Spectrum of Dirichlet Laplacian in a conical layer

Pavel Exner and Miloš Tater

Doppler Institute for Mathematical Physics and Applied Mathematics,
Břehová 7, 11519 Prague,
and Nuclear Physics Institute ASCR, 25068 Řež near Prague, Czechia

E-mail: `exner@ujf.cas.cz, tater@ujf.cas.cz`

Abstract. We study spectral properties of Dirichlet Laplacian on the conical layer of the opening angle $\pi - 2\theta$ and thickness equal to π. We demonstrate that below the continuum threshold which is equal to one there is an infinite sequence of isolated eigenvalues and analyze properties of these geometrically induced bound states. By numerical computation we find examples of the eigenfunctions.

1. Introduction

Geometrically induced bound states in infinitely stretched regions with a hard-wall (or Dirichlet) boundary attracted a lot attention in recent years, in particular, because they represent a purely quantum effect which cannot be approached from the usual semiclassical point of view. The attention was first paid to tubes where the effective attraction is caused by bending [EŠ89, GJ92, DE95], later the analogous question was addressed with one dimension added, in curved layers [DEK01, CEK04]. This question is more complicated and to the date we do not have a full analogue of the result saying that *any* local bending of an asymptotically straight tube exhibits such bound states. One reason is that geometry of surfaces is more complicated than that of curves and global properties play a more important role.

However, this is not the only difference between tubes and layers. In the case of the former we have various solvable examples. They often correspond to tubes with a non-smooth boundary being thus beyond the assumptions under which the general results are proved and illustrating the robustness of the effect; on the other hand they can be composed of simple building blocks which facilitates their solvability. In fact, some examples came already before the general theory [LLM86, EŠŠ89, SRW89] and others were found in parallel with it — see, e.g., [LCM99] for a review.

This is in contrast with our state of knowledge concerning the layers. We have reasonably strong and general existence results [DEK01, CEK04], later even extended to layers in higher dimensions [LL07], as well a particular "weak coupling" result [EK01], but to the date no solvable example is known‡. The goal of the present work is to discuss such an example in the geometrical situation which is particularly simple — apart of a trivial scaling the model depends on a single parameter.

The layer to consider is the region bordered by two conical surfaces shifted mutually along the cone axis by $\pi \csc \theta$. Using a natural separation of variables the angular component can be separated and the problem can be in each partial wave reduced to analysis of an appropriate operator on a skewed semi-infinite strip. This will be demonstrated in the next section and used in Sec. 3 to derive some properties of the corresponding discrete spectrum. To illustrate these results and to provide further insights into this binding mechanism, we present in the following section a numerical analysis of the problem: we calculate the eigenvalue dependence on the cone opening angle and present examples of eigenfunctions.

‡ By solvability we do not mean finding the eigenvalues and eigenfunctions analytically, of course, this is beyond reach even in the tubes case. What we have in mind is the possibility of bringing the analysis to the state when one can find these quantities numerically and study their dependence on the parameters of the model.
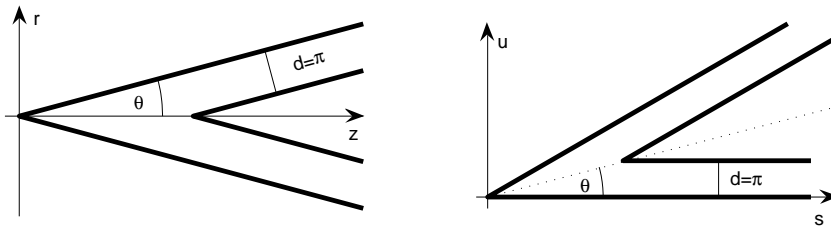
Figure 1. The problem geometry. The left part shows the setting in cylindrical coordinates as described in Secs. 2.1 and 2.2, the right part shows the cross section in rotated coordinates s, u used in Sec. 2.3.

2. The Hamiltonian

2.1. Basic definitions

As we have said the object of our study is the Hamiltonian of a quantum particle confined to a hard-wall conical layer of the opening angle $\pi - 2\theta$ for a fixed $\theta \in (0, \frac{1}{2}\pi)$, i.e.

$$\Sigma_\theta := \left\{ (r, \phi, z) \in \mathbb{R}^3 : (r, z) \in \Omega_\theta, \ \phi \in [0, 2\pi) \right\}, \tag{2.1}$$

where

$$\Omega_\theta := \left\{ (r, z) \in \mathbb{R}^2 : r \geq 0, \ 0 < z - r \sin \theta < \pi \csc \theta \right\}. \tag{2.2}$$

Without loss of generality we may assume that the layer width is π, the result for other widths being obtained by a trivial scaling. Similarly we neglect values of physical constants and identify the Hamiltonian H^θ with the Dirichlet Laplacian $-\Delta_{\mathrm{D}}^{\Sigma_\theta}$ on the Hilbert space $\mathcal{H} = L^2(\Sigma_\theta, r\mathrm{d}r\mathrm{d}\phi\mathrm{d}z)$. Using the cylindrical coordinates we can write the operator action explicitly,

$$H^\theta \psi = -\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} - \frac{\partial^2 \psi}{\partial z^2}, \tag{2.3}$$

with the domain consisting of all functions $\psi \in W^{1,2}_{\mathrm{loc}}(\Sigma_\theta)$ such that $\psi(\vec{x}) = 0$ for $\vec{x} = (r, \phi, z) \in \partial \Sigma_\theta$ and $H\psi \in L^2$. The operator is associated with the quadratic form

$$Q^\theta[\psi] = \int_{\Sigma_\theta} \left[\left| \frac{\partial \psi}{\partial r} \right|^2 + \frac{1}{r^2} \left| \frac{\partial \psi}{\partial \phi} \right|^2 + \left| \frac{\partial \psi}{\partial z} \right|^2 \right] (r, \phi, z) \, r\mathrm{d}r\mathrm{d}\phi\mathrm{d}z \tag{2.4}$$

defined on functions $\psi \in W^{1,1}_{\mathrm{loc}}(\Sigma_\theta)$ vanishing at the layer boundary and such that $Q^\theta[\psi] < \infty$. It is well known [RS78, Sec. XIII.15] that the operator H is self-adjoint and positive, $\sigma(H) \subset [0, \infty)$.

2.2. Partial wave decomposition

Using the cylindrical symmetry of the system we write state Hilbert space as

$$\mathcal{H} = L^2(\Omega_\theta, r\mathrm{d}r\mathrm{d}z) \otimes L^2(S_1) = \bigoplus_{m \in \mathbb{Z}} L^2(\Omega_\theta, r\mathrm{d}r\mathrm{d}z), \tag{2.5}$$

where $L^2(S_1)$ refers conventionally to functions on the unit circle with the orthonormal basis $\{\frac{1}{2\pi}\mathrm{e}^{im\phi} : m \in \mathbb{Z}\}$. The operator H^θ decomposes accordingly as

$$H^\theta = \bigoplus_{m\in\mathbb{Z}} H_m^\theta , \quad H_m^\theta \psi = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) - \frac{\partial^2\psi}{\partial z^2} + \frac{m^2}{r^2}\psi , \tag{2.6}$$

with the domain consisting of $\psi \in W_{\mathrm{loc}}^{1,2}(\Omega_\theta)$ satisfying the requirement $\psi(r, r\sin\theta) = \psi(r, r\sin\theta + \pi\tan\theta) = 0$ and such that $H_m\psi \in L^2$. Under these conditions the operators H_m^θ, $m \neq 0$, are self-adjoint while for $m = 0$ one has to demand additionally that $\frac{\partial\psi}{\partial r}\big|_{r=0} = 0$. All the boundary values here are understood as the appropriate limits. The quadratic forms associated with the partial wave components of the Hamiltonian are

$$Q_m^\theta[\psi] = \int_{\Sigma_\theta}\left[\left|\frac{\partial\psi}{\partial r}\right|^2 + \left|\frac{\partial\psi}{\partial z}\right|^2 + \frac{m^2}{r^2}|\psi|^2\right](r,z)\,r\mathrm{d}r\mathrm{d}z \tag{2.7}$$

being defined on all $\psi \in W_{\mathrm{loc}}^{1,1}(\Omega_\theta)$ satisfying $\psi(r, r\sin\theta) = \psi(r, r\sin\theta + \pi\csc\theta) = 0$ and such that $Q_m^\theta[\psi] < \infty$.

As usual we can pass to unitarily equivalent operators by means of the transformation $U : L^2(\Omega_\theta, r\mathrm{d}r\mathrm{d}z) \to L^2(\Omega_\theta)$ defined by $(U\psi)(r,z) := r^{1/2}\psi(r,z) \equiv \tilde{\psi}(r,z)$ and an analogous map $L^2(\Sigma_\theta, r\mathrm{d}r\mathrm{d}\phi\mathrm{d}z) \to L^2(\Sigma_\theta)$. In particular, the operator $\tilde{H}_m^\theta := UH_m^\theta U^{-1}$ acts as

$$\tilde{H}_m^\theta\tilde{\psi} = -\frac{\partial^2\tilde{\psi}}{\partial r^2} - \frac{\partial^2\tilde{\psi}}{\partial z^2} + \frac{1}{r^2}\left(m^2 - \frac{1}{4}\right)\tilde{\psi} \tag{2.8}$$

on functions $\tilde{\psi} \in W_{\mathrm{loc}}^{1,2}(\Omega_\theta)$ such that $\tilde{H}_m^\theta\tilde{\psi} \in L^2$ satisfying the Dirichlet boundary condition at the boundary of Ω_θ; the associated quadratic form is

$$\tilde{Q}_m^\theta[\tilde{\psi}] = \int_{\Sigma_\theta}\left[\left|\frac{\partial\tilde{\psi}}{\partial r}\right|^2 + \left|\frac{\partial\tilde{\psi}}{\partial z}\right|^2 + \frac{1}{r^2}\left(m^2 - \frac{1}{4}\right)|\tilde{\psi}|^2\right](r,z)\,\mathrm{d}r\mathrm{d}z \tag{2.9}$$

with the domain of all $\tilde{\psi} \in W_{\mathrm{loc}}^{1,1}(\Omega_\theta)$ satisfying the Dirichlet condition at $\partial\Omega_\theta$ and such that $Q_m^\theta[\tilde{\psi}] < \infty$.

2.3. Alternative expressions

There are other ways how to express action of our Hamiltonian. For instance, one can use rotated coordinates in the radial cross section plane according to Fig. 1,

$$s = r\cos\theta + z\sin\theta , \quad u = -r\sin\theta + z\cos\theta , \tag{2.10}$$

in which we have

$$\Omega_\theta := \{(s,u) \in \mathbb{R}^2 : s \geq 0,\, 0 < u < \min(\pi, s\cot\theta)\} . \tag{2.11}$$

The partial wave operators act now as

$$\tilde{H}_m^\theta\tilde{\psi} = -\frac{\partial^2\tilde{\psi}}{\partial s^2} - \frac{\partial^2\tilde{\psi}}{\partial u^2} + \frac{4m^2 - 1}{4(s\cos\theta - u\sin\theta)^2}\tilde{\psi} , \tag{2.12}$$

the domain being the same as above; the associated quadratic form takes in these coordinates the shape

$$\tilde{Q}_m^\theta[\tilde\psi] = \int_{\Sigma_\theta} \left[\left|\frac{\partial\tilde\psi}{\partial s}\right|^2 + \left|\frac{\partial\tilde\psi}{\partial u}\right|^2 + \frac{4m^2-1}{4(s\cos\theta - u\sin\theta)^2}|\tilde\psi|^2 \right] (s,u)\,\mathrm{d}s\mathrm{d}u\,. \quad (2.13)$$

One can also use skew coordinates,

$$y = s - u\tan\theta\,, \quad v = u\,, \quad (2.14)$$

in which the expression (2.13) acquires the form

$$\tilde{Q}_m^\theta[\tilde\psi] = \int_0^\infty \mathrm{d}y \int_0^\pi \mathrm{d}v \left[\frac{1}{\cos^2\theta}\left|\frac{\partial\tilde\psi}{\partial y}\right|^2 + \left|\frac{\partial\tilde\psi}{\partial v}\right|^2 + 2\tan\theta\,\mathrm{Re}\left(\frac{\partial\bar{\tilde\psi}}{\partial y}\frac{\partial\tilde\psi}{\partial v}\right) \right.$$

$$\left. + \frac{4m^2-1}{4y^2\cos^2\theta}|\tilde\psi|^2 \right] (y,v)\,. \quad (2.15)$$

3. Properties of the spectrum

Since the conical layer is built over a surface with a cylindrical end its basic spectral properties follow from the theory developed on [DEK01, CEK04].

Theorem 3.1. *For any fixed $\theta \in (0, \frac{1}{2}\pi)$ we have $\sigma_{\mathrm{ess}}(H^\theta) = [1, \infty)$ while the discrete spectrum of the operator is non-empty with $\dim\sigma_{\mathrm{disc}}(H^\theta) = \infty$.*

Proof. For most part of the argument we can refer to the papers quoted above. In particular, we proved there that $\inf\sigma_{\mathrm{ess}}(H^\theta) = 1$. In order to check that any $k^2 + 1$ belongs to the spectrum one has to construct a suitable Weyl sequence. Consider three dimensional Cartesian coordinates s, t, u and take the family of functions

$$g_n(s,t,u) := \sqrt{\frac{2n}{n-2}}\,\mathrm{e}^{\mathrm{i}ks}f(ns,nt)\cos\frac{\pi n u}{n-2}$$

where $|u| \le \frac{n-2}{n}$ and $f \in C_0^\infty(\mathbb{R}^2)$ has the support in $(-1,1)^2$. By construction $\|(-\Delta - k^2 - 1)g_n\| \to 0$ as $n \to \infty$, and since the vanishing quantity is invariant w.r.t. Euclidean transformations we can find a family of functions $\tilde{g}_n$ supported by rectangular blocks of the size $2n \times 2n \times (1 - \frac{2}{n})$, each of them contained in Σ_θ being naturally further and further from the layer tip as n grows, which proves the claim.

The existence of the discrete spectrum was proven in [DEK01, CEK04]. While it is not stated explicitly there that the number of the eigenvalues in $(0,1)$ is infinite, it is clear from the proof in which one constructs a sequence of trial functions supported by mutually disjoint ring-shaped domains of the conical layer, and thus orthogonal to each other; for large enough indices they give the value of the energy functional below the essential spectrum threshold. $\quad\square$

After demonstrating the existence of the discrete spectrum let us look at its properties in more detail. First we note that all the geometrically induced bound states in the layer belong to the *s*-wave sector.

Proposition 3.1. $\sigma_{\mathrm{disc}}(H_m^\theta) = \emptyset$ *holds if* $m \neq 0$.

Proof. The claim is equivalent to $\sigma_{\mathrm{disc}}(\tilde{H}_m^\theta) = \emptyset$. For any higher partial wave, $m \neq 0$, we have $\tilde{Q}_m^\theta[\tilde{\psi}] \geq \int_{\Omega_\theta} |\vec{\nabla}\tilde{\psi}|^2(r,z)\,\mathrm{d}r\mathrm{d}z$ which means that $\tilde{H}_m^\theta \geq -\Delta_{\mathrm{D}}^{\Omega_\theta}$ in the form sense where the *rhs* acts according to [RS78, Sec. XIII.15] on functions $\psi \in W_{\mathrm{loc}}^{1,2}(\Omega_\theta)$ such that $\psi(\vec{x}) = 0$ for $\vec{x} = (r,z) \in \partial\Omega_\theta$ and $-\Delta\psi \in L^2$. The result then follows by minimax principle because $\sigma(-\Delta_{\mathrm{D}}^{\Omega_\theta}) = \sigma_{\mathrm{ess}}(-\Delta_{\mathrm{D}}^{\Omega_\theta}) = [1,\infty)$. $\qquad\square$

This does not mean, of course, that the cylindrically asymmetric part of the problem is trivial — it can exhibit, for instance, an interesting scattering behaviour. Since we are dealing with the discrete spectrum here, however, we consider from now on only functions independent of the angular variable ϕ and drop the index m in the quadratic form expressions. By Theorem 3.1 the operator H^θ has a sequence of eigenvalues,

$$\lambda_1(\theta) < \lambda_2(\theta) \leq \lambda_3(\theta) \leq \cdots, \tag{3.16}$$

labelled in the ascending order; the corresponding eigenfunctions will be denoted as $\psi^{(j)} \in L^2(\Sigma_\theta)$, $j = 1, 2, \ldots$; with the symmetry in mind we use the same symbols for their radial cuts $\psi^{(j)} \in L^2(\Omega_\theta, r\mathrm{d}r\mathrm{d}z)$ and $\tilde{\psi}^{(j)}$ for their counterparts in $L^2(\Omega_\theta)$.

The main question is about the dependence of the spectrum on the cone opening angle. Let us start with demonstrating that the eigenvalues (3.16) are smooth functions of θ. To this aim we compare the cross sections (2.11) for two different values of the angle, θ and θ'. They can be mapped onto each other by a longitudinal scaling, $s \cot\theta = s' \cot\theta'$; then we can express the form (2.13) referring to the angle θ' as

$$\tilde{Q}^{\theta'}[\tilde{\psi}] = \int_{\Sigma_\theta} \left[\frac{\tan\theta}{\tan\theta'} \left|\frac{\partial\tilde{\psi}}{\partial s}\right|^2 + \frac{\tan\theta'}{\tan\theta} \left|\frac{\partial\tilde{\psi}}{\partial u}\right|^2 \right.$$
$$\left. - \frac{\sin 2\theta}{\sin 2\theta'} \frac{|\tilde{\psi}|^2}{4(s\cos\theta - u\sin\theta)^2} \right] (s,u)\,\mathrm{d}s\mathrm{d}u. \tag{3.17}$$

The integration is thus taken over the same domain and the angle dependence is moved to the coefficients of the operator.

Proposition 3.2. *The functions* $\lambda_j(\cdot)$, $j = 1, 2, \ldots$, *are* C^1 *on* $(0, \frac{1}{2}\pi)$.

Proof. To find the derivative of $\lambda_j(\cdot)$ we have Feynman-Hellmann theorem differentiate (3.17) with respect to θ' at $\theta' = \theta$. It yields

$$\lambda_j'(\theta) = \int_{\Sigma_\theta} \left[-\frac{2}{\sin 2\theta} \left|\frac{\partial\tilde{\psi}^{(j)}}{\partial s}\right|^2 + \frac{2}{\sin 2\theta} \left|\frac{\partial\tilde{\psi}^{(j)}}{\partial u}\right|^2 + \frac{|\tilde{\psi}^{(j)}|^2 \cot 2\theta}{2(s\cos\theta - u\sin\theta)^2} \right] (s,u)\,\mathrm{d}s\mathrm{d}u.$$

Since $\tilde{\psi}^{(j)} \in W^{1,1}_{\text{loc}}(\Omega_\theta)$ — in fact, it belongs to $W^{1,2}_{\text{loc}}$ — the first two contributions to the integral are finite. Combining this with the fact that $Q^\theta_m[\tilde{\psi}^{(j)}] < \infty$ we infer that also the last contribution to the integral is finite, so the derivative exists. Using then the integrand continuity with respect to θ and the dominated convergence theorem, we arrive at the result. $\qquad\square$

Note that the terms appearing in the last expression have different signs so it is not apriori clear whether the eigenvalue behave monotonously as the angle changes. Nevertheless, the geometrically induced binding becomes stronger as θ increases, in particular, we have the following result.

Proposition 3.3. *Set* $\lambda_0 := \pi^{-2} j^2_{0,1} \approx 0.58596$ *where* $j_{0,1}$ *is the first zero of Bessel function* J_0. *To any* $N \in \mathbb{N}$ *and* $\bar{\lambda} \in (\lambda_0, 1)$ *there is a* θ_0 *such that* H^θ *has at least* N *eigenvalues below* $\bar{\lambda}$ *for any* $\theta \in (\theta_0, \frac{1}{2}\pi)$.

Proof. By Dirichlet bracketing [RS78, Sec. XIII.15] it is sufficient to find a region $\Sigma \subset \Sigma_\theta$ such that $-\Delta^\Sigma_{\text{D}}$ has this property. Consider a cylinder of radius $R < \pi$ and length L. The eigenvalues of the corresponding Dirichlet Laplacian are easily found; it yields the condition

$$\lambda_0 \left(\frac{\pi}{R}\right)^2 + \left(\frac{\pi N}{L}\right)^2 < \bar{\lambda}.$$

We embed the cylinder axially into Σ_θ with one lid touching the tip of the inner surface; when θ becomes close enough to $\frac{1}{2}\pi$ we can chose R close to π and L large so that the above condition is satisfied. $\qquad\square$

4. Numerical results

In order to get a more detailed information and a better insight let us study next the spectrum numerically. As a consequence of Proposition 3.1, we may limit ourselves to two–dimensional geometry, *i.e.* to consider the spectrum of (2.6) for $m = 0$. Recall that the corresponding eigenfunctions are independent of the angular variable ϕ so we can choose $\phi = 0$ arriving at the two–dimensional problem depicted in Fig. 1. Note also that the alternative formulation (2.12) leads to the same results.

The first difficulty we meet is the absence of reliable estimates concerning the span of eigenfunctions. The knowledge of the region where the eigenfunctions are negligible is important. We have to carry out integration over an infinite region and we need to cut the integration somewhere, unless we transform the problem to a finite region. This, however, brings problems of other kinds ruling out, in particular, use of the mode-matching approach.

From general considerations we know that the smaller the gap between the eigenvalue and the continuum threshold, of course, the farther is the wavefunction stretched. Also, for small opening angles eigenfunctions spread farther. This rule of thumb may be of help as a rough estimate of integration region.
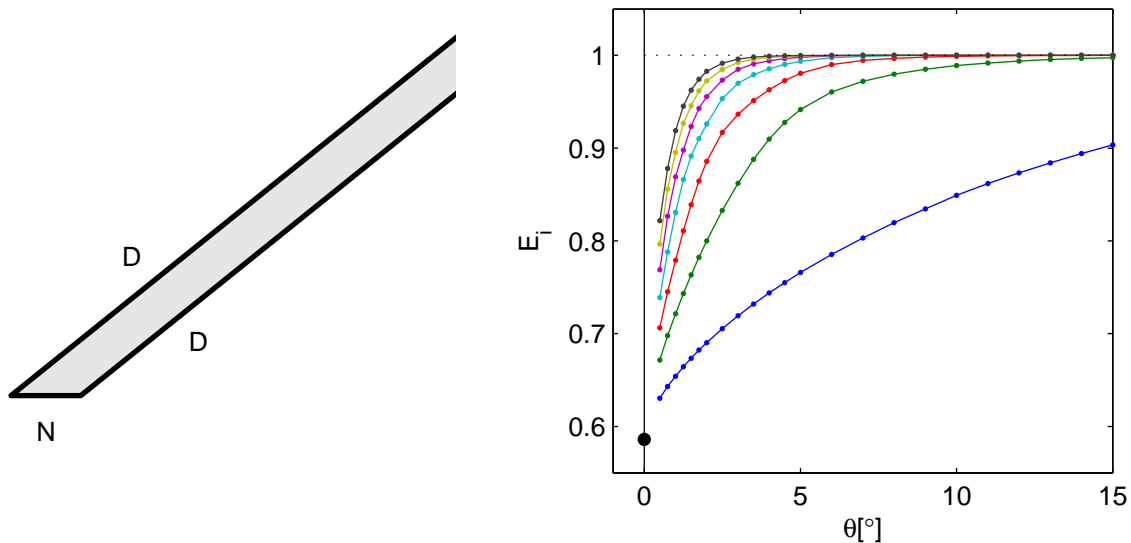
Figure 2. The left figure depicts the reduced 2D region on which the solution is sought. The right one shows the dependence of eigenvalues on the opening angle; the first seven of them are shown (colour online). The essential spectrum threshold $E = 1$ is the accumulation point of all of them and they approach it from below as $\theta \to \frac{1}{2}\pi-$. The thick dot marks the value λ_0 indicated in Proposition 3.3; the picture suggests that the eigenvalues tend to it monotonously as $\theta \to 0+$.

4.1. The method used

We must admit that we have not found a method which would be efficient enough and intuitive at the same time. To summarize our experience we prefer finite element methods. In view of the absence of knowledge of a practical span of configuration space mentioned above even this approach requires separate treating of respective levels. In order to minimize the number of elements we took further advantage of symmetry and limited the problem to the region shown in the left part of Fig. 2 with the appropriate boundary conditions. The sought eigenvalues of the conical layer are then eigenvalues of the operator (2.6) with $m = 0$ on this region.

We used 2nd order elements. The total number of degrees of freedom ranged from 50000 to about 300000 depending on eigenfunction extent. All runs for small θ (most demanding) took less than one minute each on an Intel Core 2 (1.86 GHz).

We have also tried to employ the Galerkin method. The idea was to choose a set of test functions satisfying the boundary conditions. This, however, leaves still leeway in representing the subspace to which the problem is projected, *i.e.* how to choose the particular form of the test functions. The choices we attempted led to substantially more time consuming calculation than FEM, so fitting test functions are to be found yet.

4.2. The results

The first question concerns the dependence of eigenvalues on the opening angle illustrated on Fig. 2 in the interval between 1 and 15 degrees. It agrees well with the general picture: the planar layer, $2\theta = \pi$, does not support any eigenvalue below the threshold, while for any $\theta < \pi/2$ there are by Theorem 3.1 infinitely many such isolated eigenvalues, albeit all are weakly bound for sufficiently open conical layers. As the opening angle becomes more acute the low-lying states become gradually more strongly bound. The numerical results show no degeneracies.

As we move away from the ground state, structure of the eigenfunctions acquires more and more a one-dimensional character. This is demonstrated in Fig 3 when the seven lowest eigenfunctions for the angle $\theta = 2.5°$ are calculated. As expected, the ground-state eigenfunction can be chosen positive. On the other hand, the excited states display a number of nodal lines (nodal surfaces of the full problem) that is in a one-to-one correspondence with the eigenvalue index of the sequence $\{\lambda_i(\theta)\}$, in an analogy with the classical oscillation theorem. Notice also that the node distances are increasing with the distance of the cone tip. This behaviour has to be associated with the fact that the effective-curvature induced potential in (2.12) is proportional to s^{-2} having thus the same scaling behaviour as the kinetic part of the Hamiltonian.

Another observation worth commenting is shown on Fig. 4, where we have plotted (a side view of the) squared eigenfunctions $|\psi^i|^2$ with $i = 1, \ldots, 7$ for the same opening angle θ as above. As the index i grows, dominant regions lie around the inner tip and at the "far end" of eigenfunction extent, more exactly, between the last two nodal lines. Consequently, location of the regions with the largest probability of finding the particle in the conical layer depends on the eigenvalue number in this way.

5. Concluding remarks

Our discussion has not exhausted by far all open questions concerning spectra of conical layers. We have also mentioned the eigenvalue monotonicity which is expected to hold and supported by the above numerical result. A related question concerns the simplicity of the spectrum. It is again seen in the numerical example and its validity is strongly supported by the quasi-one-dimensional character of the higher eigenfunctions, but these considerations cannot replace a rigorous proof.

Other open questions are associated with the behaviour for extreme values of θ which we left out in the previous section because these cases are numerically demanding. For small values of θ the density of eigenvalues above $\lambda_0 \approx 0.58596$ should grow in accordance with Proposition 3.3. What we see nevertheless is that already for $\theta = 2.5°$ the excited states have nodal lines (or nodal surfaces of the full problem) inside the cone "cap"; one can ask about critical values of θ for which one of those touches the inner tip. On the other hand, one can ask about the spectral asymptotics as $\theta \to \frac{1}{2}\pi-$. In contrast to [EK01] the geometric perturbation is not compactly supported for conical layers and
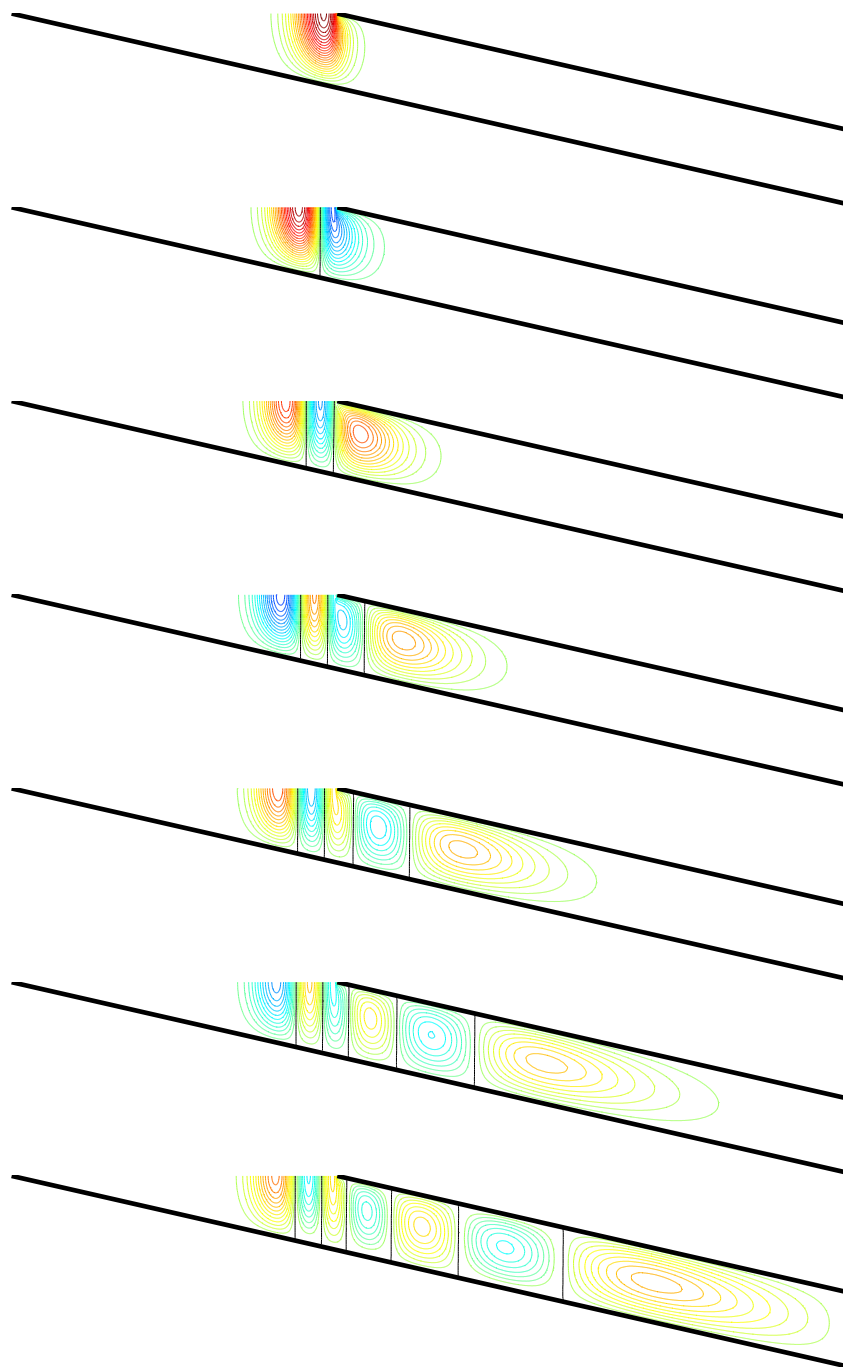
Figure 3. The contour plot of the first seven eigenfunctions (colour online) for $\theta = 2.5^{\circ}$. In view of the axial symmetry, we show only one half of $\psi(r, z)$. The the $r-$ and $z-$scales are different, the vertical axis being scaled by factor five: the transversal width is π and z ranges from 0 to 225. Nodal lines are indicated in black colour; higher eigenfuctions have clearly a quasi-onedimensional character.
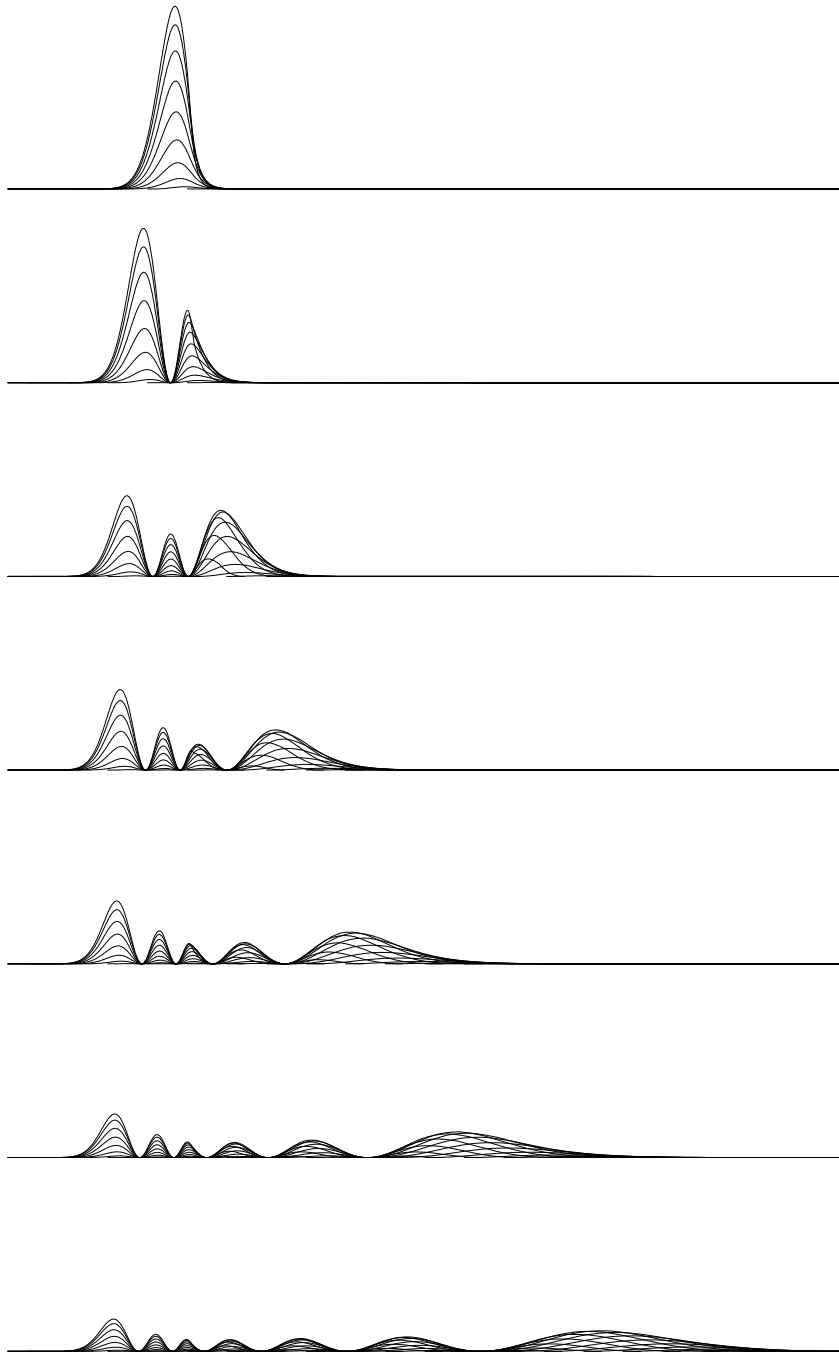
Figure 4. Side view (*i.e.*, with zero elevation angle) of $|\psi|^2$ for the same situation as in Fig. 3. All the eigenfunctions are normalized to the same constant, so that relative height ratios reflect the true situation (not distorted by different normalization of individual functions). There are two areas with higher probability, one in the vicinity of the inner tip and the other one at the "far end" of the function extent. The latter moves away from the vertex as the eigenvalue number increases.

the effective potential is on the borderline between short- and long-range cases, so it is not a priori clear what the asymptotic expansion could be.

Of interest is also the continuous spectrum of conical layers, both from the point of view of scattering — existence of wave operators and their asymptotic completeness — as well as of resonances. Being bound by the deadline of this issue, we leave all these questions to a later investigation.

Acknowledgments

References

[CEK04] G. Carron, P. Exner, D. Krejčiřík, *Topologically non-trivial quantum layers*, J. Math. Phys. **45** (2004), 774–784.

[DE95] P. Duclos, P. Exner, *Curvature-induced bound states in quantum waveguides in two and three dimensions*, Rev. Math. Phys. **7** (1995), 73–102.

[DEK01] P. Duclos, P. Exner, D. Krejčiřík, *Bound states in curved quantum layers*, Commun. Math. Phys. **223** (2001), 13–28.

[EK01] P. Exner, D. Krejčiřík, *Bound states in mildly curved layers*, J. Phys. A: Math. Gen. **A34** (2001), 5969–5985.

[EŠ89] P. Exner, P. Šeba, *Bound states in curved quantum waveguides*, J. Math. Phys. **30** (1989), 2574–2580.

[EŠŠ89] P. Exner, P. Šeba, P. Šťovíček, *On existence of a bound state in an* L*-shaped waveguide*, Czech. J. Phys. **B39** (1989), 1181–1191.

[GJ92] J. Goldstone, R.L. Jaffe, *Bound states in twisting tubes*, Phys. Rev. **B45** (1992), 14100–14107.

[LLM86] F. Lenz, J.T. Londergan, R.J. Moniz, R. Rosenfelder, M. Stingl, K. Yazaki: Quark confinement and hadronic interactions, *Ann. Phys.* **170** (1986), 65-254.

[LL07] C. Lin, Z.Q. Lu, *Existence of bound states for layers built over hypersurfaces in* $\mathbb{R}^{n+1}$, J. Funct. Anal. **244** (2007), 1–25.

[LCM99] J.T. Londergan, J.P. Carini, D.P. Murdock, *Binding and Scattering in Two-Dimensional Systems. Applications to Quantum Wires, Waveguides and Photonic Crystals*, Springer LNP m60, Berlin 1999.

[RS78] M. Reed and B. Simon: *Methods of Modern Mathematical Physics, IV. Analysis of Operators*, Academic Press, New York 1978.

[SRW89] R.L. Schult, D.G. Ravenhall, H.W. Wyld, *Quantum bound states in a classically unbounded system of crossed wires*, Phys. Rev. **B39** (1989), 5476–5479.