Examples of non-rigidity for circle homeomorphisms with breaks

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Abstract

We give examples of analytic circle maps with singularities of break type with the same rotation number and the same size of the break for which no conjugacy is Lipschitz continuous. In the second part of the paper, we discuss a class of rotation numbers for which a conjugacy is C^1-smooth, although the numbers can be strongly non-Diophantine (Liouville). For the rotation numbers in this class, we construct examples of analytic circle maps with breaks, for which the conjugacy is not $C^{1+\alpha}$ smooth, for any $\alpha > 0$.

1 Introduction

This paper concerns the rigidity of circle maps with break singularities. These are orientation-preserving homeomorphisms of the circle $\mathbb{T}^1 = \mathbb{R}\backslash\mathbb{Z}$, which are C^r-smooth outside a single point where the derivative has a jump discontinuity. Circle maps with breaks were introduced about 20 years ago as an interesting example of a one-dimensional dynamical system with rich and non-trivial renormalization behavior. Usually, non-trivial renormalizations are related to the presence of critical points, like in the case of critical circle maps. It turns out that points of break can cause behavior very similar to that of critical points. Such a "criticality" manifests itself through non-trivial scalings, complicated structure of the renormalization horseshoe, prevalence of rational rotation numbers,

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etc. At the same time, renormalization analysis of maps with breaks is simpler than in the critical case. A full renormalization theory can be constructed in this case, which is still an open problem for critical circle maps with non-analytic critical points. The simplification is related to the fact that the renormalized maps converge to a two-parameter family of linear-fractional (Möbius) maps. It is fair to say that maps with breaks form a class of maps which is situated in between circle diffeomorphisms on one side, and critical circle maps on the other.

Rigidity theory for smooth diffeomorphisms is the subject of classical Hermann theory [8, 20, 9, 17, 13]. Precise statements will be formulated below. Here, we simply point out that rigidity results for circle diffeomorphisms depend strongly on the Diophantine properties of the rotation numbers. Rigidity, in this case, refers to a statement about the smoothness of the conjugacy between a circle diffeomorphism with a Diophantine rotation number and the corresponding rigid rotation. On the other side, Arnol'd [1] has shown that such a conjugacy can be singular for Liouville numbers, even in the analytic case. Interestingly, the presence of critical points makes the situation more rigid. It was shown in [12] that C^1 rigidity of analytic critical circle maps holds for all irrational rotation numbers. Namely, two analytic critical circle maps with the same order of the critical point and the same irrational rotation number can be conjugated C^1-smoothly to each other. Since maps with breaks exhibit behavior similar to the critical one in many respects, it seemed plausible that a similar "robust" rigidity result holds in this case as well. This conjecture found a partial confirmation in [12], which suggested that for a certain class of strongly non-Diophantine rotation numbers, the conjugacy is C^1-smooth, provided that the sizes of the breaks are the same. However, as we show in this paper, robust rigidity does not hold for maps with breaks. On the contrary, we show that for certain irrational rotation numbers, the conjugacy is not even Lipschitz continuous. We also show that the conjugacy that maps one break point into another can be as "bad" as possible. Note that a similar result holds in the diffeomorphism case (see Theorem 3.6 below).

Another motivation for circle maps with breaks is related to generalized interval exchange transformations [16]. Such transformations were introduced very recently and analysis of their ergodic and rigidity properties is currently underway. The idea of this generalization is to replace the affine interval exchange with nonlinear transformations mapping corresponding subintervals into their images. It is well known that a rigid rotation can be seen as an exchange transformation of two intervals. In this sense, a circle homeomorphism can be viewed as a generalized interval exchange transformation of two intervals. Imagine, now, that the maps for both subintervals are smooth. While matching of endpoints is a natural requirement, matching of the derivatives at the end points is rather artificial. Hence, a natural generalized interval exchange of two intervals is in fact a circle homeomorphism with two points of break. Since both break points belong to one trajectory, one can piecewise smoothly conjugate such a homeomorphism to a map with one break point. This connection indicates that our results are related to the problem of rigidity for the generalized interval exchange transformations. However, it is a very special case. Indeed, Denjoy theory [4] holds in the case of circle homeomorphisms with breaks, which is not true in general. Note, finally, that circle maps with many break points can be considered as generalized interval exchanges of the corresponding number of intervals.

We proceed with precise definitions and formulation of the main results. Any orientation preserving circle homeomorphism $T : \mathbb{T}^1 \to \mathbb{T}^1$ with a break is defined uniquely by a function $\mathcal{T} : \mathbb{R} \to \mathbb{R}$ that satisfies

(i) $\mathcal{T}$ is continuous and strictly increasing on $\mathbb{R}$, with $\mathcal{T}(0) \in [0, 1)$,

(ii) $\mathcal{T}(x + 1) = \mathcal{T}(x) + 1$, for every $x \in \mathbb{R}$,

(iii) there exists a point $x_{br} \in [0, 1)$ such that $\mathcal{T}(x) \in C^r$, $r \in [1, \infty) \cup \{\infty, \omega\}$, on $[x_{br}, x_{br} + 1]$, and there exists $C > 0$ such that $\mathcal{T}'(x) > C > 0$, for every $x \in [x_{br}, x_{br} + 1]$,

(iv) the one sided derivatives $\mathcal{T}'_-(x_{br})$ and $\mathcal{T}'_+(x_{br})$ at x_{br} are such that for some $c \in \mathbb{R}^+ \backslash \{1\}$,

$$\sqrt{\frac{\mathcal{T}'_-(x_{br})}{\mathcal{T}'_+(x_{br})}} = c.$$

Such a value c will be called the size of the break.

Remark 1 The analytic case C^ω corresponds to functions $\mathcal{T}$ whose restrictions to the interval $[x_{br}, x_{br} + 1]$, denoted by $\mathcal{T}|_{[x_{br}, x_{br}+1]}$, have analytic extension on a complex disc containing $[x_{br}, x_{br} + 1]$.

The space of all such C^r circle homeomorphisms with a break of size c will be denoted by $\mathcal{B}^r_c$, and the space of corresponding lifts by $\mathcal{A}^r_c$. Size of the break plays essentially the same role as the order of the critical point (see below). Namely, it is a smooth invariant, i.e. a smooth conjugacy does not change it. It is easy to see that only maps with breaks which are of the same size have a chance to be smoothly conjugate to each other.

For any orientation-preserving circle homeomorphism T, there exists a unique rotation number ρ. It has been known since Poincaré that if any two orientation-preserving circle homeomorphisms T and $\tilde{T}$ have the same irrational rotation number, then they are topologically semi-conjugate to each other, i.e. there is a continuous circle map $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$, such that $T \circ \varphi = \varphi \circ \tilde{T}$. Denjoy theory [4] asserts that in the case of C^r-smooth circle homeomorphisms with breaks, for $r \geq 2$ (like in the case of diffeomorphisms, this condition can be slightly weakened), φ is actually a homeomorphism. In this case, φ is referred to as the (topological) conjugacy. The phenomenon that a conjugacy between two circle maps, which is just a continuous map a priori, in some cases possesses a certain degree of regularity is referred to as *rigidity*.

We present first well-known rigidity results for circle diffeomorphisms. Arnol'd proved that if an analytic circle diffeomorphism is close enough to a rigid rotation and its rotation number satisfies a certain Diophantine condition, then the conjugacy to the rotation is in fact analytic [1]. Arnol'd also conjectured a global result: there exists a subset of Lebesgue measure 1 in $(0,1)$, such that any C^∞ diffeomorphism with rotation number in this set is C^∞ conjugate to a rotation. This was proved by Herman [8]. The result of Herman [8], as well as the later extensions by Yoccoz [20], Katznelson and Orstein [9], Sinai and Khanin [17], and Khanin and Teplinsky [13], also applies to the finite differentiability case. In the case of low smoothness, one can prove [17, 13] that a $C^{2+\alpha}$-smooth circle diffeomorphism is $C^{1+\alpha-\beta}$ conjugated to a rotation if the rotation number ρ satisfies the Diophantine condition with exponent $\beta < \alpha$ (i.e. there exists $C > 0$ and $\beta \geq 0$ such that $|\rho - p/q| > C/q^{2+\beta}$, for every rational p/q). In [1], Arnol'd also gave examples of analytic circle diffeomorphisms without periodic orbits but whose rotation numbers are well-approximable by rational numbers (Liouville numbers) for which the invariant measure is singular with respect to Lebesgue measure.

The main result of this paper is the following.

Theorem 1.1 *There exist two analytic circle maps with a break $T_\rho, \tilde{T}_\rho \in \mathcal{B}_c^\omega$, with the same irrational rotation number ρ, and the same size of the break $c \neq 1$, such that no topological conjugacy φ, that satisfies*

$$\varphi^{-1} \circ T_\rho \circ \varphi = \tilde{T}_\rho, \tag{1.1}$$

is Lipschitz continuous.

Remark 2 The rotation number ρ of the maps in Theorem 1.1 belongs to a class of irrational numbers $\rho \in (0,1)$ whose odd-numbered entries k_{2n-1} in the continued fraction expansion of $\rho = [k_1, k_2, \dots]$, in the case $0 < c < 1$, or even-numbered entries k_{2n}, in the case $c > 1$, grow sufficiently fast with $n \in \mathbb{N}$.

Remark 3 In particular, Theorem 1.1 provides examples of analytic circle maps with breaks, with the same rotation number and the same size of the break, for which C^1-smooth conjugacy does not exist.

This result stands in contrast to the case of critical circle maps, that is circle homeomorphisms which are C^r-smooth everywhere and have a single point x_0 where the first derivative vanishes. Near the critical point x_0 the derivative behaves as $|x - x_0|^{\alpha-1}$, where $\alpha > 1$ is the order of the critical point. Yoccoz showed that any two analytic critical circle maps with the same irrational rotation number and the same order of the critical point are topologically conjugate to each other [21]. It has been conjectured that in the case of critical circle maps with the same order of the critical point, topological conjugacy implies C^1-conjugacy. That is, the rigidity of critical circle maps does not depend on the Diophantine properties of their rotation number. In [12], this property has been called *robust rigidity*. So far the conjecture has been proved only in the case of analytic critical circle maps. It fact, Khanin and Teplinsky [12] showed that the robust rigidity conjecture holds for all orders of the critical point, assuming that the renormalizations of such maps (see below) approach each other exponentially fast. At present, convergence of renormalizations is known only in the case when the order of critical circle maps is an odd integer larger than 1. De Faria and de Melo proved the exponential convergence of renormalizations for C^∞-smooth critical circle maps and rotation numbers of bounded type [6, 7]. In the case of analytic critical circle maps, this result has been extended to all rotation numbers by Yampolsky [19]. De Faria and de Melo also proved that, for a set of Lebesgue measure 1 in $(0,1)$, in the case of C^∞-smooth critical circle maps with odd integer order of the critical point, the conjugacy is, in fact, $C^{1+\alpha}$, for some $\alpha > 0$. They also showed that $C^{1+\alpha}$ rigidity of C^∞-smooth critical circle maps cannot be extended to all Diophantine rotation numbers. Examples of analytic critical circle maps with the same order of the critical point and the same irrational rotation number which are not $C^{1+\alpha}$ conjugated to each other for any $\alpha > 0$ have been constructed by Avila [2]. Here, we also extend the parabolic renormalization method developed in [2] and prove a similar result for the case of analytic circle maps with breaks. More precisely, we prove the following.

Theorem 1.2 *There exist $T_\rho, \widetilde{T}_\rho \in \mathcal{B}_c^\omega$ with the size of the break c and the same irrational rotation number $\rho \in (0,1)$, with bounded odd-numbered entries k_{2n-1} in the continued fraction expansion of ρ, in the case $0 < c < 1$, or even-numbered entries k_{2n}, in the case $c > 1$, such that the topological conjugacy φ between them is not $C^{1+\alpha}$, for any $\alpha > 0$.*

Remark 4 A result similar to Theorem 1.2 has been obtained independently by Dzhalilov and Teplinsky [5, 18]. Both proofs rely on Avila's construction [2] which requires only a minor modification in the break case.

The methods of proofs of Theorem 1.1 and Theorem 1.2 are very different. Both of them, however, use renormalization ideology. It has been proved in [10] that the renormalizations of circle maps with breaks with the same size of the break and with the same quadratic irrational rotation number approach each other exponentially fast. This result has been extended to all rotation numbers in [14]. It particular, this implies that renormalizations of circle maps with breaks approach a family of linear fractional maps, which is invariant under renormalizations. Within this family the renormalization operator maps convex maps into concave and vice versa. The same property is shared by renormalizations of circle maps with breaks which are not fractional linear, after sufficiently many renormalization steps. It turns out that in the case $0 < c < 1$, the concave renormalization maps correspond to even renormalization steps n, while convex renormalization maps correspond to odd n. For $c > 1$, the situation is the opposite. This explains why the behavior is very different, in the limit when $k_{n+1} \to \infty$, for even and

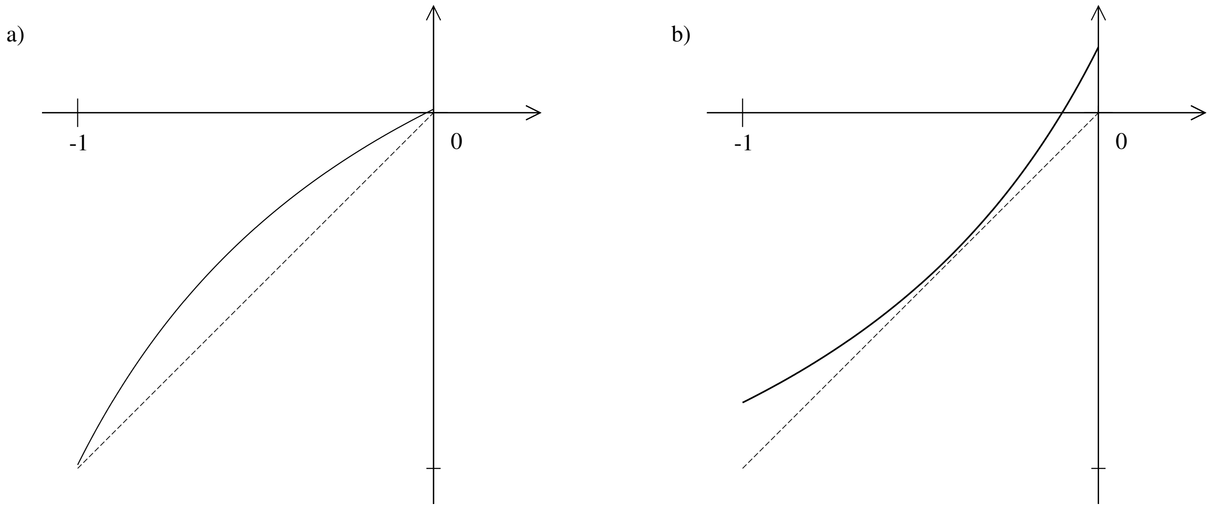

Figure 1: The graph of a renormalized map f_n for sufficiently large n and large k_{n+1}: a) Case $0 < c < 1$ and n even, or $c > 1$ and n odd; b) Case $0 < c < 1$ and n odd, or $c > 1$ and n even.

odd n. The graphs of renormalized maps f_n, defined with the marked point being the break point (see Section 2), for sufficiently large n and k_{n+1}, look like the graphs shown in Figure 1. Roughly speaking, a subsequence of renormalizations with concave graphs which in the limit $n \to \infty$ approach the diagonal very fast at the end points (Figure 1a) is characteristic of examples with the absence of rigidity that we construct in Theorem 1.1. In fact, this type of behavior is the only obstacle to C^1 rigidity. On the other hand, a subsequence of renormalizations with convex graphs which almost touch the diagonal at a point inside the interval $(-1, 0)$ (Figure 1b) provides examples of C^1-rigid maps for which rigidity cannot be extended to $C^{1+\alpha}$-smoothness as in Theorem 1.2.

The paper is organized as follows. In Section 2, we introduce the general renormalization setting for circle homeomorphisms. In Section 3, we prove Theorem 1.1. Section 4, contains a discussion of parabolic renormalization method of circle maps with breaks and the proof of Theorem 1.2.

2 General settings

2.1 Renormalization of orientation-preserving circle homeomorphisms

For every orientation-preserving homeomorphism T of the circle $\mathbb{T}^1 = \mathbb{R} \backslash \mathbb{Z}$ there is a unique rotation number ρ, given by the x-independent limit $\rho = \lim_{n\to\infty} \mathcal{T}^n(x)/n \mod 1$, for any lift $\mathcal{T}$ of T to $\mathbb{R}$. The particular renormalization that we use in this paper is closely related to the *continued fraction expansion* of the rotation number $\rho \in (0,1]$, i.e.

$$\rho = \frac{1}{k_1 + \frac{1}{k_2 + \frac{1}{k_3 + \dots}}}, \tag{2.1}$$

that we write as $\rho = [k_1, k_2, k_3, \dots]$. The sequence of integers k_n, called *partial quotients*, is infinite if and only if ρ is irrational. Every irrational ρ defines uniquely the sequence of partial quotients. Conversely, every infinite sequence of partial quotients defines uniquely an irrational number ρ as the limit of the sequence of *rational convergents* $p_n/q_n = [k_1, k_2, \dots, k_n]$. It is well-known that this sequence forms the sequence of best rational approximates of ρ, i.e. there are no rational numbers with denominators smaller or equal to q_n, that are closer to ρ than p_n/q_n. The (rational) convergents can also be defined recursively as $p_n = k_n p_{n-1} + p_{n-2}$ and $q_n = k_n q_{n-1} + q_{n-2}$, starting with $p_0 = 0$, $q_0 = 1$, $p_{-1} = 1$, $q_{-1} = 0$.

To define the renormalizations, we start with a *marked point* $x_0 \in \mathbb{T}^1$, and consider the *marked trajectory* $x_i = T^i x_0$, with $i \geq 0$. The subsequence x_{q_n}, $n \geq 0$, indexed by the denominators of the sequence of rational convergents of the rotation number ρ, will be called the sequence of *dynamical convergents*. We define $x_{q_{-1}} = x_0 - 1$. The combinatorial equivalence of all circle homeomorphisms with the same irrational rotation number implies that the order of the dynamical convergents of T is the same as the order of the dynamical convergents for the pure rotation $T_\rho : x \mapsto x + \rho$. The well-known arithmetic properties of the rational convergents now imply that dynamical convergents alternate their order in the following way:

$$x_{q_{-1}} < x_{q_1} < x_{q_3} < \dots < x_0 < \dots < x_{q_2} < x_{q_0}. \tag{2.2}$$

The interval $[x_{q_n}, x_0]$, for n odd, and $[x_0, x_{q_n}]$, for n even, will be denoted by $\Delta_0^{(n)}$, and called the n-th *renormalization segments*. We will also define $\bar{\Delta}_0^{(n)} = \Delta_0^{(n)} \cup \Delta_0^{(n+1)}$. In addition to the property (2.2), we also have the following important property: the only points of the trajectory $\{x_i : 0 < i \leq q_{n+1}\}$ that belong to $\Delta_0^{(n)}$ are $\{x_{q_n + i q_{n+1}} : 0 \leq i \leq k_{n+2}\}$.

We will use the notation $\Delta_i^{(n)}$, to denote the n-th renormalization segment associated to the marked point x_i.

The consecutive images of $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$ cover the whole circle without overlapping beyond the end points, thus forming the n-th *dynamical partition* of $\mathbb{T}^1$,

$$\mathcal{P}_n = \{T^i \Delta_0^{(n-1)} : 0 \leq i < q_n\} \cup \{T^i \Delta_0^{(n)} : 0 \leq i < q_{n-1}\}. \tag{2.3}$$

The iterates of T^{q_n} and $T^{q_{n-1}}$ restricted to $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$, respectively, are the two continuous components of the first return map for T on the interval $\bar{\Delta}_0^{(n)}$.

The n-th *renormalization* of an orientation-preserving homeomorphism T of the circle $\mathbb{T}^1$, with rotation number $\rho = [k_1, k_2, k_3, \ldots]$, with respect to the marked point $x_0 \in \mathbb{T}^1$, is a function $f_n : [-1, 0] \to \mathbb{R}$ obtained from T^{q_n}, by rescaling the coordinates. More precisely, if τ_n is the affine change of coordinates that maps x_{n-1} to -1 and x_0 to 0, then

$$f_n = \tau_n \circ T^{q_n} \circ \tau_n^{-1}. \tag{2.4}$$

If we identify x_0 with zero, then τ_n is exactly a multiplication by $(-1)^n/|\Delta_0^{(n-1)}|$. Here and in what follows, we use $|\cdot|$ to denote the length of an interval. Definition (2.4) is valid for all $n \geq 0$ if and only if ρ is irrational; otherwise, n must be less than the length of the continued fraction expansion of ρ or can be equal to it if $x_{q_{n-1}} \neq x_0$.

2.2 Modulus of continuity

A continuous real function $\omega : [0, \infty) \to [0, \infty)$ is called a modulus of continuity if it is decreasing and it vanishes at 0, i.e. if it satisfies

$$\lim_{x \to 0} \omega(x) = \omega(0) = 0. \tag{2.5}$$

We say that a function $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$, is uniformly continuous with the modulus of continuity ω, if

$$|\varphi(x) - \varphi(y)| \leq \omega(|x - y|), \tag{2.6}$$

for all $x, y \in \mathbb{T}^1$. For points on the circle the distance $|x - y|$ will be given by the minimal distance between their lifts to $\mathbb{R}$. We say that a circle homeomorphism $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$ (which is a topological conjugacy between two circle maps) admits ω as the modulus of continuity if both φ and the inverse φ^{-1} are uniformly continuous with modulus of continuity $C\omega$, for some $C > 0$.

If the homeomorphism admits $\omega(t) = t$ as the modulus of continuity, it is said to be Lipschitz continuous; if $\omega(t) = t^\alpha$, for some $\alpha \in (0, 1)$, the homeomorphism is said to be Hölder continuous with exponent α.

Note that for real-valued functions f and g of real variables we will say that f is of the order of g, and write $f = \mathcal{O}(g)$, if there exists a constant $C_1 > 1$, such that

$$C_1^{-1}|g(\cdot)| \leq |f(\cdot)| \leq C_1|g(\cdot)|, \tag{2.7}$$

everywhere.

3 A non-rigidity result

3.1 A lemma on the derivatives for rational rotation numbers

Let $T \in \mathcal{B}_c^\omega$, with the break point located at x_{br}, that satisfies $T(x_{br}) = x_{br}$. Consider the one parameter family $T_a = T + a$ of circle maps with a break in $\mathcal{B}_c^\omega$. The rotation number ρ of the maps in this family depends continuously on the parameter a. For every rational rotation number $p/q \in \mathbb{Q}$, there is a (mode-locking) interval $[a_{p/q}^{(1)}, a_{p/q}^{(2)}]$ of parameter values corresponding to p/q. If p/q has a sufficiently long continued fraction expansion, then the following properties hold. When the parameter value a is equal to $a_{p/q}^{(1)}$, in the case $c > 1$, or $a_{p/q}^{(2)}$, in the case $0 < c < 1$, the map T_a has a single periodic orbit of the type (p, q), i.e. a lift $\mathcal{T}_a : \mathbb{R} \to \mathbb{R}$ of T_a satisfies $\mathcal{T}_a^q(x_{br}) = x_{br} + p$, and the break point x_{br} belongs to the periodic orbit. Let us denote that unique value of the parameter a by $a_{p/q}$. When the parameter value a equals the other end point ($a_{p/q}^{(2)}$, in the case $c > 1$; $a_{p/q}^{(1)}$, in the case $0 < c < 1$), the map T_a has a single periodic orbit of the type (p, q), which is neutral. Obviously, the break point x_{br} does not belong to it. For all other values of the parameter inside the mode-locking interval, the map has two periodic orbits of type (p, q), one stable and one unstable [15].

Lemma 3.1 *There exist two analytic circle maps $T, \tilde{T} \in \mathcal{B}_c^\omega$, with break points at x_{br} and $\tilde{x}_{br}$, respectively, such that the following is true for the corresponding families $T_a = T + a$ and $\tilde{T}_{\tilde{a}} = \tilde{T} + \tilde{a}$, with parameters $a, \tilde{a} \in \mathbb{R}$. For every $p \in \mathbb{Z}^+$ and $q \in \mathbb{N}$ relatively prime, such that $0 \leq \frac{p}{q} < 1$, if $a_{p/q}, \tilde{a}_{p/q}$ are values of parameters such that the corresponding break point is a periodic point of type (p, q), then*

$$\prod_{i=0}^{q-1} \left(T_{a_{p/q}}\right)_+' (x_{a_{p/q},i}) \neq \prod_{i=0}^{q-1} \left(\tilde{T}_{\tilde{a}_{p/q}}\right)_+' (\tilde{x}_{\tilde{a}_{p/q},i}) . \tag{3.1}$$

Here $x_{a,i} = T_a^i(x_{br})$, $\tilde{x}_{\tilde{a},i} = \tilde{T}_{\tilde{a}}^i(\tilde{x}_{br})$, and the subscript "+" stands for the right derivative.

Proof. Let us order all rational numbers in $[0, 1)$, starting with zero, and denote the corresponding sequence by p_n/q_n, $n \in \mathbb{N}$. We will first choose two analytic circle maps T and $\tilde{T}$, with the same size of the break c, such that the corresponding lifts $\mathcal{T} : \mathbb{R} \to \mathbb{R}$ and $\tilde{\mathcal{T}} : \mathbb{R} \to \mathbb{R}$ have fixed points at the integer points (and only at these points) and have breaks at these points. We will now fix the latter map and modify the former, if necessary, in a sequence of steps, in order to produce a sequence of maps $T^{(n)}$ (with corresponding lifts $\mathcal{T}^{(n)}$), $n \in \mathbb{N}$, satisfying the condition (3.1) with $p/q = p_k/q_k$, for $1 \leq k \leq n$. We will construct this sequence inductively. The map $T^{(1)} = T$ satisfies the condition (3.1) for $p_1/q_1 = 0/1$, by our choice of T and $\tilde{T}$. Assume that the map $T^{(n)}$ satisfies the condition (3.1) with $p/q = p_k/q_k$, for $1 \leq k \leq n$, i.e. that the claim is valid for all p_k/q_k

for $1 \le k \le n$, by taking $T = T^{(n)}$. We will show that the claim is valid for all p_k/q_k, for $1 \le k \le n+1$, for some map $T = T^{(n+1)}$, that we will construct now.

In the following, the parameter values a_{p_k/q_k} associated to the map $T = T^{(n)}$, will be denoted by $a_{p_k/q_k}(n)$. To simplify the notation, denote $\mathcal{T}^{(n)}_{a_{p_k/q_k}(n)} = \mathcal{T}^{(n)}_k$ and the corresponding orbit $(\mathcal{T}^{(n)}_k)^i(x_{br}) = x_i(n,k)$, $0 \le i < q_k$. If the condition (3.1) is satisfied for $T = T^{(n)}$ and $p/q = p_{n+1}/q_{n+1}$, then $T^{(n+1)} = T^{(n)}$. Now, let $P_n : [0,1] \to \mathbb{R}$ be defined by

$$P_n(x) = x(x-1)(Ax+B)Q_n(x), \qquad Q_n(x) = \prod_{i=1}^{q_{n+1}-1}(x-x_i)^2, \tag{3.2}$$

for $x \in [0,1]$, where $x_i = \{(\mathcal{T}^{(n)}_{n+1})^i(0)\}$, $A = c^2\delta_n/Q(1) - B$ and $B = -\delta_n/Q(0)$, for some $\delta_n > 0$. Here, $\{x\} = x - [x]$ is the fractional part of a number $x \in \mathbb{R}$. Since $0 < x_i < 1$ for $1 \le i \le q_{n+1}-1$, we have $Q(0), Q(1) > 0$, and A and B are well-defined. The function P_n satisfies the conditions $P_n(0) = P_n(1) = 0$, $(P_n)'_+(0) = \delta_n$, $(P_n)'_-(1) = c^2\delta_n$, $P_n(x_i) = 0$, and $P'_n(x_i) = 0$, for all $1 \le i \le q_{n+1}-1$. Notice, that if $\delta_n > 0$ is chosen sufficiently small, then the supremum norm $\|P_n\| < C$, for some constant $C > 0$, independent of n.

Let us now extend P_n periodically to obtain a function $v_n : \mathbb{R} \to \mathbb{R}$, defined by $v_n(x) = P_n(x)$, for $x \in [0,1]$, and $v_n(x+1) = v_n(x)$, otherwise. If the condition (3.1) is not satisfied for $T = T^{(n)}$ and $p/q = p_{n+1}/q_{n+1}$, then $\mathcal{T}^{(n+1)} = \mathcal{T}^{(n)} + \epsilon_n v_n$. For sufficiently small $\epsilon_n > 0$, due to the continuity of the maps $\epsilon_n \mapsto a_{p_k/q_k}(n+1)$ and $\epsilon_n \mapsto x_i(n+1,k)$, the conditions (3.1) corresponding to $T = T^{(n+1)}$ and $p/q = p_k/q_k$ are satisfied for $1 \le k \le n$. By construction, the map $T^{(n+1)}_{n+1}$ has the same periodic orbit of type (p_{n+1}, q_{n+1}) as $T^{(n)}_{n+1}$, and the one-sided derivatives at the break point have changed. Thus, the condition (3.1) corresponding to $T = T^{(n+1)}$ is now satisfied for $p/q = p_{n+1}/q_{n+1}$.

Let

$$\left|\left((T^{(k)}_k)^{q_k}\right)'_+(x_{br}) - \left((\tilde{T}_{\tilde{a}_{p_k/q_k}})^{q_k}\right)'_+(\tilde{x}_{br})\right| = \gamma_k > 0, \tag{3.3}$$

for all $k \in \mathbb{N}$. If $\epsilon_n > 0$ is chosen sufficiently small, then

$$\left|\left((T^{(n+1)}_k)^{q_k}\right)'_+(x_{br}) - \left((T^{(n)}_k)^{q_k}\right)'_+(x_{br})\right| < \frac{\gamma_k}{2^{n+1}}, \tag{3.4}$$

for all integer $n \ge k$.

For a sufficiently fast decreasing sequence ϵ_n, the sequence of restrictions $\mathcal{T}^{(n)}|_{[0,1]}$ of functions $\mathcal{T}^{(n)}$ to $[0,1]$ converges uniformly to $\mathcal{T}^{(\infty)}|_{[0,1]}$, which can be analytically extended to a disc containing $[0,1]$. This limit defines an analytic circle map $T^{(\infty)}$ with a break. Due to estimate (3.4), we obtain

$$\left|\left((T^{(\infty)}_k)^{q_k}\right)'_+(x_{br}) - \left((T^{(k)}_k)^{q_k}\right)'_+(x_{br})\right| < \sum_{n=k}^{\infty}\frac{\gamma_k}{2^{n+1}} = \frac{\gamma_k}{2^k}. \tag{3.5}$$

Together with (3.3), this implies

$$\left| \left((T_k^{(\infty)})^{q_k} \right)'_+ (x_{br}) - \left((\tilde{T}_{\tilde{a}_{p_k/q_k}})^{q_k} \right)'_+ (\tilde{x}_{br}) \right| > \frac{\gamma_k}{2}, \tag{3.6}$$

for all $k \in \mathbb{N}$. **QED**

3.2 Distribution of iterates of the renormalized maps

Let $T \in \mathcal{B}_c^r$, for $r \geq 2$, and let $x_0 \in \mathbb{T}^1$. To prove Theorem 1.1 we will need an estimate of the distribution of iterates of the renormalized maps. The following proposition is an immediate consequence of the Denjoy lemma [4]. It is also valid in the diffeomorphism case.

Proposition 3.2 *For any $T \in \mathcal{B}_c^r$, with $r \geq 2$, we have*

$$|\Delta_{q_{n-1}}^{(n)}| = \mathcal{O}(|\Delta_{q_{n+1}-q_n}^{(n)}|) = \mathcal{O}(|\Delta_0^{(n)}|), \tag{3.7}$$

for every $n \in \mathbb{N}$.

Proof. The fact that $|\Delta_{q_{n-1}}^{(n)}| = \mathcal{O}(|\Delta_0^{(n)}|)$ follows from the fact that the former interval is the image of the latter under $T^{q_{n-1}}$. We further have $|\Delta_{q_{n+1}-q_n}^{(n)}| = \mathcal{O}(|\Delta_{q_{n+1}}^{(n)}|)$ since the former interval is the preimage of the latter under T^{q_n}. For the same reason, $|\Delta_0^{(n+1)}| = \mathcal{O}(|\Delta_{q_n}^{(n+1)}|)$. Taking into account that $|\Delta_{q_{n+1}}^{(n)}| = |\Delta_0^{(n+1)}| + |\Delta_0^{(n)}| - |\Delta_{q_n}^{(n+1)}|$, we have $|\Delta_{q_{n+1}}^{(n)}| = \mathcal{O}(|\Delta_0^{(n)}|)$. Here, we have also used that $\Delta_{q_n}^{(n+1)} \subset \Delta_0^{(n)}$. The second equality now follows directly. **QED**

In the following propositions, f_n is the n-th renormalization of $T \in \mathcal{B}_c^r$, defined by the marked point $x_0 = x_{br}$.

Proposition 3.3 $(f_n)'_-(0)/(f_n)'_+(-1) = c_n^2 + o(1)$*, when $k_{n+1} \to \infty$, where $c_n = c$ for n even and $c_n = c^{-1}$ for n odd.*

Proof. Since $(f_n)'_+(-1) = (T^{q_n})'_+(x_{q_{n-1}})$ and $(f_n)'_-(0) = (T^{q_n})'_-(x_0)$, in the limit $k_{n+1} \to \infty$, x_0 and $x_{q_{n-1}}$ belong to the same periodic orbit of T, and we have

$$\frac{(f_n)'_-(0)}{(f_n)'_+(-1)} = \frac{(T^{q_n})'_-(x_0)}{(T^{q_n})'_+(x_{q_{n-1}})} \to \frac{(T^{q_n})'_-(x_0)}{(T^{q_n})'_+(x_0)} = c_n^2 \tag{3.8}$$

Since the orientation for n even is the same as the original one, we have $c_n^2 = c^2$. In the case of odd n the orientation changes, which implies $c_n^2 = 1/c^2$. **QED**

Proposition 3.4 *Let $0 < \epsilon < 1/2$ and let n_1 and n_2 be the numbers of elements of the set $\{f_n^j(-1) : j = 1, \ldots, k_{n+1}\}$ that belong to the intervals $I_1 = [-1, -1+\epsilon]$ and $I_2 = [-\epsilon, 0]$, respectively. If $b_1 = (f_n)'_+(-1)$ and $b_2 = (f_n)'_-(0)$, then, for sufficiently large even n, if $0 < c < 1$, and odd n if $c > 1$, we have, for large k_{n+1},*

$$\begin{aligned} n_1 &= \sigma k_{n+1} + \mathcal{O}(\ln k_{n+1}), \\ n_2 &= (1-\sigma) k_{n+1} + \mathcal{O}(\ln k_{n+1}), \end{aligned} \tag{3.9}$$

where $\sigma = \frac{\ln b_2}{\ln b_1^{-1} + \ln b_2}$. Also, for any $\varkappa > 0$ and sufficiently large k_{n+1} (depending on $\varkappa$),

$$\mathcal{O}(b_1^{-(\sigma+\varkappa)k_{n+1}}) \leq |f_n(-1) + 1| \leq \mathcal{O}(b_1^{-(\sigma-\varkappa)k_{n+1}}). \tag{3.10}$$

Proof. Let us consider two subintervals of $[-1, 0]$: $I_1(k_{n+1}) = [-1, -1 + 1/k_{n+1}]$ and $I_2(k_{n+1}) = [f_n^{k_{n+1}}(-1) - 1/k_{n+1}, f_n^{k_{n+1}}(-1)]$. Let the number of points in $\{f_n^j(-1) : j = 1, \ldots, k_{n+1}\}$, that belong to these two intervals be denoted by m_1 and m_2, respectively. Then, $m_1 + m_2 = k_{n+1} + \mathcal{O}(\ln k_{n+1})$, since the number of points outside of the union of these two intervals is of the order of $\ln k_{n+1}$. If $b_1 = (f_n)'_+(-1)$ and $b_2 = (f_n)'_-(0)$, and $M = \max_{x \in [-1,0]} |f''_n(x)|$, then

$$\begin{aligned} \mathcal{O}(b_1^{-m_1}) \leq |f_n(-1) + 1| &\leq \mathcal{O}\left(b_1^{-m_1} \left(1 - \frac{M}{b_1 k_{n+1}} \right)^{-m_1} \right), \\ \mathcal{O}(b_2^{m_2}) \leq |f_n^{k_{n+1}}(-1) - f_n^{k_{n+1}-1}(-1)| &\leq \mathcal{O}\left(b_2^{m_2} \left(1 + \frac{2M}{b_2 k_{n+1}} \right)^{m_2} \right), \end{aligned} \tag{3.11}$$

where the last inequality is obtained under the assumption $|f_n^{k_{n+1}}(-1)| < 1/k_{n+1}$. Here, we have also used the fact that for sufficiently large even n, if $0 < c < 1$, and odd n, if $c > 1$, the renormalizations are concave downwards. It follows from Proposition 3.2 that $|f_n(-1)+1| = \mathcal{O}(|f_n^{k_{n+1}}(-1) - f_n^{k_{n+1}-1}(-1)|)$. Since both $m_1, m_2 < k_{n+1}$, this implies that $b_1^{-m_1} = \mathcal{O}(b_2^{m_2})$. Therefore,

$$\begin{aligned} m_1 = &\frac{\ln b_2}{\ln b_1^{-1} + \ln b_2} k_{n+1} + \mathcal{O}(\ln k_{n+1}) \\ m_2 = &\frac{\ln b_1^{-1}}{\ln b_1^{-1} + \ln b_2} k_{n+1} + \mathcal{O}(\ln k_{n+1}) \end{aligned} \tag{3.12}$$

The first inequality in (3.11) also show that $|f_n^{k_{n+1}}(-1)| < \mathcal{O}(b_1^{-m_1}) < 1/k_{n+1}$, for $m_1 > C_1 k_{n+1}$, $C_1 > 0$, and sufficiently large k_{n+1}. The claim now follows from the fact that the number of points of $\{f_n^j(-1) : j = 1, \ldots, k_{n+1}\}$, in the intervals $I_1 \backslash I_1(k_{n+1})$ and $I_2 \backslash I_2(k_{n+1})$ is of the order of $\ln k_{n+1}$.

The estimate (3.10) follows from the first inequalities in (3.11) and (3.12). **QED**

3.3 The proof of Theorem 1.1

Let $\omega : [0,\infty) \to [0,\infty)$ be a modulus of continuity. We would like to construct two analytic circle maps with the same irrational rotation number and the same size of the break for which no conjugacy admits ω as the modulus of continuity. We begin by considering the conjugacy that maps the break point of one of the maps into the break point of the other.

Lemma 3.5 *Let s_m be any sequence of positive numbers diverging to infinity. Then, there exists a sequence of natural numbers ℓ_m diverging to infinity, an $N \in \mathbb{N}$, and two analytic circle maps T_ρ and $\tilde{T}_\rho$ in $\mathcal{B}_c^\omega$, with the same irrational rotation number*

$$\rho = [\bar{1}(N), \ell_1, 1, \ell_2, 1, \ell_3 \dots],$$

and with a break of size $c \neq 1$, located at $x_{br} = x_0$ and $\tilde{x}_{br} = \tilde{x}_0$, respectively, such that the following holds. For all $m \geq 0$, there exists $j \in \mathbb{N}$, $1 \leq j \leq \ell_{m+1}$, such that for $n = N + 2m$,

$$\max\left\{ \frac{|\Delta^{(n)}_{q_{n-1}+jq_n}|}{\omega(|\tilde{\Delta}^{(n)}_{q_{n-1}+jq_n}|)}, \frac{|\tilde{\Delta}^{(n)}_{q_{n-1}+jq_n}|}{\omega(|\Delta^{(n)}_{q_{n-1}+jq_n}|)} \right\} > s_m. \tag{3.13}$$

Here, $\bar{1}(N)$ stands for a N digit string $1, \dots, 1$. If $0 < c < 1$, then N is even. If $c > 1$, then N is odd.

Proof. Let T and $\widetilde{T}$ be two maps whose existence is guaranteed in Lemma 3.1. Consider the families of maps T_a and $\tilde{T}_{\tilde{a}}$. It is well known [15] that one can choose N large enough such that for all $m \in \mathbb{N} \cup \{0\}$, the graphs of the n-th renormalizations $f_n^{(m)}$ and $\tilde{f}_n^{(m)}$ (defined with marked points x_0 and $\tilde{x}_0$ being the corresponding break points x_{br} and $\tilde{x}_{br}$), $n = N + 2m$, of the maps T_m and $\tilde{T}_m$, in these families, with parameter values corresponding to rational rotation numbers $\rho_{N,m} = [\bar{1}(N), \ell_1, 1, \ell_2, 1, \dots, \ell_m, 1]$, and the break point belonging to the periodic orbit, are concave downwards. It follows from Lemma 3.1 that $|(f_n^{(m)})'_+(-1) - (\tilde{f}_n^{(m)})'_+(-1)| = \gamma(n) > 0$. Here, we have also used the fact that $(f_n^{(m)})'_+(-1) = (T_m^{q_n})'_+(x_{br})$ and $(\tilde{f}_n^{(m)})'_+(-1) = (\tilde{T}_m^{q_n})'_+(\tilde{x}_{br})$.

Now, let T_ρ and $\widetilde{T}_\rho$ be the corresponding maps in the families T_a and $\tilde{T}_{\tilde{a}}$, with the irrational rotation number $\rho = [\bar{1}(N), \ell_1, 1, \ell_2, 1, \dots, \ell_m, 1, \dots]$. For any given m, and sufficiently large ℓ_{m+1}, the $n = N + 2m$-th renormalizations f_n and $\tilde{f}_n$ of T_ρ and $\widetilde{T}_\rho$ are also concave downwards and satisfy the estimate $|b_1 - \tilde{b}_1| > \gamma(n)/2 > 0$, where $b_1 = (f_n)'_+(-1)$ and $\tilde{b}_1 = (\tilde{f}_n)'_+(-1)$. Note that the last estimate holds uniformly in the future ℓ_j, $j > m+1$, provided that ℓ_{m+1} is large enough.

To be specific, assume, without loss of generality, that $b_1 - \tilde{b}_1 > \gamma(n)/2 > 0$. Let $\epsilon(n) > 0$ be given and let the corresponding numbers of points from Proposition 3.4 for f_n and $\tilde{f}_n$ in the interval $[-1, -1 + \epsilon(n)]$ be denoted by n_1 and $\tilde{n}_1$, respectively.

From Proposition 3.4 and Proposition 3.3, we obtain

$$\tilde{n}_1 - n_1 = \left(\frac{\ln \tilde{b}_2}{\ln(c_n^2 + o(1))} - \frac{\ln b_2}{\ln(c_n^2 + o(1))} \right) \ell_{m+1} + \mathcal{O}(\ln \ell_{m+1}), \tag{3.14}$$

and therefore

$$\tilde{n}_1 - n_1 = \frac{\ln(\tilde{b}_2/b_2)}{\ln c_n^2} \ell_{m+1} + o(\ell_{m+1}) = \frac{\ln(\tilde{b}_1/b_1)}{\ln c_n^2} \ell_{m+1} + o(\ell_{m+1}). \tag{3.15}$$

We further obtain

$$\tilde{n}_1 - n_1 > \frac{\ln\left(1 + \frac{\gamma(n)}{2\tilde{b}_1}\right)}{|\ln c^2|} \ell_{m+1} + o(\ell_{m+1}) > \frac{\gamma(n)}{4\tilde{b}_1 |\ln c^2|} \ell_{m+1} + o(\ell_{m+1}), \tag{3.16}$$

for $\gamma(n) < 2\tilde{b}_1$. This inequality gives us that, for sufficiently small $\epsilon(n) > 0$, and sufficiently large ℓ_{m+1}, we have $n_1 < \tilde{n}_1$, and that the difference $\tilde{n}_1 - n_1$ is of the order of ℓ_{m+1}.

Recall now that

$$|\Delta^{(n)}_{q_{n-1}+jq_n}| = |f_n^j(-1) - f_n^{j-1}(-1)||\Delta_0^{(n-1)}|. \tag{3.17}$$

For sufficiently small $\epsilon(n) > 0$, there exists $b > 1$ such that $\tilde{f}_n'(x) > b$, for $x \in (-1, -1 + \epsilon(n)]$. Therefore, using the monotonicity of ω, we have

$$\frac{|\Delta^{(n)}_{q_{n-1}+n_1 q_n}|}{\omega(|\widetilde{\Delta}^{(n)}_{q_{n-1}+n_1 q_n}|)} \geq \frac{|f_n^{n_1}(-1) - f_n^{n_1-1}(-1)||\Delta_0^{(n-1)}|}{\omega(|\tilde{f}_n^{\tilde{n}_1}(-1) - \tilde{f}_n^{\tilde{n}_1-1}(-1)||\widetilde{\Delta}_0^{(n-1)}|b^{-(\tilde{n}_1 - n_1)})}. \tag{3.18}$$

Now, by the definition of n_1 and $\tilde{n}_1$, and properties of geometric progressions, the lengths $|f_n^{n_1}(-1) - f_n^{n_1-1}(-1)|$ and $|\tilde{f}_n^{\tilde{n}_1}(-1) - \tilde{f}_n^{\tilde{n}_1-1}(-1)|$ are of the order of $\epsilon(n)$.

The estimates above, together with the fact that for fixed N, m and ℓ_i, $i = 1, \ldots, m$, $|\Delta_0^{(n-1)}|$ and $|\tilde{\Delta}_0^{(n-1)}|$ can be bounded uniformly in ℓ_{m+1}, imply that for every $s_m > 0$ and for sufficiently large ℓ_{m+1}, we have

$$\frac{|\Delta^{(n)}_{q_{n-1}+n_1 q_n}|}{\omega(|\widetilde{\Delta}^{(n)}_{q_{n-1}+n_1 q_n}|)} \geq s_m. \tag{3.19}$$

Here, we have also used that $\omega(|x|) \to 0$, as $|x| \to 0$. The claim follows, since the sequence ℓ_m can be constructed inductively in m. **QED**

Remark 5 Lemma 3.5 shows that for the constructed maps T_ρ and $\tilde{T}_\rho$ in $\mathcal{B}_c^\omega$, the conjugacy that maps the break point of one of the maps into the break point of the other does not admit ω as the modulus of continuity. In particular, this implies that for these two maps there is no conjugacy which is C^1-smooth.

To prove Theorem 1.1, we also need to consider conjugacies that map the break point of one of the maps into an arbitrary point of the circle. In the following, we will consider renormalizations and renormalization segments in the situations when the marked point x_0 of a map T can be different from the break point of the considered map. We emphasize this by explicitly including x_0 in the notation.

Proof of Theorem 1.1. As in the proof of Lemma 3.5, we start with two maps T and $\widetilde{T}$ whose existence is guaranteed by Lemma 3.1, and consider the corresponding families of maps T_a and $\tilde{T}_{\tilde{a}}$. We will also use the notation from the proof of Lemma 3.5. As stated in the proof of Lemma 3.5, one can choose N large enough such that for all $m \in \mathbb{N} \cup \{0\}$, the graphs of the n-th renormalizations $f_n^{(m)}$ and $\tilde{f}_n^{(m)}$ are concave downwards. Moreover, if N is large enough, then for any point $x_0 \in \mathbb{T}^1$ and all $m \in \mathbb{N} \cup \{0\}$, there exists a point $z_n^{(m)}$ such that the graph of the n-th renormalization $f_n^{(m)}(x_0)$ of T_m, defined with the marked point x_0, is concave downwards in $[-1, z_n^{(m)}]$ and $[z_n^{(m)}, 0]$. If x_0 is a point on the orbit of x_{br} under T_m, then $z_n^{(m)} = -1$; otherwise, $z_n^{(m)}$ is a point in the interior of the interval $[-1, 0]$. In fact, $z_n^{(m)}$ is just the renormalization of a point of the trajectory T_m^i, $0 \le i < q_n$, which belongs to the corresponding interval. Since $(f_n^{(m)}(x_0))'_+(z_n^{(m)}) = (T_m^{q_n})'_+(x_{br})$ and $(\tilde{f}_n^{(m)})'_+(-1) = (\tilde{T}_m^{q_n})'_+(\tilde{x}_{br})$, Lemma 3.1 implies, $|(f_n^{(m)}(x_0))'_+(z_n^{(m)}) - (\tilde{f}_n^{(m)})'_+(-1)| = \gamma(n) > 0$.

We choose now the maps T_ρ and $\tilde{T}_\rho$ in the families T_a and $\tilde{T}_{\tilde{a}}$, with the irrational rotation number $\rho = [\bar{1}(N), \ell_1, 1, \ell_2, 1, \ldots, \ell_m, 1, \ldots]$. For any fixed m, and sufficiently large ℓ_{m+1}, the $n = N + 2m$-th renormalizations f_n and $\tilde{f}_n$ of T_ρ and $\tilde{T}_\rho$ are also concave downwards. Moreover, for any $x_0 \in \mathbb{T}^1$, there exists $z_n \in [-1, 0)$ such that the graph of the n-th renormalization $f_n(x_0)$ of T_ρ, defined with the marked point x_0, is concave downwards in the intervals $[-1, z_n]$ and $[z_n, 0]$. Here, z_n is the unique point in $(-1, 0)$ where the derivative of $f_n(x_0)$ has a break, if such a point exists; otherwise, $z_n = -1$.

For a given m and $n = N + 2m$, we choose a small $\epsilon(n) > 0$ and consider three cases: (i) $-\epsilon(n) < z_n^{(m)} \le 0$, (ii) $-1 \le z_n^{(m)} < -1 + \epsilon(n)$, and (iii) $-1 + \epsilon(n) \le z_n^{(m)} \le -\epsilon(n)$.

In case (i), we first assume $(f_n^{(m)}(x_0))'_+(z_n^{(m)}) - (\tilde{f}_n^{(m)})'_+(-1) = \gamma(n)$. Therefore, we have $(f_n^{(m)}(x_0))'_+(-1) - (\tilde{f}_n^{(m)})'_+(-1) > 3\gamma(n)/4$, if $\epsilon(n)$ is small enough. Furthermore, if ℓ_{m+1} is sufficiently large, we have the estimate $b_1 - \tilde{b}_1 > \gamma(n)/2$, where $b_1 = (f_n(x_0))'_+(-1)$ and $\tilde{b}_1 = (\tilde{f}_n)'_+(-1)$, uniformly in ℓ_j, for $j > m+1$. Moreover, if $\epsilon(n)$ is sufficiently small, then there exists $b > 1$ such that $\tilde{f}_n'(x) > b$ for $x \in [-1, -1 + \epsilon(n)]$. This estimate is also uniform in ℓ_j for $j > m + 1$, if ℓ_{m+1} has been chosen sufficiently large. The number of points n_1 and $\tilde{n}_1$ of $\{f_n(x_0)^j(-1) : j = 1, \ldots, \ell_{m+1}\}$ and $\{(\tilde{f}_n)^j(-1) : j = 1, \ldots, \ell_{m+1}\}$ in the interval $[-1, -1+\epsilon(n)]$ can now be estimated using Propositions 3.2–3.4. Notice that $\tilde{n}_1$ is the same as in the proof of Lemma 3.5, while n_1 is now smaller or equal to that of the proof of Lemma 3.5, which will be here denoted by n_1^0. Since we still have the same lower bound on $\tilde{n}_1 - n_1$, we can apply the same arguments as in Lemma 3.5, to show that for any given $s_m > 0$ and sufficiently large ℓ_{m+1},

$$\frac{|\Delta^{(n)}_{q_{n-1}+n_1q_n}(x_0)|}{|\widetilde{\Delta}^{(n)}_{q_{n-1}+n_1q_n}|} \geq s_m, \tag{3.20}$$

uniformly in ℓ_j, with $j > m+1$.

Consider now the case $(\tilde{f}_n^{(m)})'_+(-1) - (f_n^{(m)}(x_0))'_+(z_n^{(m)}) = \gamma(n)$. If $n_1 \leq \tilde{n}_1$, then, for any $s_m > 0$ and for sufficiently small $\epsilon(n)$, there exists $\tilde{b} \in (\tilde{b}_1 - \gamma(n)/4, \tilde{b}_1)$, such that

$$\frac{|\Delta^{(n)}_{q_{n-1}}(x_0)|}{|\widetilde{\Delta}^{(n)}_{q_{n-1}}|} \geq \frac{b_1^{-n_1}|f_n(x_0)^{n_1}(-1) - f_n(x_0)^{n_1-1}(-1)||\Delta_0^{(n-1)}(x_0)|}{\tilde{b}^{-\tilde{n}_1}|f_n(x_0)^{\tilde{n}_1}(-1) - f_n(x_0)^{\tilde{n}_1-1}(-1)||\widetilde{\Delta}_0^{(n-1)}|} \geq s_m, \tag{3.21}$$

for sufficiently large ℓ_{m+1}. Here, we have also used the fact that all of the quantities involved, other than n_1 and $\tilde{n}_1$, are bounded uniformly in ℓ_{m+1}. If, on the other hand, $n_1 > \tilde{n}_1$, then

$$\frac{|\Delta^{(n)}_{q_{n-1}}(x_0)|}{|\widetilde{\Delta}^{(n)}_{q_{n-1}}|} \geq \frac{|\Delta^{(n)}_{q_{n-1}+\tilde{n}_1q_n}(x_0)|b_1^{-\tilde{n}_1}}{|\widetilde{\Delta}^{(n)}_{q_{n-1}+\tilde{n}_1q_n}|\tilde{b}^{-\tilde{n}_1}}, \tag{3.22}$$

and, therefore, if

$$\frac{|\widetilde{\Delta}^{(n)}_{q_{n-1}+\tilde{n}_1q_n}|}{|\Delta^{(n)}_{q_{n-1}+\tilde{n}_1q_n}(x_0)|} < s_m, \tag{3.23}$$

then the right hand side of (3.22) is greater than or equal to s_m, provided that ℓ_{m+1} is chosen sufficiently large.

Similar arguments can be applied in case (ii). The only difference is that now one has to iterate backwards $f_n(x_0)$ and $\tilde{f}_n$ starting from $[f_n(x_0)^{-1}(0), 0]$ and $[\tilde{f}_n^{-1}(0), 0]$.

Finally, in case (iii), we notice that there exists $\delta(n) > 0$ such that $|f_n^{(m)}(x_0)(-1) - (-1)| > \delta(n)$. Furthermore, if ℓ_{m+1} is sufficiently large, then $|f_n(x_0)(-1) - (-1)| > \delta(n)/2$, uniformly in ℓ_j, for $j > m+1$. Since, by Proposition 3.4,

$$|\tilde{\Delta}^{(n)}_{q_{n-1}}| \leq \mathcal{O}(\tilde{b}_1^{-(\sigma-\varkappa)\ell_{m+1}})|\tilde{\Delta}_0^{(n-1)}|, \tag{3.24}$$

we immediately obtain for any $s_m > 0$, and ℓ_{m+1} sufficiently large,

$$\frac{|\Delta^{(n)}_{q_{n-1}}(x_0)|}{|\tilde{\Delta}^{(n)}_{q_{n-1}}|} \geq \frac{|f_n(x_0)(-1)+1||\Delta_0^{(n-1)}(x_0)|}{\mathcal{O}(\tilde{b}_1^{-(\sigma-\varkappa)\ell_{n+1}}|\tilde{\Delta}_0^{(n-1)}|)} \geq \frac{\delta(n)\min_{x_0\in\mathbb{T}^1}|\Delta_0^{(n-1)}(x_0)|}{2\mathcal{O}(\tilde{b}_1^{-(\sigma-\varkappa)\ell_{n+1}}|\tilde{\Delta}_0^{(n-1)}|)} \geq s_m. \tag{3.25}$$

Now, we can choose ℓ_{m+1} large enough such that all of the above conditions are satisfied. This inductive procedure for ℓ_{m+1} provides the construction of the rotation number ρ. It is easy to see that for the two constructed maps T_ρ and $\tilde{T}_\rho$, no topological conjugacy between them is Lipshitz continuous. **QED**

3.4 A non-rigidity result for smooth diffeomorphisms

In this section, we construct examples of smooth (i.e. analytic) circle diffeomorphisms with irrational rotation numbers for which the conjugacy to the rigid rotation can be as "bad" as possible. Theorem 3.6 below is well-understood by the experts. We give a simple proof here for completeness of the presentation. Another reason for its inclusion is that we were not able to find any reference for such a result. We focus on the modulus of continuity of the conjugacy and do not discuss the singularity of the invariant measure.

Consider a circle diffeomorphism T, and the corresponding family $T_a = T + a$. As before, denote $[a^{(1)}_{p/q}, a^{(2)}_{p/q}]$ the mode-locking interval associated to an arbitrary rational rotation number $0 \leq p/q < 1$. Let us call a diffeomorphism T "regular" if for all p/q, the maps $T^q_{a^{(1)}_{p/q}}$ and $T^q_{a^{(2)}_{p/q}}$ are not the identity maps. In other words, we require that not all points of the circle are periodic points for $T_{a^{(i)}_{p/q}}$, $i = 1, 2$.

Theorem 3.6 *Let T be a "regular" circle diffeomorphism. Then, for any modulus of continuity ω, there exists an irrational rotation number ρ such that the map T_{a_ρ} has no conjugacy with the rigid rotation $R_\rho : x \mapsto x + \rho$ which admits ω as the modulus of continuity.*

Proof. Let s_n, $n \in \mathbb{N}$, be any positive sequence diverging to infinity. As in the previous section, we construct the sequence of partial quotients k_n inductively in $n \in \mathbb{N}$. For a given n, consider the rational rotation number $p_n/q_n = [k_1, \dots, k_n]$ and the corresponding map $T_n = T_{a^{(i)}_{p_n/q_n}}$, where $i = 1$ if n is odd and $i = 2$ if n is even. Let $x_n \in \mathbb{T}^1$ be any point on the circle which does not belong to a periodic orbit of T_n. Then, there exists $\delta(n) > 0$, such that the length of the interval $[x_n, T_n^{q_n} x_n]$ is bounded below by $\delta(n)$. Therefore, if k_{n+1} is chosen large enough, then the interval $\Delta_n = [x_n, T_\rho^{q_n} x_n]$ satisfies bound $|\Delta_n| \geq \delta(n)/2 > 0$, uniformly in k_j for $j > n+1$. Here, $T_\rho = T_{a_\rho}$, and ρ is an irrational number whose first n partial quotients agree with those of p_n/q_n. If $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$ is any conjugacy between the rigid rotation R_ρ and T_ρ, then the length of the corresponding interval $\widetilde{\Delta}_n = \varphi^{-1}(\Delta_n)$, $|\widetilde{\Delta}_n| = |q_n \rho - p_n| \to 0$ as $k_{n+1} \to \infty$.

Therefore, if k_{n+1} is chosen large enough, then

$$\frac{|\Delta_n|}{\omega(|\widetilde{\Delta}_n|)} \geq \frac{\delta(n)}{2\omega(|q_n \rho - p_n|)} \geq s_n, \tag{3.26}$$

uniformly in k_j for $j > n+1$. The claim follows. **QED**

4 Examples of non-$C^{1+\alpha}$ rigidity

The proof of Theorem 1.2 can be obtained by extending the parabolic renormalization scheme of Avila from the case of critical circle maps considered in [2] to the case of circle maps with breaks. Since the proofs are almost the same, we will just describe the method and direct the reader for further details to [2].

We will consider the set A of irrational rotation numbers $\rho \in (0,1)$, with bounded odd-numbered entries k_{2n-1} in the continued fraction expansion of ρ, in the case $0 < c < 1$, or bounded even-numbered entries k_{2n}, in the case $c > 1$. As mentioned in the Introduction, these are the rotation numbers for which the distance to the diagonal at the end points for the concave renormalization graphs (see Figure 1b) is bounded. It follows from the analysis conducted in [12, 14], that C^1 rigidity holds in this case. However, as we show below within set A, C^1 rigidity cannot, in general, be extended to $C^{1+\alpha}$ class of conjugacies, for some $\alpha > 0$.

We starts with all maps $f : \mathbb{R} \to \mathbb{R}$, satisfying $f(x+1) = f(x)+1$, which are C^r smooth outside the integer points at which the derivative has breaks of size c, and with the unique fixed point $p \in (-1,0)$, such that $f'(p) = 1$, and $f''(p) > 0$. For $x \in (p, p+1)$, we have $f^n(x) \to p+1$ and $f^{-n}(x) \to p$, when $n \to \infty$. We then consider the family of translated maps $f_\epsilon = f + \epsilon$, $\epsilon \geq 0$. To define the parabolic renormalization, let us first define the maps $\Phi_{f,n,\epsilon,+}$ and $\Phi_{f,n,\epsilon,-}$ from $(p, p+1)$ into $\mathbb{R}$, by

$$\begin{aligned} \Phi_{f,n,\epsilon,+}(x) &= \frac{f''(p)n^2}{2}(f_\epsilon^n(x) - f_\epsilon^n(0)), \\ \Phi_{f,n,\epsilon,-}(x) &= \frac{f''(p)n^2}{2}(f_\epsilon^{-n}(x) - f_\epsilon^{-n}(0)). \end{aligned} \tag{4.1}$$

As $n \to \infty$, the sequences $\Phi_{f,n,0,+}$ and $\Phi_{f,n,0,-}$ converge C^1-uniformly on compact sets to C^1-homeomorphisms $\Phi_{f,+} : (p, p+1) \to \mathbb{R}$ and $\Phi_{f,-} : (p, p+1) \to \mathbb{R}$, with break points in $\{f^{-j}(0) : j = 0, 1, 2, \ldots\}$ and $\{f^j(0) : j = 1, 2, \ldots\}$, respectively. The sizes of the breaks of the derivatives of $\Phi_{f,+}$ and $\Phi_{f,-}$ at each of these points are c and c^{-1}, respectively. The homeomorphisms satisfy $\Phi_{f,+}(f(x)) - \Phi_{f,+}(x) = 1$ and $\Phi_{f,-}(f(x)) - \Phi_{f,-}(x) = -1$.

We define the mapping $R_0(f) = \Phi_{f,+} \circ \Phi_{f,-}^{-1} : \mathbb{R} \to \mathbb{R}$, called the *parabolic renormalization* of f. One can show that $R_0(f)$ is a lift of a C^1 circle homeomorphism with a fixed point at 0 and breaks of size c at $\{\Phi_{f,-}(f^j(0)) : j \in \mathbb{Z}\}$. The latter observation follows from

$$(R_0(f))'(x) = \Phi'_{f,+}(\Phi_{f,+}^{-1}(x))(\Phi_{f,-}^{-1})'(x) = \frac{\Phi'_{f,+}(\Phi_{f,+}^{-1}(x))}{\Phi'_{f,-}(\Phi_{f,-}^{-1}(x))}. \tag{4.2}$$

Let $\mathcal{H}$ be the set of all C^1-diffeomorphisms $h:\mathbb{R}\to\mathbb{R}$, with $h(x+1)=h(x)$, $h(0)=0$, endowed with the natural topology. Let $\mathcal{K}$ be a compact subset of $\mathcal{H}$.

Lemma 4.1 *Let $f_0, g_0 \in \mathcal{A}_c^\omega$, with rational rotation numbers $\frac{p}{q} \in \mathbb{Q}$, and a parabolic periodic orbit. There exist sequences of maps $f_n, g_n \in \mathcal{A}_c^\omega$, such that $f_n \to f_0$ and $g_n \to g_0$ as $n \to \infty$, and for each n, f_n and g_n have the same irrational rotation number and there is no $h \in \mathcal{K}$ such that $h \circ f_n = g_n \circ h$.*

The proof of this Lemma is similar to the proof of Theorem 2.1 of [2]. It is based on the fact that arbitrarily close to a map $f \in \mathcal{A}_c^\omega$, with a rational rotation number and a parabolic periodic orbit, one can find a map $g \in \mathcal{A}_c^\omega$, whose parabolic renormalization differs from that for f.

Recall now that the set of all $h \in \mathcal{H}$ which are $C^{1+\alpha}$-smooth for some $\alpha > 0$ can be written as the union of a nested sequence of compact sets $\mathcal{K}_n \subset \mathcal{K}_{n+1}$. Let us endow $\mathcal{A}_c^\omega$ with a complete metric d, compatible with natural topology.

Lemma 4.2 *Let $f, g \in \mathcal{A}_c^\omega$, with rotation number $\rho(f) = \rho(g) \in A$. For every $\epsilon > 0$ and $k > 0$, there exists $\hat{f}, \hat{g}$ such that $\rho(\hat{f}) = \rho(\hat{g}) \in A$, $d(f, \hat{f}), d(g, \hat{g}) < \epsilon$ and if $d(\tilde{f}, \hat{f}), d(\tilde{g}, \hat{g}) < \delta$ then $k!\rho(\tilde{f}) \notin \mathbb{Z}$ and there is no $h \in \mathcal{K}_k$ such that $h \circ \tilde{f} = \tilde{g} \circ h$.*

This proof follows easily from Lemma 4.1 and is similar to the proof of Lemma 3.1 in [2]. One first chooses two maps in $\mathcal{A}_c^\omega$ with the same rational rotation number and parabolic periodic orbit, close to f and g, respectively; then, one constructs two maps $\tilde{f}, \tilde{g} \in \mathcal{A}_c^\omega$ with the same irrational rotation number in A, close to f and g, respectively, such that there is no $h \in \mathcal{K}_k$ which conjugates $\tilde{f}$ and $\tilde{g}$.

Proof of Theorem 1.2. The proof of Theorem 1.2 follows from the proof of the main theorem of [2]. We first use Lemma 4.2 to construct inductively a sequence of maps $f_n, g_n \in \mathcal{A}_c^\omega$ with the same irrational rotation numbers in A, such that there is no $h \in \mathcal{K}_n$ such that $h \circ \tilde{f}_n = \tilde{g}_n \circ h$. The desired maps are constructed as the limits of these sequences, i.e. $f = \lim_{n\to\infty} f_n$ and $g = \lim_{n\to\infty} g_n$. Clearly, they have the same irrational rotation number in A and the conjugating homeomorphism $h_{f,g} \notin \mathcal{K}_n$, for $n \in \mathbb{N} \cup \{0\}$, and is therefore not $C^{1+\alpha}$, for any $\alpha > 0$. **QED**

Remark 6 As it has been explained above, C^1 rigidity holds for rotation numbers from the set A. This set has zero Lebesgue measure, but as we prove in a separate publication [11], C^1 rigidity can be extended to Lebesgue almost all rotation numbers.

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