A NOTE ON EXISTENCE THEOREM OF PEANO

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ABSTRACT. An ODE with non-Lipschitz right hand side is considered. The set of solutions with L^p-dependence of the initial data is obtained.

1. MAIN THEOREM

Equip the space $\mathbb{R}^m = \{x = (x^1, \dots, x^m)\}$ with a norm

$$\|x\| = \max_{k=1,\dots,m} |x^k|.$$

Let B_R stands for the open ball of $\mathbb{R}^m$ with radius R and the center at the origin. By I_T denote an interval $I_T = (-T, T)$.

Introduce a vector-function $f(t, x) = (f^1, \dots, f^m)(t, x) \in C(\mathbb{R}_t \times \mathbb{R}^m_x, \mathbb{R}^m)$. Suppose that

$$\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^m} \|f(t, x)\| = M < \infty.$$

Moreover we assume that for each $t \in \overline{I}_T$ and for all $x = (x^1, \dots, x^m) \in \mathbb{R}^m$ and $y = (y^1, \dots, y^m) \in \mathbb{R}^m$ such that

$$x^j \le y^j, \quad j = 1, \dots, m$$

one has

$$f^j(t, x) \le f^j(t, y), \quad j = 1, \dots, m. \tag{1.1}$$

Our aim is to study the set of the solutions to the following initial value problem:

$$\dot{y} = f(t, y), \quad y(0) = x.$$

From Peano's existence theorem [3] we know that for all x this IVP has a solution, $y(t) \in C^1(\mathbb{R})$. It is also well known that for the same initial condition there may be several solutions.

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The condition (1.1) does not prevent the effect of non-uniqueness. To see this it is sufficient to consider the IVP with $f(t, y) = \sqrt{y}$ provided $y \geq 0$ and $f(t, y) = 0$ otherwise and $y(0) = 0$.

We study a possibility whether any initial condition x can be put in correspondence with a solution $u(t, x)$ such that the function $u(t, x)$ possesses reasonable properties.

So we look for solutions to the following IVP.

$$u_t(t, x) = f(t, u(t, x)), \quad u(0, x) = x. \tag{1.2}$$

Theorem 1. *For any positive constants T, R and $p \in [1, \infty)$ problem (1.2) has a solution $w(t, x) \in C(\overline{I}_T, L^p(B_R)) \bigcap C^1(I_T, L^p(B_R))$.*

Let μ stands for the standard Lebesgue measure in B_R.

Theorem 2. *For any $\varepsilon > 0$ there is a closed set $M_\varepsilon \subset B_R$ such that $\mu(B_R \backslash M_\varepsilon) < \varepsilon$ and $w(t, x) \in C(M_\varepsilon, C(\overline{I}_T))$.*

Proof of Theorem 2. Arrange a countable set $Z = \overline{I}_T \bigcap \mathbb{Q}$ as follows: $Z = \{t_i\}_{i \in \mathbb{N}}$.

Then by Luzin's theorem [4] we choose closed sets

$$M_i \subseteq B_R, \quad \mu(B_R \backslash M_i) < \frac{\varepsilon}{2^i}$$

such that $w(t_i, x) \in C(M_i)$.

Let us put $M_\varepsilon = \bigcap_i M_i$ then

$$\mu(B_R \backslash M_\varepsilon) = \mu\Big(\bigcup_i B_R \backslash M_i\Big) \leq \sum_i \mu(B_R \backslash M_i) < \varepsilon.$$

Take a sequence $x_k \to x, \quad \{x_k\} \subseteq M_\varepsilon$. For all $t_i \in Z$ we have

$$\|w(t_i, x_k) - w(t_i, x)\| \to 0.$$

Observe that the sequence $\{w(t, x_k)\}$ is uniformly continuous in $\overline{I}_T$:

$$\|w(t', x_k) - w(t'', x_k)\| = \Big\| \int_{t''}^{t'} f(s, w(s, x_k))\, ds \Big\| \leq M|t' - t''|, \quad t', t'' \in \overline{I}_T.$$

Thus the sequence $\{w(t, x_k)\}$ converges uniformly in Z [4]. And so as the set Z is dense in $\overline{I}_T$ this sequence converges uniformly in $\overline{I}_T$.

The Theorem is proved.

2. Proof of Theorem 1

For convenience of the reader we recall several propositions which are used in the sequel.

The following proposition is a corollary from the Vitali convergence theorem [2].

Proposition 1. *Let $(X, \mathfrak{S}, \mu)$ be a measure space, $\mu(X) < \infty$. And a sequence of measurable functions $\{f_n\}$ is such that for all $n \in \mathbb{N}$ and for almost all $x \in X$ we have $|f_n(x)| \leq \text{const}$. Assume that $\{f_n\}$ is a Cauchy sequence in measure. Then it converges in measure to a measurable function f and $\int_X (f_n - f) d\mu \to 0$.*

Formulate another fact.

Proposition 2 ([2])**.** *Let $D \subset \mathbb{R}^m$ be a measurable set with respect to the standard Lebesgue measure. Consider a function $\psi \in C(\overline{B}_R, \mathbb{R}^k)$. If $f_n \to f$ in measure in D and $\|f_n(x)\| \leq R$ almost everywhere in D then then $\psi \circ f_n \to \psi \circ f$ in measure.*

As usual we formulate our IVP in terms of the integral equation

$$u(t,x) = F(u)(t,x), \quad F(u)(t,x) = x + \int_0^t f(s, u(s,x))\, ds. \tag{2.1}$$

Definition 1. *We shall say that the function $u(t,x)$ belongs to a set X if*

1. *$u(t,x) \in C(\overline{I}_T, L^p(B_R))$,*
2. *for every $t \in \overline{I}_T$ the inequality $\|u(t,x)\| \leq R + TM$ holds almost everywhere in B_R;*
3. *for every $t', t'' \in \overline{I}_T$ the estimate*

$$\|u(t',x) - u(t'',x)\| \leq M|t' - t''|$$

holds almost everywhere in B_R.

Lemma 1. *The mapping F takes the set X to itself.*

Proof. The proof of this Lemma is straightforward. It is only not trivial to show that $t \mapsto f(t, u(t,x))$ is a strongly measurable mapping of I_T to $L^p(B_R)$.

To prove this we construct a sequence of step functions that converges in $L^p(B_R)$ to $f(t, u(t,x))$ for almost all t.

Since the function f is continuous we can approximate it with a following sequence:

$$f_n(t,x) = \sum_{j=1-n}^{n} a_{jn}(x) \chi_{\left[\frac{j-1}{n}T, \frac{j}{n}T\right]}(t), \quad \sup_{t \in \overline{I}_T} \|f_n(t,\cdot) - f(t,\cdot)\|_{C(\overline{B}_{R+TM})} \to 0.$$

In this formula χ stands for the set indicator function, $a_{jn} \in C(\overline{B}_{R+TM})$.

Since the function $u(t,x) \in C(\overline{I}_T, L^p(B_R))$ there exists a sequence

$$u_k(t,x) = \sum_{i=1-k}^{k} u_{ik}(x) \chi_{\left[\frac{i-1}{k}T, \frac{i}{k}T\right]}(t)$$

such that

$$u_{ik}(x) \in L^p(B_R), \quad \sup_{t \in \overline{I}_T} \|u_k(t,\cdot) - u(t,\cdot)\|_{L^p(B_R)} \to 0.$$

This implies the convergence in measure. For every $t \in \overline{I}_T$ and for every $\varepsilon, \sigma > 0$ there is a number N such that

$$\mu\{x \in B_R \mid \|u_k(t,x) - u(t,x)\| > \sigma\} < \varepsilon, \quad k > N.$$

By Proposition 2, for every $t \in \overline{I}_T$ we have $a_{jn}(u_k(t,x)) \to a_{jn}(u(t,x))$ in measure as $k \to \infty$ and $a_{jn}(u(t,x))$ is measurable in x [2]. Thus by Proposition 1 it follows that for every $t \in \overline{I}_T$ we obtain

$$\|a_{jn}(u_k(t,x)) - a_{jn}(u(t,x))\|_{L^p(B_R)} \to 0$$

as $k \to \infty$.

By the same argument for every $t \in \overline{I}_T$ we get

$$\|f_n(t, u_k(t,x)) - f(t, u(t,x))\|_{L^p(B_R)} \to 0, \quad n, k \to \infty.$$

Lemma is proved.

Now let us endow the space X with a partial order $\preceq$. We shall say that $u(t,x) = (u^1, \dots, u^m)(t,x) \in X$ and $v(t,x) = (v^1, \dots, v^m)(t,x) \in X$ satisfy the relation $u \preceq v$ iff for every $t \in \overline{I}_T$ the inequality $u^k(t,x) \le v^k(t,x), \quad k = 1, \dots, m$ holds almost everywhere in B_R.

Lemma 2. *A set $E = \{u \in X \mid u \preceq F(u)\}$ possesses a maximal element:*

$$w = \max E.$$

Observe that by Lemma 1 the space E is non void: $-(R + TM, \dots, R + TM) \in E$.

Proof of Lemma 2. The assertion of the Lemma is surely based on the Zorn Lemma. So it is sufficient to prove that any chain $C \subseteq E$ has an upper bound.

The space $L^p(B_R)$ is separable and the interval $\overline{I}_T$ is compact. So the space $C(\overline{I}_T, L^p(B_R))$ is separable [1].

Since the set C belongs to $C(\overline{I}_T, L^p(B_R))$, it is also separable. This implies that there is a countable set $Q \subseteq C$ such that for any element $p \in C$ there exists a sequence $\{p_n\}_{n \in \mathbb{N}} \subseteq Q$ and $\max_{t \in \overline{I_T}} \|p_n(t, \cdot) - p(t, \cdot)\|_{L^p(B_R)} \to 0$ as $n \to \infty$.

Arrange the set Q as a sequence: $Q = \{g_j\}_{j \in \mathbb{N}}$ and consider a sequence $h_l = \max\{g_1, \dots, g_l\}, \quad \{h_l\} \subseteq Q$. Here max stands in regard to the relation $\preceq$.

We claim that for each $t \in \overline{I}_T$ this sequence converges almost everywhere to a function h and this function is the desired upper bound of C.

Since for all $t \in \overline{I}_T$ and for almost all $x \in B_R$ the inequalities

$$\|h_l(t,x)\| \le R + TM, \quad h_l^n(t,x) \le h_{l+1}^n(t,x), \quad n = 1, \dots, m$$

fulfill for all $l \in \mathbb{N}$, then for every $t \in \overline{I}_T$ the sequence h_l converges to a function h almost everywhere in $x \in B_R$. And for every $t, t', t'' \in \overline{I}_T$ and almost everywhere in B_R we also get

$$\|h(t,x)\| \le R + TM, \quad \|h(t',x) - h(t'',x)\| \le M|t' - t''|. \tag{2.2}$$

By the Dominated convergence theorem for every $t \in \overline{I}_T$ the function $h(t,x) \in L^\infty(B_R)$ and $h_l(t,\cdot) \to h(t,\cdot)$ in $L^p(B_R)$.

Since the functions $h_l(t,x)$ satisfy item (3) of Definition 1 we write

$$\|h_l(t',\cdot) - h_l(t'',\cdot)\|_{L^p(B_R)} \leq M(\mu(B_R))^{1/p}|t'-t''|.$$

Thus the sequence $\{h_l\} \subset C(\overline{I}_T, L^p(B_R))$ is uniformly continuous in t and it converges to h in $C(\overline{I}_T, L^p(B_R))$ [4]. Particularly we have $h \in C(\overline{I}_T, L^p(B_R))$ and from formulas (2.2) it follows that $h \in X$.

Owing to the continuity of the function f for every $t \in \overline{I}_T$ we obtain

$$f(t, h_l(t,x)) \to f(t, h(t,x))$$

almost everywhere in B_R.

By the Dominated convergence theorem we have

$$\|f(t, h_l(t,x)) \to f(t, h(t,x))\|_{L^p(B_R)} \to 0, \quad t \in \overline{I}_T.$$

Now we apply the Dominated convergence theorem again, but this time we use its Bochner integral version :

$$\Big\| \int_0^t f(s, h_l(s,x))\,ds - \int_0^t f(s, h(s,x))\,ds \Big\|_{L^p(B_R)} \to 0.$$

From this formula it follows that there exists a subsequence $\{h_{l_i}\}$ such that

$$\int_0^t f(s, h_{l_i}(s,x))\,ds \to \int_0^t f(s, h(s,x))\,ds$$

for almost all $x \in B_R$. This states that $h \in E$.

Obviously the function h is an upper bound for Q. Check that h is an upper bound for C.

Assume the converse: there exists an element $b \in C$ such that the relation $b \preceq h$ does not hold. This implies that for some $t' \in \overline{I}_T$ and for some index k a set

$$D' = \{x \in B_R \mid b^k(t',x) - h^k(t',x) > 0\}$$

has non zero measure: $\mu(D') > 0$.

Actually there exists a set $D \subseteq D'$, $\quad \mu(D) > 0$ such that for some constant $c > 0$ one has $b^k(t',x) - h^k(t',x) \geq c, \quad x \in D$. Indeed, if it is not true then we can take a sequence

$$\{c_l\}_{l\in\mathbb{N}}, \qquad c_l > 0, \quad c_l \to 0$$

and consider sets $D_l = \{x \in D' \mid b^k(t',x) - h^k(t',x) \geq c_l\}$. By the assumption for all l we have $\mu(D_l) = 0$ but on the other hand $D' = \bigcup_l D_l$ and $\mu(D') \leq \sum_l \mu(D_l) = 0$.

Take a sequence $\{b_j\}_{j\in\mathbb{N}} \subseteq Q$ such that $b_j \to b$ in $C(\overline{I}_T, L^p(B_R))$. We obtain

$$c + h^k(t',x) - b_j^k(t',x) \leq b^k(t',x) - b_j^k(t',x) \tag{2.3}$$

almost everywhere in D. It is obvious $h^k(t',x) - b^k_j(t',x) \geq 0$ almost everywhere in B_R and from formula (2.3) we get

$$b^k(t',x) - b^k_j(t',x) \geq c \tag{2.4}$$

almost everywhere in D.

The L^p-convergence implies the convergence in measure [2] thus for every $q, \sigma > 0$ there is an index J such that if $j > J$ then

$$\mu\Big(\{x \in B_R \mid |b^k(t',x) - b^k_j(t',x)| \geq q\}\Big) < \sigma.$$

Putting in this formula $q = c$ and $\sigma = \mu(D)/2$ we obtain a contradiction with inequality (2.4).

The Lemma is proved.

Now we are ready to prove the Theorem. By Lemma 1 and inequality (1.1) it follows that $F(E) \subseteq E$. Particularly $F(w) \in E$, where $w = \max E$ is a maximal element given by Lemma 2. Consequently the relation $w \preceq F(w)$ implies that $w = F(w)$.

Now the assertion of Theorem 2 directly follows from the formula

$$w(t,x) = x + \int_0^t f(s, w(s,x))\, ds$$

if only we check that

$$f(t, w(t,x)) \in C(\bar I_T, L^p(B_R)).$$

Take a sequence $t_k \to t$. Then $w(t_k,x) \to w(t,x)$ in $L^p(B_R)$ and in measure. By Propositions 2, 1

$$f(t_k, w(t_k,x)) \to f(t, w(t,x))$$

in $L^p(B_R)$.

Theorem 2 is proved.

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