



We call  $\{a_n, b_n\}_{n=1}^{\infty}$  the Jacobi parameters for a half-line matrix and  $\{a_n, b_n\}_{n=-\infty}^{\infty}$  the Jacobi parameters for a whole-line matrix.

We will call  $J$  *measure theoretically reflectionless* on a Borel set  $\mathfrak{e} \subset \mathbb{R}$  if and only if for all  $n$ , the diagonal Green's function,

$$G_{nn}(\lambda + i0) = \lim_{\varepsilon \downarrow 0} \langle \delta_n, (J - \lambda - i\varepsilon)^{-1} \delta_n \rangle \quad (1.3)$$

is pure imaginary, that is,

$$\operatorname{Re} G_{nn}(\lambda + i0) = 0 \quad (1.4)$$

for Lebesgue a.e.  $\lambda \in \mathfrak{e}$ . Among the vast literature, we mention [6, 8, 9, 11, 12, 16, 20, 22, 25, 26, 27, 28, 29, 30, 31, 32, 36, 37, 41, 42, 43, 44, 45]. The name ‘‘reflectionless’’ is usually used without ‘‘measure theoretically’’ but we add this for reasons that will be clear shortly.

The notion first became commonly used in connection with solitons and has recently become especially important because of Remling's discovery [37] that right limits of half-line Jacobi matrices are measure theoretically reflectionless on  $\Sigma_{\text{ac}}$ , the essential support of the a.c. component of the half-line Jacobi matrix. The name comes from the fact that in the short-range case (i.e.,  $|a_n - 1| + |b_n| \rightarrow 0$  sufficiently rapidly as  $|n| \rightarrow \infty$ ), the condition is equivalent to the time-independent reflection coefficient being zero on  $\mathfrak{e}$ .

There is a second notion of reflectionless operator depending on ideas of Davies–Simon [7]. For each  $n \in \mathbb{Z}$ , let  $\chi_n^+$  be the characteristic function of  $[n, \infty)$  and  $\chi_n^-$  of  $(-\infty, n]$ . We define

$$\mathcal{H}_\ell^+ = \{ \varphi \in \mathcal{H}_{\text{ac}} \mid \text{for all } n, \lim_{t \rightarrow -\infty} \|\chi_n^+ e^{-itJ} \varphi\| = 0 \} \quad (1.5)$$

that is, states that, as  $t \rightarrow -\infty$ , are concentrated on the left.  $\mathcal{H}_\ell^-$  is the same with  $\lim_{t \rightarrow +\infty}$ , and  $\mathcal{H}_r^\pm$  are defined using  $\chi_n^-$ . Here  $\mathcal{H}_{\text{ac}}$  is the a.c. subspace for  $J$ . We let  $P_{\text{ac}}$  be the projection onto  $\mathcal{H}_{\text{ac}}$ , and let  $P_{\ell,r}^\pm$  be the orthogonal projection onto  $\mathcal{H}_{\ell,r}^\pm$ , that is,

$$P_\ell^\pm = \text{s-lim}_{t \rightarrow \mp\infty} e^{itJ} \chi_0^- e^{-itJ} P_{\text{ac}}(J) \quad P_r^\pm = \text{s-lim}_{t \rightarrow \mp\infty} e^{itJ} \chi_0^- e^{-itJ} P_{\text{ac}}(J) \quad (1.6)$$

Davies–Simon prove (they treat the analog for Schrödinger operators, but the argument is identical):

**Theorem 1.1** ([7]). *We have ( $\oplus =$  orthogonal direct sum)*

$$\mathcal{H}_{\text{ac}} = \mathcal{H}_\ell^+ \oplus \mathcal{H}_r^+ \quad (1.7)$$

$$= \mathcal{H}_\ell^- \oplus \mathcal{H}_r^- \quad (1.8)$$

*That is, any a.c. state is a sum of a state that moves entirely to the left as  $t \rightarrow -\infty$  and one that moves to the right.*

We call  $J$  *dynamically reflectionless* on a Borel set  $\mathfrak{e}$  if and only if

$$P_{\mathfrak{e}}P_{\text{ac}} = P_{\mathfrak{e}} \quad (1.9)$$

(here  $P_{\mathfrak{e}}$  is the spectral projection for  $J$ ) and

$$P_{\mathfrak{e}}[\mathcal{H}_{\ell}^+] = P_{\mathfrak{e}}[\mathcal{H}_{\ell}^-] \quad (1.10)$$

Before stating our main theorem, we want to define a third notion of reflectionless operator for reasons that will become clear momentarily. For any  $n \in \mathbb{Z}$ , let  $J_n^+$  be the Jacobi matrix obtained from dropping the row and column with  $b_n$  and keeping the lower right piece, that is,  $J_n^+$  is the one-sided Jacobi matrix with Jacobi parameters

$$b_{\ell}^{(n),+} = b_{n+\ell} \quad a_{\ell}^{(n),+} = a_{n+\ell} \quad (1.11)$$

$J_n^-$  has parameters

$$b_{\ell}^{(n),-} = b_{n+1-\ell} \quad a_{\ell}^{(n),-} = a_{n-\ell} \quad (1.12)$$

Thus, if  $a_n$  is replaced by 0, the whole-line Jacobi matrix  $J$  breaks into a direct sum of  $J_n^+$  and a matrix unitarily equivalent to  $J_n^-$  after reordering the indices in inverse order.

For any half-line Jacobi matrix,  $J$ , we define its  $m$ -function by

$$m(z, J) = \langle \delta_1, (J - z)^{-1} \delta_1 \rangle \quad (1.13)$$

and for a whole-line Jacobi matrix,

$$m_n^{\pm}(z, J) = m(z, J_n^{\pm}) \quad (1.14)$$

These are related to the Green's function (1.3) by

$$G_{nn}(z) = -\frac{1}{a_n^2 m_n^+(z) - m_n^-(z)^{-1}} \quad (1.15)$$

We call a whole-line Jacobi matrix *spectrally reflectionless* on a Borel set  $\mathfrak{e}$  if for a.e.  $\lambda \in \mathfrak{e}$  and all  $n$ ,

$$a_n^2 m_n^+(\lambda + i0) \overline{m_n^-(\lambda + i0)} = 1 \quad (1.16)$$

By (1.15), (1.16) implies  $\text{Re } G_{nn} = 0$ , so

$$(1.16) \text{ for } \lambda \text{ and } n \Rightarrow (1.4) \text{ for } \lambda \text{ and } n$$

and so

$$\begin{aligned} J \text{ is spectrally reflectionless on } \mathfrak{e} &\Rightarrow \\ J \text{ is measure theoretically reflectionless on } \mathfrak{e} & \end{aligned} \quad (1.17)$$

Moreover, as we will see below,

$$(1.16) \text{ for } \lambda \text{ and one } n \Rightarrow (1.16) \text{ for } \lambda \text{ and all } n \quad (1.18)$$

This set of ideas is rounded out by the following theorem:

**Theorem 1.2** (Gesztesy–Krishna–Teschl [11]; Sodin–Yuditskii [45]). *If (1.4) holds for a.e.  $\lambda \in \mathfrak{e}$  and three consecutive values of  $n$ , then (1.16) holds for a.e.  $\lambda \in \mathfrak{e}$  and all  $n$ .*

In particular, in (1.17),  $\Rightarrow$  can be replaced by  $\Leftrightarrow$ . However, this is not true for CMV matrices [4].

Here is our main result:

**Theorem 1.3.** *For any whole-line Jacobi matrix  $J$  and Borel set  $\mathfrak{e}$  of positive Lebesgue measure,  $J$  is spectrally reflectionless on  $\mathfrak{e}$  if and only if it is dynamically reflectionless on  $\mathfrak{e}$ .*

This verifies a 25-year old conjecture of Deift–Simon [9], namely

**Corollary 1.4.** *The a.c. spectrum for two-sided ergodic Jacobi matrices is dynamically reflectionless.*

*Proof.* By Kotani theory [25, 38], such operators are spectrally reflectionless on the a.c. spectrum.  $\square$

This is a special case of a more general result that we will prove concerning reflection probability. Let  $\Sigma_{\text{ac}}^{(2)}$  be the set of  $\lambda \in \mathbb{R}$  where  $J$  has multiplicity 2, so automatically a.c. spectrum (see [21, 23, 24, 39]).  $P_{\ell,r}^{\pm}$  commute with  $J$ , so they take  $\text{Ran}(P_{\Sigma_{\text{ac}}^{(2)}}(J))$  to itself.  $J$  restricted to  $\text{Ran}(P_{\ell,r}^{\pm} P_{\Sigma_{\text{ac}}^{(2)}}(J))$  is of multiplicity 1. Thus,

$$R = P_{\ell}^{+} P_{\ell}^{-} P_{\ell}^{+} \upharpoonright \text{Ran}(P_{\ell}^{+} P_{\Sigma_{\text{ac}}^{(2)}}(J)) \quad (1.19)$$

is a scalar function of  $J$ , and so there is a function  $R(E)$  on  $\Sigma_{\text{ac}}^{(2)}$  so that

$$R = R(J) \upharpoonright \text{Ran}(P_{\ell}^{+} P_{\Sigma_{\text{ac}}^{(2)}}(J)) \quad (1.20)$$

As defined by Davies–Simon [7],  $R(\lambda)$  is the *dynamic reflection probability*, the probability that a state of energy  $\lambda$  that comes in from the left at very negative times goes out on the left. There is a time-reversal symmetry, namely that one gets the same function,  $R$ , with  $P_{\ell}^{-} P_{\ell}^{+} P_{\ell}^{-} \upharpoonright \text{Ran}(P_{\ell}^{-})$ . Similarly, there is a left-right symmetry, so one gets the same function with  $P_r^{+} P_r^{-} P_r^{+} \upharpoonright \text{Ran}(P_r^{+})$ .

Define the *spectral reflection probability* by (see Theorem 2.4 below for why this is a good definition)

$$\left| \frac{a_0^2 m_0^+(\lambda + i0) \overline{m_0^-(\lambda - i0)} - 1}{a_0^2 m_0^+(\lambda + i0) m_0^-(\lambda - i0) - 1} \right|^2 \quad (1.21)$$

We will prove

**Theorem 1.5.**  *$R(\lambda)$  is given by (1.21) on  $\Sigma_{\text{ac}}^{(2)}(J)$ .*

Theorem 1.5 implies Theorem 1.3 since

$$R(J) \upharpoonright \mathfrak{e} = 0 \Leftrightarrow P_\ell^+ P_\mathfrak{e} = P_r^- P_\mathfrak{e} \quad (1.22)$$

and

$$(1.21) = 0 \Leftrightarrow (1.16) \text{ holds} \quad (1.23)$$

The various formulae involving  $m_n^\pm$  are complicated, in part because the simple formulae are given by Weyl solutions. It pays to rewrite them here since the rewriting is critical to our proof.

We are interested in solutions of

$$a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1} = z u_n \quad (1.24)$$

For any  $z \in \mathbb{C}_+ = \{z \mid \text{Im } z > 0\}$ , there are solutions  $u_n^\pm(z)$  which are  $\ell^2$  at  $\pm\infty$ , unique up to a constant. We will normalize by

$$u_0^\pm = 1 \quad (1.25)$$

By general principles (see, e.g., [46, Chap. 2], though our notation is slightly different from his), for Lebesgue a.e.  $\lambda$ ,  $u_n^\pm(\lambda + i\varepsilon)$  has a limit as  $\varepsilon \downarrow 0$ , which we denote by  $u_n^\pm(\lambda + i0)$  which solves (1.24) at  $\lambda$ .

$m^\pm$  can be expressed in terms of  $u^\pm$  by ([46])

$$m_n^+(\lambda + i0) = -\frac{u_{n+1}^+(\lambda + i0)}{a_n u_n^+(\lambda + i0)} \quad (1.26)$$

$$m_n^-(\lambda + i0) = -\frac{u_n^-(\lambda + i0)}{a_n u_{n+1}^-(\lambda + i0)} \quad (1.27)$$

The Green's function, (1.3), which is symmetric, is given for  $n \leq m$  by

$$G_{nm}(\lambda + i0) = \frac{u_n^-(\lambda + i0) u_m^+(\lambda + i0)}{W(\lambda + i0)} \quad (1.28)$$

where

$$W(z) = a_n [u_{n+1}^+(z) u_n^-(z) - u_{n+1}^-(z) u_n^+(z)] \quad (1.29)$$

is  $n$ -independent.

From these formulae, (1.15) is immediate. Moreover, with the normalization  $u_{n=0}^\pm = 1$ , we see that (1.16) is equivalent to  $u_{n=1}^+(\lambda + i0) = \overline{u_{n=1}^-(\lambda + i0)}$  which, by uniqueness of solutions, implies

$$u_n^+(\lambda + i0) = \overline{u_n^-(\lambda + i0)} \quad (1.30)$$

for all  $n$ . This explains why (1.18) holds. It shows that

$$J \text{ is spectrally reflectionless for } \lambda \in \mathfrak{e} \Leftrightarrow (1.30) \text{ for } \lambda \in \mathfrak{e} \quad (1.31)$$

The key to our proof of Theorem 1.3 (and also Theorem 1.5) will be

**Almost-Theorem 1.6.**  $\text{Ran}(P_\ell^+ P_{\Sigma_{\text{ac}}^{(2)}})$  is spanned by  $\{u_n^+(\lambda + i0) \mid \lambda \in \Sigma_{\text{ac}}^{(2)}\}$  and  $\text{Ran}(P_r^+ P_{\Sigma_{\text{ac}}^{(2)}})$  by  $\{u_n^-(\lambda + i0) \mid \lambda \in \Sigma_{\text{ac}}^{(2)}\}$ .

We call this an almost-theorem because we are, for now, vague about what we mean by “span.” The  $u_n^\pm$  are only continuum eigenfunctions, so by span we will mean suitable integrals.

We can now understand why the almost-theorem will imply Theorem 1.3. By time-reversal invariance,

$$P_r^- = \overline{P_r^+} \quad (1.32)$$

Thus,

$$J \text{ is dynamically reflectionless for } \lambda \in \mathfrak{e} \Leftrightarrow \overline{P_r^+} P_{\Sigma_{\text{ac}}^{(2)}} = P_\ell^+ P_{\Sigma_{\text{ac}}^{(2)}} \quad (1.33)$$

and the almost-theorem says the right side is the same as (1.30).

For short-range perturbations of the free Jacobi matrix ( $b_n \equiv 0$ ,  $a_n \equiv 1$ ), the almost-theorem follows from suitable stationary phase/integration by parts ideas as noted in Davies–Simon [7]. Such methods cannot work for general Jacobi matrices where  $\Sigma_{\text{ac}}^{(2)}$  might be a positive measure Cantor set. What we will see is by replacing the limit

$$P_\ell^+ = \text{s-lim}_{t \rightarrow -\infty} e^{itJ} \chi_0^- e^{-itJ} P_{\text{ac}}(J) \quad (1.34)$$

that Davies–Simon [7] use by an abelian limit, a simple calculation will yield the almost-theorem.

Section 2 proves all the above results for Jacobi matrices. Section 3 discusses (continuum) Schrödinger operators and Section 4 CMV matrices.

## 2. THE JACOBI CASE

In this section, we prove Almost-Theorem 1.6 and use it to prove Theorem 1.5, and thereby Theorem 1.3. To make sense of Almost-Theorem 1.6, we need to begin with an eigenfunction expansion. While this expansion can be viewed as a rephrasing of Section 2.5 of Teschl [46], it is as easy to establish it from first principles as to manipulate the results of [46] to the form we need. Our use of Stone’s formula is similar to that of Gesztesy–Zinchenko [18].

Fundamental to this is the matrix for  $\lambda \in \mathbb{R}$ ,

$$S(\lambda)_{nm} = \lim_{\varepsilon \downarrow 0} (2\pi i)^{-1} [(J - \lambda - i\varepsilon)^{-1} - (J - \lambda + i\varepsilon)^{-1}]_{nm} \quad (2.1)$$

defined for a.e.  $\lambda \in \mathbb{R}$  and all  $n, m$ . We use  $S$  for “Stone” or “spectral” since Stone’s formula (Thm. VII.13 of [34]) and the spectral theorem imply that for any  $\varphi, \psi$  of finite support on  $\mathbb{Z}$  and any Borel set,  $\mathfrak{e}$ ,

$$\langle \varphi, P_{\mathfrak{e}} P_{\text{ac}} \psi \rangle = \int_{\lambda \in \mathfrak{e}} \left( \sum_{n,m} \bar{\varphi}_n \psi_m S(\lambda)_{nm} \right) d\lambda \quad (2.2)$$

Define for  $\lambda \in \Sigma_{\text{ac}}^{(2)}$ ,

$$f_{\pm}(\lambda) = \pm \frac{a_0 \operatorname{Im}(u_1^{\mp}(\lambda + i0))}{\pi |W(\lambda + i0)|^2} \quad (2.3)$$

where  $u_n^{\pm}$  is normalized by (1.25) and  $W$  is given by (1.29). This looks asymmetric in  $\pm$ , but

$$f_+(\lambda) = -\frac{a_{-1} \operatorname{Im}(u_{-1}^-(\lambda + i0))}{\pi |W(\lambda + i0)|^2} \quad (2.4)$$

$$= \frac{a_{-1}^2 \operatorname{Im}(m_{-1}^-(\lambda + i0))}{\pi |W(\lambda + i0)|^2} \quad (2.5)$$

while

$$f_-(\lambda) = \frac{a_0^2 \operatorname{Im}(m_0^+(\lambda + i0))}{\pi |W(\lambda + i0)|^2} \quad (2.6)$$

symmetric under reflection about  $n = 0$ . This makes it clear that

$$f_{\pm}(\lambda) > 0 \quad \text{a.e. } \lambda \in \Sigma_{\text{ac}}^{(2)} \quad (2.7)$$

The key to our eigenfunction expansion is

$$S_{nm}(\lambda) = \overline{u_n^+(\lambda + i0)} u_m^+(\lambda + i0) f_+(\lambda) + \overline{u_n^-(\lambda + i0)} u_m^-(\lambda + i0) f_-(\lambda) \quad (2.8)$$

for all  $n, m$  and a.e.  $\lambda \in \Sigma_{\text{ac}}^{(2)}$ .

**Theorem 2.1.** (2.8) holds for all  $n, m$  and a.e.  $\lambda \in \Sigma_{\text{ac}}^{(2)}$ .

*Proof.* By general principles on limits of Stieltjes transforms, for a.e.  $\lambda \in \Sigma_{\text{ac}}^{(2)}$ ,  $\lim_{\varepsilon \downarrow 0} u_n^{\pm}(\lambda + i\varepsilon) = u_n^{\pm}(\lambda + i0)$  exists. We will prove (2.8) for such  $\lambda$ . It is easy to see that  $S_{nm}(\lambda) = S_{mn}(\lambda)$ , so it suffices to consider the case  $n \leq m$ .

By the resolvent formula, for  $\operatorname{Im} z > 0$ ,

$$\begin{aligned} \pi S_{nm}(z) &\equiv (2i)^{-1} [(J - z)^{-1} - (J - \bar{z})^{-1}]_{nm} \\ &= (\operatorname{Im} z) \sum_k (J - \bar{z})_{nk}^{-1} (J - z)_{km}^{-1} \end{aligned} \quad (2.9)$$

$$= (\operatorname{Im} z) |W(z)|^{-2} (t_{nm}^{(1)} + t_{nm}^{(2)} + t_{nm}^{(3)}) \quad (2.10)$$

by (1.28), where

$$t_{nm}^{(1)} = \left[ \sum_{k \leq n} |u_k^-(z)|^2 \right] \overline{u_n^+(z)} u_m^+(z) \quad (2.11)$$

$$t_{nm}^{(2)} = \left[ \sum_{k \geq m+1} |u_k^+(z)|^2 \right] \overline{u_n^-(z)} u_m^-(z) \quad (2.12)$$

$$t_{nm}^{(3)} = \left[ \sum_{k=n+1}^m \overline{u_k^+(z)} u_k^-(z) \right] \overline{u_n^-(z)} u_m^+(z) \quad (2.13)$$

Because of the  $\text{Im } z$  in front of (2.10),  $\lim(\text{Im } z)t_{nm}^{(3)}(\lambda + iy) = 0$  since the limit exists (the sum is finite). Similarly, we can change the summation limits of the  $k$  sums in  $t^{(1)}, t^{(2)}$  to any other finite value, since in the limit, finite sums multiplied by  $\text{Im } z$  go to zero. The result is

$$S_{nm}(\lambda + i0) = q^{(1)}(\lambda) \overline{u_n^+(\lambda + i0)} u_m^+(\lambda + i0) + q^{(2)}(\lambda) \overline{u_n^-(\lambda + i0)} u_m^-(\lambda + i0) \quad (2.14)$$

where

$$\pi q^{(1)}(\lambda) = \lim_{\varepsilon \downarrow 0} |W(\lambda + i0)|^{-2} \varepsilon \sum_{k \leq -1} |u_k^-(\lambda + i\varepsilon)|^2 \quad (2.15)$$

$$\pi q^{(2)}(\lambda) = \lim_{\varepsilon \downarrow 0} |W(\lambda + i0)|^{-2} \varepsilon \sum_{k \geq 1} |u_k^+(\lambda + i0)|^2 \quad (2.16)$$

By the resolvent formula for  $J_0^+$  and the analog of (1.28) (with the normalization (1.25)),

$$\begin{aligned} \text{Im } m_0^+(z) &= \text{Im}(J_0^+ - z)_{11}^{-1} \\ &= (\text{Im } z) \sum_{k=1}^{\infty} (J_0^+ - \bar{z})_{1k}^{-1} (J_0^+ - z)_{k1}^{-1} \\ &= (\text{Im } z) a_0^2 \sum_{k=1}^{\infty} |u_k^+(z)|^2 \end{aligned} \quad (2.17)$$

so

$$q^{(2)}(\lambda) = f_-(\lambda) \quad (2.18)$$

and similarly,

$$q^{(1)}(\lambda) = f_+(\lambda) \quad (2.19)$$

This proves (2.8).  $\square$

From (2.8), we immediately get an eigenfunction expansion.

**Theorem 2.2.** *For any  $\varphi \in \ell^2(\mathbb{Z})$  of finite support, define*

$$\widehat{\varphi}_{\pm}(\lambda) = \sum_n \overline{u_n^{\pm}(\lambda)} \varphi_n \quad (2.20)$$

as functions on  $\Sigma_{\text{ac}}^{(2)}$ . Then

$$\int_{\Sigma_{\text{ac}}^{(2)}} [|\widehat{\varphi}_+(\lambda)|^2 f_+(\lambda) + |\widehat{\varphi}_-(\lambda)|^2 f_-(\lambda)] d\lambda = \|P_{\text{ac}} P_{\Sigma_{\text{ac}}^{(2)}} \varphi\|^2 \quad (2.21)$$

So  $\widehat{\varphi}_\pm$  extend to continuous maps of  $\ell^2(\mathbb{Z})$  to  $L^2(\Sigma_{\text{ac}}^{(2)}, f_\pm d\lambda)$ . Moreover, if  $\widehat{\varphi} = (\widehat{\varphi}_+, \widehat{\varphi}_-)$ , then

$$\widehat{(J\varphi)}_\pm(\lambda) = \lambda \widehat{\varphi}_\pm(\lambda) \quad (2.22)$$

For each  $n$ ,

$$\int_{\Sigma_{\text{ac}}^{(2)}} |u_n^\pm(\lambda)|^2 f_\pm(\lambda) d\lambda \leq 1 \quad (2.23)$$

In particular, for any

$$g = (g_+, g_-) \in L^2(\Sigma_{\text{ac}}^{(2)}, f_+ d\lambda) \oplus L^2(\Sigma_{\text{ac}}^{(2)}, f_- d\lambda) \equiv \mathcal{H}_J$$

and any  $n$ , we can define

$$\check{g}_n = \int g_+(\lambda) u_n^+(\lambda + i0) f_+(\lambda) d\lambda + \int g_-(\lambda) u_n^-(\lambda + i0) f_-(\lambda) d\lambda \quad (2.24)$$

$\check{g}$  lies in  $\ell^2(\mathbb{Z})$ , and for any  $\varphi \in \ell^2$ ,

$$\langle \check{g}, \varphi \rangle = \langle g, \widehat{\varphi} \rangle \quad (2.25)$$

and

$$\widehat{\check{g}} = g \quad (2.26)$$

We have  $\check{g} \in \text{Ran}(P_{\text{ac}} P_{\Sigma_{\text{ac}}^{(2)}})$  and  $\check{\cdot}$  is a bijection of this range and  $\mathcal{H}_J$ .

*Proof.* (2.21) is immediate from (2.2) and (2.8). (2.22) follows from a summation by parts and

$$\sum_m J_{nm} u_m^\pm(\lambda + i0) = \lambda u_n^\pm(\lambda + i0) \quad (2.27)$$

(2.23) comes from putting  $\delta_n$  into (2.21).

By (2.23), the integrals in (2.24) converge for all  $g \in \mathcal{H}_J$ . For  $\varphi$  of finite support, (2.25) is an interchange of integration and finite sum. In particular, if  $\chi_N$  is the characteristic function of  $\{j \in \mathbb{Z} \mid |j| \leq N\}$  and  $\varphi = \chi_N \check{g}$ , (2.25) implies

$$\begin{aligned} \sum_{|j| \leq N} |\check{g}_j|^2 &\leq \|g\| \|\widehat{\varphi}\| \\ &\leq \|g\| \|\varphi\| \\ &= \|g\| \left( \sum_{|j| \leq N} |g_j|^2 \right)^{1/2} \end{aligned} \quad (2.28)$$

so for all  $N$ ,

$$\|\chi_N \check{g}\| \leq \|g\| \quad (2.29)$$

so  $\check{g} \in \ell^2$  and

$$\|\check{g}\| \leq \|g\| \quad (2.30)$$

Thus, (2.25) extends to all  $\varphi$  by continuity.

By (2.21) and (2.22),  $\hat{\cdot}$  is a unitary spectral representation for  $\tilde{J} = J \upharpoonright \text{Ran}(P_{\text{ac}}P_{\Sigma_{\text{ac}}^{(2)}})$  on  $\text{Ran}(\hat{\cdot})$ . Since  $\tilde{J}$  has uniform multiplicity 2,  $\text{Ran}(\hat{\cdot})$  must be all  $\mathcal{H}_J$ . It follows that  $(\hat{\cdot})(\hat{\cdot})^* = \mathbf{1}$  on  $\mathcal{H}_J$ . Since  $\check{\cdot} = (\hat{\cdot})^*$ , this is (2.26).  $\square$

We can now prove a precise version of Almost-Theorem 1.6. Let

$$\mathcal{H}_J^\pm = \{g \in \mathcal{H}_J \mid g_\mp = 0\} \quad (2.31)$$

and let  $P^\pm$  be the projection in  $\ell^2(\mathbb{Z})$  onto the image of  $\mathcal{H}_J^\pm$  under  $\check{\cdot}$ . Then

**Theorem 2.3.** *We have*

$$P_r^+ P_{\Sigma_{\text{ac}}^{(2)}}(J) = P^- \quad (2.32)$$

$$P_\ell^+ P_{\Sigma_{\text{ac}}^{(2)}}(J) = P^+ \quad (2.33)$$

$$P_r^- P_{\Sigma_{\text{ac}}^{(2)}}(J) = \overline{P^-} \quad (2.34)$$

$$P_\ell^- P_{\Sigma_{\text{ac}}^{(2)}}(J) = \overline{P^+} \quad (2.35)$$

*Remark.* Let  $C$  be complex conjugation on  $\ell^2$ . By  $\bar{A}$  we mean  $CAC$ .

*Proof.* We claim that it suffices to prove for  $\varphi \in \text{Ran}(P^+)$  that

$$P_\ell^+ \varphi = \varphi \quad (2.36)$$

for then, by reflection in  $n = 0$ , we see that for  $\psi \in \text{Ran}(P^-)$ ,

$$P_r^+ \psi = \psi \quad (2.37)$$

and

$$(P_\ell^+ + P_r^+) P_{\Sigma_{\text{ac}}^{(2)}}(J) = P_{\Sigma_{\text{ac}}^{(2)}}(J) = P^+ + P^- \quad (2.38)$$

implies (2.32)/(2.33). Since  $e^{-itJ} = e^{itJ}$ , (2.34)/(2.35) then follow.

Clearly, it suffices to prove (2.36) for a dense set of  $\varphi \in \text{Ran}(P^+)$ ; equivalently, for a dense set of  $g \in L^2(\Sigma_{\text{ac}}^{(2)}, f_+ d\lambda)$ , where

$$\varphi_n = \int g(\lambda) u_n^+(\lambda + i0) f_+(\lambda) d\lambda \quad (2.39)$$

By Egoroff's theorem, for a dense set of  $g$ , we can suppose  $g \in L^\infty$ , and for each fixed  $m, n$ ,  $G_{nm}(\lambda + ik^{-1}) \rightarrow G_{nm}(\lambda + i0)$  as  $k \rightarrow \infty$ , uniformly for  $\lambda \in \text{supp}(g)$ . We henceforth assume these properties for  $g$ .

By (1.6) and an abelian theorem [35, Sect. XI.6, Lemma 5],

$$P_\ell^+ \varphi = \chi_0^- \varphi - i \lim_{t \rightarrow -\infty} \int_t^0 e^{isJ} [J, \chi_0^-] e^{-isJ} \varphi ds$$

$$= \chi_0^- \varphi - i \lim_{\varepsilon \downarrow 0} \int_{-\infty}^0 e^{\varepsilon s} e^{isJ} [J, \chi_0^-] e^{-isJ} \varphi ds$$

Since the limit exists, we can replace  $\varepsilon$  by  $1/k$  and do the  $s$  integral

$$(P_\ell^+ \varphi)_n = \chi_0^-(n) \varphi_n - \lim_{k \rightarrow \infty} \sum_{m, n=-\infty}^{\infty} \int G_{nm} \left( \lambda + \frac{i}{k} \right)^{-1} [J, \chi_0^+]_{m\ell} g(\lambda) u_\ell^+(\lambda + i0) f_+(\lambda) d\lambda$$

But  $[J, \chi_0^-]$  is rank two. In fact,  $[J, \chi_0^+]_{m\ell} \neq 0$  only for  $(m, \ell) = (0, 1)$  or  $(1, 0)$ , so the sum is finite, and by the uniform convergence of  $G_{nm}(\lambda + \frac{i}{k})$  for  $\lambda \in \text{supp}(g)$  and  $u_\ell^+ \in L^2(\mathbb{R}, f_+ d\lambda)$ , we see that we can take the limit inside the integral. The result is

$$(P_\ell^+ \varphi)_n = \chi_0^-(n) \varphi_n - \int a_0 g(\lambda) [G_{n1}(\lambda + i0) u_0^+(\lambda + i0) - G_{n0}(\lambda + i0) u_1^+(\lambda + i0)] f_+(\lambda) d\lambda \quad (2.40)$$

If  $n > 0$ , using (1.28),

$$\begin{aligned} & a_0 [G_{n1}(\lambda + i0) u_0^+(\lambda + i0) - G_{n0}(\lambda + i0) u_1^+(\lambda + i0)] \\ &= \frac{a_0 [u_0^+(\lambda + i0) u_1^-(\lambda + i0) - u_1^+(\lambda + i0) u_0^-(\lambda + i0)]}{W(\lambda)} u_n^+(\lambda + i0) \\ &= u_n^+(\lambda + i0) \end{aligned} \quad (2.41)$$

so (2.40) says

$$(P_\ell^+ \varphi)_n = \varphi_n \quad (2.42)$$

If  $n \leq 0$ , the  $u_{0,1}^+$  in (2.41) becomes  $u_{0,1}^-$  and  $u_n^+$  becomes  $u_n^-$ , so the factor in [ ] is zero, and again (2.42) holds.  $\square$

*Remark.*  $\chi_0^-$  can be replaced by any  $\chi_\ell^-$ . So in the analog of (2.40) (where  $G_{n1}, G_{n0}$  become  $G_{n\ell+1}, G_{n\ell}$ ), one can even take  $\ell$  to be  $n$ -dependent. Using this, one can use either the argument we used for  $n > 0$  (by picking  $\ell < n$ ) or for  $n \leq 0$  (by picking  $\ell \geq n$ ) rather than needing both calculations!

The above implies  $P_\ell^+ P_\varepsilon = P_r^- P_\varepsilon$  if and only if for a.e.  $\lambda \in \mathfrak{e}$ ,  $u_n^+ = \overline{u_n^-}$ , which holds if and only if, by (1.26)/(1.27), (1.16) holds for a.e.  $\lambda \in \mathfrak{e}$ . Thus, one has Theorem 1.3.

The following proves Theorem 1.5, and thereby completes the proofs of the results stated in Section 1.

**Theorem 2.4.** For a.e.  $\lambda \in \Sigma_{ac}^{(2)}$ , we can write

$$u_n^+(\lambda + i0) = \alpha(\lambda) \overline{u_n^+(\lambda + i0)} + \beta(\lambda) \overline{u_n^-(\lambda + i0)} \quad (2.43)$$

and the function  $R$  of (1.20) is given by

$$R(\lambda) = |\alpha(\lambda)|^2 \quad (2.44)$$

Moreover,  $R(\lambda)$  is given by (1.21).

*Proof.* For a.e.  $\lambda \in \Sigma_{ac}^{(2)}$ ,  $\text{Im } u_n^+(\lambda) < 0$ ,  $\text{Im } u_n^-(\lambda) > 0$ , so  $u^\pm(\lambda)$  are linearly independent solutions of  $Ju = \lambda u$ . It follows that (2.43) holds. If

$$\varphi = \int g(\lambda) u_n^+(\lambda + i0) f_+(\lambda) d\lambda \in \text{Ran}(P_\ell^+) \quad (2.45)$$

then (2.43) implies that

$$(P_\ell^- \varphi)_n = \int g(\lambda) \alpha(\lambda) \overline{u_n^+(\lambda + i0)} f_+(\lambda) d\lambda \quad (2.46)$$

from which

$$\|P_\ell^- \varphi\|^2 = \int |\alpha(\lambda) g(\lambda)|^2 f_+(\lambda) d\lambda \quad (2.47)$$

This implies (2.44).

If

$$W(f, g) = a_0(g_1 f_0 - f_1 g_0) \quad (2.48)$$

then (2.43) implies

$$\alpha(\lambda) = \frac{W(u_n^+(\lambda + i0), \overline{u_n^-(\lambda + i0)})}{W(u_n^+(\lambda + i0), u_n^-(\lambda + i0))} \quad (2.49)$$

Since

$$u_0^\pm = 1 \quad u_1^+ = -a_0 m_0^+ \quad u_1^- = -(a_0 m_0^-)^{-1} \quad (2.50)$$

(2.49) implies (1.27).  $\square$

### 3. THE SCHRÖDINGER CASE

In this section, we consider a Schrödinger operator on  $\mathbb{R}$ ,

$$H = -\frac{d^2}{dx^2} + V(x) \quad (3.1)$$

where  $V$  is in  $L_{loc}^1$  and limit point at both  $+\infty$  and  $-\infty$ , so  $H$  is the usual selfadjoint operator (see, e.g., [14, Appendix A]). Because it is limit point, there are, for any  $z \in \mathbb{C}_+$ , unique solutions  $u_\pm(x, z)$  obeying

$$-u'' + Vu = zu \quad (3.2)$$

$$u_\pm(0, z) = 1 \quad (3.3)$$

$$u_{\pm} \in L^2(0, \pm\infty) \quad (3.4)$$

For Lebesgue a.e.  $\lambda \in \mathbb{R}$ ,

$$\lim_{\varepsilon \downarrow 0} u_{\pm}(x, \lambda + i\varepsilon) \equiv u_{\pm}(x, \lambda + i0) \quad (3.5)$$

exists for all  $x \in \mathbb{R}$ . Moreover,  $\Sigma_{\text{ac}}^{(2)}$ , the a.c. spectrum of multiplicity 2, is determined by

$$\text{Im}(\mp u'_{\pm}(0, \lambda + i0)) > 0 \quad (3.6)$$

(it is always  $\geq 0$ ) for a.e.  $x \in \Sigma_{\text{ac}}^{(2)}$ , that is, positivity for both  $u_+$  and  $u_-$ .

The Weyl  $m$ -functions (see [14, Appendix A]) are defined by

$$m^{\pm}(x, \lambda + i0) = \mp \left[ \frac{u'(x, \lambda + i0)}{u(x, \lambda + i0)} \right] \quad (3.7)$$

and for  $\lambda \in \mathbb{C}_+$  if  $\lambda + i0$  is replaced by  $\lambda$ . We define  $m(\lambda) \equiv m(x = 0, \lambda)$ . The Green's function is given by (for  $x \leq y$ )

$$G(x, y; \lambda) = \frac{u_-(x, \lambda)u_+(x, \lambda)}{W(\lambda)} \quad (3.8)$$

where

$$W(\lambda) = u_-(x, \lambda)u'_+(x, \lambda) - u'_-(x, \lambda)u_+(x, \lambda) \quad (3.9)$$

is  $x$ -independent so that

$$W(\lambda) = -(m^+(\lambda) + m^-(\lambda)) \quad (3.10)$$

and

$$G(x, x; \lambda) = -(m^+(x, \lambda) + m^-(x, \lambda))^{-1} \quad (3.11)$$

$H$  is called *spectrally reflectionless* on  $\mathfrak{e} \subset \Sigma_{\text{ac}}^{(2)}$  if and only if for a.e.  $\lambda \in \mathfrak{e}$  and all  $x$ ,

$$m^+(x, \lambda + i0) = \overline{-m^-(x, \lambda + i0)} \quad (3.12)$$

As proven in Davies–Simon [7], if  $\chi_y^{\pm}$  is the characteristic function of  $[y, \pm\infty)$ , then

$$P_{\ell}^{\pm} = \text{s-lim}_{t \rightarrow \mp\infty} e^{itH} \chi_y^- e^{-itH} P_{\text{ac}} \quad (3.13)$$

exists and is  $y$ -independent. Indeed,  $\chi_y^-$  can be replaced by any continuous function,  $j$ , which goes to 1 at  $-\infty$  and 0 at  $+\infty$ . If  $\chi_y^-$  is replaced by  $\chi_y^+$ , we get  $P_r^{\pm}$ . If  $\mathcal{H}_{\ell,r}^{\pm}$  is  $\text{Ran}(P_{\ell,r}^{\pm})$ , then (1.7) and (1.8) hold. If (1.9) and (1.10) hold, we say  $H$  is dynamically reflectionless on  $\mathcal{H}$ .

Following [7], the dynamic reflection probability is given by (1.19)/(1.20) with  $J$  replaced by  $H$ . The spectral reflection probability (see, e.g., Gesztesy–Nowell–Pötz [13] or Gesztesy–Simon [15]) is given on  $\Sigma_{ac}^{(2)}$  by

$$\left| \frac{m^+(\lambda + i0) + \overline{m^-(\lambda + i0)}}{m^+(\lambda + i0) + m^-(\lambda + i0)} \right|^2 \quad (3.14)$$

Our main theorems in this case are:

**Theorem 3.1.**  *$H$  is dynamically reflectionless on  $\epsilon \in \Sigma_{ac}^{(2)}$  if and only if it is spectrally reflectionless.*

**Theorem 3.2.**  *$R(\lambda)$  is given by (3.14).*

The proofs closely follow those of Section 2, so we settle for a series of remarks explaining the differences:

1.  $S$  is now defined as

$$S(x, y; \lambda) = \pi^{-1} \operatorname{Im} G(x, y; \lambda + i0) \quad (3.15)$$

and there is still a Stone formula like (2.2). One defines

$$f_{\pm}(\lambda) = \frac{\operatorname{Im} m^{\pm}(\lambda + i0)}{\pi |m^+(\lambda + i0) + m^-(\lambda + i0)|^2} \quad (3.16)$$

One proves

$$\begin{aligned} S(x, y; \lambda) &= \overline{u_+(x, \lambda + i0)} u_+(y, \lambda + i0) f_+(\lambda) \\ &\quad + \overline{u_-(x, \lambda + i0)} u_-(y, \lambda + i0) f_-(\lambda) \end{aligned} \quad (3.17)$$

The proof is the same as that of Theorem 2.1, except sums over  $k$  become integrals over  $w \in \mathbb{R}$ .

2. Once one has (3.17), one can develop eigenfunction expansions analogously to Theorem 2.2. The one difference is that since  $\delta(x)$  is not in  $L^2$ , we do not have the analog of (2.23). However,

$$\operatorname{Im} G(x, x; \lambda = i) = \int \frac{\operatorname{Im} G(x, x; \lambda + i0)}{\lambda^2 + 1} d\lambda \quad (3.18)$$

which implies that

$$\int_{\Sigma_{ac}^{(2)}} \frac{|u^{\pm}(x, \lambda + i0)|^2}{\lambda^2 + 1} f_{\pm}(\lambda) d\lambda < \infty \quad (3.19)$$

and that suffices to define an inverse transform on  $L^2(\Sigma_{ac}^{(2)}, d\lambda)$  functions of compact support.

3. As a preliminary to the next step, we note that if  $\eta$  is a function of compact support with a continuous derivative and  $q$  is  $C^\infty$ , then by an integration by parts,

$$\int \eta(x) \left[ q(x) \frac{d}{dx} \eta(x) + \frac{d}{dx} (q\eta)(x) \right] dx = 0 \quad (3.20)$$

4. In computing  $(P_\ell^+ \varphi)(x_0)$  for  $x_0 < 0$ , we can compute  $\lim_{t \rightarrow \infty} (e^{itH} j e^{-itH} \varphi)$  with a  $C^\infty j$  which is 1 if  $x < 0$  and 0 if  $x > 1$ . Thus, in following the calculation in the proof of Theorem 2.3, we start with

$$(P_\ell^- \varphi)(x_0) = \varphi(x_0) - i \lim_{\varepsilon \downarrow 0} \int_{-\infty}^0 e^{\varepsilon s} (e^{isH} [H, j] \varphi)(x_0) ds \quad (3.21)$$

Since  $[H, j]$  involves  $j'$  and  $j''$ , we can instead write  $F[H, j]F$  where  $F$  is multiplication by a  $C^\infty$  function supported in  $(x_0, 2)$  which is 1 on  $[0, 1]$ . When we put in the eigenfunction expansion, we get

$$\int u^-(x_0, \lambda + i0) f_+(\lambda) g(\lambda) h(\lambda) d\lambda \quad (3.22)$$

where  $h$  has the form of the left side of (3.20) with

$$\eta(x) = F(x)u^+(x, \lambda + i0) \quad q(x) = -j'(x) \quad (3.23)$$

yielding  $(P_\ell^- \varphi)(x_0) = \varphi(x_0)$  for  $x_0 < 0$ . By shifting  $j$  to the right, we get this for all  $x_0$  (as in the remark following the proof of Theorem 2.3).

#### 4. THE CMV CASE

The basic objects in this section are two-sided CMV matrices,  $\mathcal{C}$ , depending on a sequence  $\{\alpha_n\}_{n=-\infty}^\infty$  of Verblunsky coefficients. One-sided CMV matrices appeared first in the numeric matrix literature [1, 2, 33] and were rediscovered by the OPUC community [5]. Two-sided CMV matrices were defined first in [40], although related objects appeared earlier in [3, 10]. For further study, we mention [39, 17, 19, 4].

$\mathcal{C}$  is defined as follows. Given  $\alpha \in \mathbb{D}$ , we let  $\rho = (1 - |\alpha|^2)^{1/2}$  and we let  $\Theta(\alpha)$  be the  $2 \times 2$  matrix,

$$\Theta(\alpha) = \begin{pmatrix} -\alpha & \rho \\ \rho & \bar{\alpha} \end{pmatrix}$$

and let  $\Theta_j$  be  $\Theta$  acting on  $\delta_{j-1}, \delta_j$  in  $\ell^2(\mathbb{Z})$ . Then

$$\mathcal{C} = \mathcal{LM} \quad (4.1)$$

where

$$\mathcal{L} = \bigoplus_{n=-\infty}^{\infty} \Theta_{2n}(\alpha_{2n}) \quad \mathcal{M} = \bigoplus_{n=-\infty}^{\infty} \Theta_{2n+1}(\alpha_{2n+1}) \quad (4.2)$$

First, one can develop a unitary analog of the Davies–Simon theory [7]. It is not hard to show that the Pearson theorem on two-space scattering (see, e.g., [35, Thm. XI.7]) extends to the unitary case. That is, if  $U$  and  $V$  are unitary,  $J$  is bounded, and  $UJ - JV$  is trace class, then

$$\text{s-lim}_{t \rightarrow \mp\infty} U^{-n} J V^n P_{\text{ac}}(V) \quad (4.3)$$

exists. Thus, if  $\chi_n^\pm$  are defined as in Section 1, one defines

$$\begin{aligned} P_\ell^\pm &= \text{s-lim}_{n \rightarrow \mp\infty} \mathcal{C}^{-n} \chi_0^\mp \mathcal{C}^n P_{\text{ac}}(\mathcal{C}) \\ P_r^\pm &= \text{s-lim}_{n \rightarrow \pm\infty} \mathcal{C}^{-n} \chi_0^\pm \mathcal{C}^n P_{\text{ac}}(\mathcal{C}) \end{aligned} \quad (4.4)$$

As in Section 1, we define

$$\mathcal{H}_{\ell,r}^\pm = \text{Ran}(P_{\ell,r}^\pm) \quad (4.5)$$

and we say  $\mathcal{C}$  is *dynamically reflectionless* on  $\mathfrak{e}$  if (1.9) and (1.10) hold.

If  $\alpha_{n-1}$  is replaced by 1, the CMV matrix breaks into a direct sum of two CMV matrices,  $\mathcal{C}_n^+$  on  $\ell^2(\{n, n+1, \dots\})$  and  $\mathcal{C}_{n-1}^-$  on  $\ell^2(\{n-2, n-3, \dots\})$ .  $F_+(z, n)$  is defined for  $z \notin \partial\mathbb{D}$  by setting

$$F_+(z, n) = \left\langle \delta_n, \left( \frac{\mathcal{C}_n^+ + z}{\mathcal{C}_n^+ - z} \right) \delta_n \right\rangle \quad (4.6)$$

and  $F_-(z, n-1)$  by

$$F_-(z, n-1) = \left\langle \delta_{n-1}, \left( \frac{\mathcal{C}_{n-1}^- + z}{\mathcal{C}_{n-1}^- - z} \right) \delta_{n-1} \right\rangle \quad (4.7)$$

It is known (see, e.g., [40]) that when restricted to  $z \in \mathbb{D}$ ,  $F_+(z, n)$  is the Carathéodory function whose Verblunsky coefficients are  $\{\alpha_n, \alpha_{n+1}, \dots\}$  and  $F_-(z, n-1)$  has Verblunsky coefficients  $\{-\bar{\alpha}_{n-2}, -\bar{\alpha}_{n-3}, \dots\}$ . We will let  $F_\pm(z) = F_\pm(z, n=0)$ .

As Carathéodory functions,  $F_\pm(z, n)$  have a.e. boundary values on  $\partial\mathbb{D}$  which we denote by  $F_\pm(e^{i\theta}, n) = \lim_{r \uparrow 1} F_\pm(re^{i\theta}, n)$ .  $\mathcal{C}$  is called *spectrally reflectionless* on  $\mathfrak{e} \subset \partial\mathbb{D}$  if and only if for a.e.  $e^{i\theta} \in \mathfrak{e}$  and all  $n \in \mathbb{Z}$ ,

$$F_+(e^{i\theta}, n) = \overline{F_-(e^{i\theta}, n)} \quad (4.8)$$

There is an equivalent definition using Schur functions (see, e.g., [4]). The equivalence is an easy computation using the relations between the Carathéodory and Schur functions (see, e.g., [17]). It is known [17] that (4.8) for one  $n$  implies it for all  $n$ . It is also known [4] that while (4.8) implies  $\langle \delta_n, (\mathcal{C} + z)/(\mathcal{C} - z)\delta_n \rangle$  has purely real boundary values a.e. on  $\mathfrak{e}$ , the converse can be false.

The dynamic reflection probability  $R(e^{i\theta})$  is given by (1.19)/(1.20) with  $J$  replaced by  $\mathcal{C}$ . The spectral reflection probability is given on  $\Sigma_{\text{ac}}^{(2)}$  by

$$\left| \frac{F_+(e^{i\theta}) - \overline{F_-(e^{i\theta})}}{F_+(e^{i\theta}) + \overline{F_-(e^{i\theta})}} \right|^2 \quad (4.9)$$

Our main theorems in this case are:

**Theorem 4.1.**  $\mathcal{C}$  is dynamically reflectionless on  $\mathfrak{e}$  if and only if it is spectrally reflectionless on  $\mathfrak{e}$ .

**Theorem 4.2.**  $R(e^{i\theta})$  is given by (4.9).

The proofs closely follow those of Section 2, so we again settle for a series of remarks:

1. The analysis requires us to simultaneously study solutions of  $\mathcal{C}$  and  $\mathcal{C}^T$ . To do so, let

$$\mathcal{E} = \begin{pmatrix} \mathcal{C} & 0 \\ 0 & \mathcal{C}^T \end{pmatrix}$$

acting on two sequences labeled by all of  $\mathbb{Z}$ . Following Gesztesy–Zinchenko [19], let

$$\begin{pmatrix} p(z, n) \\ r(z, n) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q(z, n) \\ s(z, n) \end{pmatrix}$$

be the two (Laurent polynomial) solutions to the equation

$$\mathcal{E} \begin{pmatrix} u \\ v \end{pmatrix} = z \begin{pmatrix} u \\ v \end{pmatrix} \quad (4.10)$$

satisfying the initial conditions

$$\begin{pmatrix} p(z, 0) \\ r(z, 0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q(z, 0) \\ s(z, 0) \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

That is, for one solution, the components of  $u$  are  $p$  and of  $v$  are  $r$ , and this solution is uniquely determined by the initial conditions given (see (4.12)). Similarly, the components for the second solution are given by  $q$  and  $s$ . Finally, we let

$$\begin{pmatrix} u_{\pm}(z, n) \\ v_{\pm}(z, n) \end{pmatrix} = \begin{pmatrix} q(z, n) \\ s(z, n) \end{pmatrix} \pm F_{\pm}(z) \begin{pmatrix} p(z, n) \\ r(z, n) \end{pmatrix}$$

be the unique solutions that are  $\ell^2$  at  $\pm\infty$ , normalized by

$$\begin{pmatrix} u_{\pm}(z, 0) \\ v_{\pm}(z, 0) \end{pmatrix} = \begin{pmatrix} -1 \pm F_{\pm}(z) \\ 1 \pm F_{\pm}(z) \end{pmatrix} \quad (4.11)$$

We note that there are a number of relations between  $u_{\pm}$  and  $v_{\pm}$  that we will need (see [19]). First, (4.10) is equivalent to

$$\begin{pmatrix} u(z, n) \\ v(z, n) \end{pmatrix} = T(z, n) \begin{pmatrix} u(z, n-1) \\ v(z, n-1) \end{pmatrix} \quad (4.12)$$

where

$$T(z, n) = \begin{cases} \frac{1}{\rho_n} \begin{pmatrix} \alpha_n & z \\ 1/z & \overline{\alpha_n} \end{pmatrix}, & n \text{ odd} \\ \frac{1}{\rho_n} \begin{pmatrix} \overline{\alpha_n} & 1 \\ 1 & \alpha_n \end{pmatrix}, & n \text{ even} \end{cases} \quad (4.13)$$

Similarly, (4.10) implies

$$\begin{aligned} \begin{pmatrix} u(z, 2n-1) \\ u(z, 2n) \end{pmatrix} &= \Theta_{2n}(\alpha_{2n}) \begin{pmatrix} v(z, 2n-1) \\ v(z, 2n) \end{pmatrix} \\ \begin{pmatrix} u(z, 2n-2) \\ u(z, 2n-1) \end{pmatrix} &= \Theta_{2n-1}(\alpha_{2n-1}) \begin{pmatrix} v(z, 2n-2) \\ v(z, 2n-1) \end{pmatrix} \end{aligned} \quad (4.14)$$

Finally, for all  $n \in \mathbb{Z}$ , we have

$$v_{\pm}(1/\bar{z}, n) = -\overline{u_{\pm}(z, n)} \quad (4.15)$$

This is because

$$\mathcal{C}u = zu \text{ holds} \quad \Leftrightarrow \quad \mathcal{C}^T \bar{u} = (1/\bar{z})\bar{u} \text{ holds}$$

and (4.6)/(4.7) imply  $F_{\pm}(1/\bar{z}) = -\overline{F_{\pm}(z)}$ , and because the solutions to (4.10) that are  $\ell^2$  at  $\pm\infty$  are unique up to normalization.

2. Using the solutions  $u_{\pm}(z, n)$  and  $v_{\pm}(z, n)$  we can write the analog of (1.28) (see [19]):

$$(\mathcal{C} - z)_{nm}^{-1} = \frac{-1}{zW(z)} \begin{cases} u_{-}(z, n)v_{+}(z, m), & n < m \text{ or } n = m = 2k + 1 \\ v_{-}(z, m)u_{+}(z, n), & m < n \text{ or } n = m = 2k \end{cases} \quad (4.16)$$

where

$$W(z) = u_{+}(z, n)v_{-}(z, n) - v_{+}(z, n)u_{-}(z, n) \quad (4.17)$$

is independent of  $n \in \mathbb{Z}$ .

3. Next we find the analog of  $[J, \chi_0^+]$ . Due to the structure of (4.1), the results are different depending on whether  $n$  is even or odd. For  $n$  even:

$$[\mathcal{C}, \chi_n^+] = -\rho_n(\rho_{n-1}|\delta_n\rangle\langle\delta_{n-2}| + \overline{\alpha_{n-1}}|\delta_n\rangle\langle\delta_{n-1}| + \alpha_{n+1}|\delta_{n-1}\rangle\langle\delta_n| - \rho_{n+1}|\delta_{n-1}\rangle\langle\delta_{n+1}|) \quad (4.18)$$

while if  $n$  is odd we get the same thing but transposed and with a minus sign:

$$[\mathcal{C}, \chi_n^+] = \rho_n(\rho_{n-1}|\delta_{n-2}\rangle\langle\delta_n| + \overline{\alpha_{n-1}}|\delta_{n-1}\rangle\langle\delta_n| + \alpha_{n+1}|\delta_n\rangle\langle\delta_{n-1}| - \rho_{n+1}|\delta_{n+1}\rangle\langle\delta_{n-1}|) \quad (4.19)$$

4.  $S$  is defined (using a.e. boundary values) as

$$S(n, m; e^{i\theta}) = \frac{1}{2\pi} \lim_{r \uparrow 1} ((\mathcal{C} + r e^{i\theta})(\mathcal{C} - r e^{i\theta})^{-1} - (\mathcal{C} + r^{-1} e^{i\theta})(\mathcal{C} - r^{-1} e^{i\theta})^{-1})_{nm} \quad (4.20)$$

and there is a Stone formula like (2.2). Proceeding as in Section 2 and using (4.15) and (4.16), one can deduce the analog of (2.8):

$$S(n, m; e^{i\theta}) = u_+(e^{i\theta}, n) \overline{u_+(e^{i\theta}, m)} f_+(e^{i\theta}) + u_-(e^{i\theta}, n) \overline{u_-(e^{i\theta}, m)} f_-(e^{i\theta}) \quad (4.21)$$

where  $u_{\pm}(e^{i\theta}, n) = \lim_{r \uparrow 1} u_{\pm}(r e^{i\theta}, n)$  and

$$\pi f_{\pm}(e^{i\theta}) = \lim_{r \uparrow 1} \frac{1}{r e^{i\theta} |W(r e^{i\theta})|^2} \langle u_{\mp}(r^{-1} e^{i\theta}), [\mathcal{C}, \chi_k^{\mp}] u_{\mp}(r^{-1} e^{i\theta}) \rangle$$

As before, this is independent of  $k$ , and choosing  $k = 0$  one may use (4.11), (4.12), and (4.18)/(4.19) to find the analog of (2.3):

$$f_{\pm}(e^{i\theta}) = \frac{4 \operatorname{Re} F_{\mp}(e^{i\theta})}{\pi |W(e^{i\theta})|^2} \quad (4.22)$$

Once one has (4.21), one may develop eigenfunction expansions exactly as in Theorem 2.2.

5. To prove Theorem 4.1, we first define  $P^{\pm}$  and  $\overline{P^{\pm}}$  as in Section 2 but using  $\lim_{r \uparrow 1} u_{\pm}(r e^{i\theta}, n)$  and  $\lim_{r \uparrow 1} u_{\pm}(r^{-1} e^{i\theta}, n)$  respectively. As before, we consider  $P_{\ell}^+ = \text{s-lim}_{n \rightarrow -\infty} \mathcal{C}^{-n} \chi_0^- \mathcal{C}^n P_{\text{ac}}(\mathcal{C})$ . Because

$$\mathcal{C}^{-n} \chi_0^- \mathcal{C}^n - \mathcal{C}^{-(n-1)} \chi_0^- \mathcal{C}^{n-1} = \mathcal{C}^{-n} [\chi_0^-, \mathcal{C}] \mathcal{C}^{n-1}$$

and the strong limit defining  $P_{\ell}^+$  exists, we see

$$P_{\ell}^+ = \chi_0^- + \text{s-lim}_{n \rightarrow -\infty} \sum_{k=1}^n \mathcal{C}^{-k} [\chi_0^-, \mathcal{C}] \mathcal{C}^{k-1}$$

Choosing a dense set of  $\varphi \in \operatorname{Ran}(P^+)$  as before, and using the eigenfunction expansion and an abelian theorem, we find

$$\begin{aligned} (P_{\ell}^+ \varphi)_m &= (\chi_0^- \varphi)_m + \left( \lim_{n \rightarrow -\infty} \sum_{k=1}^n \mathcal{C}^{-k} [\chi_0^-, \mathcal{C}] \mathcal{C}^{k-1} \varphi \right)_m \\ &= (\chi_0^- \varphi)_m + \lim_{r \uparrow 1} \lim_{n \rightarrow -\infty} \int \sum_{k=1}^n \mathcal{C}^{-k} [\chi_0^-, \mathcal{C}] (r e^{i\theta})^{k-1} u_+(e^{i\theta}, m) g(e^{i\theta}) f_+(e^{i\theta}) \frac{d\theta}{2\pi} \end{aligned}$$

$$= (\chi_0^- \varphi)_m + \int (\mathcal{C} - e^{i\theta})^{-1} [\chi_0^-, \mathcal{C}] u_+(e^{i\theta}, m) g(e^{i\theta}) f_+(e^{i\theta}) \frac{d\theta}{2\pi}$$

The proof of Theorem 4.1 then proceeds exactly as the proof of Theorem 1.3, but now using (4.11)–(4.22). The proof of Theorem 4.2 follows that of Theorem 2.4 but with  $\lim_{r \uparrow 1} u_{\pm}(re^{i\theta}, n)$  and  $\lim_{r \uparrow 1} u_{\pm}(r^{-1}e^{i\theta}, n)$  replacing  $u_{\pm}(x + i0, n)$  and  $\overline{u_{\pm}(x + i0, n)}$ .

## REFERENCES

- [1] J. Baik, P. Deift, K. T.-R. McLaughlin, P. Miller, and X. Zhou, *Optimal tail estimates for directed last passage site percolation with geometric random variables*, Adv. Theor. Math. Phys. **5** (2001), 1207–1250.
- [2] S. Bernstein, *Sur une classe de polynomes orthogonaux*, Commun. Kharkow **4** (1930), 79–93.
- [3] O. Bourget, J. S. Howland, and A. Joye, *Spectral analysis of unitary band matrices*, Comm. Math. Phys. **234** (2003), 191–227.
- [4] J. Breuer, E. Ryckman, and M. Zinchenko, *Right limits and reflectionless measures for CMV matrices*, to appear in Comm. Math. Phys.
- [5] M. J. Cantero, L. Moral, and L. Velázquez, *Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle*, Linear Algebra Appl. **362** (2003), 29–56.
- [6] W. Craig, *The trace formula for Schrödinger operators on the line*, Comm. Math. Phys. **126** (1989), 379–407.
- [7] E. B. Davies and B. Simon, *Scattering theory for systems with different spatial asymptotics on the left and right*, Comm. Math. Phys. **63** (1978), 277–301.
- [8] C. De Concini and R. A. Johnson, *The algebraic-geometric AKNS potentials*, Ergod. Th. Dynam. Sys. **7** (1987), 1–24.
- [9] P. Deift and B. Simon, *Almost periodic Schrödinger operators, III. The absolutely continuous spectrum in one dimension*, Comm. Math. Phys. **90** (1983) 389–411.
- [10] J. S. Geronimo and A. Teplyaev, *A difference equation arising from the trigonometric moment problem having random reflection coefficients—an operator-theoretic approach*, J. Funct. Anal. **123** (1994), 12–45.
- [11] F. Gesztesy, M. Krishna, and G. Teschl, *On isospectral sets of Jacobi operators*, Comm. Math. Phys. **181** (1996), 631–645.
- [12] F. Gesztesy, K. A. Makarov, and M. Zinchenko, *Local ac spectrum for reflectionless Jacobi, CMV, and Schrödinger operators*, Acta Appl. Math. **103** (2008), 315–339.
- [13] F. Gesztesy, R. Nowell, and W. Pötz, *One-dimensional scattering theory for quantum systems with nontrivial spatial asymptotics*, Differential Integral Equations **10** (1997), 521–546.
- [14] F. Gesztesy and B. Simon, *Uniqueness theorems in inverse spectral theory for one-dimensional Schrödinger operators*, Trans. Amer. Math. Soc. **348** (1996), 349–373.
- [15] F. Gesztesy and B. Simon, *Inverse spectral analysis with partial information on the potential, I. The case of an a.c. component in the spectrum*, Helv. Phys. Acta **70** (1997), 66–71.

- [16] F. Gesztesy and P. Yuditskii, *Spectral properties of a class of reflectionless Schrödinger operators*, J. Funct. Anal. **241** (2006), 486–527.
- [17] F. Gesztesy and M. Zinchenko, *A Borg-type theorem associated with orthogonal polynomials on the unit circle*, J. London Math. Soc. (2) **74** (2006), 757–777.
- [18] F. Gesztesy and M. Zinchenko, *On spectral theory for Schrödinger operators with strongly singular potentials*, Math. Nachr. **279** (2006), 1041–1082.
- [19] F. Gesztesy and M. Zinchenko, *Weyl–Titchmarsh theory for CMV operators associated with orthogonal polynomials on the unit circle*, J. Approx. Theory **139** (2006), 172–213.
- [20] F. Gesztesy and M. Zinchenko, *Local spectral properties of reflectionless Jacobi, CMV, and Schrödinger operators*, J. Diff. Eqs. **246** (2009), 78–107.
- [21] D. J. Gilbert, *On subordinacy and spectral multiplicity for a class of singular differential operators*, Proc. Roy. Soc. Edinburgh Sect. A **128** (1998), 549–584.
- [22] R. A. Johnson, *The recurrent Hill’s equation*, J. Diff. Eqs. **46** (1982), 165–193.
- [23] I. S. Kac, *On the multiplicity of the spectrum of a second-order differential operator*, Soviet Math. Dokl. **3** (1962), 1035–1039; Russian original in Dokl. Akad. Nauk SSSR **145** (1962), 510–513.
- [24] I. S. Kac, *Spectral multiplicity of a second-order differential operator and expansion in eigenfunction*, Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), 1081–1112 [Russian]. Erratum: Izv. Akad. Nauk SSSR **28** (1964), 951–952.
- [25] S. Kotani, *Ljapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators*, in “Stochastic Analysis,” K. Itô (ed.), North-Holland, Amsterdam, 1984, pp. 225–247.
- [26] S. Kotani, *One-dimensional random Schrödinger operators and Herglotz functions*, in “Probabilistic Methods in Mathematical Physics,” K. Itô and N. Ikeda (eds.), Academic Press, New York, 1987, pp. 219–250.
- [27] S. Kotani and M. Krishna, *Almost periodicity of some random potentials*, J. Funct. Anal. **78** (1988), 390–405.
- [28] M. Melnikov, A. Poltoratski, and A. Volberg, *Uniqueness theorems for Cauchy integrals*, Publ. Mat. **52** (2008), 289–314.
- [29] F. Nazarov, A. Volberg, and P. Yuditskii, *Reflectionless measures with a point mass and singular continuous component*, preprint, 2007.  
<http://arxiv.org/abs/0711.0948>
- [30] F. Peherstorfer and P. Yuditskii, *Asymptotic behavior of polynomials orthonormal on a homogeneous set*, J. Anal. Math. **89** (2003), 113–154.
- [31] A. Poltoratski and C. Remling, *Reflectionless Herglotz functions and Jacobi matrices*, to appear in Comm. Math. Phys.
- [32] A. Poltoratski, B. Simon, and M. Zinchenko, *The Hilbert transform of a measure*, preprint.
- [33] M. Praehofer and H. Spohn, *Universal distributions for growth processes in  $1 + 1$  dimensions and random matrices*, Phys. Rev. Lett. **84** (2000), 4882–4885.
- [34] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, I: Functional Analysis*, Academic Press, New York, 1972.
- [35] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, III: Scattering Theory*, Academic Press, New York, 1979.

- [36] C. Remling, *The absolutely continuous spectrum of one-dimensional Schrödinger operators*, Math. Phys. Anal. Geom. **10** (2007), 359–373.
- [37] C. Remling, *The absolutely continuous spectrum of Jacobi matrices*, preprint, 2007. [arXiv:math-sp/0706.1101](https://arxiv.org/abs/math-sp/0706.1101)
- [38] B. Simon, *Kotani theory for one dimensional stochastic Jacobi matrices*, Comm. Math. Phys. **89** (1983), 227–234.
- [39] B. Simon, *On a theorem of Kac and Gilbert*, J. Funct. Anal. **223** (2005), 109–115.
- [40] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, AMS Colloquium Series, **54.1**, American Mathematical Society, Providence, RI, 2005.
- [41] R. Sims, *Reflectionless Sturm–Liouville equations*, J. Comp. Appl. Math. **208** (2007), 207–225.
- [42] M. Sodin and P. Yuditskii, *Almost periodic Sturm–Liouville operators with Cantor homogeneous spectrum*, Comment. Math. Helv. **70** (1995), 639–658.
- [43] M. Sodin and P. Yuditskii, *Almost periodic Sturm–Liouville operators with Cantor homogeneous spectrum and pseudo-continuable Weyl functions*, Russian Acad. Sci. Dokl. Math. **50** (1995), 512–515; Russian original in Dokl. Akad. Nauk **339** (1994), 736–738.
- [44] M. Sodin and P. Yuditskii, *Almost periodic Sturm–Liouville operators with homogeneous spectrum*, in “Algebraic and Geometric Methods in Mathematical Physics,” A. Boutet de Monvel and A. Marchenko (eds.), Kluwer, Dordrecht, 1996, pp. 455–462.
- [45] M. Sodin and P. Yuditskii, *Almost periodic Jacobi matrices with homogeneous spectrum, infinite-dimensional Jacobi inversion, and Hardy spaces of character-automorphic functions*, J. Geom. Anal. **7** (1997), 387–435.
- [46] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Mathematical Surveys and Monographs, **72**, American Mathematical Society, Providence, R.I., 2000.