WEYL-TITCHMARSH TYPE FORMULA FOR HERMITE OPERATOR WITH SMALL PERTURBATION

SERGEY SIMONOV

Abstract. Small perturbations of the Jacobi matrix with weights $\sqrt{n}$ and zero diagonal are considered. A formula relating the asymptotics of polynomials of the first kind to the spectral density is obtained, which is analogue of the classical Weyl-Titchmarsh formula for the Schrödinger operator on the half-line with summable potential. Additionally a base of generalized eigenvectors for "free" Hermite operator is studied and asymptotics of Plancherel-Rotach type are obtained.

1. Introduction

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers and $\{b_n\}_{n=1}^{\infty}$ be a sequence of real numbers, $\{e_n\}_{n=1}^{\infty}$ be the canonical basis in the space $l^2(\mathbb{N})$ (i.e., each vector e_n has zero components except the n-th which is 1), let also l_{fin} be the linear set of sequences with finite number of non-zero components. One can define an operator $\mathcal{J}$ in l^2, which acts in l_{fin} by the rule

$$\begin{aligned}(\mathcal{J}u)_n &= a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1}, \ n \geq 2,\\ (\mathcal{J}u)_1 &= b_1 u_1 + a_1 u_2.\end{aligned}$$

The operator is first defined on l_{fin} and then the closure is considered. Then $\mathcal{J}$ is self-adjoint in l^2 provided $\sum\limits_{n=0}^{\infty} \frac{1}{a_n} = \infty$ [3] (Carleman condition), and it has the following matrix representation with respect to the canonical basis:

$$\mathcal{J} = \begin{pmatrix} b_1 & a_1 & 0 & \cdots \\ a_1 & b_2 & a_2 & \cdots \\ 0 & a_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

1991 *Mathematics Subject Classification.* 47A10, 47B36.

Key words and phrases. Jacobi matrices, Absolutely continuous spectrum, Subordinacy theory, Weyl-Titchmarsh theory.

Consider the spectral equation for $\mathcal{J}$:

$$a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1} = \lambda u_n, \ n \geq 2. \tag{1}$$

Solution $P_n(\lambda)$ of (1) such that $P_1(\lambda) \equiv 1$, $P_2(\lambda) = \frac{\lambda - b_1}{a_1}$ is a polynomial in λ of degree $n-1$ and is called the polynomial of the first kind. Correspondingly the solution $Q_n(\lambda)$ such that $Q_1(\lambda) \equiv 0$, $Q_2(\lambda) \equiv \frac{1}{a_1}$ is a polynomial of degree $n-2$ and is called the polynomial of the second kind. For two solutions of (1) u_n and v_n, the expression

$$W(u,v) := W(\{u_n\}_{n=1}^\infty, \{v_n\}_{n=1}^\infty) := a_n(u_n v_{n+1} - u_{n+1}v_n)$$

is independent of n and is called the (discrete) Wronskian of u and v. One always has

$$W(P(\lambda), Q(\lambda)) \equiv 1.$$

The spectrum of every Jacobi matrix is simple and the vector e_1 from the standard basis is the generating vector [3]. Let dE be the operator-valued spectral measure associated with $\mathcal{J}$. Polynomials of the first kind are orthogonal with respect to the measure $d\rho := (dEe_1, e_1)$, which is also called the spectral measure [2]. For non-real values of λ solutions of (1) that belong to l^2 are proportional to $Q_n(\lambda)+m(\lambda)P_n(\lambda)$ [2], where

$$m(\lambda) := \int_{\mathbb{R}} \frac{d\rho(x)}{x-\lambda}, \ \lambda \in \mathbb{C}\backslash\mathbb{R}$$

is the Weyl function. By Fatou's Theorem [6],

$$\rho'(\lambda) = \frac{1}{\pi}\operatorname{Im} m(\lambda + i0),$$

for a.a. $\lambda \in \mathbb{R}$.

In the present paper we consider small perturbations of the operator $\mathcal{J}_0$, which is defined by the sequences $\{\sqrt{n}\}_{n=1}^\infty$ and $\{0\}_{n=1}^\infty$:

$$\mathcal{J}_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & \sqrt{2} & \cdots \\ 0 & \sqrt{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let us call $\mathcal{J}_0$ the "free" Hermite operator. We will call (following [11]) $\mathcal{J}$ the Hermite operator, if it can be considered close to $\mathcal{J}_0$ in some sense. Let us call $\mathcal{J}$ the "small" perturbation of $\mathcal{J}_0$, if $\mathcal{J}$ is defined by sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ such that (let $c_n := a_n - \sqrt{n}$)

$$c_n = o(\sqrt{n}) \text{ as } n \to \infty \text{ and } \sum_{n=1}^\infty \left(\frac{|c_n|}{n} + \frac{|c_{n+1} - c_n| + |b_n|}{\sqrt{n}} \right) < \infty. \tag{2}$$

Denote the following expression by Λ: for any sequence $\{u_n\}_{n=1}^\infty$,

$$(3)\qquad \begin{aligned}(\Lambda u)_n &:= c_{n-1}u_{n-1} + b_n u_n + c_n b_{n+1},\ n \geq 2,\\ (\Lambda u)_1 &:= b_1 u_1 + c_1 u_2.\end{aligned}$$

Although Λ is not a Jacobi matrix, we will write $\mathcal{J} = \mathcal{J}_0 + \Lambda$. The spectrum of $\mathcal{J}_0$ is purely absolutely continuous on $\mathbb{R}$ with the spectral density

$$\rho_0'(\lambda) = \frac{e^{-\frac{\lambda^2}{2}}}{\sqrt{2\pi}}.$$

As it will be shown, the spectrum of $\mathcal{J}$ is also purely absolutely continuous under assumption (2).

Our goal in the present paper is to study the spectral density of $\mathcal{J}$ using the asymptotic analysis of generalized eigenvectors of $\mathcal{J}$ (i.e., solutions of the spectral equation (1)). The method is based upon the comparison of solutions of (1) to solutions of the spectral equation for the free Hermite operator,

$$(4)\qquad \sqrt{n-1}u_{n-1} + \sqrt{n}u_{n+1} = \lambda u_n,\ n \geq 2.$$

This is analogous to the Weyl-Titchmarsh theory for the Schrödinger operator on the half-line with the summable potential. The following results will be proven (Theorem 1 in Section 2 and Theorem 2 in Section 3). Let w be the standard error function [1]

$$(5)\qquad w(z) := \frac{1}{\pi i}\int_{\Gamma_z^-} \frac{e^{-\zeta^2}d\zeta}{\zeta - z} = -\frac{1}{\pi i}\int_{\Gamma_{-z}^+} \frac{e^{-\zeta^2}d\zeta}{\zeta + z},$$

where the contours $\Gamma_z^\pm$ are shown on Figure 1. Function w is entire.

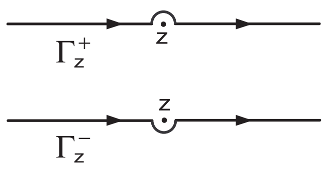

FIGURE 1. Contours $\Gamma_z^\pm$

Theorem 1. *For every $\lambda \in \mathbb{C}$ equation (4) has a basis of solutions*

$$I_n^+(\lambda) := \frac{(-1)^{n-1}e^{\frac{\lambda^2}{2}}w^{(n-1)}\left(\frac{\lambda}{\sqrt{2}}\right)}{\sqrt{(n-1)!2^{n+1}}}$$

and

$$I_n^-(\lambda) := \frac{e^{\frac{\lambda^2}{2}} w^{(n-1)}\left(-\frac{\lambda}{\sqrt{2}}\right)}{\sqrt{(n-1)!2^{n+1}}},$$

which have the following asymptotics as $n \to \infty$:

$$I_n^{\pm}(\lambda) = \frac{(\mp i)^{n-1} e^{\frac{\lambda^2}{4} \pm i\lambda\sqrt{n}}}{(8\pi n)^{1/4}}\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right).$$

These asymptotics are uniform with respect to λ in every bounded set in $\mathbb{C}$. Polynomials of the first kind for $\mathcal{J}_0$ are related to $I_n^{\pm}$ in the following way:

$$P_{0n}(\lambda) = I_n^+(\lambda) + I_n^-(\lambda).$$

Theorem 2. *Let the conditions* (2) *hold for $\mathcal{J}$. Then*
1. For every $\lambda \in \overline{\mathbb{C}_+}$ there exists

$$F(\lambda) := 1 + i\sqrt{2\pi} e^{-\frac{\lambda^2}{2}} \sum_{n=1}^{\infty} (\Lambda I^+(\lambda))_n P_n(\lambda)$$

(the Jost function), which is analytic function in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}_+}$.
2. Polynomials of the first kind have the following asymptotics as $n \to \infty$:

- *For $\lambda \in \mathbb{C}_+$,*

$$P_n(\lambda) = F(\lambda) I_n^-(\lambda) + o\left(\frac{e^{Im\lambda\sqrt{n}}}{n^{1/4}}\right) \text{ as } n \to \infty,$$

- *For $\lambda \in \mathbb{R}$,*

$$P_n(\lambda) = F(\lambda) I_n^-(\lambda) + \overline{F(\lambda)} I_n^+(\lambda) + o(n^{-\frac{1}{4}}) \text{ as } n \to \infty.$$

3. The spectrum of $\mathcal{J}$ is purely absolutely continuous, and for a.a. $\lambda \in \mathbb{R}$

$$\rho'(\lambda) = \frac{e^{-\frac{\lambda^2}{2}}}{\sqrt{2\pi}|F(\lambda)|^2}$$

(the Weyl-Titchmarsh type formula).

The idea of the Weyl-Titchmarsh type formula is the relation between the spectral density and the behavior of $P_n(\lambda)$ for large values of n. We can formulate this in the form of the corollary.

Corollary 1. *Let the conditions* (2) *hold for* $\mathcal{J}$*. Then the spectrum of* $\mathcal{J}$ *is purely absolutely continuous and the spectral density equals for a.a.* $\lambda \in \mathbb{R}$

$$\rho'(\lambda) = \frac{1}{\pi} \lim_{n\to\infty} \frac{1}{\sqrt{n}(P_n^2(\lambda) + P_{n+1}^2(\lambda))},$$

the right-hand side being finite and non-zero for every $\lambda \in \mathbb{R}$*.*

Theorem 2 can be proven by another method, based on the Levinson-type analytical and smooth theorem, cf. [5] and papers of Bernzaid-Lutz [8], Janas-Moszyński [13] and Silva [16], [17]. None of their results is directly applicable here, and the approach of the present paper is different.

The considered situation is parallel to the Weyl-Titchmarsh theory for Schrödinger operator on the half-line with summable potential. Let q be a real-valued function on $\mathbb{R}_+$ and $q \in L_1(\mathbb{R}_+)$. Consider the Schrödinger operator on $\mathbb{R}_+$

$$\mathcal{L} = -\frac{d^2}{dx^2} + q(x)$$

with the Dirichlet boundary condition. The purely absolutely continuous spectrum of $\mathcal{L}$ coincides with $\mathbb{R}_+$ [7]. Let $\varphi(x, \lambda)$ be a solution of the spectral equation for $\mathcal{L}$,

$$-u''(x, \lambda) + q(x)u(x, \lambda) = \lambda u(x, \lambda),$$

such that $\varphi(0, \lambda) \equiv 0$, $\varphi'(0, \lambda) \equiv 1$ (satisfying the boundary condition). The following result holds [7].

Proposition 1. *If* $q \in L_1(\mathbb{R}_+)$*, then for every* $k > 0$ *there exist* $a(k)$ *and* $b(k)$ *such that*

$$\varphi(x, k^2) = a(k)\cos(kx) + b(k)\sin(kx) + o(1) \text{ as } x \to +\infty,$$

and for a.a. $\lambda > 0$

$$\rho'(\lambda) = \frac{1}{\pi\sqrt{\lambda}(a^2(\sqrt{\lambda}) + b^2(\sqrt{\lambda}))}$$

(the classical Weyl-Titchmarsh formula).

Solutions $I_n^+(\lambda)$ and $I_n^-(\lambda)$ are the direct analogues to the solutions $\frac{e^{ikx}}{2ik}$ and $\frac{e^{-ikx}}{-2ik}$ of the spectral equation for "free" Schrödinger operator,

$$-u''(x, k^2) = k^2 u(x, k^2).$$

The main technical difficulty of our problem is non-triviality of solutions $I_n^\pm(\lambda)$ compared to $\frac{e^{\pm ikx}}{\pm 2ik}$. The model of the Hermite operator was studied in the paper of Brown-Naboko-Weikard [11], but solutions $I_n^\pm(\lambda)$ were not introduced there.

2. The free Hermite operator

In this section we study asymptotic properties of generalized eigenvectors for J_0 and prove Theorem 1. Let us give its formulation again.

Theorem. *For every $\lambda \in \mathbb{C}$ equation (4) has a basis of solutions*

$$I_n^+(\lambda) := \frac{(-1)^{n-1} e^{\frac{\lambda^2}{2}} w^{(n-1)}\left(\frac{\lambda}{\sqrt{2}}\right)}{\sqrt{(n-1)!2^{n+1}}} \tag{6}$$

and

$$I_n^-(\lambda) := \frac{e^{\frac{\lambda^2}{2}} w^{(n-1)}\left(-\frac{\lambda}{\sqrt{2}}\right)}{\sqrt{(n-1)!2^{n+1}}}, \tag{7}$$

which have the following asymptotics as $n \to \infty$:

$$I_n^\pm(\lambda) = \frac{(\mp i)^{n-1} e^{\frac{\lambda^2}{4} \pm i\lambda\sqrt{n}}}{(8\pi n)^{1/4}} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right). \tag{8}$$

These asymptotics are uniform with respect to λ in every bounded set in $\mathbb{C}$. Polynomials of the first kind for $\mathcal{J}_0$ are related to $I_n^\pm$ in the following way:

$$P_{0n}(\lambda) = I_n^+(\lambda) + I_n^-(\lambda). \tag{9}$$

Proof. The spectral equation (4) for $\mathcal{J}_0$,

$$\sqrt{n-1}u_{n-1} + \sqrt{n}u_{n+1} = \lambda u_n, \ n \geq 2,$$

can be transformed to the recurrence relation

$$2nv_{n-1}(x) + v_{n+1}(x) = 2xv_n(x), \ n \geq 1 \tag{10}$$

if one takes $v_n := \sqrt{2^n n!}u_{n+1}$ and $x := \frac{\lambda}{\sqrt{2}}$. Equation (10) is satisfied by Hermite polynomials [1], and this means (together with initial values: $H_0(x) \equiv 1$ and $H_1(x) = 2x$), that the polynomials of the first kind for J_0 equal

$$P_{0n}(\lambda) = \frac{H_{n-1}(\frac{\lambda}{\sqrt{2}})}{\sqrt{2^{n-1}(n-1)!}}. \tag{11}$$

Equation (10) has two other linearly independent solutions, $w^{(n)}(-x)$ and $(-1)^n w^{(n)}(x)$ [1]. This can be checked by substituting them into (10) using the formula

$$w^{(n)}(z) = \frac{n!}{\pi i} \int_{\Gamma_z^-} \frac{e^{-\zeta^2} d\zeta}{(\zeta - z)^{n+1}}$$

and integrating by parts. From the integral representation for Hermite polynomials [1],

$$H_n(x) = \frac{n!}{2\pi i}\oint_0 \frac{e^{2xz-z^2}}{z^{n+1}}dz$$
$$= \frac{n!e^{x^2}}{2\pi i}\left(\int_{\Gamma^-_{-x}} \frac{e^{-z^2}dz}{(z+x)^{n+1}} - \int_{\Gamma^+_{-x}} \frac{e^{-z^2}dz}{(z+x)^{n+1}}\right)$$
$$= \frac{e^{x^2}}{2}w^{(n)}(-x) + \frac{e^{x^2}}{2}(-1)^n w^{(n)}(x),$$

where the contour Γ^+_z is shown on Figure 1. Correspondingly, equation (4) has two linearly independent solutions of the form (6) and (7) and relation (9) holds. Asymptotics of these solutions follow immediately from Corollary 2 from the appendix. □

In what follows we will need to know the Wronskian of the solutions.

Lemma 1.

$$W(I^+(\lambda), I^-(\lambda)) = i\frac{e^{\frac{\lambda^2}{2}}}{\sqrt{2\pi}}.$$

Proof. One has from (6) and (7):

$$W(I^+(\lambda), I^-(\lambda)) = I^+_1(\lambda)I^-_2(\lambda) - I^+_2(\lambda)I^-_1(\lambda)$$
$$= \frac{e^{\lambda^2}}{4\sqrt{2}}\left(w\left(\frac{\lambda}{\sqrt{2}}\right)w'\left(-\frac{\lambda}{\sqrt{2}}\right) + w\left(-\frac{\lambda}{\sqrt{2}}\right)w'\left(\frac{\lambda}{\sqrt{2}}\right)\right) = i\frac{e^{\frac{\lambda^2}{2}}}{\sqrt{2\pi}},$$

using the following properties of the error function [1]:

$$w'(z) = -2zw(z) + \tfrac{2i}{\sqrt{\pi}},$$
$$w(z) + w(-z) = 2e^{-z^2}.$$

□

3. The perturbed Hermite operator

In this section, we consider the Hermite operator $\mathcal{J}$ with "small" perturbation, i.e., satisfying conditions (2), and prove Theorem 2. We study asymptotics of polynomials of the first and second kind using the Volterra-type equation and derive from these asymptotics a formula for the Weyl function. The desired Weyl-Titchmarsh type formula follows from this. We start with proving a formula of variation of parameters. Remind that $P_n(\lambda)$ are polynomials of the first kind for $\mathcal{J}$, $P_{0n}(\lambda)$ are polynomials of the first kind for $\mathcal{J}_0$, Λ is the expression given by (3), $a_n=\sqrt{n}+c_n$. Let us denote

$$W(\lambda):=W(I^+(\lambda),I^-(\lambda)).$$

Lemma 2. *For $n\geq 2$,*

(12)

$$\frac{a_{n-1}}{\sqrt{n-1}}P_n(\lambda)=P_{0n}(\lambda)-\sum_{k=1}^{n-1}\frac{(\Lambda I^+(\lambda))_kI_n^-(\lambda)-I_n^+(\lambda)(\Lambda I^-(\lambda))_k}{W(\lambda)}P_k(\lambda).$$

Proof. Let us omit the dependence on λ everywhere. First let us prove that

$$P_n=u_n-\sum_{k=2}^{n-1}\frac{I_k^+I_n^- - I_k^-I_n^+}{W}(\Lambda P)_k,\ n\geq 3, \tag{13}$$

where u is the solution of (4) such that $u_1=P_1$ and $u_2=P_2$. Let us denote

$$\widetilde{P}_n:=\begin{cases} u_n-\sum_{k=2}^{n-1}\frac{I_k^+I_n^- - I_k^-I_n^+}{W}(\Lambda P)_k,\ n\geq 3,\\ P_n,\ n=1,2\end{cases}$$

I fact, one has to check that

$$\sqrt{n-1}\widetilde{P}_{n-1}-\lambda\widetilde{P}_n+\sqrt{n}\widetilde{P}_{n+1}=-(\Lambda P)_n,\ n\geq 2,$$

(this non-homogeneous equation has only one solution with fixed two first values, so $\widetilde{P}$ should coincide with P). Since u, I^+ and I^- are solutions to (4) and

$$\sqrt{n}\sum_{k=n}^{n}\frac{I_k^+I_{n+1}^- - I_{n+1}^+I_k^-}{W}(\Lambda P)_k=(\Lambda P)_n,$$

the previous is equivalent to

$$-\lambda\sum_{k=n-1}^{n-1}\frac{I_k^+I_n^- - I_n^+I_k^-}{W}(\Lambda P)_k+\sqrt{n}\sum_{k=n-1}^{n-1}\frac{I_k^+I_{n+1}^- - I_{n+1}^+I_k^-}{W}(\Lambda P)_k=0$$

for $n\geq 3$. The latter is true, because $-\lambda I_n^\pm+\sqrt{n}I_{n+1}^\pm=-\sqrt{n-1}I_{n-1}^\pm$.

After shifting indices in different parts of the sum in (13) one obtains:

$$P_n=u_n+\frac{I_1^+I_n^- - I_n^+I_1^-}{W}(b_1P_1+c_1P_2)\\ -\sum_{k=1}^{n-1}\frac{(\Lambda I^+)_kI_n^- - I_n^+(\Lambda I^-)_k}{W}P_k-\frac{c_{n-1}}{\sqrt{n-1}}P_n.$$

Since

$$u_n = \frac{I_1^+ P_2 - I_2^+ P_1}{W} I_n^- - \frac{I_1^- P_2 - I_2^- P_1}{W} I_n^+, \ P_1 = 1, \ P_2 = \frac{\lambda - b_1}{a_1},$$

one has:

$$u_n + \frac{I_1^+ I_n^- - I_n^+ I_1^-}{W}(b_1 P_1 + c_1 P_2) = \frac{\lambda I_1^+ - I_2^+}{W} I_n^- - \frac{\lambda I_1^- - I_2^-}{W} I_n^+$$
$$= \frac{I_1^+ P_{02} - I_2^+ P_{01}}{W} I_n^- - \frac{I_2^+ P_{01} - I_1^+ P_{02}}{W} I_n^+ = P_{0n}.$$

Therefore

$$\frac{a_{n-1}}{\sqrt{n-1}} P_n = P_{0n} - \sum_{k=1}^{n-1} \frac{(\Lambda I^+)_k I_n^- - I_n^+ (\Lambda I^-)_k}{W} P_k.$$

□

Equation (12) is of Volterra type. We need the following standard lemma to deal with it. Consider the Banach space

$$\mathcal{B} := \left\{ \{u_n\}_{n=1}^\infty : \sup_n \left(\frac{|u_n| n^{1/4}}{e^{|\operatorname{Im}\lambda|\sqrt{n}}} \right) < \infty \right\}$$

with the norm

$$\|u\|_{\mathcal{B}} := \sup_n \left(\frac{|u_n| n^{1/4}}{e^{|\operatorname{Im}\lambda|\sqrt{n}}} \right)$$

(we omit the dependence on λ in the notation for $\mathcal{B}$). Let $\mathcal{V}$ be the expression

$$(\mathcal{V}u)_n := \begin{cases} 0, \ n = 1 \\ \sum_{k=1}^{n-1} V_{nk} u_k, \ n \geq 2 \end{cases} \tag{14}$$

for any sequence $\{u\}_{n=1}^\infty$. Let

$$\nu := \sup_{n>1} \sum_{k=1}^{n-1} |V_{nk}| e^{|\operatorname{Im}\lambda|(\sqrt{k}-\sqrt{n})} \left(\frac{n}{k} \right)^{1/4}. \tag{15}$$

Lemma 3. *If $\nu < \infty$, then $\mathcal{V}$ is a bounded operator in $\mathcal{B}$, $(I - \mathcal{V})^{-1}$ exists and $\|\mathcal{V}\|_{\mathcal{B}} \leq \nu$, $\|(I - \mathcal{V})^{-1}\|_{\mathcal{B}} \leq e^\nu$.*

Proof. By definition of the operator norm we have to check the finiteness of the following:

$$\sup_{u\neq 0}\frac{\|\mathcal{V}u\|_{\mathcal{B}}}{\|u\|_{\mathcal{B}}}=\sup_{u\neq 0}\frac{\sup\limits_{n>1}\frac{\left|\sum_{k=1}^{n-1}V_{nk}u_k\right|n^{1/4}}{e^{|\operatorname{Im}\lambda|\sqrt{n}}}}{\sup\limits_{n}\frac{|u_n|n^{1/4}}{e^{|\operatorname{Im}\lambda|\sqrt{n}}}}$$

$$\leq \sup_{u\neq 0}\frac{\sup\limits_{n>1}\sum\limits_{k=1}^{n-1}|V_{nk}|\frac{|u_k|k^{1/4}}{e^{|\operatorname{Im}\lambda|\sqrt{k}}}\left(\frac{n}{k}\right)^{1/4}e^{|\operatorname{Im}\lambda|(\sqrt{k}-\sqrt{n})}}{\sup\limits_{n}\frac{|u_n|n^{1/4}}{e^{|\operatorname{Im}\lambda|\sqrt{n}}}}.$$

Denoting $\widetilde{u}_n:=u_n\frac{n^{1/4}}{e^{\operatorname{Im}\lambda\sqrt{n}}}$, we have:

$$\sup_{u\neq 0}\frac{\|\mathcal{V}u\|_{\mathcal{B}}}{\|u\|_{\mathcal{B}}}\leq\sup_{\widetilde{u}\neq 0}\frac{\sup\limits_{n>1}\sum\limits_{k=1}^{n-1}|V_{nk}||\widetilde{u}_k|\left(\frac{n}{k}\right)^{1/4}e^{|\operatorname{Im}\lambda|(\sqrt{k}-\sqrt{n})}}{\sup\limits_{n}|\widetilde{u}_n|}$$

$$\leq\sup_{n>1}\sum_{k=1}^{n-1}|V_{nk}|\left(\frac{n}{k}\right)^{1/4}e^{|\operatorname{Im}\lambda|(\sqrt{k}-\sqrt{n})},$$

hence $\mathcal{V}$ is bounded. Quite similarly,

$$\|\mathcal{V}^l\|_{\mathcal{B}}\leq\sup_{n>1}\sum_{k=1}^{n-1}\left|\sum_{1\leq k_1<k_2<\ldots<k_{l-1}<k}V_{nk_1}V_{k_1k_2}\ldots V_{k_{l-1}k}\right|\left(\frac{n}{k}\right)^{1/4}e^{|\operatorname{Im}\lambda|(\sqrt{k}-\sqrt{n})}$$

$$\leq\sup_{n>1}\frac{\left(\sum_{k=1}^{n-1}|V_{nk}|\left(\frac{n}{k}\right)^{1/4}e^{|\operatorname{Im}\lambda|(\sqrt{k}-\sqrt{n})}\right)^l}{l!}.$$

Therefore

$$1+\|\mathcal{V}\|_{\mathcal{B}}+\|\mathcal{V}^2\|_{\mathcal{B}}+\ldots\leq\exp\left\{\sup_{n>1}\sum_{k=1}^{n-1}|V_{nk}|e^{|\operatorname{Im}\lambda|(\sqrt{k}-\sqrt{n})}\left(\frac{n}{k}\right)^{1/4}\right\},$$

and hence the operator $(I-\mathcal{V})^{-1}$ exists, is bounded, and its norm is estimated by the same expression. □

Now we can prove the uniform estimate on the growth of polynomials.

Lemma 4. *Let the condition* (2) *hold for* $\mathcal{J}$*. Then*

$$P_n(\lambda)=O\left(\frac{e^{|Im\lambda|\sqrt{n}}}{n^{1/4}}\right)\ \text{as } n\to\infty \tag{16}$$

uniformly with respect to λ *on every bounded set in* $\mathbb{C}$.

Proof. Let us rewrite (12) as

$$P(\lambda) = v(\lambda) + \mathcal{V}(\lambda)P(\lambda),$$

where

$$v_n(\lambda) := \begin{cases} 1, \ n = 1, \\ \frac{\sqrt{n-1}}{a_{n-1}} P_{0n}(\lambda), \ n \geq 2, \end{cases}$$

$$(\mathcal{V}(\lambda))_n := \begin{cases} 0, \ n = 1, \\ -\frac{\sqrt{n-1}}{a_{n-1}} \sum_{k=1}^{n-1} \frac{(\Lambda I^+(\lambda))_k I_n^-(\lambda) - I_n^+(\lambda)(\Lambda I^-(\lambda))_k}{W(\lambda)} u_k, \ n \geq 2. \end{cases}$$

What we need to prove is that $P(\lambda) \in \mathcal{B}$ and $\|P(\lambda)\|_{\mathcal{B}}$ is bounded on every bounded set in $\mathbb{C}$. It will suffice to prove the same for $\|v(\lambda)\|_{\mathcal{B}}$ and for $\nu(\lambda)$ related to $\mathcal{V}(\lambda)$ by (15), due to Lemma 3. First follows from the asymptotics given by Theorem 1, so consider the second. The kernel of $\mathcal{V}(\lambda)$ is

$$V_{nk}(\lambda) := -\frac{\sqrt{n-1}}{a_{n-1}} \frac{(\Lambda I^+(\lambda))_k I_n^-(\lambda) - I_n^+(\lambda)(\Lambda I^-(\lambda))_k}{W(\lambda)}, \ 1 \leq k \leq n-1.$$

Fix a bounded set $K \subset \mathbb{C}$. It follows from (8) that

(17)

$$\begin{aligned}(\Lambda I^\pm(\lambda))_k &= \frac{\left|e^{\frac{\lambda^2}{4}}\right|}{(8\pi)^{\frac{1}{4}}} \left| c_{k-1} \frac{i^{k-1} e^{i\lambda\sqrt{k-1}}}{(k-1)^{\frac{1}{4}}} + b_k \frac{i^k e^{i\lambda\sqrt{k}}}{k^{\frac{1}{4}}} + c_k \frac{i^{k+1} e^{i\lambda\sqrt{k+1}}}{(k+1)^{\frac{1}{4}}} \right| \\ &\quad + O\left(\frac{|c_{k-1}| + |b_k| + |c_k|}{k^{\frac{3}{4}}} e^{\mp \mathrm{Im}\lambda\sqrt{k}} \right) \\ &= O\left(\left(|b_k| + |c_k - c_{k-1}| + \frac{|c_k|}{\sqrt{k}} \right) \frac{e^{\mp \mathrm{Im}\lambda\sqrt{k}}}{k^{\frac{1}{4}}} \right) \quad \text{as } k \to \infty\end{aligned}$$

uniformly with respect to $\lambda \in K$. Hence there exists C_1 such that

$$\begin{aligned}&|(\Lambda I^+(\lambda))_k I_n^-(\lambda)|, |I_n^+(\lambda)(\Lambda I^-(\lambda))_k| \\ &\qquad < C_1(\left(|b_k| + |c_k - c_{k-1}| + \frac{|c_k|}{\sqrt{k}} \right) \frac{e^{|\mathrm{Im}\,\lambda||\sqrt{n}-\sqrt{k}|}}{(nk)^{1/4}}\end{aligned}$$

for every $n, k \in \mathbb{N}$. Therefore there exists C_2 such that

$$\nu(\lambda) < C_2 \sum_{k=1}^{\infty} \left(\frac{|c_k|}{k} + \frac{|c_k - c_{k-1}| + |b_k|}{\sqrt{k}} \right),$$

and this estimate is uniform with respect to $\lambda \in K$. This completes the proof. □

It is possible now to introduce the Jost function and to find the asymptotics of the polynomials.

Lemma 5. *Let the condition* (2) *hold for* $\mathcal{J}$*. Then the function*

$$F(\lambda) := 1 + i\sqrt{2\pi}e^{-\frac{\lambda^2}{2}}\sum_{n=1}^{\infty}(\Lambda I^+(\lambda))_n P_n(\lambda) \tag{18}$$

is analytic in $\mathbb{C}_+$ *and continuous in* $\overline{\mathbb{C}_+}$*. Polynomials of the first kind for* $\mathcal{J}$*,* $P_n(\lambda)$*, have the following asymptotics as* $n \to \infty$*:*

- *For* $\lambda \in \mathbb{C}_+$*,*

$$P_n(\lambda) = F(\lambda)I_n^-(\lambda) + o\left(\frac{e^{Im\lambda\sqrt{n}}}{n^{1/4}}\right) \text{ as } n \to \infty, \tag{19}$$

- *For* $\lambda \in \mathbb{R}$*,*

$$P_n(\lambda) = F(\lambda)I_n^-(\lambda) + \overline{F(\lambda)}I_n^+(\lambda) + o(n^{-\frac{1}{4}}) \text{ as } n \to \infty. \tag{20}$$

Proof. Let us rewrite (12) as

$$P_n(\lambda)\frac{a_{n-1}}{\sqrt{n-1}} \tag{21}$$
$$= \left(1 + \sum_{k=1}^{n-1}\frac{(\Lambda I^-(\lambda))_k P_k(\lambda)}{W(\lambda)}\right) I_n^+(\lambda) + \left(1 - \sum_{k=1}^{n-1}\frac{(\Lambda I^+(\lambda))_k P_k(\lambda)}{W(\lambda)}\right) I_n^-(\lambda).$$

From the estimates on $(\Lambda I^+(\lambda))_k$ and $P_k(\lambda)$ (17) and (16) it follows that

$$(\Lambda I^+(\lambda))_k P_k(\lambda) = O\left(\frac{|c_k|}{k} + \frac{|c_{k+1} - c_k| + |b_k|}{\sqrt{k}}\right) \text{ as } k \to \infty$$

uniformly with respect to λ on every compact subset of $\overline{\mathbb{C}_+}$. Hence the expression

$$F_n(\lambda) := 1 - \sum_{k=1}^{n-1}\frac{(\Lambda I^+(\lambda)_k P_k(\lambda)}{W(\lambda)}$$

converges as $n \to \infty$ to the function

$$F(\lambda) := 1 - \sum_{k=1}^{\infty}\frac{(\Lambda I^+(\lambda))_k P_k(\lambda)}{W(\lambda)}$$

analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}_+}$.

Consider $\lambda \in \mathbb{C}_+$. The first term in (21) is relatively small. Indeed,

$$\left| \frac{I_n^+(\lambda)\left(1+\sum\limits_{k=1}^{n-1}\frac{(\Lambda I^-(\lambda)_k P_k(\lambda)}{W(\lambda)}\right)}{I_n^-(\lambda)} \right| =$$

$$= O\left(e^{-2\mathrm{Im}\lambda\sqrt{n}} + \sum_{k=1}^{n-1} e^{2\mathrm{Im}\lambda(\sqrt{k}-\sqrt{n})}\left(\frac{|c_k|}{k} + \frac{|c_{k+1}-c_k|+|b_k|}{\sqrt{k}}\right)\right) = o(1)$$

as $n \to \infty$. This means that $\frac{P_n(\lambda)}{I_n^-(\lambda)} \to F(\lambda)$ as $n \to \infty$.

Consider $\lambda \in \mathbb{R}$. Equation (21) yields:

$$P_n(\lambda) = \left(F_n(\lambda)\frac{\sqrt{n-1}}{a_{n-1}}\right) I_n^-(\lambda) + \left(\overline{F_n(\lambda)}\frac{\sqrt{n-1}}{a_{n-1}}\right) I_n^+(\lambda).$$
$$= F(\lambda)I_n^-(\lambda) + \overline{F(\lambda)}I_n^+(\lambda) + o(n^{-\frac{1}{4}}) \text{ as } n \to \infty$$

due to asymptotics (8) of $I_n^{\pm}(\lambda)$ and the convergence of $F_n(\lambda)$. The proof is complete. □

The final step is the proof of the absolute continuity of the spectrum of $\mathcal{J}$ and the formula for the spectral density.

Lemma 6. *Let the condition* (2) *hold for* $\mathcal{J}$. *Then the spectrum of* $\mathcal{J}$ *is purely absolutely continuous and for a.a.* $\lambda \in \mathbb{R}$ *the following formula holds:*

$$\rho'(\lambda) = \frac{e^{-\frac{\lambda^2}{2}}}{\sqrt{2\pi}|F(\lambda)|^2}, \tag{22}$$

where $F(\lambda)$ *defined by* (18) *and does not vanish on* $\mathbb{R}$.

Proof. Polynomials of the second kind have asymptotics of the same type as polynomials of the first kind. Cropped Jacobi matrix $\mathcal{J}_1$ being the original one $\mathcal{J}$ with the first row and the first column removed,

$$\mathcal{J}_1 = \begin{pmatrix} b_2 & a_2 & 0 & \cdots \\ a_2 & b_3 & a_3 & \cdots \\ 0 & a_3 & b_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

satisfies conditions of Lemma 5. And the polynomials $a_1 Q_n(\lambda)$ are the polynomials of the first kind for $\mathcal{J}_1$, so there exists a function $F_1(\lambda)$, analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}_+}$, such that

- For $\lambda \in \mathbb{C}_+$, $Q_n(\lambda) = F_1(\lambda)I_n^-(\lambda) + o\left(\frac{e^{\mathrm{Im}\lambda\sqrt{n}}}{n^{1/4}}\right)$ as $n \to \infty$,

- For $\lambda \in \mathbb{R}$, $Q_n(\lambda) = F_1(\lambda)I_n^-(\lambda) + \overline{F_1(\lambda)}I_n^+(\lambda) + o(n^{-\frac{1}{4}})$ as $n \to \infty$.

The combination $Q_n(\lambda) + m(\lambda)P_n(\lambda)$ belongs to l^2 for $\lambda \in \mathbb{C}_+$, hence

$$m(\lambda) = -\frac{F_1(\lambda)}{F(\lambda)} \text{ for } \lambda \in \mathbb{C}_+.$$

Consider $\lambda \in \mathbb{R}$. One has:

$$\begin{aligned} 1 = W(P,Q) &= (\sqrt{n} + c_n)(P_nQ_{n+1} - P_{n+1}Q_n) \\ &= \sqrt{n}(I_n^+I_{n+1}^- - I_{n+1}^+I_n^-)(\overline{F}F_1 - F\overline{F}_1) + o(1) \\ &= W(I^+, I^-)(\overline{F}F_1 - F\overline{F}_1), \end{aligned}$$

therefore

$$F_1(\lambda)\overline{F(\lambda)} - \overline{F_1(\lambda)}F(\lambda) = -i\sqrt{2\pi}e^{-\frac{\lambda^2}{2}}$$

for $\lambda \in \mathbb{R}$, and hence for every $\lambda \in \overline{\mathbb{C}_+}$. It follows that $F(\lambda)$ and $F_1(\lambda)$ do not have zeros in $\overline{\mathbb{C}_+}$. For every $\lambda \in \mathbb{R}$ there exists the finite limit

$$m(\lambda + i0) = -\frac{F_1(\lambda)}{F(\lambda)},$$

which is continuous in λ. It follows then [14] that the spectrum of $\mathcal{J}$ is purely absolutely continuous and the spectral density equals for a.a. $\lambda \in \mathbb{R}$

$$\rho'(\lambda) = \frac{1}{\pi}\mathrm{Im}\, m(\lambda + i0) = \frac{F(\lambda)\overline{F_1(\lambda)} - \overline{F(\lambda)}F_1(\lambda)}{2\pi i|F(\lambda)|^2} = \frac{e^{-\frac{\lambda^2}{2}}}{\sqrt{2\pi}|F(\lambda)|^2},$$

which completes the proof. □

Theorem 2 follows directly from Lemmas 5 and 6. Let us repeat its formulation.

Theorem. *Let the conditions* (2) *hold for* $\mathcal{J}$. *Then*
1. For every $\lambda \in \overline{\mathbb{C}_+}$ *there exists*

$$F(\lambda) := 1 + i\sqrt{2\pi}e^{-\frac{\lambda^2}{2}}\sum_{n=1}^{\infty}(\Lambda I^+(\lambda))_n P_n(\lambda)$$

(the Jost function), which is analytic function in $\mathbb{C}_+$ *and continuous in* $\overline{\mathbb{C}_+}$.
2. Polynomials of the first kind have the following asymptotics as $n \to \infty$*:*

- *For* $\lambda \in \mathbb{C}_+$,

$$P_n(\lambda) = F(\lambda)I_n^-(\lambda) + o\left(\frac{e^{Im\lambda\sqrt{n}}}{n^{1/4}}\right) \text{ as } n \to \infty,$$

- *For* $\lambda \in \mathbb{R}$,

$$P_n(\lambda) = F(\lambda) I_n^-(\lambda) + \overline{F(\lambda)} I_n^+(\lambda) + o(n^{-\frac{1}{4}}) \text{ as } n \to \infty.$$

3. The spectrum of $\mathcal{J}$ *is purely absolutely continuous, and for a.a.* $\lambda \in \mathbb{R}$

$$\rho'(\lambda) = \frac{e^{-\frac{\lambda^2}{2}}}{\sqrt{2\pi} |F(\lambda)|^2}.$$

It remains to prove the following corollary.

Corollary. *Let the conditions* (2) *hold for* $\mathcal{J}$. *Then the spectrum of* $\mathcal{J}$ *is purely absolutely continuous and the spectral density equals for a.a.* $\lambda \in \mathbb{R}$

$$\rho'(\lambda) = \frac{1}{\pi} \lim_{n\to\infty} \frac{1}{\sqrt{n}(P_n^2(\lambda) + P_{n+1}^2(\lambda))}, \tag{23}$$

the right-hand side being finite and non-zero for every $\lambda \in \mathbb{R}$.

Proof. From the asymptotics (20) and (8) one has for $\lambda \in \mathbb{R}$:

$$P_n^2(\lambda) + P_{n+1}^2(\lambda) = \frac{4|F(\lambda)|^2 e^{\frac{\lambda^2}{2}}}{\sqrt{8\pi n}} + o\left(\frac{1}{\sqrt{n}}\right),$$

so

$$\frac{1}{|F(\lambda)|^2} = \sqrt{\frac{2}{\pi}} \frac{e^{\frac{\lambda^2}{2}}}{\lim\limits_{n\to\infty} \sqrt{n}(P_n^2(\lambda) + P_{n+1}^2)}.$$

Substituting into (22) gives the answer and completes the proof. □

4. Appendix. Asymptotics of derivatives of the error function

This section is devoted to finding asymptotics of $w^{(n)}(z)$ as $n \to \infty$. It is natural to prove a little wider result: asymptotics of $w^{(n-1)}(\mu\sqrt{2n})$ as $n \to \infty$ uniform with respect to the parameter μ in some neighbourhood of the point 0. Such asymptotics (with the scaled parameter) are called asymptotics of Plancherel-Rotach type, after [15], where the authors proved such asymptotics for Hermite polynomials. Let

$$\varphi(z) := z + \sqrt{z^2 - 1}$$

be the inverse Zoukowski function with the branch chosen such that $\varphi(0) = i$.

Theorem 3. *There exist μ_0 such that*

(24)

$$w^{(n-1)}(\mu\sqrt{2n}) = \sqrt{\frac{2}{n}}^{\,n} \frac{(n-1)!(-1)^{n-1}}{\sqrt{\pi}\sqrt{1-\varphi^2(\mu)}} \frac{e^{-\frac{n}{2}(\varphi(\mu)-2\mu)^2}}{(\varphi(\mu))^{n-1}} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

as $n \to \infty$ uniformly with respect to $|\mu| < \mu_0$.

Proof. One has from (5):

$$w^{(n-1)}(\mu\sqrt{2n}) = \frac{(-1)^n(n-1)!}{\pi i} \int_{\Gamma^+_{-\mu\sqrt{2n}}} \frac{e^{-\zeta^2} d\zeta}{(\zeta + \mu\sqrt{2n})^n},$$

for the contour Γ^+_z see Figure 1. Taking $\zeta = (z-\mu)\sqrt{\frac{n}{2}}$, one obtains:

$$w^{(n-1)}(\mu\sqrt{2n}) = (-1)^n \sqrt{\frac{2}{n}}^{\,n-1} \frac{(n-1)!}{\pi i} \int_{\Gamma^+_0} \frac{e^{-\frac{n}{2}(z-2\mu)^2}}{z^n} dz.$$

Let us denote

$$f(z,\mu) := -\frac{(z-2\mu)^2}{2} - \ln z.$$

This function has a critical point $z = \varphi(\mu)$ (the point where its derivative with respect to z turns to zero). Due to Taylor's expansion, for every μ

$$f(z,\mu) = f(\varphi(\mu),\mu) + \frac{f''(\varphi(\mu),\mu)}{2}(z-\varphi(\mu))^2 + O(z-\varphi(\mu))^3$$

as $z \to \varphi(\mu)$. Let us denote

$$a(\mu) := \sqrt{\tfrac{-2}{f''(\varphi(\mu),\mu)}} = \sqrt{\tfrac{2\varphi^2(\mu)}{\varphi^2(\mu)-1}},$$
$$s := \tfrac{z-\varphi(\mu)}{a(\mu)},$$
$$h(s,\mu) := f(a(\mu)s + \varphi(\mu),\mu) - f(\varphi(\mu),\mu)$$

and change the variable in the integral. Then one has to integrate over the contour $\{s = \frac{z-\varphi(\mu)}{a(\mu)}, z \in \Gamma^+_0\}$, which can be transformed into the real line for values of μ small enough (since $\varphi(\mu) \to i$ and $a(\mu) \to 1$ as $\mu \to 0$, so the point $s = -\frac{\varphi(\mu)}{a(\mu)} \to -i$ corresponds to the point $z = 0$). One will have:

$$w^{(n-1)}(\mu\sqrt{2n}) = (-1)^n \sqrt{\frac{2}{n}}^{\,n-1} \frac{(n-1)!}{\pi i} a(\mu) e^{nf(\varphi(\mu),\mu)} \int_{-\infty}^{+\infty} e^{nh(s,\mu)} ds$$
$$= (-1)^{n-1} \sqrt{\frac{2}{n}}^{\,n-1} \frac{\sqrt{2}(n-1)!}{\pi\sqrt{1-\varphi^2(\mu)}} \frac{e^{-\frac{n}{2}(\varphi(\mu)-2\mu)^2}}{(\varphi(\mu))^{n-1}} \int_{-\infty}^{+\infty} e^{nh(s,\mu)} ds.$$

It remains to prove the following lemma.

Lemma 7. *There exists* μ_1 *such that*

$$\int_{-\infty}^{+\infty} e^{nh(s,\mu)}ds = \sqrt{\frac{\pi}{n}} + O\left(\frac{1}{n}\right) \text{ as } n \to \infty$$

uniformly with respect to $|\mu| < \mu_1$.

Proof. We divide the proof into three parts.

1. Let us see that

$$\int_{-n^{-3/8}}^{n^{-3/8}} e^{nh(s,\mu)}ds = \sqrt{\frac{\pi}{n}} + O\left(\frac{1}{n}\right) \text{ as } n \to \infty$$

uniformly with respect to μ in some neighbourhood of 0. One has:

$$h(s,\mu) = -\frac{(a(\mu)s)^2}{2} - a(\mu)s(\varphi(\mu) - 2\mu) - \ln\left(1 + \frac{a(\mu)s}{\varphi(\mu)}\right).$$

Note that

$$h(0,\mu) \equiv 0,\ h'_s(0,\mu) \equiv 0,\ h''_{ss}(0,\mu) \equiv -2.$$

Hence for every $k \geq 0$

$$\frac{\partial^k h}{\partial \mu^k}(0,0) = \frac{\partial^{k+1} h}{\partial s \partial \mu^k}(0,0) = \frac{\partial^{k+3} h}{\partial s^2 \partial \mu^{k+1}}(0,0) = 0.$$

The function $h(s,\mu)$ is C^∞ at $(0;0)$, so

$$h(s,\mu) = -s^2 + O(s^3) \text{ as } s, \mu \to 0$$

(i.e., there exist C_1, δ_1 such that if $|s|, |\mu| < \delta_1$, then $|h(s,\mu) + s^2| < C_1|s|^3$). This obviously in particular means that there exists $\delta_0 > 0$ such that if $-\delta_0 < s < \delta_0$ and $|\mu| < \delta_0$, then

$$\begin{cases} |h(s,\mu) + s^2| < C_1|s|^3 \\ \text{Re } h(s,\mu) < -\frac{s^2}{2}. \end{cases} \tag{25}$$

One has:

$$\int_{-n^{-3/8}}^{n^{-3/8}} e^{nh(s,\mu)}ds - \sqrt{\frac{\pi}{n}}$$
$$= \int_{-n^{-3/8}}^{n^{-3/8}} (e^{nh(s,\mu)} - e^{-ns^2})ds - \left(\int_{-\infty}^{-n^{-3/8}} + \int_{n^{-3/8}}^{+\infty}\right) e^{-ns^2}ds.$$

Since for every $\alpha, \beta > 0$,

$$\int_x^{+\infty} t^\alpha e^{-\beta t^2}dt = O(x^{\alpha+1}e^{-\beta x^2}) \text{ as } x \to +\infty, \tag{26}$$

one has:

$$\left(\int_{-\infty}^{-n^{-3/8}}+\int_{n^{-3/8}}^{+\infty}\right)e^{-ns^2}ds=O\left(\frac{1}{n}\right)\quad\text{as } n\to\infty.$$

Let $|s|<\min\{n^{-\frac{3}{8}};\delta_0\}$ and $|\mu|<\delta_0$. Then

$$n|h(s,\mu)+s^2|<C_1n|s|^3<\frac{C_1}{n^{\frac{1}{8}}}$$

and there exists N_1 such that

if $n>N_1$, $|s|<n^{-3/8}$ and $|\mu|<\delta_0$,
then $|e^{n(h(s,\mu)+s^2)}-1|<2C_1n|s|^3$.

Hence we arrive at the following (uniform for $|\mu|<\delta_0$) estimate:

$$\left|\int_{-n^{-3/8}}^{n^{-3/8}}(e^{nh(s,\mu)}-e^{-ns^2})ds\right|\le\int_{-n^{-3/8}}^{n^{-3/8}}e^{-ns^2}|e^{n(h(s,\mu)+s^2)}-1|ds$$
$$<2C_1n\int_{-n^{-3/8}}^{n^{-3/8}}|s|^3e^{-ns^2}ds=O\left(\frac{1}{n}\right)\quad\text{as } n\to\infty$$

from (26).

2. The following is an immediate consequence of (25) and (26): for $|\mu|<\delta_0$,

$$\left|\left(\int_{-\delta_0}^{-n^{-3/8}}+\int_{n^{-3/8}}^{\delta_0}\right)e^{nh(s,\mu)}ds\right|<2\int_{n^{-3/8}}^{\delta_0}e^{-\frac{ns^2}{2}}ds$$
$$<\frac{2}{\sqrt{n}}\int_{n^{1/8}}^{+\infty}e^{-\frac{t^2}{2}}dt=O\left(\frac{1}{n}\right)\quad\text{as } n\to\infty$$

uniformly with respect to μ.

3. Let us prove that

$$\left(\int_{-\infty}^{-\delta_0}+\int_{\delta_0}^{+\infty}\right)e^{nh(s,\mu)}ds=O\left(\frac{1}{n}\right)\quad\text{as } n\to\infty$$

uniformly with respect to μ in some neighbourhood of 0. Consider the real part of the last term in

$$h(s,\mu)=-\frac{(a(\mu)s)^2}{2}-a(\mu)s(\varphi(\mu)-2\mu)-\ln\left(1+\frac{a(\mu)s}{\varphi(\mu)}\right).\tag{27}$$

One has

$$\operatorname{Re}\ln\left(1+\frac{a(\mu)s}{\varphi(\mu)}\right)=\ln|i+\Gamma(\mu)s|,$$

where

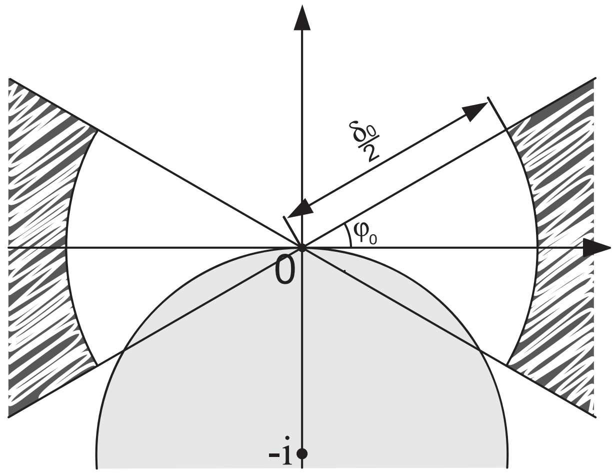

FIGURE 2. The plane of the parameter s

$$\gamma(\mu) := \frac{ia(\mu)}{\varphi(\mu)}.$$

Consider $s \in (-\infty; -\delta_0) \cup (\delta_0; +\infty)$. There exists an angle φ_0 (small enough) such that the domains shown on Figure 2 do not intersect. Since $\gamma(\mu) \to 1$ as $\mu \to 0$, there exists $\mu_1 < \delta_0$ such that if $|\mu| < \mu_1$, then $|\gamma(\mu)| > \frac{1}{2}$ and $|\arg\gamma(\mu)| < \varphi_0$. Then $|i + \gamma(\mu)s| > 1$. Let $\theta := \frac{1}{3}$. By the choice of μ_1 we can also ensure that if $|\mu| < \mu_1$, then

$$\begin{cases} \operatorname{Re} a^2(\mu) > \frac{1}{2}, \\ \operatorname{Re}[a(\mu)(\varphi(\mu) - 2\mu)] > -\frac{\delta_0\theta}{2} \end{cases}$$

and hence

$$\operatorname{Re} h(s,\mu) < -\frac{1}{4}(s^2 - 2s\delta_0\theta)$$

for every real s such that $|s| > \delta_0$. One has:

$$\left|\left[\int_{-\infty}^{-\delta_0} + \int_{\delta_0}^{+\infty}\right] e^{nh(s,\mu)} ds\right| < 2\int_{\delta_0}^{+\infty} e^{-\frac{n}{4}(s^2 - 2s\delta_0\theta)} ds$$

$$= 2e^{\frac{n}{4}\delta_0^2\theta^2} \int_{\delta_0(1-\theta)}^{+\infty} e^{-\frac{n}{4}s^2} ds = O\left(e^{\frac{n}{4}\delta_0^2(2\theta-1)}\right) = O\left(\frac{1}{n}\right) \text{ as } n \to \infty$$

uniformly with respect to μ. This completes the proof of the lemma. □

□

As a corollary we have asymptotics of $w^{(n-1)}(z)$ as $n \to \infty$ for fixed z.

Corollary 2.

$$w^{(n-1)}(z)=\sqrt{\frac{2}{n}}^{\,n}\frac{(n-1)!i^{n-1}}{\sqrt{2\pi}}e^{\frac{n}{2}+iz\sqrt{2n}-\frac{z^2}{2}}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)\ \textit{as } n\to\infty$$

uniformly with respect to z *in every bounded set in* $\mathbb{C}$.

Proof. We just need to substitute $\mu=\frac{z}{\sqrt{2n}}$ into (24) and go through tedious calculation, using that

$$\varphi(z)=i+z-\frac{iz^2}{2}+O(z^4)\text{ as }z\to 0.$$

□

5. Acknowledgement

The author expresses his deep gratitude to Dr. A.V. Kiselev for valuable discussions of the problem and to Prof. S.N. Naboko for his constant attention to this work and for many fruitful discussions of the subject. The author also wishes to thank Prof. V.S. Buslaev for constructive criticism. The work was supported by grants RFBR-06-01-00249 and INTAS-05-1000008-7883, and also by Vladimir Deich prize.

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Department of Mathematical Physics, Institute of Physics, St. Petersburg University, Ulianovskaia 1, 198904, St. Petergoff, St. Petersburg, Russia

E-mail address: sergey_simonov@mail.ru