This implies that the negative spectrum of one of the operators $H_\pm$ is not discrete. So we come to the conclusion that $H_\pm > 0$ on

$$\mathbb{R}^d \setminus \cup_n \Omega_n.$$

Now let us observe that $\sum_n \epsilon_n^{1/2} < \infty$, because the domains ω_n are disjoint and our operators have an eigenvalue lower than $-\epsilon_N/2$ on each of ω_N. Also, it is clear that any bounded ball B_r of radius $r < \infty$ intersects only finite number of Ω_n, otherwise a Schrödinger operator on B_r would have infinite number of eigenvalues below zero. $\square$

Lemma 4.4 allows one to estimate the potential on the union of the sets Ω_n However, these sets might not exhaust the whole space $\mathbb{R}^d$, so we have to consider the case when

$$\mathbb{R}^d \setminus \cup_n \Omega_n \neq \emptyset.$$

The set $\mathbb{R}^d \setminus \cup_n \Omega_n$ might contain unfilled gaps, where both operators H_+ and H_- are positive. Suppose that a spherical layer Λ is a connected component of the set $\mathbb{R}^d \setminus \cup_n \Omega_n$. Then either the width of the layer Λ is too large or it is small relative to the widths of the sets Ω_{n_1} and Ω_{n_2} having a common boundary with the set Λ. So we can formulate

Lemma 4.5. *Enlarging some of the sets Ω_n in Lemma 4.4 one can achieve that*

$$\mathbb{R}^d = \Big(\cup_n \Omega_n\Big) \cup \Big(\cup_n \Lambda_n\Big). \tag{4.3}$$

where Λ_n are such spherical layers on which both operators H_+ and H_- are positive. Each Λ_m intersects exactly two sets Ω_n. If Λ_n intersects Ω_{n_1} and Ω_{n_2}, then

$$d(\Lambda_n \cap \Omega_{n_j}) \geq 5\epsilon_{n_j}^{-1/2}, \qquad j = 1, 2,$$

where $d(G)$ denotes the width of G. Moreover, Λ_n in (4.3) *can be so chosen that both operators H_+ and H_- are positive on*

$$\Lambda_n \cup \Big(\cup_{j=1,2}\{x \in \Omega_{n_j} : \quad \mathrm{dist}(x, \Lambda_n) < \epsilon_{n_j}^{-1/2}\}\Big).$$

Proof. Indeed, each time when there is a spherical layer Λ having a common boundary with Ω_{n_1} and Ω_{n_2} and the property that $H_\pm \geq 0$ on Λ, we compare $d(\Lambda)$ with the numbers $6\epsilon_{n_1}^{-1/2}$ and $6\epsilon_{n_2}^{-1/2}$. If $d(\Lambda)$ is small (i.e. $d(\Lambda) \leq 6(\epsilon_{n_1}^{-1/2} + \epsilon_{n_2}^{-1/2})$) we keep enlarging both sets Ω_{n_1} and Ω_{n_2} until Λ disappears. However if $d(\Lambda) > 6(\epsilon_{n_1}^{-1/2} + \epsilon_{n_2}^{-1/2})$, then we enlarge Ω_{n_1}

and Ω_{n_2} giving them the pieces of Λ of the width $6\epsilon_{n_1}^{-1/2}$ and $6\epsilon_{n_2}^{-1/2}$, correspondingly. Finally, we make the set Λ smaller by removing two pieces of the width $\epsilon_{n_1}^{-1/2}$ and $\epsilon_{n_2}^{-1/2}$. $\square$

Lemma 4.6. *Let Ω_n, ϵ_n and Λ_n be the same as in Lemma 4.4 and Lemma 4.5. There exists a sequence of H^1-functions $\phi_n \geq 0$ supported in Ω_n and a sequence of H^1-functions $\psi_n \geq 0$ supported in Λ_n such that*

$$\sum_n \phi_n(x) + \sum_n \psi_n(x) = 1,$$

$$\sum_n \int (|\nabla\phi_n(x)|^2 + |\nabla\psi_n(x)|^2)|x|^{1-d}dx \leq C\sum_n \epsilon_n^{1/2}.$$

Moreover, one can find vector potentials A_n and $\tilde{A}_n$ such that

$$V + \epsilon_{j(n)} = \mathrm{div} A_n + |A_n|^2 \text{ on } \Omega_n, \qquad V = \mathrm{div}\tilde{A}_n + |\tilde{A}_n|^2 \text{ on } \Lambda_n \tag{4.4}$$

and

$$\sum_n \Big(\int_{\Omega_n} |A_n|^2|x|^{1-d}dx + \int_{\Lambda_n} |\tilde{A}_n|^2|x|^{1-d}dx \Big) \leq C\Big(1 + \sum_n \epsilon_n^{1/2}\Big).$$

Proof. According to our constructions, if $\Omega_j \cap \Omega_n \neq \emptyset$ then the width of $\Omega_j \cap \Omega_n$ is bounded from below by $6(1-20^{-1})\epsilon_k^{-1/2}$, where $k = \min\{j,n\}$, while the width of the intersections $\Lambda_i \cap \Omega_n \neq \emptyset$ are not less than $5\epsilon_n^{-1/2}$. We define the functions ϕ_n in such a way that the slopes of their graphs will be inversely proportional to the width of these intersections. Therefore the functions ϕ_n will satisfy the relations

$$\int_{\Omega_j\cap\Omega_n} |\nabla\phi_n|^2|x|^{1-d}dx \leq C\epsilon_k^{1/2}, \qquad k = \min\{j,n\};$$

$$\int_{\Omega_n\cup\Lambda_j} |\nabla\phi_n|^2|x|^{1-d}dx \leq C\epsilon_n^{1/2}.$$

We can additionally require that $\phi_n + \phi_j = 1$ on the intersection $\Omega_n \cap \Omega_j$ and $\phi_n + \psi_i = 1$ on the intersection $\Omega_n \cap \Lambda_i$. The representations (4.4) could be obtained by application of Lemma 4.3, since the operators $H_\pm$ satisfy the inequalities

$$H_\pm \geq -\epsilon_{j(n)} \text{ on } \mathbb{R}^d \setminus \Big(\cup_{m<j(n)}\Omega_m\Big),$$

and since both operators H_+ and H_- are positive on

$$\Lambda_n \cup \Big(\cup_{j=1,2}\{x \in \Omega_{n_j} : \quad \mathrm{dist}(x,\Lambda_n) < \epsilon_{n_j}^{-1/2}\}\Big),$$

where $\Omega_{n_j} \cap \Lambda_n \neq \emptyset$. The estimates for A_n in this construction follow from Lemma 4.3 with $\phi = |x|^{(1-d)/2}\tilde{\phi_n}$, where $\tilde{\phi_n} = 1$ on the set

$$\Omega_n \subset \mathbb{R}^d \setminus \Big(\cup_{m<j(n)}\Omega_m\Big).$$

The slopes of the remaining part of the graph of this function should be inversely proportional to the number

$$\mathrm{dist}\Big(\Omega_n, \cup_{m<j(n)}\Omega_m\Big) \geq \frac{3}{10\epsilon_{j(n)}^{1/2}}$$

The latter inequality was mentioned in Lemma 4.4. The integral estimates for the vector potentials $\tilde{A}_n$ can be obtained in a similar way. $\square$

The end of the proof of Theorem 1.3. Let us define

$$A = \sum_n(\phi_n A_n + \psi_n\tilde{A}_n), \qquad W = -\sum_n \epsilon_{j(n)}\phi_n \quad V_1 = W + \mathrm{div}(A) + |A|^2$$

Then one can easily see that

$$V_1 = V + \sum_n(A_n\nabla\phi_n + \tilde{A}_n\nabla\psi_n) + |A|^2 - \sum_n(\phi_n|A_n|^2 + \psi_n|\tilde{A}_n|^2),$$

which implies

$$\int |V - V_1||x|^{1-d}dx < \infty.$$

If we define $V_0 = V - V_1 + W$, then we will represent the potential in the form $V = V_0 + \mathrm{div}(A) + |A|^2$. Finally, since

$$\int |W||x|^{1-d}dx < \infty, \qquad \int |A|^2|x|^{1-d}dx < \infty,$$

we conclude that

$$\int (|V_0| + |A|^2)|x|^{1-d}dx < \infty. \quad \square$$

5. Proof of Theorem 1.4

Now, we use the results of [21] to establish the presence of the absolutely continuous spectrum of H_+. As it is known the operator $-\Delta$ is unitary equivalent to the sum $-\partial^2/\partial r^2 + \alpha_d/r^2 + (-\Delta_\theta)/r^2$ where Δ_θ is the Laplace-Beltrami operator on the unit sphere and α_d is a certain constant. Since the values of V on compact subsets do not influence the presence of absolutely continuous spectrum, we can assume that $V(x) = -\alpha_d/|x|^2$ for $1 < |x| < 2$. According to [21], there is a probability measure μ on the real line $\mathbb{R}$, whose a.c. component is essentially supported by a subset of the a.c. spectrum of the operator H_+. Namely, one constructs an operator A_+ having the same a.c. spectrum as H_+ in the following way:

$$A_+ = -\Delta + V, \qquad D(A_+) = \{u \in H^2(\mathbb{R}^d \backslash B_1): \quad u(\theta) = 0,\ \theta \in \mathbb{S}^{d-1}\}.$$

Then one sets $\mu(\delta) = (E_{A_+}(\delta)f, f)$ for a spherically symmetric function f having the property

$$\mathrm{supp} f \subset \{x \in \mathbb{R}^d: \quad 1 < |x| < 2\}.$$

Theorem 5.1. [21] *Let V be a compactly supported potential and let $\lambda_j(V)$ be the negative eigenvalues of H_+. Then for any continuous compactly supported function $\phi \geq 0$ on the positive half-line $(0, \infty)$, we have*

$$\begin{aligned} \int_0^\infty \log\Big(\frac{\mu'(\lambda)}{\phi(\lambda)}\Big)\phi(\lambda)d\lambda \geq -C\Big(\sum_j \sqrt{|\lambda_j(V)|} \\ + \sqrt{||V||_\infty} + \int V(x)|x|^{1-d}dx + 1\Big) \end{aligned} \tag{5.1}$$

where $||V||_\infty$ denotes the L^∞-norm of the function V and $C > 0$ depends on ϕ and the choice of $f \in L^2$.

One of the important properties of the measure μ is that (see [31])

$$V_n \to V \text{ in } L^2_{loc} \Rightarrow \mu_n \to \mu \text{ weakly}.$$

Using are going to combine this property with the lower semi-continuity of the entropy (see the paper [18] by Killip and Simon)

Theorem 5.2. *Let the sequence of probability measures μ_n converge to μ weakly on the real line $\mathbb{R}$. Then for any $0 < a < b < \infty$ and any positive continuous (on the real line) function ϕ with the property* $\mathrm{supp}\phi \subset [a, b]$, *we have*

$$\int_a^b \log\Big(\frac{\mu'(\lambda)}{\phi(\lambda)}\Big)\phi(\lambda)d\lambda \geq \liminf_{n\to\infty} \int_a^b \log\Big(\frac{\mu_n'(\lambda)}{\phi(\lambda)}\Big)\phi(\lambda)d\lambda.$$

Let $[V_0]_+$ and $[V_0]_-$ be the positive and negative parts of the function V_0 and let χ_n be the characteristic function of the ball of radius n with the center at the point 0. Consider the Schrödinger operator with the potential

$$V_{n,\varepsilon} = [V_0]_+ - \chi_n[V_0]_- + \mathrm{div}(A) + (1-\varepsilon)^{-1}|A|^2, \qquad \varepsilon \in (0, 1).$$

The number of negative eigenvalues of the Schrödinger operator with the potential $V_{n,\varepsilon}$ can be estimated by a quantity that depends only on $\varepsilon^{-1}\chi_n[V_0]_-$. Indeed, define $N(W)$ as the number of negative eigenvalues of the operator $-\Delta + W$, where W is a real potential. Then

$$N(W_1 + W_2) \leq N(\varepsilon^{-1}W_1) + N((1-\varepsilon)^{-1}W_2).$$

Set now $W_1 = -\chi_n[V_0]_-$ and $W_2 = [V_0]_+ + \mathrm{div}(A) + (1-\varepsilon)^{-1}|A|^2$. Since $N((1-\varepsilon)^{-1}W_2) = 0$, we will obtain that

$$N(V_{n,\varepsilon}) \le N(-\varepsilon^{-1}\chi_n[V_0]_-).$$

The right hand side of this inequality is independent of A and $[V_0]_+$. Therefore both of these functions can be approximated by compactly supported functions in such a way that $N(V_{n,\varepsilon})$ stays bounded. Let $\mu_{n,\varepsilon}$ be the corresponding spectral measure of the operator $-\Delta + V_{n,\varepsilon}$. Then by the lower semi-continuity of the entropy (see [18]) we obtain

$$\begin{aligned}\int_0^\infty \log\Big(\frac{\mu'_{n,\varepsilon}(\lambda)}{\phi(\lambda)}\Big)\phi(\lambda)d\lambda \ge -C(\sum_j |\lambda_j(V_{n,\varepsilon})|^{1/2} + \sqrt{||V||_\infty} \\ + \int (|V_0| + (1-\varepsilon)^{-1}|A|^2)|x|^{1-d}dx + 1)\end{aligned} \tag{5.2}$$

for any continuous function $\phi \ge 0$ with supp$\phi \subset (0,\infty)$. Since $V_{n,\varepsilon}$ is monotone in both n and ε, we have $\lambda_j(V) \le \lambda_j(V_{n,\varepsilon})$. Therefore, the relation (5.2) leads to the inequality

$$\begin{aligned}\int_0^\infty \log\Big(\frac{\mu'_{n,\varepsilon}(\lambda)}{\phi(\lambda)}\Big)\phi(\lambda)d\lambda \ge -C(\sum_j |\lambda_j(V)|^{1/2} + \sqrt{||V||_\infty} \\ + \int (|V_0| + (1-\varepsilon)^{-1}|A|^2)|x|^{1-d}dx + 1).\end{aligned}$$

Now we can extend (5.1) to the general case due to the upper semi-continuity of the entropy:

$$\begin{aligned}\int_0^\infty \log\Big(\frac{\mu'(\lambda)}{\phi(\lambda)}\Big)\phi(\lambda)d\lambda \ge -C(\sum_j \sqrt{|\lambda_j(V)|} + \sqrt{||V||_\infty} \\ + \int (|V_0| + |A|^2)|x|^{1-d}dx + 1).\end{aligned}$$

Convergence of the integral in the left hand side implies that $\mu' > 0$ almost everywhere on the set where ϕ is positive. This completes the proof of Theorem 1.4. □

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