ON WEAKLY NONLINEAR BACKWARD PARABOLIC PROBLEM

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Abstract. We consider weakly nonlinear backward parabolic problem with Dirichlet homogeneous boundary conditions and prove that this problem has a solution. This solution is defined and bounded for all $t \geq 0$. The conditions of the uniqueness for such a solution are also considered.

1. Main theorem

Let $M \subset \mathbb{R}^m = \{(x_1, \dots, x_m)\}$ be an open bounded domain with smooth boundary $\partial M = \overline{M} \backslash M$. By $\mathbb{R}_+$ denote the set of nonnegative real numbers. Introduce a scalar function

$$f(t, x, y, z) \in C(D), \quad D = \mathbb{R}_+ \times \overline{M} \times \mathbb{R} \times \mathbb{R}^m$$

and suppose this function to be bounded:

$$\sup_{(t,x,y,z) \in D} |f(t, x, y, z)| = K < \infty$$

and locally Lipschitz in (y, z) i.e. if (t, x, y, z) and (t, x, y', z') belong to a bounded set $U \subset D$ then

$$|f(t, x, y, z) - f(t, x, y', z')| \leq C(|y - y'| + \|z - z'\|) \tag{1.1}$$

with some positive constant C depending only on U. Here $\| \cdot \|$ is a norm in $\mathbb{R}^m$.

Consider the following backward parabolic problem for the scalar function $u(t, x), \quad x \in M$

$$u_t = -\Delta u + f(t, x, u, \nabla u), \quad u(t, \partial M) = 0, \quad t \geq 0. \tag{1.2}$$

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Theorem 1. *Problem (1.2) has a solution $u(t,x)\in C(\mathbb{R}_+,H^1_0(M))$ such that*

$$\sup_{t\ge 0}\|u(t,\cdot)\|_{H^1(M)}<\infty. \tag{1.3}$$

If the function f satisfies the monotonicity condition i.e. for any $u,v\in H^1_0(M)$ and for any $t\ge 0$ one has

$$(f(t,\cdot,u,\nabla u)-f(t,\cdot,v,\nabla v),u-v)_{L^2(M)}\ge 0 \tag{1.4}$$

then there is no more than one solution to problem (1.2) in $C(\mathbb{R}_+,H^1_0(M))$ with feature (1.3).

If the function f does not depend on the last argument then formula (1.4) follows from the standard monotonicity in y that is, formula

$$(f(t,x,y_1)-f(t,x,y_2))(y_1-y_2)\ge 0$$

is satisfied for all admissible t,x,y_1,y_2.

Note that the existence statement of Theorem 1 remains valid in case when M is a smooth compact Riemannian manifold without boundary. This statement is proved by the same means.

2. Proof of Theorem 1

In the sequel all the inessential positive constants we denote by the same letter c.

2.1. Preliminaries on functional analysis.

In this section we collect several useful theorems.

Theorem 2 (Brouwer fixed point theorem)**.** *Let B be a closed ball in $\mathbb{R}^m$. Then any continuous mapping $f:B\to B$ has a fixed point $\hat x\in B$ that is $f(\hat x)=\hat x$.*

Let $(X,\|\cdot\|_X)$ be a Banach space. Equip the space $C(\mathbb{R}_+,X)$ with the topology of compact convergence. That is, a sequence $\{w_j(t)\}\subset C(\mathbb{R}_+,X)$ is convergent to an element $w(t)\in C(\mathbb{R}_+,X)$ as $j\to\infty$ iff for any compact set $I\subset\mathbb{R}_+$ we have

$$\max_{t\in I}\|w_j(t)-w(t)\|_X\to 0.$$

Recall the Arzela-Ascoli theorem [4].

Theorem 3. *Assume that a set $H\subset C(\mathbb{R}_+,X)$ is closed, bounded, uniformly continuous and for every $t\in\mathbb{R}_+$ the set $\{u(t)\in X\}$ is a compact set in the space X. Then the set H is a compact set in the space $C(\mathbb{R}_+,X)$.*

Proposition 1. *Let X,Y be Banach spaces. Suppose that $A_a:X\to Y,\quad a'>a>0$ is a collection of bounded linear operators such that for each $x\in X$ we have*

$$\sup_{a'>a>0}\|A_a x\|_Y<\infty,\quad \|A_a x\|_Y\to 0\quad\text{as}\quad a\to 0.$$

Then for any compact set $B \subset X$ it follows that

$$\sup_{x\in B} \|A_a x\|_Y \to 0 \quad \text{as} \quad a \to 0.$$

This result is a direct consequence of the Banach-Steinhaus theorem [4], [2].

Denote by $e^{\Delta t}u$ the solution to the following parabolic problem

$$v_t = \Delta v, \quad v(0,x) = u(x), \quad v(t,\partial M) = 0, \quad t \geq 0.$$

Collecting the results from [6] formulate a proposition on the semigroup $e^{\Delta t}$.

Proposition 2. *Define a function $\eta(t,\delta)$ by the following rule: if $0 < t < 1$ then*

$$\eta(t,\delta) = \frac{c}{t^{\delta/2}},$$

if $t \geq 1$ then

$$\eta(t,\delta) = ce^{-\lambda t}.$$

Here c, λ are positive constants.

For any ν, $\delta \in [0,2]$ one has

$$\|e^{t\Delta}u\|_{H^{\nu+\delta}(M)} \leq \eta(t,\delta)\|u\|_{H^\nu(M)}.$$

In $L^2(M)$ there exists an orthogonal basis

$$\{e_i(x)\}_{i\in\mathbb{N}} \subset H^1_0(M), \quad (e_i, e_j)_{L^2(M)} = \delta_{ij}$$

of the eigenfunctions of the Laplacian:

$$\Delta e_i = -\lambda_i e_i, \quad e^{\Delta t}e_i = e^{-\lambda_i t}e_i, \quad 0 < \lambda_1 < \lambda_2 < \dots .$$

Actually the functions e_j belong to $C^\infty(\overline{M})$ [5].

Proposition 3. *The semigroup $e^{\Delta t}$ is a strongly continuous semigroup in the space $H^1_0(M)$.*

Proof. Recall that $H^1_0(M)$ is a separable Hilbert space with respect to the scalar product $(u,v)_{H^1_0(M)} = (\nabla u, \nabla v)_{L^2(M)}$. From this formula one can see that the functions

$$\psi_i = \frac{e_i}{\|e_i\|_{H^1_0(M)}}, \quad i \in \mathbb{N}$$

form an orthogonal basis in $H^1_0(M)$. Indeed, if a function $u \in H^1_0(M)$ is orthogonal to all the functions ψ_i:

$$0 = (\nabla\psi_i, \nabla u)_{L^2(M)} = -(\Delta\psi_i, u)_{L^2(M)} = \frac{\lambda_i(e_i,u)_{L^2(M)}}{\|e_i\|_{H^1_0(M)}}, \quad i \in \mathbb{N}$$

then $u = 0$.

If $u = \sum_{k=1}^\infty u_k\psi_k$ then $e^{\Delta t}u = \sum_{k=1}^\infty u_k e^{-\lambda_k t}\psi_k$. Thus for each $u \in H^1_0(M)$ we have

$$\|e^{\Delta t}u - u\|^2_{H^1_0(M)} = \sum_{k=1}^\infty u_k^2(e^{-\lambda_k t} - 1)^2 \to 0, \quad t \searrow 0.$$

□

2.2. **The approximate system.** Let the subspace $L^2_n(M)\subset L^2(M)$ consist of finite linear combinations of the form $\sum_{k=1}^n u_k e_k$. The elements of $L^2_n(M)$ we denote by u^n. Obviously the space $L^2_n(M)$ is isomorphic to $\mathbb{R}^n$.

Take a function $u(x)=\sum_{k=1}^\infty u_k e_k(x)\in L^2(M)$ and define the projection operation $S_n : L^2(M)\to L^2_n(M)$ by the formula

$$S_n u=\sum_{k=1}^n u_k e_k,\quad \|S_n u\|_{L^2(M)}\le \|u\|_{L^2(M)},\quad (S_n u,v)_{L^2(M)}=(u,S_n v)_{L^2(M)}.$$

In this section we study the following system

$$u^n_t=-\Delta u^n+S_n f(t,x,u^n,\nabla u^n). \tag{2.1}$$

This system approximates problem (1.2).

System (2.1) is an $n-$dimensional system of ordinary differential equations with respect to the vectors $(u^n_1,\ldots,u^n_n)$ from the expansion $u^n=\sum_{j=1}^n u^n_j e_j\in L^2_n(M)$.

Let us show that the right side of system (2.1) satisfies Lipschitz conditions, so the Cauchy existence and uniqueness theorem and another standard theorems on ODE are applied. Indeed, using formula (1.1) we have

$$\begin{aligned}\|S_n f(t,\cdot,u^n,\nabla u^n)&-S_n f(t,\cdot,v^n,\nabla v^n)\|_{L^2(M)}\\ &\le \|f(t,\cdot,u^n,\nabla u^n)-f(t,\cdot,v^n,\nabla v^n)\|_{L^2(M)}\\ &\le c\|f(t,\cdot,u^n,\nabla u^n)-f(t,\cdot,v^n,\nabla v^n)\|_{L^\infty(M)}\\ &\le c(\|u^n-v^n\|_{L^\infty(M)}+\|u^n-v^n\|_{C^1(\overline{M})}).\end{aligned}$$

To this end it remains to note that all the norms in a finite dimensional space are equivalent and thus

$$\|u^n-v^n\|_{L^\infty(M)}+\|u^n-v^n\|_{C^1(\overline{M})}\le c\|u^n-v^n\|_{L^2(M)}.$$

Since the function f is bounded it is easy to show that all the solutions to system (2.1) are defined for all $t\ge 0$.

Denote by $u^n(t,v^n)$ the solution to system (2.1) with initial condition $u^n(0,v^n)=v^n$.

For brevity sake we do not write x in the arguments of the functions.

Consider a mapping

$$F(v^n)=-\int_0^{+\infty} e^{\Delta s}S_n f(s,u^n(s,v^n),\nabla u^n(s,v^n))\,ds.$$

From Proposition 2 it follows that the mapping $F : L^2_n(M) \to L^2_n(M)$ is well defined. Indeed,

$$\begin{aligned}\|F(v^n)\|_{L^2(M)} &\le \int_0^{+\infty} \eta(s,0)\|S_n f(s,u^n(s,v^n),\nabla u^n(s,v^n))\|_{L^2(M)}\,ds\\ &\le \int_0^{+\infty} \eta(s,0)\|f(s,u^n(s,v^n),\nabla u^n(s,v^n))\|_{L^2(M)}\,ds\\ &\le K|M|^{1/2}\int_0^{+\infty}\eta(s,0)\,ds, \quad |M| = \int_M dx.\end{aligned}$$

Moreover, as a consequence from this estimate we see that the map F takes the whole space $L^2_n(M)$ to the closed ball $B_R \subset L^2_n(M)$ with center at the origin and with the radius $R = K|M|^{1/2}\int_0^{+\infty}\eta(s,0)\,ds$.

Lemma 1. *The mapping F has a fixed point $\hat{u}^n \in B_R$.*

Proof. By virtue of the Brouwer fixed point theorem and observations above, it is sufficient to check that the mapping F is continuous.

Take a sequence $\{v^n_j\}_{j\in\mathbb{N}} \subset L^2_n(M)$ such that $v^n_j \to v^n$ in $L^2(M)$ as $j \to \infty$. Since the solutions to system (2.1) continuously depend on initial data [3] and due to the equivalence of the norms, for any $s \ge 0$ we obtain $\|u^n(s,v^n_j) - u^n(s,v^n)\|_{C^1(\overline{M})} \to 0$.

Consequently, for any $s \ge 0$ it follows that

$$\|f(s,u^n(s,v^n_j),\nabla u^n(s,v^n_j)) - f(s,u^n(s,v^n),\nabla u^n(s,v^n))\|_{L^\infty(M)} \to 0. \quad (2.2)$$

Observe that

$$\begin{aligned}&\|e^{\Delta s}S_n(f(s,u^n(s,v^n_j),\nabla u^n(s,v^n_j)) - f(s,u^n(s,v^n),\nabla u^n(s,v^n)))\|_{L^2(M)}\\ &\quad\le c\eta(s,0)\|f(s,u^n(s,v^n_j),\nabla u^n(s,v^n_j)) - f(s,u^n(s,v^n),\nabla u^n(s,v^n))\|_{L^\infty(M)}.\end{aligned}$$

This inequality gives us two things. As a first, by (2.2) we conclude that the sequence

$$\|e^{\Delta s}S_n(f(s,u^n(s,v^n_j),\nabla u^n(s,v^n_j)) - f(s,u^n(s,v^n),\nabla u^n(s,v^n)))\|_{L^2(M)}$$

tends pointwise in $s \ge 0$ to zero. And the second, this sequence is estimated from above by the function $c\eta(s,0) \in L^1(\mathbb{R}_+)$. By the Dominated convergence theorem we deduce

$$\|F(v^n_j) - F(v^n)\|_{L^2(M)} \to 0.$$

□

Any solution $u^n(t,v^n)$ to system (2.1) satisfies the integral equation

$$u^n(t,v^n) = e^{-\Delta t}\Big(v^n + \int_0^t e^{\Delta s}S_n f(s,u^n(s,v^n),\nabla u^n(s,v^n))\,ds\Big).$$

Consequently, the solution with initial condition $v^n = \hat{u}^n$ (see Lemma 1) satisfies the equation

$$u^n(t,\hat{u}^n) = \int_{+\infty}^t e^{\Delta(s-t)}S_n f(s,u^n(s,\hat{u}^n),\nabla u^n(s,\hat{u}^n))\,ds. \quad (2.3)$$

Denote this solution by $u^n(t)$, if it is needed we shall write $u^n(t,x)$.

Lemma 2. *For any* $0\le r<2$ *the sequence* $\{u^n(t)\}$ *is uniformly bounded in* $H^r(M)$*:*

$$\sup_{t\ge 0,\, n\in\mathbb{N}}\|u^n(t)\|_{H^r(M)}=c_r<\infty.$$

Proof. By (2.3) we have

$$\begin{aligned}\|u^n(t)\|_{H^r(M)}&\le\int_t^{+\infty}\|e^{\Delta(s-t)}S_nf(s,u^n(s),\nabla u^n(s))\|_{H^r(M)}\,ds\\&\le\int_t^{+\infty}\eta(s-t,r)\|S_nf(s,u^n(s),\nabla u^n(s))\|_{L^2(M)}\,ds\\&\le\int_t^{+\infty}\eta(s-t,r)\|f(s,u^n(s),\nabla u^n(s))\|_{L^2(M)}\,ds\\&\le K|M|^{1/2}\int_t^{+\infty}\eta(s-t,r)\,ds=K|M|^{1/2}\int_0^{+\infty}\eta(s,r)\,ds.\end{aligned}$$

It is easy to check that the last term is not greater than $c(2-r)^{-1}$ with some positive constant c independent on t and r. □

Lemma 3. *The sequence* $\{u^n(t)\}$ *is uniformly continuous in* $C(\mathbb{R}_+,H^1(M))$*:*

$$\sup_{n\in\mathbb{N}}\|u^n(t_1)-u^n(t_2)\|_{H^1(M)}\to 0$$

as $|t_1-t_2|\to 0,\quad t_1,t_2\ge 0.$

Proof. Assume for definiteness that $t_2\ge t_1$ and taking into account (2.3) write down the identity

$$\begin{aligned}u^n(t_2)-u^n(t_1)&=\int_{t_1}^{t_2}e^{\Delta(s-t_1)}S_nf(s,u^n(s),\nabla u^n(s))\,ds\\&+\Big(\mathrm{id}_{L^2(M)}-e^{(t_2-t_1)\Delta}\Big)\int_{+\infty}^{t_2}e^{\Delta(s-t_2)}S_nf(s,u^n(s),\nabla u^n(s))\,ds.\end{aligned}\tag{2.4}$$

Estimate the first term from the right side of this formula:

$$\begin{aligned}\Big\|\int_{t_1}^{t_2}&e^{\Delta(s-t_1)}S_nf(s,u^n(s),\nabla u^n(s))\,ds\Big\|_{H^1(M)}\\&\le\int_{t_1}^{t_2}\|e^{\Delta(s-t_1)}S_nf(s,u^n(s),\nabla u^n(s))\|_{H^1(M)}\,ds\\&\le\int_{t_1}^{t_2}\eta(s-t_1,1)\|S_nf(s,u^n(s),\nabla u^n(s))\|_{L^2(M)}\,ds\\&\le c\int_{t_1}^{t_2}\eta(s-t_1,1)\,ds=c\int_0^{t_2-t_1}\eta(\xi,1)\,d\xi.\end{aligned}\tag{2.5}$$

In this formula the constant c does not depend on anything.

Now we proceed with the second term from the right side of (2.4). Observe that the set

$$W = \Big\{ \int_{+\infty}^{t_2} e^{\Delta(s-t_2)} S_n f(s, u^n(s), \nabla u^n(s))\, ds \Big| n \in \mathbb{N}, \quad t_2 \geq 0 \Big\} \subset H_0^1(M)$$

is bounded in $H^r(M)$, $1 < r < 2$. Indeed, this follows from our usual estimate

$$\Big\| \int_{+\infty}^{t_2} e^{\Delta(s-t_2)} S_n f(s, u^n(s), \nabla u^n(s))\, ds \Big\|_{H^r(M)} \leq$$

$$\leq c \int_{t_2}^{+\infty} \eta(s - t_2, r)\, ds = c \int_0^{+\infty} \eta(s, r)\, ds.$$

Since the embedding $H^r(M) \subset H^1(M)$ is completely continuous [1], it follows that the set W is relatively compact in $H^1(M)$, and since $W \subset H_0^1(M)$ it is equivalent to say that W is relatively compact in $H_0^1(M)$. Consequently by Propositions 1 and 3 we have

$$\sup_{\xi \in W} \|(\mathrm{id}_{L^2(M)} - e^{(t_2-t_1)\Delta})\xi\|_{H^1(M)} \to 0, \quad t_2 - t_1 \to 0.$$

Combining this formula and formula (2.5) with (2.4) we obtain the assertion of the Lemma. □

From Lemmas 2 and 3 we obtain the following

Corollary 1. *The set* $\{u^n(t)\}_{n\in\mathbb{N}}$ *is relatively compact in* $C(\mathbb{R}_+, H_0^1(M))$.

Indeed, for every fixed $t \geq 0$ the set $\{u^n(t)\}_{n\in\mathbb{N}}$ is bounded in $H^r(M)$, $r > 1$ and thus it is relatively compact in $H^1(M)$. So Theorem 3 implies the Corollary.

By Corollary 1 the sequence $\{u^n(t)\}_{n\in\mathbb{N}}$ contains a subsequence that is convergent in $C(\mathbb{R}_+, H_0^1(M))$. For this subsequence we shall use the same notation. So we have

$$u^n(t) \to u(t)$$

in $C(\mathbb{R}_+, H_0^1(M))$ as $n \to \infty$.

In the next section we show that the function $u(t)$ is a desired solution to problem (1.2).

2.3. **The solution to problem (1.2).** Since the functions u^n solve system (2.1), for any $\phi \in H_0^1(M)$ we have

$$(u^n(t), \phi)_{L^2(M)} = (u^n(0), \phi)_{L^2(M)} + \int_0^t (\nabla u^n(s), \nabla\phi)_{L^2(M)}\, ds$$

$$+ \int_0^t (S_n f(s, u^n(s), \nabla u^n(s)), \phi)_{L^2(M)}\, ds. \tag{2.6}$$

Our goal is to pass in this formula to the limit as $n \to \infty$. Evidently we have $(u^n(0), \phi)_{L^2(M)} \to (u(0), \phi)_{L^2(M)}$ and $(u^n(t), \phi)_{L^2(M)} \to (u(t), \phi)_{L^2(M)}$.

By Lemma 2 for all $s \in [0,t]$ the sequence $(\nabla u^n(s), \nabla\phi)_{L^2(M)}$ is bounded:

$$|(\nabla u^n(s), \nabla\phi)_{L^2(M)}| \le \|u^n(s)\|_{H^1(M)}\|\phi\|_{H^1(M)} \le c_1\|\phi\|_{H^1(M)}.$$

Having the convergence $(\nabla u^n(s), \nabla\phi)_{L^2(M)} \to (\nabla u(s), \nabla\phi)_{L^2(M)}$ and due to the Dominated convergence theorem we obtain

$$\int_0^t (\nabla u^n(s), \nabla\phi)_{L^2(M)}\,ds \to \int_0^t (\nabla u(s), \nabla\phi)_{L^2(M)}\,ds.$$

Consider the last term from the right side of (2.6)

$$(S_n f(s, u^n(s), \nabla u^n(s)), \phi)_{L^2(M)} = (f(s, u^n(s), \nabla u^n(s)), \phi)_{L^2(M)} + (f(s, u^n(s), \nabla u^n(s)), S_n\phi - \phi)_{L^2(M)}. \tag{2.7}$$

The last term of this formula is processed as follows

$$\begin{aligned} |(f(s, u^n(s), \nabla u^n(s)), S_n\phi - \phi)_{L^2(M)}| &\le \|f(s, u^n(s), \nabla u^n(s))\|_{L^2(M)}\|S_n\phi - \phi\|_{L^2(M)} \\ &\le K|M|^{1/2}\|S_n\phi - \phi\|_{L^2(M)} \to 0 \end{aligned}$$

as $n \to \infty$.

Since for all $s \in [0,t]$ it follows that $u^n(s,x) \to u(s,x)$ in $H^1(M)$ then

$$u^n(s,x) \to u(s,x), \quad \nabla u^n(s,x) \to \nabla u(s,x)$$

in $L^2(M)$. Consequently, the sequence $\{u^n(s,x)\}$ contains a subsequence, which we shall denote by the same manner, such that for almost all $x \in M$ we have

$$u^n(s,x) \to u(s,x), \quad \nabla u^n(s,x) \to \nabla u(s,x).$$

This implies that for almost all $x \in M$

$$f(s, x, u^n(s,x), \nabla u^n(s,x)) \to f(s, x, u(s,x), \nabla u(s,x)).$$

And by the Dominated convergence theorem for all $s \in [0,t]$ we obtain

$$(f(s, u^n(s), \nabla u^n(s)), \phi)_{L^2(M)} \to (f(s, u(s), \nabla u(s)), \phi)_{L^2(M)}.$$

From this formula and by formula (2.7) for all $s \in [0,t]$ we have

$$(S_n f(s, u^n(s), \nabla u^n(s)), \phi)_{L^2(M)} \to (f(s, u(s), \nabla u(s)), \phi)_{L^2(M)}.$$

Applying the Dominated convergence theorem again we get

$$\int_0^t (S_n f(s, u^n(s), \nabla u^n(s)), \phi)_{L^2(M)}\,dx \to \int_0^t (f(s, u(s), \nabla u(s)), \phi)_{L^2(M)}\,dx.$$

Gathering all these observations and from formula (2.6) we finally have

$$\begin{aligned} (u(t), \phi)_{L^2(M)} = (u(0), \phi)_{L^2(M)} &+ \int_0^t (\nabla u(s), \nabla\phi)_{L^2(M)}\,ds \\ &+ \int_0^t (f(s, u(s), \nabla u(s)), \phi)_{L^2(M)}\,ds. \end{aligned}$$

This proves Theorem 1 in the part of existence.

2.4. **Proof of the uniqueness.** The proof of the second part of the theorem is completely standard.

Indeed, assume the converse: there exist two solutions $u(t,x), v(t,x) \in C(\mathbb{R}_+, H_0^1(M))$ such that for some $t_0 \geq 0$ we have $u(t_0,x) \neq v(t_0,x)$. From (1.2) it follows that

$$u_t - v_t = -\Delta(u-v) + f(t,x,u,\nabla u) - f(t,x,v,\nabla v).$$

(Sometimes, when it can not bring an ambiguity, we shall not write the arguments of the functions. It is just for the brevity sake.)

Now we multiply in $L^2(M)$ both sides of this equality by $u(t,x) - v(t,x)$ and obtain:

$$\frac{1}{2}\frac{d}{dt}\|u-v\|^2_{L^2(M)}$$
$$= (\nabla(u-v), \nabla(u-v))_{L^2(M)} + (f(t,\cdot,u,\nabla u) - f(t,\cdot,v,\nabla v), u-v)_{L^2(M)}.$$

By the standard facts on Sobolev spaces [1] and inequality (1.4) the right side of this equality is estimated from below by the following expression:

$$\|u(t,\cdot) - v(t,\cdot)\|^2_{H^1(M)}.$$

This expression in its part is estimated as

$$\|u(t,\cdot) - v(t,\cdot)\|^2_{H^1(M)} \geq c\|u(t,\cdot) - v(t,\cdot)\|^2_{L^2(M)}.$$

Finally we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t,\cdot) - v(t,\cdot)\|^2_{L^2(M)} \geq c\|u(t,\cdot) - v(t,\cdot)\|^2_{L^2(M)},$$

and thus

$$\|u(t,\cdot) - v(t,\cdot)\|^2_{L^2(M)} \geq \|u(t_0,\cdot) - v(t_0,\cdot)\|^2_{L^2(M)} e^{c(t-t_0)}.$$

This inequality provides the contradiction with condition (1.3).

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