ON THE MULTISUMMABILITY OF DIVERGENT SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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Abstract. We consider the Cauchy problem for the general linear partial differential equations in two complex variables with constant coefficients. We obtain the necessary and sufficient conditions for the multisummability of formal solution in terms of analytic continuation with an appropriate growth condition of the Cauchy data.

1. Introduction and notation

The application of the theory of multisummability to the formal power series solutions of ordinary differential equations has given very fruitful results. In particular, it was proved that every formal solution of meromorphic ordinary differential equation is multisummable (see B.L.J. Braaksma [7] and [8]).

For partial differential equation, we usually can also obtain the formal solutions, which are power series in one variable, whose coefficients are functions of additional variables. But in this case the characterisation of multisummability of formal solutions is much more complicated and depends not only on the equation but also on the Cauchy data.

In the first such result Lutz, Miyake and Schäfke [10] showed that the formal solution of the heat equation is 1-summable in a direction d if and only if the Cauchy data can be analytically continued to infinity in directions $d/2$ and $d/2+\pi$ with an exponential growth of order 2.

This result was extended to more general equations by authors such as W. Balser [1], [3]–[4], Balser and Malek [5], Balser and Miyake [6], K. Ichinobe [9], S. Malek [11], S. Michalik [12]–[13] and M. Miyake [14].

The most general result was given by W. Balser [3], who considered the Cauchy problem for general linear partial differential equations in two variables with constant coefficients

$$P(\partial_t,\partial_z)u(t,z)=0,\quad \partial_t^n u(0,z)=\varphi_n(z)\in\mathcal{O}(D)\quad n=0,...,m-1, \tag{1}$$

where D is some complex neighbourhood of origin and a polynomial $P(\lambda,\xi)$ satisfies

$$P(\lambda,\xi)=\lambda^m P(\xi)-\sum_{j=1}^{m}\lambda^{m-j}P_j(\xi)=P(\xi)(\lambda-\lambda_1(\xi))^{m_1}...(\lambda-\lambda_l(\xi))^{m_l}. \tag{2}$$

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W. Balser has constructed the normalized formal solution of (1) and he has found the sufficient condition for multisummability of that solution in terms of analytic continuation with appropriate growth conditions of the Cauchy data.

In the paper we will show that this sufficient condition is also necessary. We will also give another construction of normalized formal solution and another proof of Balser's result in a more general framework of fractional equations.

Namely, we will consider the general $1/p$-partial differential equation in two variables with constant coefficients

$$P(\partial_t^{1/p}, \partial_z^{1/p})u(t,z) = 0, \quad (\partial_t^{1/p})^n u(0,z) = \varphi_n(z) \quad n = 0, ..., m-1, \tag{3}$$

where $p \in \mathbb{N}$ and the Cauchy data are $1/p$-analytic (i.e. the functions $z \mapsto \varphi_n(z^p)$ are analytic) in some complex neighbourhood of origin.

We will show that the normalized formal solution $\hat{u}(t,z) = \hat{u}_1(t,z) + ... + \hat{u}_l(t,z)$ of (3) satisfies

$$(\partial_t^{1/p} - \lambda_j(\partial_z^{1/p}))^{m_j} \hat{u}_j(t,z) = 0 \quad \text{for} \quad j = 1, ..., l,$$

where $\lambda_j(\partial_z^{1/p})$ is a kind of pseudodifferential operator introduced in our previous paper [13] and $\lambda_j(\xi)$ is a function defined by (2) with $q_j \in \mathbb{Q}$ and $\lambda_j \in \mathbb{C} \setminus \{0\}$ satisfying

$$\lim_{\xi \to \infty} \frac{\lambda_j(\xi)}{\xi^{q_j}} = \lambda_j.$$

We will show that the behaviour of formal solution $\hat{u}_j(t,z)$ depends on q_j and λ_j as follows

- For $q_j < 1$ the function $t \mapsto \hat{u}_j(t,z)$ is $1/p$-entire function with an exponential growth of order $1/(1-q_j)$ (see Theorem 1).
- For $q_j = 1$ the function $t \mapsto \hat{u}_j(t,z)$ is $1/p$-analytic in some complex neighbourhood of origin. Moreover this function is $1/p$-analytically continued to infinity in a direction d with an exponential growth of order $s > 1$ if and only if the Cauchy data $\varphi_n(z)$ are $1/p$-analytically continued in a direction $d + p \arg \lambda_j$ with the same growth at infinity (see Theorem 2).
- For $q_j > 1$ the series $\hat{u}_j(t,z)$ is $(q_j - 1)$-Gevrey formal power series in $t^{1/p}$. Moreover $\hat{u}_j(t,z)$ is $(q_j-1)^{-1}$-summable in a direction d with respect to $t^{1/p}$ if and only if the Cauchy data $\varphi_n(z)$ are $1/p$-analytically continued in directions $(d + p(\arg \lambda_j + 2\pi k))/q_j$ with the growth of order $q_j/(q_j - 1)$ at infinity (see Theorem 3).

As a consequence, we will obtain the sufficient and necessary condition for multisummability of normalized formal solution of (3) in terms of analytic continuation with an appropriate growth condition of the Cauchy data. The precise formulation of this main result of our paper is given in Theorem 4.

This result one can treat as a generalisation of our previous paper [13], where k-summability of some restricted linear partial differential equations has been studied.

In the paper we use the following notation. The complex disc in $\mathbb{C}^n$ with a centre at origin and a radius $r > 0$ is denoted by $D_r^n := \{z \in \mathbb{C}^n : |z| < r\}$. To simplify notation, we write D_r for $n = 1$. A sector in a direction d with an opening ε in the universal covering space $\widetilde{\mathbb{C}}$ of $\mathbb{C} \setminus \{0\}$ is denoted by

$$S(d, \varepsilon, R) := \{z \in \widetilde{\mathbb{C}} : z = re^{i\theta}, \ d - \varepsilon/2 < \theta < d + \varepsilon/2, \ 0 < r < R\}$$

for $d \in \mathbb{R}$, $\varepsilon > 0$ and $0 < R \leq +\infty$. In the case of $R = +\infty$, we denote it briefly by $S(d, \varepsilon)$. Moreover, if the value of opening angle ε is not essential, then we write S_d for short. A sector S' is called a *proper subsector* of $S(d, \varepsilon, R)$ if its closure in $\widetilde{\mathbb{C}}$ is contained in $S(d, \varepsilon, R)$.

By $\mathcal{O}(D)$ we denote the space of analytic functions on a domain $D \subseteq \mathbb{C}^n$. The Banach space of analytic functions on D_r, continuous on its closure and equipped with the norm $\|\varphi\|_r := \max_{|z|\leq r} |\varphi(z)|$ is denoted by $\mathbb{E}(r)$.

The space of formal power series

$$\hat{u}(t,z) = \sum_{j=0}^{\infty} u_j(z)t^j \quad \text{with} \quad u_j(z) \in \mathbb{E}(r)$$

is denoted by $\mathbb{E}(r)[[t]]$. Moreover, we set $\mathbb{E}[[t]] := \bigcup_{r>0} \mathbb{E}(r)[[t]]$.

2. Gevrey formal power series and Borel summability

In this section we recall some definitions and fundamental facts about the Gevrey formal power series, Borel summability and multisummability. For more details we refer the reader to [2].

Definition 1. A function $u(t, z) \in \mathcal{O}(S(d, \varepsilon) \times D_r)$ is of *exponential growth of order at most $s > 0$ as $t \to \infty$ in $S(d, \varepsilon)$* if and only if for any $r_1 \in (0, r)$ and any $\varepsilon_1 \in (0, \varepsilon)$ there exist $A, B < \infty$ satisfying

$$\max_{|z|\leq r_1} |u(t,z)| < Ae^{B|t|^s} \quad \text{for every} \quad t \in S(d, \varepsilon_1).$$

The space of such functions will be denoted by $\mathcal{O}^s(S(d, \varepsilon) \times D_r)$ (or $\mathcal{O}^s(S_d \times D_r)$ for short)

Analogously, a function $\varphi(z) \in \mathcal{O}(S(d, \varepsilon))$ is of *exponential growth of order at most $s > 0$ as $z \to \infty$ in $S(d, \varepsilon)$* if and only if for any $\varepsilon_1 \in (0, \varepsilon)$ there exist $A, B < \infty$ such that

$$|\varphi(z)| < Ae^{B|z|^s} \quad \text{for every} \quad z \in S(d, \varepsilon_1).$$

The space of such functions will be denoted by $\mathcal{O}^s(S(d, \varepsilon))$ (or $\mathcal{O}^s(S_d)$ for short).

Definition 2. Let $k > 0$. A formal power series

$$\hat{u}(t,z) := \sum_{j=0}^{\infty} u_j(z)t^j \quad \text{with} \quad u_j(z) \in \mathbb{E}(r) \tag{4}$$

is $1/k$-*Gevrey formal power series in t* if its coefficients satisfy

$$\max_{|z|\leq r} |u_j(z)| \leq AB^j\Gamma(1 + j/k) \quad \text{for} \quad j = 0, 1, \dots$$

with some positive constants A and B.

The set of $1/k$-Gevrey formal power series in t over $\mathbb{E}(r)$ is denoted by $\mathbb{E}(r)[[t]]_{1/k}$. We also set $\mathbb{E}[[t]]_{1/k} := \bigcup_{r>0} \mathbb{E}(r)[[t]]_{1/k}$.

Definition 3. Let $k > 0$ and $d \in \mathbb{R}$. A formal series $\hat{u}(t, z) \in \mathbb{E}[[t]]_{1/k}$ defined by (4) is called *k-summable in a direction d* if and only if its *k-Borel transform*

$$\tilde{v}(t,z) := \sum_{j=0}^{\infty} u_j(z)\frac{t^j}{\Gamma(1 + j/k)} \in \mathcal{O}^k(S_d \times D_r).$$

The *k-sum of $\hat{u}(t,z)$ in the direction d* is represented by the Laplace transform of $\tilde{v}(t,z)$

$$u^\theta(t,z) := \frac{1}{t^k}\int_0^{\infty(\theta)} e^{-(s/t)^k}\tilde{v}(s,z)\,ds^k,$$

where the integration is taken over any ray $e^{i\theta}\mathbb{R}_+ := \{re^{i\theta} : r \geq 0\}$ with $\theta \in (d-\varepsilon/2, d+\varepsilon/2)$.

For every $k>0$ and $d\in\mathbb{R}$, according to the general theory of moment summability (see Section 6.5 in [2]), a formal series (4) is k-summable in the direction d if and only if the same holds for the series

$$\sum_{j=0}^{\infty} u_j(z)\frac{j!\Gamma(1+j/k)}{\Gamma(1+j(1+1/k))}t^j.$$

Consequently, we obtain a characterisation of k-summability (analogous to Definition 3), if we replace the k-Borel transform by the *modified k-Borel transform*

$$v(t,z) := \mathcal{B}^k\hat{u}(t,z) := \sum_{j=0}^{\infty} u_j(z)\frac{j!t^j}{\Gamma(1+j(1+1/k))}$$

and the Laplace transform by the *Ecalle acceleration operator*

$$u^\theta(t,z) = t^{-k/(1+k)}\int_0^{\infty(\theta)} v(s,z)C_{1+1/k}((s/t)^{k/(1+k)})\,ds^{k/(1+k)}$$

with $\theta\in(d-\varepsilon,d+\varepsilon)$. Here integration is taken over the ray $e^{i\theta}\mathbb{R}_+$ and $C_{1+1/k}(\zeta)$ is defined by

$$C_{1+1/k}(\zeta) := \frac{1}{2\pi i}\int_\gamma u^{-1/(k+1)}e^{u-\zeta u^{k/(k+1)}}\,du$$

with a path of integration γ as in the Hankel integral for the inverse Gamma function (from ∞ along $\arg u = -\pi$ to some $u_0<0$, then on the circle $|u|=|u_0|$ to $\arg u = \pi$, and back to ∞ along this ray).

Hence the k-summability can be characterised as follows

Proposition 1. *Let $k>0$ and $d\in\mathbb{R}$. A formal series $\hat{u}(t,z)$ given by (4) is k-summable in a direction d if and only if its modified k-Borel transform*

$$\mathcal{B}^k\hat{u}(t,z) = \sum_{j=0}^{\infty} u_j(z)\frac{j!t^j}{\Gamma(1+j(1+1/k))}$$

satisfies conditions:

a) $\mathcal{B}^k\hat{u}(t,z)\in\mathcal{O}(D_r^2)$ *(for some $r>0$), i.e. $\hat{u}(t,z)\in\mathbb{E}(r)[[t]]_{1/k}$.*
b) $\mathcal{B}^k\hat{u}(t,z)$ *is analytically continued to $S_d\times D_r$ (for some $r>0$).*
c) $\mathcal{B}^k\hat{u}(t,z)$ *is of exponential growth of order at most k as $t\to\infty$ in S_d.*

We are now ready to define multisummability in some multidirection.

Definition 4. Let $k_1>...>k_n>0$. We say that a real vector $(d_1,...,d_n)$ is an *admissible multidirection* if and only if

$$|d_j-d_{j-1}|\leq\pi(1/k_j-1/k_{j-1})/2 \quad \text{for} \quad j=2,...,n.$$

Let $\mathbf{k}=(k_1,...,k_n)\in\mathbb{R}_+^n$ and let $\mathbf{d}=(d_1,...,d_n)\in\mathbb{R}^n$ be an admissible multidirection. We say that a formal power series $\hat{u}(t,z)$ given by (4) is **k**-*multisummable* *in a multidirection* $\mathbf{d}$ if and only if $\hat{u}(t,z)=\hat{u}_1(t,z)+...+\hat{u}_n(t,z)$, where $\hat{u}_j(t,z)$ is k_j-summable in a direction d_j for $j=1,...,n$.

3. α-Derivatives, α-analytic functions and operators $B^{\alpha,\beta}$

In this section, in a similar way to [13], we introduce some tools to study divergent solutions of linear partial differential equations. First, we define some kind of fractional derivatives ∂_z^α of the formal power series in $\mathbb{C}[[z^\alpha]]$. These operators are the natural generalisation of the derivative ∂_z defined into the space $\mathbb{C}[[z]]$. Namely, we have

Definition 5. Let $\alpha\in\mathbb{Q}_+$. The linear operator on the space of formal power series

$$\partial_z^\alpha\colon \mathbb{C}[[z^\alpha]]\to\mathbb{C}[[z^\alpha]]$$

given by the formula

$$\partial_z^\alpha\Big(\sum_{n=0}^\infty\frac{u_n}{\Gamma(1+\alpha n)}z^{\alpha n}\Big)=\sum_{n=0}^\infty\frac{u_{n+1}}{\Gamma(1+\alpha n)}z^{\alpha n} \tag{5}$$

is called an *α-derivative.*

Definition 6. We say that a function $u(z)$ is *α-analytic* on $D\subset\mathbb{C}$ (or, generally, on $D\subset\mathbb{C}^n$) if and only if the function $z\mapsto u(z^{1/\alpha})$ is analytic for every $z^{1/\alpha}\in D$. The space of α-analytic functions will be denoted by $\mathcal{O}_\alpha(D)$.

Moreover, analogously to Definition 1, we will denote by $\mathcal{O}_\alpha^s(S_d\times D_r)$ (resp. $\mathcal{O}_\alpha^s(S_d)$) the space of α-analytic functions on $S_d\times D_r$ (resp. S_d) with an exponential growth of order s.

If the formal power series $\hat{u}(z)\in\mathbb{C}[[z^\alpha]]$ is convergent in some complex neighbourhood of origin, then its sum $u(z)$ is the α-analytic function near the origin. For such functions we have well defined α-derivative given by (5), which coincides with the Caputo fractional derivative.

We may also define the *α-Taylor series* of $u(z)\in\mathcal{O}_\alpha(D)$ by the formula

$$u(z)=\sum_{n=0}^\infty\frac{(\partial_z^\alpha)^n u(0)}{\Gamma(1+\alpha n)}z^{\alpha n}.$$

In the case of α-analytic functions, the role of the exponential function e^z is played by

$$e_\alpha(z):=E_\alpha(z^\alpha)=\sum_{n=0}^\infty\frac{z^{\alpha n}}{\Gamma(1+\alpha n)},$$

where $E_\alpha(z)$ denotes the Mittag-Leffler function. By the definition of $e_\alpha(z)$ and by the results on the Mittag-Leffler function (see [15]), we have

Proposition 2. *The function $e_\alpha(z)$ satisfies the following properties:*

a) *$e_\alpha(z)\in\mathcal{O}_\alpha(\mathbb{C})$ and there exists $C<\infty$ such that $|e_\alpha(z)|\le Ce^{|z|}$ for every $z\in\mathbb{C}$,*
b) *for every $a\in\mathbb{C}$ we have $\partial_z^\alpha e_\alpha(az)=a^\alpha e_\alpha(az)$,*
c) *if $\alpha<2$ and $\arg z\in(\pi/2,2\pi/\alpha-\pi/2)$ then $e_\alpha(z)\to 0$ as $z\to\infty$.*

Since every q/p-analytic function is also $1/p$-analytic, without loss of generality we may take $\alpha=1/p$, where $p\in\mathbb{N}$. Observe that $1/p$-analytic function is in fact an analytic function defined on the Riemann surface of $\sqrt[p]{z}$. Hence we have the following integral representation

Proposition 3 (see Lemma 1 in [13])**.** *Let $\varphi(z) \in \mathcal{O}_{1/p}(D_r)$. Then for every $|z| < \varepsilon < r$ and $k \in \mathbb{N}$ we have*

$$(6)\qquad (\partial_z^{1/p})^k\varphi(z) = \frac{1}{2p\pi i}\oint_{|w|=\varepsilon}^{p}\varphi(w)\int_0^{\infty(\theta)}\zeta^{k/p}e_{1/p}(z\zeta)e^{-w\zeta}\,d\zeta\,dw$$

for $\theta \in (\arg w - \pi/2, \arg w + \pi/2)$, where $\oint_{|w|=\varepsilon}^{p}$ denotes that we integrate p times around the positively oriented circle of radius ε.

Moreover, there exist $\varrho > 0$ and $A, B < \infty$ satisfying

$$\sup_{|z|<\varrho}|(\partial_z^{1/p})^k\varphi(z)| \leq AB^{k/p}\Gamma(1+k/p) \quad \text{for} \quad k = 0, 1, \ldots$$

The formula (6) motivates the introduction of some kind of pseudodifferential operators on the space of $1/p$-analytic functions. To this end, let $q(\xi)$ be an analytic function for $|\xi| > |\zeta_0^{1/p}|$ with polynomial growth at infinity. Following [13] we define

$$q(\partial_z^{1/p})e_{1/p}(z\zeta) := q(\zeta^{1/p})e_{1/p}(z\zeta).$$

Hence for every $\varphi(z) \in \mathcal{O}_{1/p}(D_r)$ we have

$$(7)\qquad q(\partial_z^{1/p})\varphi(z) := \frac{1}{2p\pi i}\oint_{|w|=\varepsilon}^{p}\varphi(w)\int_0^{\infty(\theta)}q(\zeta^{1/p})e_{1/p}(z\zeta)e^{-w\zeta}\,d\zeta\,dw$$

with $\theta \in (\arg w - \pi/2, \arg w + \pi/2)$. Since $q_n(\xi)$ is a holomorphic function with polynomial growth at infinity, the left-hand side of (7) is a well-defined $1/p$-analytic function in some complex neighbourhood of origin.

Now we introduce the operators $B^{\alpha,\beta}$, which are related to the modified k-Borel operators $\mathcal{B}^k$. Using the operators $B^{\alpha,\beta}$ we can reduce the question about summability to the study of the solution of the appropriate Kowalevskaya type equation.

Definition 7. Let $\alpha, \beta \in \mathbb{Q}_+$. We define a linear operator on the space of formal power series

$$B^{\alpha,\beta}\colon \mathbb{E}[[t^\alpha]] \to \mathbb{E}[[t^\beta]]$$

by the formula

$$B^{\alpha,\beta}\big(\hat{u}(t,z)\big) = B^{\alpha,\beta}\Big(\sum_{n=0}^{\infty}\frac{u_n(z)}{\Gamma(1+\alpha n)}t^{\alpha n}\Big) := \sum_{n=0}^{\infty}\frac{u_n(z)}{\Gamma(1+\beta n)}t^{\beta n}.$$

Observe that for any formal series $\hat{u}(t,z) \in \mathbb{E}[[t]]$ and $\mu, \nu \in \mathbb{N}$, $\mu > \nu$, we get

$$\mathcal{B}^k\hat{u}(t,z) = (B^{1,\mu/\nu}\hat{u})(t^{\nu/\mu}, z) \quad \text{with} \quad \mu/\nu = 1 + 1/k.$$

Hence for $k \in \mathbb{Q}_+$ we can reformulate Proposition 1 as follows

Proposition 4. *Let $\mu, \nu \in \mathbb{N}$, $\mu > \nu$, $k = (\mu/\nu - 1)^{-1}$. Then the formal series $\hat{u}(t,z) \in \mathbb{E}[[t]]$ is k-summable in a direction d if and only if the function $v(t,z) := B^{1,\mu/\nu}\hat{u}(t,z)$ satisfies the following properties:*

a) *$z \mapsto v(t,z)$ is analytic in some complex neighbourhood of origin,*
b) *$t \to v(t,z)$ is μ/ν-analytic in some complex neighbourhood of origin,*
c) *$t \to v(t,z)$ is μ/ν-analytically continued to infinity in directions $(d+2j\pi)\nu/\mu$ ($j = 0, \ldots, \mu-1$) with an exponential growth of order $k+1$.*

We recall the important properties of the operators $B^{\alpha,\beta}$, which play crucial role in our study of summability. Namely, immediately from definition we have

Proposition 5 (see also Proposition 4 in [13]). *Let $\alpha, \beta \in \mathbb{Q}_+$ and $\hat{u}(t,z) \in \mathbb{E}[[t^\alpha]]$. Then operators $B^{\alpha,\beta}$ and derivatives satisfy the following commutation formulas:*

a) $B^{\alpha,\beta}\partial_t^\alpha \hat{u}(t,z) = \partial_t^\beta B^{\alpha,\beta}\hat{u}(t,z)$;
b) $B^{\alpha,\beta}\partial_z \hat{u}(t,z) = \partial_z B^{\alpha,\beta}\hat{u}(t,z)$;
c) $B^{\alpha,\beta}P(\partial_t^\alpha, \partial_z)\hat{u}(t,z) = P(\partial_t^\beta, \partial_z)B^{\alpha,\beta}\hat{u}(t,z)$ *for any polynomial $P(\tau,\zeta)$ with constant coefficients.*

At the end of this section we extend the notion of Gevrey orders and Borel summability to formal power series in $t^{1/p}$.

Definition 8. Let $\gamma \in \mathbb{Q}_+$. The Banach space of γ-analytic functions on D_r, continuous on its closure and equipped with the norm $\|\varphi\|_r := \max_{|z|\leq r}|\varphi(z)|$ is denoted by $\mathbb{E}_\gamma(r)$.

Definition 9. Let $k > 0$ and $\gamma \in \mathbb{Q}_+$. A formal power series

$$\hat{u}(t,z) := \sum_{j=0}^\infty u_j(z)t^{j/p} \quad \text{with} \quad u_j(z) \in \mathbb{E}_\gamma(r)$$

is $1/k$-*Gevrey formal power series in* $t^{1/p}$ if its coefficients satisfy

$$\max_{|z|\leq r}|u_j(z)| \leq AB^{j/p}\Gamma(1+j/kp) \quad \text{for} \quad j = 0,1,\dots$$

with some positive constants A and B.

The set of $1/k$-Gevrey formal power series in $t^{1/p}$ over $\mathbb{E}_\gamma(r)$ is denoted by $\mathbb{E}_\gamma(r)[[t^{1/p}]]_{1/k}$. We also set $\mathbb{E}_\gamma[[t^{1/p}]]_{1/k} := \bigcup_{r>0}\mathbb{E}_\gamma(r)[[t^{1/p}]]_{1/k}$.

Definition 10. Let $k > 0$ and $d \in \mathbb{R}$. A formal series $\hat{u}(t,z) \in \mathbb{E}_\gamma[[t^{1/p}]]_{1/k}$ is called k-*summable in a direction* d if and only if the series

$$\hat{w}(t,z) := \hat{u}(t^p, z)$$

is kp-summable in a direction d/p.

Let us suppose that

$$\hat{u}(t,z) = \sum_{j=0}^\infty \frac{u_j(z)}{\Gamma(1+j/p)}t^{j/p}.$$

Then

$$\hat{w}(t,z) = \sum_{j=0}^\infty \frac{u_j(z)}{\Gamma(1+j/p)}t^{j}.$$

Using kp-Borel transform of $\hat{w}(t,z)$ we obtain the series

$$\sum_{j=0}^\infty \frac{u_j(z)}{\Gamma(1+j/p)\Gamma(1+j/kp)}t^{j}.$$

By the general theory of moment summability, we may replace this one by the following $1/p$-*modified* kp-*Borel transform* of $\hat{w}(t,z)$, which is defined by

$$\mathcal{B}^{kp}_{1/p}\hat{w}(t,z) := \sum_{j=0}^\infty \frac{u_j(z)}{\Gamma(1+j(1+1/k)/p)}t^{j}.$$

Observe that this modified transform is connected with the operator $B^{1/p,(1+1/k)/p}$ by the formula

$$\mathcal{B}^{kp}_{1/p}\hat{w}(t,z) = (B^{1/p,(1+1/k)/p}\hat{u})(t^{kp/(k+1)},z).$$

Hence, similarly to Proposition 5, we have the following characterisation of k-summability by the operators $B^{\alpha,\beta}$.

Proposition 6. *Let $\mu,\nu\in\mathbb{N}$, $\mu>\nu$, $k=(\mu/\nu-1)^{-1}$ and $d\in\mathbb{R}$. The formal series $\hat{u}(t,z)\in\mathbb{E}_{1/p}[[t^{1/p}]]$ is k-summable in a direction d if and only if the function $v(t,z):=B^{1/p,\mu/\nu p}\hat{u}(t,z)$ satisfies the following conditions:*

a) *$z\mapsto v(t,z)$ is $1/p$-analytic in some complex neighbourhood of origin in $\mathbb{C}$,*
b) *$t\mapsto v(t,z)$ is $\mu/\nu p$-analytic in some complex neighbourhood of origin in $\mathbb{C}$,*
c) *$t\mapsto v(t,z)$ is $\mu/\nu p$-analytically continued to infinity in directions $(d+2j\pi)\nu/\mu$ ($j=0,...,\mu-1$) with an exponential growth of order $k+1$.*

4. Normalized formal solutions

In this section we construct some special solution of (3), which is called the *normalized formal solutions.* Another construction of such solutions (in case $p=1$) was given earlier by W. Balser [3]–[4].

Fix $p\in\mathbb{N}$. We consider the general fractional linear partial differential equation in two variables with constant coefficients

$$P(\partial_t^{1/p},\partial_z^{1/p})u(t,z)=0. \tag{8}$$

It means that

$$P(\partial_t^{1/p},\partial_z^{1/p}) := (\partial_t^{1/p})^m P(\partial_z^{1/p}) - \sum_{j=1}^{m}(\partial^{1/p})_t^{m-j}P_j(\partial_z)$$

with some $m\in\mathbb{N}$ and polynomials $P(\xi)$, $P_j(\xi)$. Without loss of generality we may assume that $P(\xi)$ and $P_m(\xi)$ are not identically zero. Let $g:=\deg P(\xi)$. Observe that the formal power series solution of (8) with the Cauchy data on $t=0$

$$(\partial_t^{1/p})^n u(0,z)=\varphi_n(z)\in\mathcal{O}_{1/p}(D_r)\quad\text{for}\quad n=0,...,m-1 \tag{9}$$

is uniquely determined if and only if $g=0$ (see Proposition 1 in [4] for more details). For $g\geq 1$, in a similar way to W. Balser [4], we will construct the normalized formal solution of (8) satisfying the initial data (9).

First, we consider the difference equation

$$P(\xi)q_n(\xi)=\sum_{j=1}^{m}P_j(\xi)q_{n-j}(\xi). \tag{10}$$

with the initial conditions

$$q_0(\xi)=1\quad\text{and}\quad q_{-1}(\xi)=...=q_{-m+1}(\xi)=0.$$

Observe that the solution $q_n(\xi)$ is a rational function, so we may assume that it is a holomorphic function for sufficiently large $|\xi|$ (say, $|\xi|>|\zeta_0^{1/p}|$).

Fix $\varphi(z)\in\mathcal{O}_{1/p}(D_r)$. Applying (7) we define the coefficients $u_n(z)$ ($n=0,1,...$) by

$$u_n(z)=q_n(\partial_z^{1/p})\varphi(z)=\frac{1}{2p\pi i}\oint_{|w|=\varepsilon}^{p}\varphi(w)\int_{\zeta_0}^{\infty(\theta)}q_n(\zeta^{1/p})e_{1/p}(z\zeta)e^{-w\zeta}\,d\zeta\,dw \tag{11}$$

with $\theta \in (-\arg w - \pi/2, -\arg w + \pi/2)$.

Observe that the coefficients $u_n(z)$ satisfy the recursion formula

$$P(\partial_z^{1/p})u_n(z) = \sum_{j=1}^{m} P_j(\partial_z^{1/p})u_{n-j}(z) \quad \text{for} \quad n \geq m.$$

It means that

$$\hat{u}(t,z) = \sum_{n=0}^{\infty} \frac{u_n(z)}{\Gamma(1+n/p)} t^{n/p}$$

is a normalized formal solution of (8) with the initial data

$$\varphi_n(z) = q_n(\partial_z^{1/p})\varphi(z) \quad \text{for} \quad n = 0, 1, ..., m-1.$$

Moreover, by the principle of superpositions of solutions of linear equations, we may construct the normalized formal solution for any initial condition (9).

To show more exactly the shape of normalized formal solution, we will consider the characteristic equation of (10)

$$P(\xi)\lambda^m = \sum_{j=1}^{m} P_j(\xi)\lambda^{m-j}. \tag{12}$$

We may assume that for sufficiently large $|\xi|$, say $|\xi| > |\zeta_0^{1/p}|$, the characteristic equation has exactly l distinct holomorphic solutions $\lambda_1(\xi), ..., \lambda_l(\xi)$ of multiplicity $m_1, ..., m_l$ $(m_1 + ... + m_l = m)$. According to the theory of difference equations, we have

$$q_n(\xi) = \sum_{j=1}^{l} \sum_{k=0}^{m_j - 1} c_{jk}(\xi) n^k \lambda_j^n(\xi),$$

where the coefficients $c_{jk}(\xi)$ are holomorphic with polynomial growth for sufficiently large $|\xi|$ ($|\xi| > |\zeta_0^{1/p}|$, say).

It means that

$$\hat{u}(t,z) = \sum_{j=1}^{l} \hat{u}_j(t,z) := \sum_{j=1}^{l} \sum_{k=1}^{m_j} r^k(t, \partial_t^{1/p}) \hat{u}_{jk}(t,z), \tag{13}$$

where

$$r(t, \partial_t^{1/p}) := p((\partial_t^{1/p})^p t - 1)$$

and

$$\begin{aligned} \hat{u}_{jk}(t,z) \quad = \quad & \sum_{n=0}^{\infty} \frac{t^{n/p}}{\Gamma(1+n/p)} \times \\ & \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^{p} \varphi(w) \int_{\zeta_0}^{\infty(\theta)} c_{jk}(\zeta^{1/p}) \lambda_j^n(\zeta^{1/p}) e_{1/p}(z\zeta) e^{-w\zeta}\, d\zeta\, dw. \end{aligned} \tag{14}$$

By the direct computation we obtain

Lemma 1 (see Lemma 4 in [13]). *The formal power series $\hat{u}_j(t,z)$ defined by (13) and (14) satisfies the pseudodifferential equation*

$$(\partial_t^{1/p} - \lambda_j(\partial_z^{1/p}))^{m_j} u_j(t,z) = 0.$$

Hence we may define the normalized formal solution as follows

Definition 11. The solution $\hat{u}(t,z)$ of (8) with the initial data (9) is called a *normalized formal series solution* if and only if $\hat{u}(t,z)$ satisfies the pseudodifferential equation

$$(\partial_t^{1/p}-\lambda_1(\partial_z^{1/p}))^{m_1}...(\partial_t^{1/p}-\lambda_l(\partial_z^{1/p}))^{m_l}u(t,z)=0.$$

5. Gevrey estimates

In this section we study the Gevrey order of normalized formal solution. First, we define a *pole order* $q_j\in\mathbb{Q}$ and a *leading term* $\lambda_j\in\mathbb{C}\setminus\{0\}$ of the characteristic root $\lambda_j(\xi)$ as the numbers satisfying formula

$$\lim_{\xi\to\infty}\frac{\lambda_j(\xi)}{\xi^{q_j}}=\lambda_j \quad \text{for} \quad j=1,...,l.$$

We are now ready to show

Theorem 1. *Let $\hat{u}(t,z)=\sum_{j=1}^{l}\hat{u}_j(t,z)$ be a normalized formal solution of (8) with $\hat{u}_j(t,z)$ satisfying the pseudodifferential equation*

$$(\partial_t^{1/p}-\lambda_j(\partial_z^{1/p}))^{m_j}u_j(t,z)=0.$$

and let $q_j\in\mathbb{Q}$ be a pole order of characteristic root $\lambda_j(\xi)$. Then the formal power series $\hat{u}_j(t,z)$ for $j=1,...,l$ is characterised as follows:

- *For $q_j<1$ the series $\hat{u}_j(t,z)$ is convergent to the $1/p$-entire function of order $1/(1-q_j)$.*
- *For $q_j=1$ the series $\hat{u}_j(t,z)$ is convergent in some neighbourhood of origin.*
- *For $q_j>1$ the series $\hat{u}_j(t,z)$ is a Gevrey series of order q_j-1.*

Proof. Without loss of generality we may assume that $\hat{u}_j(t,z)$ is defined by (13) and (14). So, it is sufficient to estimate the coefficients of the formal series

$$\hat{u}_{jk}(t,z):=\sum_{n=0}^{\infty}\frac{u_{jkn}(z)}{\Gamma(1+n/p)}t^{n/p}$$

given by

$$u_{jkn}(z):=\frac{1}{2p\pi i}\oint_{|w|=\varepsilon}^{p}\varphi(w)\int_{\zeta_0}^{\infty}c_{jk}(\zeta^{1/p})\lambda_j^n(\zeta^{1/p})e_{1/p}(z\zeta)e^{-w\zeta}\,d\zeta\,dw.$$

Since $c_{jk}(\zeta^{1/p})$ is of polynomial growth at infinity, we may assume that for $|\zeta|>|\zeta_0|$ there exists $a\in\mathbb{N}$ such that $|c_{jk}(\zeta^{1/p})|\le|\zeta|^a$. In a similar way we may assume that $|\lambda_j(\zeta^{1/p})|\le 2|\lambda_j||\zeta|^{q_j/p}$ for $|\zeta|>|\zeta_0|$. Hence, by Proposition 2, we have

$$\begin{aligned}
&\Big|\int_{\zeta_0}^{\infty(\theta)}c_{jk}(\zeta^{1/p})\lambda_j^n(\zeta^{1/p})e_{1/p}(z\zeta)e^{-w\zeta}\,d\zeta\Big|\\
\le\ &\int_{|\zeta_0|}^{\infty}s^a2^n|\lambda_j^n|s^{nq_j/p}e_{1/p}(|z|s)e^{-|w|s}\,ds\\
\le\ &AB^n\int_0^{\infty}s^as^{nq_j/p}e^{(|z|-|w|)s}\,ds\le AB^n\frac{\Gamma(1+a+nq_j/p)}{(|w|-|z|)^{a+nq_j/p}}\\
\le\ &\tilde{A}\tilde{B}^n\frac{\Gamma(1+nq_j/p)}{(|w|-|z|)^{a+nq_j/p}}.
\end{aligned}$$

It means that for $z \in D_{\varepsilon/2}$ we have

$$\begin{aligned} |u_{jkn}(z)| &\leq \frac{1}{2p\pi} \oint_{|w|=\varepsilon}^{p} |\varphi(z)| \tilde{A}\tilde{B}^n \frac{\Gamma(1+nq_j/p)}{(|w|-|z|)^{a+nq_j/p}}\, d|w| \\ &\leq \tilde{A}\tilde{B}^n \frac{\Gamma(1+nq_j/p)}{(\varepsilon/2)^{a+nq_j/p}} \leq CD^{n/p}\Gamma(1+nq_j/p). \end{aligned}$$

In a consequence we see that the formal series

$$\hat{u}_{jk}(t,z) = \sum_{n=0}^{\infty} \frac{u_{jkn}(z)}{\Gamma(1+n/p)} t^{n/p}$$

is a Gevrey series of order $q_j - 1$. It means that this one is divergent for $q_j > 1$, convergent in some neighbourhood of origin for $q_j = 1$ and $1/p$-entire function for $q_j < 1$. In the last case, by Proposition 2, we have

$$\begin{aligned} |u_{jk}(t,z)| &\leq \sum_{n=0}^{\infty} \frac{CD^{n/p}\Gamma(1+nq_j/p)}{\Gamma(1+n/p)} |t|^{n/p} \leq \sum_{n=0}^{\infty} \frac{CD^n}{\Gamma(1+(1-q_j)n/p)} |t|^{n/p} \\ &\leq Ce_{(1-q_j)/p}(\tilde{D}|t|^{1/(1-q_j)}) \leq \tilde{C}e^{\tilde{D}|t|^{1/(1-q_j)}}. \end{aligned}$$

Finally, observe that the similar properties satisfies also the formal series $\hat{u}_j(t,z)$. □

6. Analytic solution

In this section we study the properties of terms $\hat{u}_j(t,z)$ of the normalized formal solution $\hat{u}(t,z)$, which are determined by the characteristic roots $\lambda_j(\xi)$ with the pole order $q_j = 1$. In this case, by Theorem 1, $\hat{u}_j(t,z)$ satisfies the Cauchy-Kowalevskaya type theorem. Moreover, we show that $t \mapsto \hat{u}_j(t,z)$ is analytically continued in some direction with an exponential growth of order $s > 1$ if and only if the Cauchy data satisfy the similar properties. To this end, we shall use two auxiliary lemmas, following [13].

Lemma 2 (see Lemma 3 in [13])**.** *Let us assume that $\lambda(\xi)$ is analytic for $|\xi| > |\zeta_0|$ and $\lim_{\xi\to\infty} \lambda(\xi)/\xi = \lambda \in \mathbb{C}\setminus\{0\}$. Moreover, let $\varphi(z) \in \mathcal{O}^s_{1/p}(D_r \cup S_{d+p\arg\lambda})$. Then the function*

$$v(t,z) := \frac{1}{2p\pi i} \oint_{|w|=\varepsilon}^{p} \varphi(w) \int_{\zeta_0}^{\infty(\theta)} e_{1/p}(t\lambda^p(\zeta^{1/p}))e_{1/p}(z\zeta)e^{-w\zeta}\, d\zeta\, dw$$

is $1/p$-analytic in some complex neighbourhood of origin and is $1/p$-analytically continued to the set $S_d \times D_{r'}$ with an exponential growth of order s.

Lemma 3 (see Lemma 6 in [13])**.** *Let $u(t,z) \in \mathcal{O}_{1/p}(D_r^2)$ with some $r > 0$. Then for every $n \in \mathbb{N}$, $u(t,z)$ satisfies the pseudodifferential equation*

$$\left(\partial_t^{1/p} - \lambda_j(\partial_z^{1/p})\right)^n u(t,z) = 0$$

if and only if $u(t,z)$ is a solution of

$$\left(\partial_z^{1/p} - \lambda_j^{-1}(\partial_t^{1/p})\right)^n u(t,z) = 0.$$

Now, we are ready to prove

Theorem 2. *Let $s>1$, $d\in\mathbb{R}$ and let $\hat{u}(t,z)=\hat{u}_1(t,z)+\ldots+\hat{u}_l(t,z)$ be a normalized formal solution of (8) with the initial data (9), where $\hat{u}_j(t,z)$ satisfies the pseudodifferential equation*

$$(\partial_t^{1/p}-\lambda_j(\partial_z^{1/p}))^{m_j}u_j(t,z)=0$$

and $\lambda_1(\xi)$,...,$\lambda_l(\xi)$ are the characteristic roots of (12). We also assume that there exists $\tilde{l}\in\{1,\ldots,l\}$ such that

$$\lim_{\xi\to\infty}\frac{\lambda_j(\xi)}{\xi}=\lambda_j\in\mathbb{C}\setminus\{0\}\quad \textit{for}\quad j=1,\ldots,\tilde{l}. \tag{15}$$

Then the formal series $\hat{u}_j(t,z)$ is convergent to $u_j(t,z)\in\mathcal{O}_{1/p}(D_r^2)$ for $j=1,\ldots,\tilde{l}$. Moreover, $\varphi_n(z)\in\mathcal{O}^s_{1/p}(S_{d+p\arg\lambda_j})$ $(n=0,\ldots,m-1,\ j=1,\ldots,\tilde{l})$ if and only if $\tilde{u}(t,z):=u_1(t,z)+\ldots+u_{\tilde{l}}(t,z)\in\mathcal{O}^s_{1/p}(S_d\times D_r)$.

Proof. The first part of the proof is given by (15) and Theorem 1.

($\Longrightarrow$) Without loss of generality we may assume that the Cauchy data satisfy

$$\varphi_n(z)=q_n(\partial_z^{1/p})\varphi(z)\quad\text{for } \varphi(z)\in\mathcal{O}^s_{1/p}(S_{d+p\arg\lambda_j}),\quad n=0,\ldots,m-1,\ j=1,\ldots,\tilde{l},$$

where $q_n(\partial_z^{1/p})$ is a pseudodifferential operator defined by (11).

Repeating the construction of normalized formal solution we see that $\hat{u}(t,z)=\hat{u}_1(t,z)+\ldots+\hat{u}_l(t,z)$, where

$$\hat{u}_j(t,z)=\sum_{k=1}^{m_j}r(t,\partial_t^{1/p})^k\hat{u}_{jk}(t,z)$$

and

$$\hat{u}_{jk}(t,z)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{1}{2p\pi i}\oint_{|w|=\varepsilon}^{p}\varphi(w)\int_{\zeta_0}^{\infty(\theta)}c_{jk}(\zeta^{1/p})\lambda_j^n(\zeta^{1/p})e_{1/p}(z\zeta)e^{-w\zeta}\,d\zeta\,dw.$$

By Theorem 1, the formal power series $\hat{u}_j(t,z)$ is convergent in D_r^2 to the function

$$\begin{aligned}u_j(t,z)\quad:=\quad&\frac{1}{2p\pi i}\oint_{|w|=\varepsilon}^{p}\varphi(w)\times\\&\times\sum_{k=0}^{m_j-1}\int_{\zeta_0}^{\infty(\theta_j)}c_{jk}(\zeta^{1/p})r^k(t,\partial_t^{1/p})e_{1/p}(t\lambda_j^p(\zeta^{1/p}))e_{1/p}(z\zeta)e^{-w\zeta}\,d\zeta\,dw\end{aligned}$$

for $j=1,..,\tilde{l}$.

Furthermore, by Lemma 2, if $\varphi(z)\in\mathcal{O}^s_{1/p}(S_{d+p\arg\lambda_j})$ then $u_j(t,z)\in\mathcal{O}^s_{1/p}(S_d\times D_r)$. Hence also $\tilde{u}(t,z)=u_1(t,z)+\ldots+u_{\tilde{l}}(t,z)\in\mathcal{O}^s_{1/p}(S_d\times D_r)$.

($\Longleftarrow$) Fix $j\in\{1,\ldots,\tilde{l}\}$. Since $u_j(t,z)\in\mathcal{O}_{1/p}(D_r^2)$ satisfies the equation

$$(\partial_t^{1/p}-\lambda_j(\partial_z^{1/p}))^{m_j}u_j(t,z)=0,$$

by Lemma 3 the function $u_j(t,z)$ satisfies also

$$(\partial_z^{1/p}-\lambda_j^{-1}(\partial_t^{1/p}))^{m_j}u_j(t,z)=0.$$

Hence the function $\tilde{u}(t,z) = u_1(t,z) + ... + u_{\tilde{l}}(t,z)$ is a solution of the Cauchy problem in z-direction

$$\tilde{P}(\partial_t^{1/p}, \partial_z^{1/p})\tilde{u}(t,z) = 0,$$
$$(\partial_t^{1/p})^n \tilde{u}(t,0) = \psi_n(t) \quad \text{with some} \quad \psi_n(t) \in \mathcal{O}^s_{1/p}(S_d) \quad (n = 0, ..., \tilde{m}-1),$$

where

$$\begin{aligned} \tilde{P}(\partial_t^{1/p}, \partial_z^{1/p}) &:= (\partial_z^{1/p})^{\tilde{m}} - \sum_{j=1}^{\tilde{m}} (\partial_z^{1/p})^{\tilde{m}-j} \tilde{P}_j(\partial_t^{1/p}) \\ &= (\partial_z^{1/p} - \lambda_1^{-1}(\partial_t^{1/p}))^{m_1} ... (\partial_z^{1/p} - \lambda_{\tilde{l}}^{-1}(\partial_t^{1/p}))^{m_{\tilde{l}}} \end{aligned}$$

and $\tilde{m} := m_1 + ... + m_{\tilde{l}}$.

Without loss of generality we may assume that $\psi_0(t) = \psi(t) \in \mathcal{O}^s_{1/p}(S_d)$ and $\psi_n(t) = \sum_{j=1}^n \tilde{P}_j(\partial_t^{1/p})\psi_{n-j}(t)$ for $n = 1, ..., \tilde{m}-1$. Hence repeating the construction of normalized formal solution with replaced variables we conclude that

$$\tilde{u}(t,z) = \tilde{u}_1(t,z) + ... + \tilde{u}_{\tilde{l}}(t,z),$$

where

$$\begin{aligned} \tilde{u}_j(t,z) &:= \sum_{k=0}^{m_j-1} r^k(z, \partial_z^{1/p}) \frac{1}{2p\pi i} \oint_{|s|=\varepsilon}^{p} \psi(s) \times \\ &\times \int_{\tau_0}^{\infty(\tilde{\theta}_j)} \tilde{c}_{jk}(\tau^{1/p}) e_{1/p}(z\lambda_j^{-p}(\tau^{1/p})) e_{1/p}(t\tau) e^{-s\tau}\, d\tau\, ds. \end{aligned}$$

Since $\lim\limits_{\xi\to\infty} \lambda_j^{-1}(\xi)/\xi = \lambda_j^{-1}$, we have $\tilde{u}_j(t,z) \in \mathcal{O}^s_{1/p}(D_r \times S_{d+p\arg\lambda_j})$ by Lemma 2. Moreover, by Lemmas 1 and 3, $\tilde{u}_j(t,z)$ satisfies the formula

$$(\partial_t^{1/p} - \lambda_j(\partial_z^{1/p}))^{m_j} \tilde{u}_j(t,z) = 0 \quad \text{for} \quad j = 1, ..., \tilde{l}.$$

In a similar way to [9] we define for $j = 1, ..., \tilde{l}$

$$P_j(\partial_t^{1/p}, \partial_z^{1/p}) := (\partial_t^{1/p} - \lambda_j(\partial_z^{1/p}))^{m_j-1} \prod_{k=1,\ k\neq j}^{\tilde{l}} (\partial_t^{1/p} - \lambda_k(\partial_z^{1/p}))^{m_k}$$

and

$$\overline{u}_j(t,z) := P_j(\partial_t^{1/p}, \partial_z^{1/p})\tilde{u}(t,z) = P_j(\partial_t^{1/p}, \partial_z^{1/p})\tilde{u}_j(t,z) \in \mathcal{O}^s_{1/p}(D_r \times S_{d+p\arg\lambda_j}).$$

Without loss of generality we may assume that

$$(\partial_t^{1/p})^n \tilde{u}(0,z) = 0 \text{ for } n < \tilde{m}-1, \qquad (\partial_t^{1/p})^{m-1}\tilde{u}(0,z) = \varphi(z).$$

Hence also $\overline{u}_j(0,z) = \varphi(z)$ and we conclude that $\varphi(z) \in \mathcal{O}^s_{1/p}(S_{d+p\arg\lambda_j})$, which proves the theorem. □

7. Multisummability of normalized formal solutions

In the last section we consider the terms $\hat{u}_j(t,z)$ of the normalized formal solution, which are determined by the characteristic roots $\lambda_j(\xi)$ with the pole order $q_j > 1$. In this case, by Theorem 1, $\hat{u}_j(t,z)$ is a (q_j-1)-Gevrey formal power series in $t^{1/p}$. In this section, we shall be concerned with summability properties of the formal series $\hat{u}_j(t,z)$.

To this end, we apply the operators $B^{\alpha,\beta}$ to the formal solution $\hat{u}(t,z)$. By Proposition 5 with ∂_z replaced by $\partial_z^{1/\nu p}$, we have

Proposition 7. *Let $\mu,\nu\in\mathbb{N}$, $\mu>\nu$. A series $\hat{u}(t,z)$ is a normalized formal solution of (8) with the initial data (9) if and only if the formal series $\hat{v}(t,z) := B^{1/p,\mu/\nu p}\hat{u}(t,z)$ satisfies the following fractional equation*

$$(16)\qquad \begin{aligned} &\tilde{P}(\partial_t^{1/\nu p},\partial_z^{1/\nu p})v(t,z)=0,\\ &(\partial_t^{1/\nu p})^j v(0,z)=\varphi_n(z)\in\mathcal{O}_{1/p}(D_r)\ \text{for } j=n\mu,\ n=0,...,m-1,\\ &(\partial_t^{1/\nu p})^j v(0,z)=0\ \text{for } j\neq n\mu,\ j<m\mu,\ n=0,...,m-1,\end{aligned}$$

where

$$\begin{aligned}\tilde{P}(\partial_t^{1/\nu p},\partial_z^{1/\nu p}) &= P((\partial_t^{1/\nu p})^{\mu},(\partial_z^{1/\nu p})^{\nu})\\ &= (\partial_t^{1/\nu p})^{\mu m}P((\partial_z^{1/\nu p})^{\nu})-\sum_{j=1}^{m}(\partial_t^{1/\nu p})^{\mu(m-j)}P_j((\partial_z^{1/\nu p})^{\nu}).\end{aligned}$$

Now we are ready to prove

Proposition 8. *Let $\mu,\nu\in\mathbb{N}$, $\mu>\nu$, $s>1$, $d\in\mathbb{R}$ and let $\hat{u}(t,z)=\hat{u}_1(t,z)+...+\hat{u}_l(t,z)$ be a normalized formal solution of (8) with the initial data (9), where $\hat{u}_j(t,z)$ satisfies the pseudodifferential equation*

$$(\partial_t^{1/p}-\lambda_j(\partial_z^{1/p}))^{m_j}u_j(t,z)=0$$

and $\lambda_1(\xi)$,...,$\lambda_l(\xi)$ are the characteristic roots of (12). We also assume that there exists $\tilde{l}\in\{1,...,l\}$ such that

$$\lim_{\xi\to\infty}\frac{\lambda_j(\xi)}{\xi^{\mu/\nu}}=\lambda_j\in\mathbb{C}\setminus\{0\}\quad for\quad j=1,...,\tilde{l}.$$

Then the formal series $\hat{v}_j(t,z) := B^{1/p,\mu/\nu p}\hat{u}_j(t,z)$ is convergent to a function $v_j(t,z)$, where $t\mapsto v_j(t,z)\in\mathcal{O}_{\mu/\nu p}(D_r)$ and $z\mapsto v_j(t,z)\in\mathcal{O}_{1/p}(D_r)$. Moreover, $\varphi_n(z)\in\mathcal{O}^s_{1/p}(S_{(d+p\arg\lambda_j+2k\pi)\nu/\mu})$ ($n=0,...,m-1$, $j=1,...,\tilde{l}$, $k=0,...,\mu-1$) if and only if $t\mapsto\tilde{v}(t,z)\in\mathcal{O}^s_{\mu/\nu p}(S_{(d+2k\pi)\nu/\mu})$ ($k=0,...,\mu-1$) and $z\mapsto\tilde{v}(t,z)\in\mathcal{O}_{1/p}(D_r)$, where $\tilde{v}(t,z):=v_1(t,z)+...+v_{\tilde{l}}(t,z)$.

Proof. By Proposition 7, the series $\hat{v}(t,z):=B^{1/p,\mu/\nu p}\hat{u}(t,z)$ is a normalized formal solution of (16). Moreover, $\hat{v}(t,z)=\hat{v}_1(t,z)+...+\hat{v}_l(t,z)$, where $\hat{v}_j(t,z):=B^{1/p,\mu/\nu p}\hat{u}_j(t,z)$ satisfies the equation

$$((\partial_t^{1/\nu p})^{\mu}-\lambda_j(\partial_z^{1/p}))^{m_j}v_j(t,z)=0$$

On the other hand

$$\begin{aligned}(\partial_t^{1/\nu p})^{\mu}-\lambda_j(\partial_z^{1/p}) &= (\partial_t^{1/\nu p}-\sigma_0\lambda_j^{1/\mu}((\partial_z^{1/\nu p})^{\nu}))...(\partial_t^{1/\nu p}-\sigma_{\mu-1}\lambda_j^{1/\mu}((\partial_z^{1/\nu p})^{\nu}))\\ &= (\partial_t^{1/\nu p}-\tilde{\lambda}_{j1}(\partial_z^{1/\nu p}))...(\partial_t^{1/\nu p}-\tilde{\lambda}_{j\mu}(\partial_z^{1/\nu p})),\end{aligned}$$

where $\sigma_0, ..., \sigma_{\mu-1}$ are the complex roots of $z^\mu = 1$ and $\tilde{\lambda}_{jk}(\xi) := \sigma_k \lambda_j^{1/\mu}(\xi^\nu)$ for $j = 1, ..., l$ and $k = 0, ..., \mu - 1$.

It means that

$$\hat{v}(t,z) := \sum_{j=1}^{l} \sum_{k=0}^{\mu-1} \hat{v}_{jk}(t,z),$$

where $\hat{v}_{jk}(t,z)$ satisfies

$$(\partial_t^{1/\nu p} - \tilde{\lambda}_{jk}(\partial_z^{1/\nu p}))^{m_j} v_{jk}(t,z) = 0.$$

Moreover, we have for $j = 1, ..., \tilde{l}$, $k = 0, ..., \mu - 1$

$$\lim_{\xi\to\infty} \frac{\tilde{\lambda}_{jk}(\xi)}{\xi} = \lim_{\xi\to\infty} \sigma_k \Big(\frac{\lambda_j(\xi^\nu)}{\xi^\mu}\Big)^{1/\mu} = \sigma_k \lambda_j^{1/\mu} =: \tilde{\lambda}_{jk}$$

and

$$\arg \tilde{\lambda}_{jk} = (\arg \lambda_j + 2k\pi)/\mu.$$

Applying Theorem 2 to $\hat{v}(t,z)$, we conclude that $\hat{v}_j(t,z)$ is convergent to $v_j(t,z) \in \mathcal{O}_{1/\nu p}(D_r)$. On the other hand $t \mapsto \hat{v}_j(t,z)$ is a formal power series in $t^{\mu/\nu p}$ and $z \mapsto \hat{v}_j(t,z)$ is a formal power series in $z^{1/p}$. Hence $t \mapsto v_j(t,z) \in \mathcal{O}_{\mu/\nu p}(D_r)$ and $z \mapsto v_j(t,z) \in \mathcal{O}_{1/p}(D_r)$. Moreover, also by Theorem 2, we have $\varphi_n(z) \in \mathcal{O}^s_{1/\nu p}(S_{(d+p\arg\lambda_j+2k\pi)\nu/\mu})$ ($n = 0, ..., m-1$, $j = 1, ..., \tilde{l}$, $k = 0, ..., \mu - 1$) if and only if $v(t,z) \in \mathcal{O}^s_{1/\nu p}(S_{(d+2k\pi)\nu/\mu} \times D_r)$ ($k = 0, ..., \mu - 1$). Since $t \mapsto \hat{v}(t,z)$ is a formal power series in $t^{\mu/\nu p}$ and $z \mapsto \hat{v}(t,z)$ is a formal power series in $z^{1/p}$, we obtain the desired conclusion. □

Combining Propositions 6 and 8 we have

Theorem 3. *Let $\mu, \nu \in \mathbb{N}$, $\mu > \nu$, $k = (\mu/\nu - 1)^{-1}$, $d \in \mathbb{R}$ and let $\hat{u}(t,z) = \hat{u}_1(t,z) + ... + \hat{u}_l(t,z)$ be a normalized formal solution of (8) with the initial data (9), where $\hat{u}_j(t,z)$ satisfies the pseudodifferential equation*

$$(\partial_t^{1/p} - \lambda_j(\partial_z^{1/p}))^{m_j} \hat{u}_j(t,z) = 0$$

and $\lambda_1(\xi)$,...,$\lambda_l(\xi)$ are the characteristic roots of (12). We also assume that there exists $\tilde{l} \in \{1, ..., l\}$ such that

$$\lim_{\xi\to\infty} \frac{\lambda_j(\xi)}{\xi^{\mu/\nu}} = \lambda_j \in \mathbb{C} \setminus \{0\} \quad \textit{for} \quad j = 1, ..., \tilde{l}.$$

Then $\varphi_n(z) \in \mathcal{O}^{k+1}_{1/p}(S_{(d+p\arg\lambda_j+2k\pi)\nu/\mu})$ ($n = 0, ..., m-1$, $j = 1, ..., \tilde{l}$, $k = 0, ..., \mu - 1$) if and only if $\tilde{u}(t,z) := \hat{u}_1(t,z) + ... + \hat{u}_{\tilde{l}}(t,z)$ is k-summable in a direction d.

Hence, finally we obtain the main theorem

Theorem 4. *Let us assume that*

$$\{\lambda_{ji}(\xi) : \ j = 1, ..., \tilde{n}, \ i = 1, ..., l_j\}$$

is the set of characteristic roots of $P(\lambda, \xi) = 0$ satisfying

$$\lim_{\xi\to\infty} \frac{\lambda_{ji}(\xi)}{\xi^{q_j}} = \lambda_{ji} \in \mathbb{C} \setminus \{0\} \quad \textit{for} \quad j = 1, ..., \tilde{n}, \ i = 1, ..., l_j.$$

We also assume that there exist exactly n pole orders of characteristic roots, which are greater than 1, say $1 < q_1 < ... < q_n < \infty$. Moreover, let $\mu_j, \nu_j \in \mathbb{N}$ and

$k_j > 0$ be such that $\mu_j/\nu_j = q_j$ and $k_j = (q_i - 1)^{-1}$ for $j = 1, ..., n$. Then the normalized formal solution $\hat{u}(t,z)$ of (8) is $(k_1, ..., k_n)$-multisummable in an admissible multidirection $(d_1, ..., d_n)$ if and only if the initial values $\varphi_k(z)$ satisfy

$$\varphi_k(z) \in \mathcal{O}_{1/p}^{k_j+1}(S(j)) \quad \textit{for} \quad j = 1, ..., n,\ k = 0, ..., m-1,$$

where

$$S(j) := D_r \cup \bigcup_{i=1}^{l_j} \bigcup_{\alpha=0}^{\mu_j - 1} S_{(d_j + p \arg \lambda_{ji} + 2\alpha\pi)/q_j}.$$

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