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OBSTRUCTION ARGUMENT FOR TRANSITION CHAINS OF TORI INTERSPERSED WITH GAPS

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Abstract. We consider a dynamical system whose phase space contains a two-dimensional normally hyperbolic invariant manifold diffeomorphic to an annulus. We assume that the dynamics restricted to the annulus is given by an area preserving monotone twist map. We assume that in the annulus there exist finite sequences of primary invariant Lipschitz tori of dimension 1, with the property that the unstable manifold of each torus has a topologically crossing intersection with the stable manifold of the next torus in the sequence. We assume that the dynamics along these tori is topologically transitive. We assume that the tori in these sequences, with the exception of the tori at the ends of the sequences, can be C^0-approximated from both sides by other primary invariant tori in the annulus. We assume that the region in the annulus between two successive sequences of tori is a Birkhoff zone of instability. We prove the existence of orbits that follow the sequences of invariant tori and cross the Birkhoff zones of instability.

1. **Introduction.** This paper is a continuation to [23], in which we describe a topological method for proving the existence of orbits that shadow transition chains of primary invariant tori interspersed with Birkhoff zones of instability. In [23] the boundaries of the Birkhoff zones of instabilities were assumed to be smooth. In this paper we consider the general case when all the primary invariant tori in the transition chains and at the boundaries of the Birkhoff zones of instabilities are only Lipschitz.

Here by a primary torus in an annulus we mean a 1-dimensional torus that cannot be homotopically deformed into a point in the annulus. Given a normally hyperbolic invariant manifold diffeomorphic to an annulus, by a transition chain of primary invariant tori in the annulus we mean a finite or countable sequence of primary invariant C^1-smooth (Lipschitz) tori with the following properties: (i) the unstable manifold of each torus intersects transversally (topologically crossing) the stable manifold of the subsequent torus in the sequence, (ii) the motion on each torus is topologically transitive. The definition of topological crossing is given in Section 5.

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Also, by a Birkhoff zone of instability in an annulus, we mean a region between two primary invariant tori that does not contain any other primary invariant torus in between.

In this paper we consider a discrete dynamical system whose phase space contains a normally hyperbolic invariant manifold diffeomorphic to an annulus. The dynamics restricted to the annulus is assumed to be an area preserving monotone twist map.

In general, if the twist map is close to integrable, the KAM theorem yields many invariant tori in the annulus, close to the integrable ones. Besides the KAM tori, there also exist other primary invariant tori. Due to normal hyperbolicity, all primary invariant tori possess stable and unstable manifolds. Under some generic nondegeneracy conditions on the dynamics, the stable and unstable manifold of nearby tori have transverse or topologically crossing intersections. One can link together nearby tori through their heteroclinic connections and form transition chains of such tori. However, gaps are also formed between the primary invariant tori. Some of the gaps can be large, in the sense that one may not be able to show that the transition chains extend across those gaps by using standard analytical arguments. Thus, we obtain transition chains of tori alternating with gaps.

In this paper we assume that these gaps are Birkhoff zones of instability with Lipschitz boundaries, and that the transition chains can be extended all the way to the boundaries of these gaps. We use topological arguments to prove that there exist orbits that follow infinitely many transition chains and also cross the Birkhoff zones of instability between successive transition chains.

The motivation of this work resides within the Arnold diffusion problem. In 1964, Arnold [1] proposed a model of a Hamiltonian system consisting of a rotator and a pendulum with a small periodic perturbation of special type. He proved the existence of orbits along which the action variable of the rotator changes by some arbitrarily large quantity, for all sufficiently small perturbations. Arnold conjectured that this phenomenon is generic in the whole of Hamiltonian systems.

Arnold's example has two parameters, with one of the parameters corresponding to the pendulum and the other corresponding to the small coupling between the rotator and the pendulum. When the parameters are both set to zero, the system describes the motion of the rotator alone and is completely integrable. Changing the parameter corresponding to the pendulum to some non-zero value introduces hyperbolicity to the system. The phase space of the rotator turns into a normally hyperbolic invariant manifold and is foliated by invariant tori. Each torus has stable and unstable manifolds that coincide. When the second parameter is changed to some non-zero value, much smaller than the first parameter, the phase space of the rotator survives as a normally hyperbolic invariant manifold. Moreover, the perturbation in [1] is chosen of a special type so that all the invariant tori survive the perturbation. (This is not the case for general perturbations.) The effect of the small perturbation is that the stable and unstable manifolds of the invariant tori split, and so the unstable (stable) manifold of a torus intersects transversally the stable (unstable) manifolds of neighboring tori. Transverse heteroclinic connections between nearby tori are thus formed. One can proceed by constructing transition chains of tori that travel arbitrarily far in the phase space of the rotator. Then Arnold uses the "obstruction property" to show that there exist orbits that follow the transition chains (see also [2]). A transition torus T is said to satisfy the obstruction property if for every invariant manifold V intersecting transversely the stable manifold of T, the unstable manifold of T is contained in the closure of V. Often, one applies the obstruction property by taking an open neighborhood B of a point on the stable manifold, and inferring that the closure of the set $\{\phi_t(B) \mid t \geq 0\}$ contains the unstable manifold of T, where ϕ_t denotes the Hamiltonian flow. In [20], a version of the Lambda Lemma is used to prove that the transition tori of Arnold satisfy the obstruction property. Then the obstruction property is used in combination with simple point-set topology to provide an argument for the existence of orbits shadowing the transition chain. We briefly describe this argument here. Suppose that we have a sequence of invariant tori $\{T_1, T_2, \dots, T_n, \dots\}$ such that each successive pair of tori in the sequence is linked by a transverse heteroclinic connection. We choose a closed ball B_1 centered at a point on the stable manifold of T_1 and contained in some small neighborhood of T_1. Since $W^u(T_1) \pitchfork W^s(T_2) \neq \emptyset$, the obstruction property (and implicitly the Lambda Lemma) implies that the stable manifold of T_2 intersects B_1. Then there exists a small closed ball $B_2 \subseteq B_1$, centered at a point on the stable manifold of T_2, that is taken by the flow ϕ_t into some small neighborhood of T_2. Applying again the obstruction argument we infer that the stable manifold of T_3 intersects the image of B_2 through the flow ϕ_t. Thus, there is a smaller closed ball $B_3 \subseteq B_2$ that is taken by the flow into some small neighborhood of T_3. This construction can be repeated inductively for an arbitrarily large number of steps, resulting in a sequence of closed balls $B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$. Since the balls are compact, their intersection is non-empty. Any point in the intersection will shadow the prescribed sequence of tori.

The emphasis of this paper is on how to extend the obstruction argument in the case of transition chains of tori interspersed with gaps, modeled here as Birkhoff zones of instability. The obstruction mechanism as described above breaks down at the zones of instability. However, there exist connecting orbits that go from near one boundary of a zone of instability to near the other boundary of the zone of instability. The difficulty is how to link orbits that shadow the transition chains with orbits that shadow the connecting orbits, where the standard obstruction argument does not apply. To overcome this, we use a topological approach inspired by Easton's method of correctly aligned windows. Instead of closed balls as in the above argument, we use closed rectangular boxes (windows) that, under the dynamics, cross one another along some unstable-like directions. The unstable like-directions correspond to the hyperbolic unstable directions plus one distinguished direction in the annulus which shears under the twist map on the annulus. One particular feature of windows is that they are robust objects, so we are able to make adjustments to the geometry of these windows to compensate for the lack of control on the dynamics within the zones of instability.

Topological methods were previously applied to the Arnold diffusion problem in [35, 28, 21, 23]. Transition chains of tori formed with topologically crossing heteroclinic connections were considered in [22]. Mechanisms for producing diffusing orbits by combining the dynamics along heteroclinic connections with the dynamics across zones of instability appeared in [13, 32] through geometric methods, and in [38, 9, 10, 4, 27, 3] through variational methods.

The novelty of the approach in this paper consists of the following: we consider transition chains of tori that are not necessarily smooth; we consider that the stable and unstable manifolds of the consecutive tori in the chain have topologically crossing intersections; we consider a non-perturbative setting, in which the dynamics on Λ is not assumed to be nearly integrable; we consider dynamical systems that are not necessarily Hamiltonian, in which the stable and unstable manifolds of the transition tori can have different dimensions.

2. **Background and notation.** In this section we recall some results on normally hyperbolic invariant manifolds, on the scattering map, and on twist maps, following [25, 16, 31].

If $F : M \to M$ is a C^r-diffeomorphism, with $r \geq 1$, on a smooth manifold M, a submanifold Λ of M is said to be a normally hyperbolic invariant manifold for F if Λ is invariant under F, there exists a splitting of the tangent bundle of TM into sub-bundles

$$TM = E^u \oplus E^s \oplus T\Lambda,$$

that are invariant under dF, and there exist a constant $C > 0$ and rates $0 < \lambda < \mu^{-1} < 1$, such that for all $x \in \Lambda$ we have

$$\begin{aligned} v \in E^s_x &\Leftrightarrow \|DF^k_x(v)\| \leq C\lambda^k \|v\| \text{ for all } k \geq 0, \\ v \in E^u_x &\Leftrightarrow \|DF^k_x(v)\| \leq C\lambda^{-k} \|v\| \text{ for all } k \leq 0, \\ v \in T_x\Lambda &\Leftrightarrow \|DF^k_x(v)\| \leq C\mu^{|k|} \|v\| \text{ for all } k \in \mathbb{Z}. \end{aligned}$$

In the sequel we will assume that Λ is a compact and connected manifold. Then the dimensions of E^s_x and E^u_x are independent of x. We denote $n_s = \dim(E^s_x)$, $n_u = \dim(E^u_x)$, $n_c = \dim(T_x\Lambda)$, where $n_s + n_u + n_c = \dim(M)$. The manifold Λ is C^ℓ-smooth where ℓ is a positive integer with $1 \leq \ell < \min\{r, (\log \lambda^{-1})(\log \mu)^{-1}\}$; in the sequel we will assume that the rates are so that there exists such an integer $\ell \geq 2$. The stable and unstable manifolds of Λ, as well as the stable and unstable manifolds of each point $x \in \Lambda$, are defined by

$$\begin{aligned} W^s(\Lambda) =& \{y \in M \,|\, d(F^k(y), \Lambda) \leq C_y\lambda^k \text{ for all } k \geq 0\}, \\ W^u(\Lambda) =& \{y \in M \,|\, d(F^k(y), \Lambda) \leq C_y\lambda^{-k} \text{ for all } k \leq 0\}, \\ W^s(x) =& \{y \in M \,|\, d(F^k(y), F^k(x)) \leq C_{x,y}\lambda^k \text{ for all } k \geq 0\}, \\ W^u(x) =& \{y \in M \,|\, d(F^k(y), F^k(x)) \leq C_{x,y}\lambda^{-k} \text{ for all } k \leq 0\}. \end{aligned}$$

The map F takes fibers into corresponding fibers, i.e.

$$F(W^s(x)) = W^s(F(x)), \quad F(W^u(x)) = W^u(F(x)).$$

We have $W^s(\Lambda) = \bigcup_{x\in\Lambda} W^s(x)$ and $W^u(\Lambda) = \bigcup_{x\in\Lambda} W^u(x)$. For each $x \in W^s(\Lambda)$ there exists a unique $x^+ \in \Lambda$ such that $x \in W^s(x^+)$, and for each $x \in W^u(\Lambda)$ there exists a unique $x^- \in \Lambda$ such that $x \in W^u(x^-)$. The manifolds $W^s(x)$, $W^u(x)$ are C^r-smooth, and the manifolds $W^s(\Lambda)$, $W^u(\Lambda)$ are $C^{\ell-1}$-smooth. Also, if L is an submanifold of Λ that is invariant under F, we can define the invariant manifolds $W^s(L) = \bigcup_{x\in L} W^s(x)$ and $W^u(L) = \bigcup_{x\in L} W^u(x)$.

We define the wave maps $\Omega^+ : W^s(\Lambda) \to \Lambda$ by $\Omega^+(x) = x^+$, and $\Omega^- : W^u(\Lambda) \to \Lambda$ by $\Omega^-(x) = x^-$. The maps Ω^+ and Ω^- are C^ℓ-smooth.

We now describe a map, called the scattering map, acting on the normally hyperbolic invariant manifold Λ by following the heteroclinic excursions. Assume that $W^u(\Lambda)$ and $W^s(\Lambda)$ have a differentiably transverse intersection along a homoclinic n_c-dimensional $C^{\ell-1}$-smooth manifold Γ. This means that $\Gamma \subseteq W^u(\Lambda) \cap W^s(\Lambda)$ and, for each $x \in \Gamma$, we have

$$\begin{aligned} T_xM &= T_xW^u(\Lambda) + T_xW^s(\Lambda), \\ T_x\Gamma &= T_xW^u(\Lambda) \cap T_xW^s(\Lambda). \end{aligned}$$

Let us assume the additional condition that for each $x \in \Gamma$ we have

$$T_xW^s(\Lambda) = T_xW^s(x^+) \oplus T_x(\Gamma),$$
$$T_xW^u(\Lambda) = T_xW^u(x^-) \oplus T_x(\Gamma),$$

where x^-, x^+ are the uniquely defined points in Λ corresponding to x.

The restrictions of the wave maps Ω^+, Ω^- to Γ are local $C^{\ell-1}$-diffeomorphisms. By restricting Γ to a submanifold of it, if necessary, we can ensure that Ω^+, Ω^- are $C^{\ell-1}$-diffeomorphisms. A homoclinic manifold Γ for which the corresponding restrictions of the wave maps are $C^{\ell-1}$-diffeomorphisms will be referred as a homoclinic channel. Thus, we can define the $C^{\ell-1}$-diffeomorphism $S = \Omega^+ \circ (\Omega^-)^{-1}$ from an open subset D^- in Λ to an open subset D^+ in Λ. We will refer to S as the scattering map associated to the homoclinic channel Γ. In the sequel we will regard S as a partially defined map, so the image of a set A by S means the set $S(A \cap D^-)$.

The scattering map has the following simple property: If L_1 and L_2 are two invariant C^1-smooth invariant manifolds of complementary dimensions in Λ, and if $S(L_1)$ has a differentiably transverse intersection with L_2, then $W^u(L_1)$ has a differentiably transverse intersection with $W^s(L_2)$.

In Section 5 we give the definition of topological crossing. Intuitively, two manifolds of complementary dimensions are topologically crossing if they can be made differentiably transverse with non-zero oriented intersection number by the means of a sufficiently small homotopy. The original manifolds do not need to be smooth, they can be topological manifolds. Therefore, the previous property of the scattering map has an immediate consequence in the case of topologically crossing manifolds. If L_1 and L_2 are two invariant C^0-manifolds of complementary dimensions in Λ and $S(L_1)$ has a topologically crossing with L_2, then $W^u(L_1)$ has a topologically crossing intersection with $W^s(L_2)$.

We now recall some facts on twist maps. Suppose that Λ is a 2-dimensional C^1-smooth manifold diffeomorphic with an annulus $[0,1] \times \mathbb{T}^1$, and is described through a system of action-angle coordinate (I, ϕ), with $I \in [0,1]$ and $\phi \in \mathbb{T}^1$. A C^1-smooth map $f : \Lambda \to \Lambda$ is said to be a monotone twist map provided that $\partial(\mathrm{pr}_\phi \circ f)/\partial I > 0$, where pr_ϕ is the projection onto the ϕ-coordinate. By a primary invariant torus (or, equivalently, an essential invariant circle) we mean a 1-dimensional torus invariant under f in Λ that cannot be homotopically deformed into a point inside Λ. Since f is a monotone twist map, each primary invariant torus T is the graph of some Lipschitz function (see [5, 6]).

A region in Λ between two primary invariant tori is called a Birkhoff zone of instability provided that there is no invariant primary torus in the interior of the region. It is known that, for an area preserving monotone twist map f on Λ, given a Birkhoff zone of instability, if its boundary tori are topologically transitive, then there exist Birkhoff connecting orbits that go from any prescribed neighborhood of an arbitrary point on one boundary torus to any prescribed neighborhood of an arbitrary point on the other boundary torus (see [5, 6]).

3. **Main Result.** We now describe the assumptions for the main theorem of this paper.

We consider a C^1-diffeomorphism $F : M \to M$ of a smooth $(n_s + n_u + 2)$-dimensional manifold M, where $n_s, n_u > 0$.

We assume that there exists an invariant submanifold $\Lambda \subseteq M$ that is diffeomorphic to an annulus $[0,1] \times \mathbb{T}^1$.

We assume that Λ is normally hyperbolic, where $\dim(E^s_x) = n_s$ and $\dim(E^u_x) = n_u$ at every point $x \in \Lambda$.

We assume that the restriction $F_{|\Lambda}$ of F to Λ is an area preserving monotone twist map.

We assume that $W^u(\Lambda)$ and $W^s(\Lambda)$ have a differentiably transverse intersection along a homoclinic 2-dimensional manifold Γ that is C^1-smooth. Additionally, we assume that Γ is a homoclinic channel and so the wave maps $\Omega^+ : W^s(\Lambda) \to \Lambda$ and $\Omega^- : W^u(\Lambda) \to \Lambda$ are C^1-diffeomorphisms when restricted to Γ. Let $S = \Omega^+ \circ (\Omega^-)^{-1}$ be the corresponding scattering map, defined from some open subset D^- in Λ to an open subset D^+ in Λ. The scattering map is also a C^1-diffeomorphism.

We consider a bi-infinite sequence of invariant primary tori $(T_i)_{i\in\mathbb{Z}}$ in Λ. These tori are assumed to be Lipschitz tori. Below we will assume that the sequence of tori $(T_i)_{i\in\mathbb{Z}}$ can be partitioned into finite sequences with special properties; we will describe this partition by considering a certain increasing bi-infinite subsequence of indices $(i_k)_{k\in\mathbb{Z}}$ in $\mathbb{Z}$.

We assume that the tori in the sequence satisfy the following properties.

(A1) Each torus T_i in the given sequence intersects the domain D^- of the scattering map S associated to Γ.

(A2) The restriction of F to each torus T_i is topologically transitive.

(A3) Each subsequence of tori $(T_i)_{i=i_k+1,\dots,i_{k+1}}$, with $k \in \mathbb{Z}$, is a topological transition chain in the sense following sense: there exists a curve segment $\mathcal{T}_i \subseteq T_i$ in the domain of the scattering map S such that the image $S(\mathcal{T}_i)$ of $\mathcal{T}_i$ under S intersects T_{i+1} at exactly one point, in a topologically crossing manner, for $i = i_k+2, \dots, i_{k+1}-2$, and the image $S(\mathcal{T}_i)$ intersects T_{i+1} at exactly three points, in a topologically crossing manner, for $i = i_k+1, i_{k+1}-1$.

(A4) The region in Λ between T_{i_k} and T_{i_k+1}, with $k \in \mathbb{Z}$, is a Birkhoff zone of instability.

(A5) Each torus T_i which is not at the boundary of one of the Birkhoff zones of instability specified in (A4), can be C^0-approximated from both sides with invariant tori from Λ, i.e., there exist two sequences of primary invariant tori $(T_{j^-_l(i)})_{l\geq 1} \subseteq \Lambda$ and $(T_{j^+_l(i)})_{l\geq 1} \subseteq \Lambda$ that approach T_i in the C^0-topology, such that the annulus bounded by $T_{j^-_l(i)}$ and $T_{j^+_l(i)}$ contains T_i in its interior for all l.

Some motivation for considering the above structures is given in [23]. We plan to expand on this motivation in some future papers. The main result of this paper is the following:

Theorem 3.1. *We consider a discrete dynamical system $F : M \to M$ as above. Given a sequence of primary invariant tori $(T_i)_{i\in\mathbb{Z}}$ in Λ satisfying the properties (A1) – (A5) from above, for each sequence $(\epsilon_i)_{i\in\mathbb{Z}}$ of positive real numbers, there exist an orbit $(z_i)_{i\in\mathbb{Z}}$ and positive integers $(n_i)_{i\in\mathbb{Z}}$ such that*

$$\begin{aligned} z_{i+1} &= F^{n_i}(z_i), \quad \text{for all } i \in \mathbb{Z}, \\ d(z_i, T_i) &< \epsilon_i, \quad \text{for all } i \in \mathbb{Z}. \end{aligned}$$

4. **Verification of the hypothesis of Theorem 3.1 in models.** In this section we outline some possible methods to verify the assumptions of Theorem 3.1 in some models.

4.1. **Normal hyperbolicity, twist map property, computation of t he scattering map.** One class of systems of interest is the class of a priori unstable nearly integrable Hamiltonian systems. These are perturbed Hamiltonian systems for which the unperturbed integrable part possesses separatrices (following [11]).

In Arnold's example discussed in Section 1, if one fixes the parameter corresponding to the pendulum, the resulting system with the small perturbation depending on the other parameter is an a priori unstable nearly integrable Hamiltonian system. The small perturbation in [1] is however non-generic. Some examples of a priori unstable nearly integrable Hamiltonian systems in which the perturbations are generic can be found in [14, 16, 17].

In these examples, if we consider the dynamics of the time-one map F of the Hamiltonian flow, the phase space for the unperturbed system contains a normally hyperbolic invariant manifold Λ_0, which is diffeomorphic to an annulus, and whose stable and unstable manifolds coincide. The restriction of F to the annulus is an integrable area preserving twist map. When a small perturbation is added to the system, Λ_0 is survived by a normally hyperbolic invariant manifold Λ. Under some non-degeneracy conditions on the perturbation, the stable and unstable manifolds of Λ intersect transversally. These non-degeneracy condition can be verified through a Melnikov function or Melnikov potential associated to the perturbation. Melnikov theory can also be used to verify the existence of a homoclinic channel Γ, and to compute explicitly the scattering map S. The dynamics on Λ is still given by an area preserving twist map, however no longer integrable. If the dynamics on Λ_0 satisfies the conditions required by the KAM theorem, then there exist many invariant primary tori in Λ surviving the perturbation. Also, the KAM theorem leaves between invariant primary tori some 'large gaps' whose size is bigger than the distance the stable and unstable manifolds move.

For example, we consider a mechanical system consisting of one pendulum and one rotator with a weak, periodic coupling described by the following time-dependent Hamiltonian (following [14]):

$$H_\varepsilon(p,q,I,\phi,t) \quad = \quad \frac{1}{2}p^2 + (1-\cos(q)) + \frac{1}{2}I^2 + \varepsilon h(p,q,I,\phi,t;\varepsilon), \tag{1}$$

where $(p,q,I,\phi,t) \in \mathbb{R}\times\mathbb{T}^1\times\mathbb{R}\times\mathbb{T}^1\times\mathbb{T}^1$. The pendulum has a homoclinic orbit to $(0,0)$. Let $(p^0(\sigma), q^0(\sigma))$ be a parametrization of such a homoclinic orbit, where $\sigma\in\mathbb{R}$ represents the time for the motion of the pendulum. The Melnikov potential for this homoclinic orbit is defined by

$$\begin{aligned}\mathcal{M}(\tau,I,\phi,t) = \quad -\int_{-\infty}^{\infty} \quad & \left[h(p^0(\sigma), q^0(\sigma), I, \phi + I\sigma, t+\sigma; 0)\right. \\ & \left. -h(0,0,I,\phi+I\sigma,t+\sigma;0)\right] d\sigma.\end{aligned}$$

Assume the following non-degeneracy conditions on the Melnikov potential $\mathcal{M}$:

(i) For each I in some interval (I^-, I^+), and each (ϕ,t) in some open set in $\mathbb{T}\times\mathbb{T}$, the map

$$\tau\in\mathbb{R}\to\mathcal{M}(\tau,I,\phi,t)\in\mathbb{R}$$

has a non-degenerate critical point τ^*, which can be parameterized as

$$\tau^* = \tau^*(I,\phi,t).$$

(ii) For each (I, ϕ, t) as above, the function

$$(I, \phi, t) \to \frac{\partial \mathcal{M}}{\partial \phi}(\tau^*(I, \phi, t), I, \phi, t)$$

is non-constant and positive.

For the Hamiltonian flow of (1), condition (i) implies that the unstable and stable manifolds of the annulus $\tilde{\Lambda} = \{(I, \phi, t) \,|\, I \in (I^-, I^+),\, \phi \in \mathbb{T}^1,\, t \in \mathbb{T}^1\}$ intersect transversally along a homoclinic 3-dimensional manifold $\tilde{\Gamma}$ that is described by the implicit equation $\tau^* = \tau^*(I, \phi, t)$, for (I, ϕ, t) in some open domain in $(I^-, I^+) \times \mathbb{T}^1 \times \mathbb{T}^1$. When we restrict $\tilde{\Gamma}$ to some appropriate domain of (I, ϕ, t), we obtain a homoclinic channel, which we still denote $\tilde{\Gamma}$. The scattering map S associated to this homoclinic channel $\tilde{\Gamma}$ can be computed in terms of the Melnikov potential $\mathcal{M}$. If $S(x^-) = x^+$, then the change in the I-coordinate by S is given by

$$I(x^+) - I(x^-) = \varepsilon[I, \mathcal{M}](\tau^*(I, \phi, t), I, \phi, t) + O_{C^1}(\varepsilon^{1+\varrho}),$$

for some $\varrho > 0$. Condition (ii) implies that there are points in the domain of the scattering map S whose I-coordinate is increased by S. When we discretize the Hamiltonian flow by the time-one map F, we obtain that $\Lambda = \{(I, \phi) \,|\, I \in (I^-, I^+),\, \phi \in \mathbb{T}^1, \}$ is a normally hyperbolic invariant manifold. Its stable and unstable manifolds intersect transversally along some 2-dimensional homoclinic channel Γ corresponding to $\tilde{\Gamma}$.

We want to point out that the existence of a normally hyperbolic invariant manifold, the verification of the twist map property, and the computation of the scattering map can be done for many other types of systems besides (1) – see, for example, the survey [16].

Condition (ii) also implies that the image of each torus T under the scattering map S has an intersection point with a torus T' which is $O(\varepsilon)$-close to T. If $S(T)$ has a transverse intersection with T', then $W^u(T)$ has a transverse intersection with $W^s(T')$. If $S(T)$ has a topologically crossing intersection with T', then $W^u(T)$ has a topologically crossing intersection with $W^s(T')$.

4.2. **Topological assumptions (A1)–(A5).** We return to the general class of a priori unstable nearly integrable Hamiltonian systems described at the beginning of Subsection 4.1. We will restrict our attention to systems for which the scattering map S associated to some homoclinic channel Γ can be computed explicitly.

In general, the domain of the scattering map S is an open set D^- in Λ, of size of order $O(1)$ with respect to the size of the perturbation. Thus condition (A1) amounts to choosing a sequence of transition tori that intersect the domain D^-.

The KAM primary tori are topologically transitive, thus by selecting the intermediate tori in a transition chain $(T_i)_{i=i_k+1, i_{k+1}}$, i.e. the tori T_i with $i_k + 1 < i < i_{k+1}$, to be KAM tori, one ensures condition (A2). Condition (A2) also requires that the tori at the ends of a transition chain, i.e. T_{i_k+1} and $T_{i_{k+1}}$ on the boundaries of Birkhoff zones of instability, should be topologically transitive. A sufficient condition for this is that these boundary tori can be obtained as C^0-limits of some KAM tori.

The scattering map S associated to the homoclinic channel Γ can be used to verify that the stable and unstable manifolds of sufficiently close invariant primary tori intersect. More precisely, if the image of a curve segment of the torus T_i under the scattering map intersects transversally (topologically crossing) another torus T_{i+1}, then the unstable manifold of T_i intersects transversally (topologically crossing) the stable manifold of T_{i+1}. In this way one can verify condition (A3). (The definition of topological crossing is given in Section 5.) Condition (A3) also requires the existence of not only one but three such intersection points for the tori at both ends of a transition chain. The fact that the image of a torus under the scattering map intersects another torus at several points is automatically verified in many examples, such as in [14, 15]. The reason we require this condition is that when the image of a curve segment of the torus T_i under the scattering map S intersects another torus T_{i+1} three times in a topologically crossing manner, it determines two open regions bounded by $S(T_i)$ and T_{i+1} in Λ, one region on one side and the other region on the other side of T_{i+1}. The existence of these regions is being used for applying the existence of Birkhoff connecting orbits property to cross the Birkhoff zones of instability of (A4). See Figure 4. (Here we note that the condition (A3) in the case $i = i_k + 1$, saying that there is a curve segment $\mathcal{T}_{i_k+1} \subset T_{i_k+1}$ in the domain of S such that $S(\mathcal{T}_{i_k+1})$ intersects T_{i_k+2} at exactly three points in a topologically crossing manner, implies that there is a curve segment $\mathcal{T}_{i_k+2} \subset T_{i_k+2}$ in the domain of S^{-1} such that $S^{-1}(\mathcal{T}_{i_k+2})$ intersects T_{i_k+1} at exactly three points in a topologically crossing manner.) One can form transition chains of primary KAM tori by joining successive heteroclinic connections. If the boundary torus $T_{i_{k+1}}$ in a transition chain $(T_i)_{i=i_k+1,i_{k+1}}$ is the C^0-limit of some KAM tori $(T_{j_l(i_{k+1})})_l$, and if the image of $T_{i_{k+1}-1}$ under the scattering map intersects transversally, in an uniform manner, all the tori $T_{j_l(i_{k+1})}$, then the image of $T_{i_{k+1}-1}$ under the scattering map intersects in a topologically crossing manner the boundary torus $T_{i_{k+1}}$. A similar type of criterion can be formulated in regard to the boundary torus T_{i_k+1}. Thus, one can extend the transition chain of KAM tori to the boundary of the 'large gaps' specified in (A3).

In Theorem 3.1 we assume that the 'large gaps' are Birkhoff zones of instability as in (A4). The verification of Birkhoff zones of instability is related to the so called converse KAM theory. Some criteria to establish that a certain region of the phase space contains no primary torus can be found in [24, 26, 29, 30]. Also, C. Simó communicated to us a scheme to verify the existence of Birkhoff zones of instability in some models of dynamics between separatrices associated to different fixed points or periodic orbits [36]. It is conceivable that the verification of the assumption (A4) can be done in various examples through a combination of analytical and numerical methods.

Finally, we can ensure condition (A5) by selecting the intermediate tori T_i with $i_k + 1 < i < i_{k+1}$ in a transition chain $(T_i)_{i=i_k+1,i_{k+1}}$, to be KAM tori, as discussed earlier. The fact that the KAM tori form a Cantor set ensures that each torus T_i with $i_k + 1 < i < i_{k+1}$ can be C^0-approximated from both sides by other KAM tori.

5. Topological shadowing.

In this section we present a simple topological method to detect orbits with prescribed itineraries in a discrete dynamical system. This method is inspired from the works of C. Conley, R. Easton and R. McGehee [12, 18, 19], and some of its subsequent developments [8, 22, 37]. The proofs of the statements in this section can be found or follow immediately from similar statements in [23, 37].

Definition 5.1. Two immersed C^0-manifolds N_1 and N_2 in M, of complementary dimensions in M i.e., $\dim(N_1) + \dim(N_2) = \dim(M)$, cross topologically provided that there exist compact embedded C^0-submanifolds with boundary $\bar{N}_1 \subseteq N_1$ and

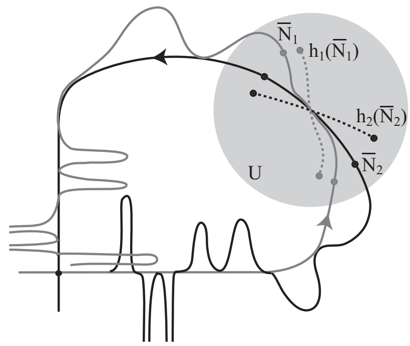

FIGURE 1. Topologically crossing manifolds.

$\bar{N}_2 \subseteq N_2$, and an orientable open subset U of M with $\partial\bar{N}_1 \cap U = \partial\bar{N}_2 \cap U = \emptyset$ such that the following conditions hold

(i) $\dim(\bar{N}_1) = \dim(N_1)$ and $\dim(\bar{N}_2) = \dim(N_2)$,
(ii) $\partial\bar{N}_1 \cap \bar{N}_2 = \emptyset$ and $\partial\bar{N}_2 \cap \bar{N}_1 = \emptyset$,
(iii) $\bar{N}_1 \cap \bar{N}_2 \subseteq U$,
(iv) there exists a homotopy $h : [0,1] \times M \to M$ such that:
  (iv.a) $h_0(\bar{N}_1) = \bar{N}_1$ and $h_0(\bar{N}_2) = \bar{N}_2$,
  (iv.b) the homotopy h_t moves points by less than $\varepsilon/2$, where

$$\varepsilon = \min(\mathrm{dist}(\bar{N}_1 \setminus U, \bar{N}_2), \mathrm{dist}(\bar{N}_2 \setminus U, \bar{N}_1)),$$

  (iv.c) $h_1(\bar{N}_1)$ and $h_1(\bar{N}_2)$ are orientable C^1-smooth manifolds in M,
  (iv.d) there is a choice of orientation on $h_1(\bar{N}_1), h_1(\bar{N}_2)$ and U such that the oriented intersection number relative to U is non-zero, i.e.,

$$\#_U(h_1(\bar{N}_1), h_1(\bar{N}_2)) \neq 0.$$

At the intuitive level, the above definition says that two manifolds are topologically crossing if they can be made differentiably transverse with non-zero oriented intersection number by the means of a sufficiently small homotopy. Since the embedded manifolds in the above definition are manifolds with boundaries, one has to require that the homotopy does not let the boundary of one manifold cross the other manifold. See Figure 1. We note that in Definition 5.1 the homotopy h_t can be chosen to be arbitrarily small. Since the oriented intersection number is a homotopy invariant, topological transversality is stable under small C^0-perturbations.

Definition 5.2. A window W in M is a homeomorphism $w : B^u \times B^s \to M$, together with its image $w(B^u \times B^s)$ in M, where B^u and B^s are the closed unit balls in $\mathbb{R}^u$ and $\mathbb{R}^s$ respectively, with $u + s = \dim(M)$.

In the sequel, we will refer to any topological disk in W of the type $w(B^u \times \{y_0\})$ with $y_0 \in B^s$ as an unstable-like leaf, and to any topological disk of the type $w(\{x_0\} \times B^s)$ with $x_0 \in B^u$ as a stable-like leaf, respectively.

Definition 5.3. Let W_1 and W_2 be two windows in M, and $w_1 : B^u \times B^s \to M$, $w_2 : B^u \times B^s \to M$ be their corresponding homeomorphisms. We say that W_1 is correctly aligned with W_2 if for each $x_0 \in B^u$ and $y_0 \in B^s$, the unstable-like leaf $w_1(B^u, y_0)$ topologically crosses the stable-like leaf $w_2(x_0, B^s)$, with the same non-zero oriented intersection number for all pairs of leaves.

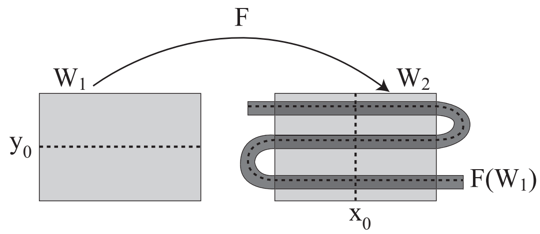

FIGURE 2. Correctly aligned windows.

See Figure 2.

Remark 5.4. The above definition is equivalent to Definition 6 in [37]. In that version of correct alignment, the union of the boundaries of all unstable-like leaves of a window W is referred as the exit set of W, and the union of the boundaries of all stable-like leaves is referred as the entry set of W. Definition 6 in [37] requires that the exit set of W_1 is disjoint from W_2, W_1 is disjoint from the entry set of W_2, and there exists a homotopy, which does not alter the above conditions on the exit and entry sets, that deforms W_1 into a u-dimensional surface that projects onto the unstable-like direction of W_2 with non-zero Brouwer degree. Here, for convenience, we opted for a version of this definition which is expressed in terms of the topological transversality of leaves. This version of the definition is also closer in spirit to the original version of correct alignment formulated in [18].

Given two windows W_1 and W_2 and a homeomorphism F on M, if $F(W_1)$ is correctly aligned with W_2, we say that W_1 is correctly aligned with W_2 under F. Note that the correct alignment of windows is robust, in the sense that if two windows are correctly aligned under a map, then they remain correctly aligned under a sufficiently C^0-small perturbation of that map.

The following result can be viewed as a topological version of the Shadowing Lemma that applies to systems which are not hyperbolic.

Theorem 5.5. *Let $(W_i)_{i\in\mathbb{Z}}$ be a bi-infinite sequence of windows in M, with u-dimensional unstable-like leaves and s-dimensional stable-like leaves, where $u+s = dim(M)$. Let F_i be a collection of homeomorphisms on M. If W_i is correctly aligned with W_{i+1} under F_i for all i, then there exists a point $p \in W_0$ such that*

$$F_i \circ \ldots \circ F_0(p) \in W_{i+1}, \text{ for all } i.$$

In the context of this paper, the maps F_i will represent different powers of the map F which defines the dynamical system under consideration.

We now discuss certain subsets of a window that they are themselves windows.

Definition 5.6. Let W be a window in M, and let $w : B^u \times B^s \to M$ be the associated homeomorphism. A subset $\hat{W}$ of W is said to be a horizontal sub-window of W if

$$\hat{W} = \bigcup_{x\in B^u} w(x, B^s_x),$$

where $\{B^s_x\}_x$ is a family of topological disks in B^s that depends continuously with $x \in B^u$.

A subset $\tilde{W}$ of W is said to be vertical sub-window of W if

$$\tilde{W} = \bigcup_{y \in B^s} w(B^u_y, y),$$

where $\{B^u_y\}_y$ is a family of topological disks in B^u that depends continuously with $y \in B^s$.

Note that by restricting w to the $\bigcup_{x \in B^u} \{x\} \times B^s_x$ we obtain a homeomorphism from a topological rectangle to $\hat{W}$, thus $\hat{W}$ together with this restriction of w is itself a window. Similarly, $\tilde{W}$ together with the restriction of w to $\bigcup_{y \in B^s} B^u_y \times \{y\}$ is also a window.

We have the following straightforward result.

Lemma 5.7. *If the window W_1 is correctly aligned with the window W_2, and $\tilde{W}_2$ is a vertical sub-window of W_2, then W_1 is also correctly aligned with $\tilde{W}_2$. If W_1 is correctly aligned with W_2, and $\hat{W}_1$ is a horizontal sub-window of W_1, then $\hat{W}_1$ is also correctly aligned with W_2.*

It is however not true that if W_1 is correctly aligned with W_2, and $\hat{W}_2$ is a horizontal sub-window of W_2, then W_1 is correctly aligned with $\hat{W}_2$. It is also not true that if W_1 is correctly aligned with W_2, and $\tilde{W}_1$ is a vertical sub-window of W_1, then $\tilde{W}_1$ is correctly aligned with W_2.

The following statement provides a method of construction of correctly aligned windows about the topologically crossing intersection of two manifolds.

Proposition 5.8. *Suppose that N_1 and N_2 are two Lipschitz-manifolds in M, of complementary dimensions, that are topologically crossing at a point p. Then, for every neighborhood V of p, there exists a pair of windows W_1 and W_2 contained in V, with distinguished homeomorphisms $w_1 : B^{\dim N_1} \times B^{\dim N_2} \to M$ and $w_2 : B^{\dim N_1} \times B^{\dim N_2} \to M$ respectively, such that the following hold true:*

(i) W_1 is correctly aligned with W_2,
(ii) the unstable-like leaf $w_1(B^{\dim N_1}, 0)$ is contained in N_1, and each unstable-like leaf $w_1(B^{\dim N_1}, y_0)$ topologically crosses N_2, for all $y_0 \in B^{\dim N_2}$,
(iii) the stable-like leaf $w_2(0, B^{\dim N_2})$ is contained in N_2, and each stable-like leaf $w_2(x_0, B^{\dim N_2})$ topologically crosses N_1, for all $x_0 \in B^{\dim N_1}$.

Proof. The idea is to define two tubular neighborhoods of the manifolds N_1 and N_2 so that they form a pair of correctly aligned windows under the identity mapping. Choose the embedded submanifolds $\bar{N}_1$ and $\bar{N}_2$ given by Definition 5.1 to be disks contained in V. If the manifolds N_1 and N_2 are C^1-smooth, the standard tubular neighborhood theorem [7] gives two tubular neighborhoods W_1 of $\bar{N}_1$ and W_2 of $\bar{N}_2$, respectively, that are diffeomorphic to disks of dimension $\dim(M)$. If the manifolds N_1 and N_2 are only Lipschitz, then the local result on the existence two Lipschitz tubular neighborhoods W_1 of $\bar{N}_1$ and W_2 of $\bar{N}_2$, respectively, is direct. See [33] for results about global smoothing.

Define two homeomorphisms $w_1, w_2 : B^{\dim N_1} \times B^{\dim N_2} \to V$ whose images are tubular neighborhoods W_1, W_2, respectively, such that $w_1(B^{\dim N_1}, 0) \subseteq \bar{N}_1$ and $w_2(0, B^{\dim N_2}) \subseteq \bar{N}_2$. By choosing w_1 and w_2 so that $w_1(x_0, B^{\dim N_2})$ and $w_2(B^{\dim N_1}, y_0)$ are sufficiently small for all $x_0 \in B^{\dim N_1}$ and all $y_0 \in B^{\dim N_2}$, the stability of topological transversality under small perturbations implies that the image of $w_1(\cdot, y_0)$ topologically crosses N_2 for each $y_0 \in B^{\dim N_2}$, and the image of $w_2(x_0, \cdot)$ topologically crosses N_1 for each $x_0 \in B^{\dim N_1}$. □

6. **Proof of the main theorem.** We are under the assumptions of Theorem 3.1. We are given a bi-infinite sequence $(T_i)_{i\in\mathbb{Z}}$ of invariant primary tori in Λ satisfying the properties (A1)–(A5) from Section 3. We would like to show that there is an orbit (z_i) that (ε_i)-shadows this sequence of tori. For this purpose, we construct a sequence of correctly aligned windows along the transition chains and across the Birkhoff zones of instability.

6.1. **Linearized coordinate system.** Since Λ is normally hyperbolic, the map F is topologically conjugate to its linearization near Λ (see [34]). More precisely, there exists a homeomorphism h from a neighborhood of the zero section in $T_\Lambda M \subset \mathbb{R}^{n_u+1} \times \mathbb{R}^{n_s+1}$ to a neighborhood of Λ in M such that

$$F(h(x, v)) = h(F(x), (DF)_x(v)),$$

for all $x \in \Lambda$ and all $v \in T_x M$ sufficiently small. The map h induces a system of linearized coordinates in a neighborhood of Λ in M. Such a coordinate system is in general not smooth. The stable manifolds $W^s(x)$ will correspond through this linearized coordinate system to the stable fibers E^s_x, and the unstable manifolds $W^u(x)$ will correspond to the unstable fibers E^u_x, where E^s and E^u denote the stable and unstable bundles associated to the normally hyperbolic manifold Λ, respectively. The map h can be used to define windows in a neighborhood of Λ in M. In our constructions below, the unstable-like leaves of the windows will correspond to the hyperbolic unstable directions plus one extra direction from the center directions, and the stable-like leaves will correspond to the hyperbolic stable directions plus one extra direction from the center directions.

6.2. **Construction of windows along a heteroclinic orbit.** We consider two invariant tori T_{i-1} and T_i in the sequence $(T_i)_{i\in\mathbb{Z}}$, such that $W^u(T_{i-1})$ topologically crosses $W^s(T_i)$ at a point $x_{i-1,i}$ in the homoclinic channel Γ. Due to normal hyperbolicity, there exist $x^-_{i-1} \in T_{i-1}$ and $x^+_i \in T_i$ such that $x_{i-1,i} \in W^u(x^-_{i-1}) \cap W^s(x^+_i)$. From the definition of the homoclinic channel Γ, the restriction to Γ of the wave maps $\Omega^\pm : \Gamma \to \Lambda$ are diffeomorphisms onto their images. There exists a curve $\gamma^-_{i-1,i}$ in Γ corresponding through Ω^- to T_{i-1}, and there exists a curve $\gamma^+_{i-1,i}$ in Γ corresponding through Ω^- to T_i. These curves are topologically crossing at $x_{i-1,i}$ in Γ since $W^u(T_{i-1})$ and $W^s(T_i)$ are topologically crossing at $x_{i-1,i}$ in M. By the definition of the scattering map, we have that $\Omega^+(\gamma^-_{i-1,i}) = S(T_{i-1})$ and $\Omega^-(\gamma^+_{i-1,i}) = S^{-1}(T_i)$. (Note that $\gamma^-_{i-1,i}$ and $\gamma^+_{i-1,i}$ are not homeomorphic to the tori T_{i-1} and T_i, but only to some curve segments of these tori, since the maps $(\Omega^\pm)^{-1}$ are not defined on the whole of Λ.)

We will now construct a window $W_{i-1,i}$ about $x_{i-1,i}$; we will propagate $W_{i-1,i}$ backwards in time to $F^{-m^-_{i-1}}(W_{i-1,i})$ about a point $F^{-m^-_{i-1}}(x_{i-1,i})$ that is $(\varepsilon_{i-1}/2)$-close to $F^{-m^-_{i-1}}(x^-_{i-1}) \in T_{i-1}$; also, we will propagate $W_{i-1,i}$ forward in time to $F^{m^+_i}(W_{i-1,i})$ about a point $F^{m^+_i}(x_{i-1,i})$ that is $(\varepsilon_i/2)$-close to $F^{m^+_i}(x^+_i) \in T_i$.

The distance between $F^m(x_{i-1,i})$ and $F^m(x^+_i)$, in the intrinsic metric of $W^s(T_i)$, tends to 0 as $m \to \infty$. Also, the curve $F^m(\gamma^+_{i-1,i})$ approaches T_i in the C^0-topology, and the curve $F^m(\gamma^-_{i-1,i})$ approaches $F^m(S(T_{i-1}))$ in the C^0-topology, as $m \to \infty$. We choose and fix an m^+_i sufficiently large such that $F^{m^+_i}(\gamma^+_{i-1,i})$ is within a distance of $(\varepsilon_i/2)$ from T_i, and $F^{m^+_i}(\gamma^-_{i-1,i})$ is within a distance of $(\varepsilon_i/2)$ from $F^{m^+_i}(S(T_{i-1}))$. Consequently, $F^{m^+_i}(x_{i-1,i})$ is within a distance of $(\varepsilon_i/2)$ from $F^{m^+_i}(x^+_i)$.

The iterate $F^{m_i^+}(W^u(T_{i-1}))$ of $W^u(T_{i-1})$ is topologically crossing $W^s(T_i)$ at $F^{m_i^+}(x_{i-1,i})$. We choose an (n_s+1)-dimensional topological disk D_i in $W^s(T_i)$, centered at $F^{m_i^+}(x_i^+)$ and contained in an $(\varepsilon_i/2)$-neighborhood of $F^{m_i^+}(x_i^+)$, such that $F^{m_i^+}(x_i)$ is an interior point to D_i. By replacing $W^u(T_{i-1})$ with some small topological disk centered at $x_{i-1,i}$, we can assume that $F^{m_i^+}(W^u(T_{i-1}))$ is itself an (n_u+1)-dimensional topological disk contained in an $(\varepsilon_i/2)$-neighborhood of $F^{m_i^+}(x_i^+)\in T_i$, and is topologically crossing $D_i\subseteq W^s(T_i)$ at $F^{m_i^+}(x_{i-1,i})$.

We define a homoeomorphism $w_{i-1,i}$ on $B^{n_u+1}\times B^{n_s+1}$, whose image is a window $W_{i-1,i}$, satisfying the following properties: the image of $W_{i-1,i}$ under $F^{m_i^+}$ is contained in an (ε_i)-neighborhood of $F^{m_i^+}(x_i^+)\in T_i$, $w_{i-1,i}(B^1\times\{0\},0)\subseteq\gamma^-_{i-1,i}$, $w_{i-1,i}(B^{n_u+1},0)\subseteq W^u(T_{i-1})$, $w_{i-1,i}(0,B^1\times\{0\})\subseteq\gamma^+_{i-1,i}$, and $w_{i-1,i}(0,B^{n_s+1})\subseteq W^s(T_i)$, and that the image of each unstable-like leaf $w_{i-1,i}(B^{n_u+1},y_0)$ under $F^{m_i^+}$ is a topological disk topologically crossing D_i, at a point interior to both disks, for all $y_0\in B^{n_s+1}$. The latter property is ensured by the stability of topological crossing under small perturbations, provided that we choose the stable-like and the unstable-like leaves of $W_{i-1,i}$ sufficiently small.

Near Λ we have a conjugacy h between F and DF. We define a homeomorphism $w_i^+ : B_\delta^{n_u+1}\times B_\delta^{n_s+1}\to M$, where the radius $\delta>0$ of the balls is small enough so that the image of w_i^+ is contained in an (ε_i)-neighborhood of $F^{m_i^+}(x_i^+)\in T_i$. We can choose h, and implicitly w_i^+, so that $w_i^+(0,B^{n_s+1})=D_i\subseteq W^s(T_i)$. By choosing $\delta>0$ sufficiently small, we can ensure that each leaf $w_i^+(x_0,B_\delta^{n_s+1})$ is topologically crossing the image of each unstable-like leaf $w_{i-1,i}(B_\delta^{n_u+1},y_0)$ under $F^{m_i^+}$, for all $x_0\in B^{n_u+1}$ and all $y_0\in B_\delta^{n_s+1}$. For convenience, by using a coordinate change we can assume that w_i^+ is defined on $B^{n_u+1}\times B^{n_s+1}$ rather than on $B_\delta^{n_u+1}\times B_\delta^{n_s+1}$.

We require one additional condition on W_i^+. Noting that the intersection between the image of w_i^+ and Λ is a 2-dimensional topological rectangle R_i^+ that contains a segment of T_i, we require that two of the sides of the rectangle R_i^+ lie on some pair of primary invariant tori near T_i, on opposite sides of T_i. (We will use these tori to control the evolution of R_i^+ under the dynamics of F restricted to Λ.) To make this requirement precise, we introduce some notation.

Each stable-like leaf $w_i^+(x_0,B^{n_s+1})$ of W_i^+ is topologically crossing each unstable-like leaf $w_i^+(B^{n_u+1},y_0)$ at a point. The intersection between the stable-like leaf $w_i^+(0,B^{n_s+1})$ and Λ is a curve segment of T_i contained in R_i^+. The intersections between the stable-like leaves $w_i^+(x_0,B^{n_s+1})$ and Λ, where $x_0\in B^{n_u+1}$, form a continuous family $\{\mathcal{T}^+_{a(i)}\}_a$ of disjoint curve segments that approach T_i in the C^0 topology, with each $\mathcal{T}^+_{a(i)}$ corresponding to some $w_i^+(x_0,B^1\times\{0\})$. The intersections between the unstable-like leaves $w_i^+(B^{n_u+1},y_0)$ and Λ, where $y_0\in B^{n_s+1}$, form a continuous family of disjoint curve segments $\{\mathcal{S}^+_{b(i)}\}_b$ that are topologically crossing T_i, with each $\mathcal{S}^+_{b(i)}$ corresponding to some $w_i^+(B^1\times\{0\},y_0)$. We require that the curve $w_i^+(B^1\times\{0\},0)\subseteq F^{m_i^+}(S(T_{i-1}))$. Thus, one of the curves from the family $\{\mathcal{S}^+_{b(i)}\}_b$, say $\mathcal{S}^+_{b_0(i)}$, is contained in the image under $F^{m_i^+}$ of $S(T_{i-1})$.

Since to construct W_i^+ we used the linearized coordinates near Λ, each stable-like leaf $w_i^+(x_0, B^{n_s+1})$ of W_i^+ is a union of fibers of the form

$$w_i^+(x_0, B^{n_s+1}) = \bigcup_{p\in\mathcal{T}^+_{a(i)}} \left(W^s_{\rm loc}(p)\cap W_i^+\right),$$

for some curve segment $\mathcal{T}_{a(i)}$ in R_i^+. Also, each unstable-like leaf $w_i^+(B^{n_u+1}, y_0)$ of W_i^+ is a union of fibers of the form

$$w_i^+(B^{n_u+1}, y_0) = \bigcup_{q\in\mathcal{S}^+_{b(i)}} \left(W^u_{\rm loc}(q)\cap W_i^+\right),$$

for some curve segment $\mathcal{S}^+_{b(i)}$ in R_i^+.

Now we make precise the aforementioned additional requirement that we impose on W_i^+. Since T_i can be C^0-approximated from both sides by invariant tori, there exist a pairs of invariant tori $T_{j_l^-}(i)$ and $T_{j_l^+}(i)$, both within a distance of $(\varepsilon_i/2)$ from T_i in Λ, such that T_i is in the interior of the annulus bounded by $T_{j_l^-(i)}$ and $T_{j_l^+(i)}$. We require that each curve segment $\mathcal{S}^+_{b(i)}$ in R_i^+ has its endpoints lying on the tori $T_{j_l^-(i)}$ and $T_{j_l^+(i)}$. That is, the rectangle R_i^+ has a pair of sides made of the endpoints of the curves $\mathcal{S}^+_{b(i)}$ on opposite sides of T_i in Λ, and lying on some invariant tori neighboring T_i.

This completes the construction of a homeomorphism $w_i^+ : B^{n_u+1}\times B^{n_s+1}\to M$ defining a second window W_i^+ contained in an (ε_i)-neighborhood of $F^{m_i^+}(x_i^+)\in T_i$. Note that the stable-like leaf $w_i^+(0, B^{n_s+1})$ of W_i^+ is contained in $W^s(T_i)$. Also, the window $W_{i-1,i}$ is correctly aligned with the window W_i^+ under $F^{m_i^+}$.

In the case when when T_i is at the boundary of a Birkhoff zone of instability as in (A4), we only require that the rectangle R_i^+ has the pair of sides made of the endpoints of the curves $\mathcal{S}^+_{b(i)}$ lying on opposite sides of T_i in Λ, and we do not require anymore that these sides are lying on some pair of primary invariant tori near T_i, on opposite sides of T_i. (Indeed, on one side of T_i there is a Birkhoff zone of instability, and there is no primary invariant torus inside the zone.)

The above construction concerns the propagation of the window $W_{i-1,i}$ forward in time to a point $F^{m_i^+}(W_{i-1,i})$ near T_i, and the construction of a window W_i^+ about T_i such that $W_{i-1,i}$ is correctly aligned with W_i^+ under $F^{m_i^+}$.

In a similar fashion we propagate the window $W_{i-1,i}$ backwards in time to $F^{-m^-_{i-1}}(W_{i-1,i})$ about the point $F^{-m^-_{i-1}}(x_{i-1,i})\in W^u(T_{i-1})$, and construct a window W^-_{i-1} about the point $F^{-m^-_{i-1}}(x^-_{i-1})\in T_{i-1}$, such that W^-_{i-1} is correctly aligned with the window $F^{-m^-_{i-1}}(W_{i-1,i})$. Moreover, the window W^-_{i-1} is chosen to be inside an (ε_{i-1})-neighborhood of T_{i-1}.

We now list the key features of W^-_{i-1}, that are analogous to the corresponding features of W_i^+. The window W^-_{i-1} is the image of a homeomorphism $w^-_{i-1} : B^{n_u+1}_\delta\times B^{n_s+1}_\delta\to M$, where $\delta>0$ is sufficiently small; through a change of coordinates we can assume that w^-_{i-1} is defined on $B^{n_u+1}\times B^{n_s+1}$. The unstable-like leaf $w^-_{i-1}(B^{n_u+1}, 0)$ of W^-_{i-1} is contained in $W^u(T_{i-1})$. The intersection between the image of w^-_{i-1} and Λ is a 2-dimensional topological rectangle R^-_{i-1} that contains a segment of T_{i-1}. The intersections between the unstable-like leaves $w^-_{i-1}(B^{n_u+1}, y_0)$ and Λ, where $y_0\in B^{n_s+1}$, form a continuous family of disjoint curve segments $\{\mathcal{T}^-_{a(i-1)}\}_a$ that approach T_{i-1} in the C^0-topology, with each $\mathcal{T}^-_{a(i-1)}$ corresponding to some curve $w^-_{i-1}(B^1\times\{0\},y_0)$. The intersections between the stable-like leaves $w^-_{i-1}(x_0,B^{n_s+1})$ and Λ, where $x_0\in B^{n_u+1}$, form a continuous family $\{\mathcal{S}^-_{b(i-1)}\}_a$ of disjoint curve segments that are topologically crossing T_{i-1}, with each $\mathcal{S}^-_{b(i-1)}$ corresponding to some $w^-_{i-1}(x_0,B^1\times\{0\})$. It is also required that the curve $w^-_{i-1}(0,B^1\times\{0\})$ is contained in $F^{-m^-_{i-1}}(S^{-1}(T_i))$, and each curve segment $\mathcal{S}^-_{b(i-1)}$ in R^-_{i-1} has its points lying on a pair of tori $T_{j_l^-(i-1)}$ and $T_{j_l^+(i-1)}$, each within $(\varepsilon_{i-1}/2)$ from T_{i-1} (so the rectangle R^-_{i-1} has a pair of sides lying on $T_{j_l^-(i-1)}$ and $T_{j_l^+(i-1)}$).

We have that each stable-like leaf $w^-_{i-1}(x_0,B^{n_s+1})$ of W^-_{i-1} is a union of fibers of the form

$$w^-_{i-1}(x_0,B^{n_s+1})=\bigcup_{p\in\mathcal{S}^-_{b(i-1)}}\left(W^s_{\rm loc}(p)\cap W^-_{i-1}\right),$$

for some curve segment $\mathcal{S}^-_{b(i-1)}$ in R^-_{i-1}. Similarly, each unstable-like leaf $w^-_{i-1}(B^{n_u+1},y_0)$ of W^-_{i-1} is a union of fibers of the form

$$w^-_{i-1}(B^{n_u+1},y_0)=\bigcup_{q\in\mathcal{T}^-_{a(i-1)}}\left(W^u_{\rm loc}(q)\cap W^-_{i-1}\right),$$

for some curve segment $\mathcal{T}^-_{a(i-1)}$ in R^-_{i-1}. In the special case when T_{i-1} is at the boundary of a Birkhoff zone of instability of (A4), the rectangle R^-_{i-1} is constructed to have the pair of sides made of the endpoints of the curves $\mathcal{S}^-_{b(i-1)}$ just lying on opposite sides of T_{i-1} in Λ, rather than lying on some invariant tori neighboring T_{i-1}.

6.3. **Construction of windows along a transition chain.** We consider a transition chain of tori $T_{i_{k-1}+1},T_{i_{k-1}+2},\dots,T_{i_k-1},T_{i_k}$, where k is some positive integer, as in (A3). We have that $W^u(T_{i-1})$ has a topologically crossing intersection with $W^u(T_i)$ at a point $x_{i-1,i}$, where T_{i-1} and T_i are two consecutive tori in the above sequence. The point $x_{i-1,i}$ lies on the unstable manifold of $x^-_{i-1}\in T_{i-1}$ and also on the stable manifold of $x^+_i\in T_i$. Each torus in the transition chain, except for the tori at both ends, can be approximated from both sides, relative to the C^0-topology, by other primary invariant tori.

We would like to construct a finite sequence of windows along these tori such that any two consecutive windows in the sequence are correctly aligned under some power of F. We will perform this construction inductively starting at $T_{i_{k-1}+1}$, at the beginning of the transition chain.

The initial step of the construction goes as described in Subsection 6.2. This consists in constructing a window $W_{i_{k-1}+1,i_{k-1}+2}$ about $x_{i_{k-1}+1,i_{k-1}+2}$, and two windows, $W^-_{i_{k-1}+1}$ about the point $F^{-m^-_{i_{k-1}+1}}(x^-_{i_{k-1}+1})\in T_{i_{k-1}+1}$, and $W^+_{i_{k-1}+2}$ about the point $F^{-m^+_{i_{k-1}+2}}(x^+_{i_{k-1}+2})\in T_{i_{k-1}+2}$, such that $W^-_{i_{k-1}+1}$ is correctly aligned with $W_{i_{k-1}+1,i_{k-1}+2}$ under some iterate $F^{m^-_{i_{k-1}+1}}$, and $W_{i_{k-1}+1,i_{k-1}+2}$ is correctly aligned with $W^+_{i_{k-1}+2}$ under some iterate $F^{m^+_{i_{k-1}+2}}$. Also, all points of the window $W^-_{i_{k-1}+1}$ are within $(\varepsilon_{i_{k-1}+1})$ from $T_{i_{k-1}+1}$, and all points of the window $W^+_{i_{k-1}+2}$ are within $(\varepsilon_{i_{k-1}+2})$ from $T_{i_{k-1}+2}$.

We now assume that the inductive construction has been completed up to the heteroclinic connection between T_{i-1} and T_i, where the tori T_{i-1} and T_i are two consecutive tori in the chain $T_{i_{k-1}+1}, T_{i_{k-1}+2}, \dots, T_{i_k-1}, T_{i_k}$. The inductive construction yields a window W_i^+ within (ε_i) from T_i, which is of the form

$$W_i^+ = \bigcup_{(I,\phi)\in R_i^+} Q_i^+(I,\phi),$$

where the rectangle R_i^+ has a pair of sides lying on some invariant tori $T_{j^-(i)}$ and $T_{j^+(i)}$ on opposite sides of T_i, and $Q_i^+(I,\phi)$ is a topological (n_s+n_u)-dimensional rectangle corresponding to the hyperbolic directions, for each $(I,\phi)\in R_i^+$. The unstable-like leaves of this window are leaves of the form $w_i^+(B^{n_u}\times B^1, y_0)$, that intersect R_i^+ along curves $\mathcal{S}^+_{b(i)}$ which are topologically crossing T_i and have their endpoints on the tori $T_{j^-(i)}$ and $T_{j^+(i)}$.

We consider the subsequent heteroclinic connection in the chain, between T_i and T_{i+1}. About the corresponding heteroclinic point $x_{i,i+1}$ we construct a test window $W_{i,i+1}$. As in Subsection 6.2, we construct the windows W_i^- about T_i, and W_{i+1}^+ about the T_{i+1}, such that W_i^- is correctly aligned with $F^{-m_i^-}(W_{i,i+1})$, and $F^{m_{i+1}^+}(W_{i,i+1})$ is correctly aligned with W_{i+1}^+. The window W_i^- is contained in an (ε_i)-neighborhood of T_i, and is of the form

$$W_i^- = \bigcup_{(I,\phi)\in R_i^-} Q_i^-(I,\phi).$$

Here we chose the rectangle R_i^- so that it has a pair of sides on $T_{j^-(i)}$ and $T_{j^+(i)}$, which are the same neighboring tori of T_i as in the construction of W_i^+. Also, $Q_i^-(I,\phi)$ is a topological (n_s+n_u)-dimensional rectangle corresponding to the hyperbolic directions, for each $(I,\phi)\in R_i^-$. The stable-like leaves of this window are leaves of the form $w_i^-(x_0, B^{n_s}\times B^1)$, that intersect R_i^- along curves $\mathcal{S}^-_{b(i)}$ that are topologically crossing T_i and have their endpoints on the tori $T_{j^-(i)}$ and $T_{j^-(i)}$ neighboring T_i.

We will use the twist map property of F restricted to Λ to make the windows W_i^+ and W_i^- correctly aligned under some iterate of F. For this purpose, we will first make R_i^+ correctly aligned with R_i^- under some iterate of F.

By the twist condition, and by the fact that F is topologically transitive on T_i (assumption (A2)), there exists m_i such that each curve $\mathcal{S}^+_{b(i)}$ in R_i^+ connecting $T_{j^-(i)}$ and $T_{j^+(i)}$ is mapped by F^{m_i} onto a curve that intersects in a topologically crossing manner each curve $\mathcal{S}^-_{b(i)}$ in R_i^-, also connecting $T_{j^-(i)}$ and $T_{j^+(i)}$. Since the curves $\mathcal{S}^+_{b(i)}$ in R_i^+ connecting $T_{j^-(i)}$ and $T_{j^+(i)}$ represent the unstable-like leaves of R_i^+, and the curves $\mathcal{S}^-_{b(i)}$ in R_i^- connecting $T_{j^-(i)}$ and $T_{j^+(i)}$ represent the stable-like leaves of R_i^-, we obtain that the window R_i^+ is correctly aligned with the window R_i^- under F^{m_i}. See Figure 3.

We need to impose additional restrictions W_i^+ (implicitly on the $(2n)$-dimensional rectangles $Q_i^+(I,\phi)$), and on W_i^- (implicitly on $Q_i^-(I,\phi)$), such that the whole window W_i^+ is correctly aligned with the window W_i^- under F^{m_i}. We want that the image of each leaf of the type $w_i^+(B^{n_u}\times B^1, y_0)$ under F^{m_i} should cross topologically each leaf of the type $w_i^-(x_0, B^{n_s}\times B^1)$. The alignment in the hyperbolic direction is due to the fact that the unstable directions will contract exponentially and the stable

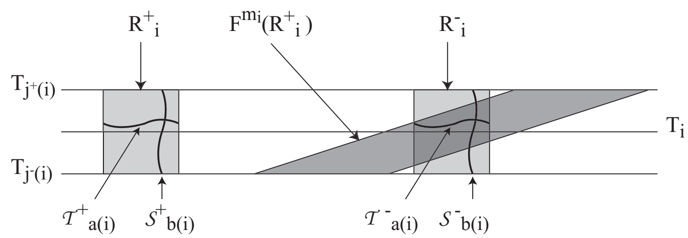

FIGURE 3. Windows in the annulus correctly aligned under the twist map.

directions will expand exponentially. (For the correct alignment in the hyperbolic directions, we will restrict $F^{m_i}(W_i^+)$ to the neighborhood of Λ where the linearized coordinates from Subsection 6.1 are well defined.) Thus one obtains W_i^+ correctly aligned with W_i^- under F^{m_i}. This completes the induction step.

This construction is continued inductively forward along the transition chain $T_{i_{k-1}+1}, T_{i_{k-1}+2}, \ldots, T_{i_k-1}, T_{i_k}$. When the upper boundary T_{i_k} of the Birkhoff zone of instability between T_{i_k} and T_{i_k+1} is reached, we need to cross it.

We emphasize that this construction uses as an essential feature the fact that the tori in the transition chain can be approximated from both sides, relative to the C^0-topology, by other primary invariant tori. We note that this type of tori also play an important role in the variational method in [9, 10] for proving the existence of drift orbits.

6.4. **Construction of windows across a Birkhoff zone of instability.** We consider a Birkhoff zones of instability bounded by the tori T_{i_k} and T_{i_k+1} for some k, as in assumption (A4).

The torus T_{i_k} is the last torus in a transition chain $T_{i_{k-1}+1}, T_{i_{k-1}+2}, \ldots, T_{i_k-1}, T_{i_k}$. We assume that correctly windows have already been constructed inductively along this transition chain. These windows form a sequence of the type:

$$W^-_{i_{k-1}+1}, W_{i_{k-1}+1, i_{k-1}+2}, W^+_{i_{k-1}+2}, \ldots, W^-_{i_k-1}, W_{i_k-1, i_k}, W^+_{i_k},$$

where each window is correctly aligned with the following window in the sequence under some iterate of F.

We consider the torus T_{i_k+1} at the other boundary of the Birkhoff zone of instability. Corresponding to the heteroclinic connection between T_{i_k+1} and T_{i_k+2}, the next invariant torus in the sequence $(T_i)_{i\in\mathbb{Z}}$, we construct the correctly aligned windows $W^-_{i_k+1}$ near T_{i_k+1}, W_{i_k+1,i_k+2} near x_{i_k+1,i_k+2}, and $W^+_{i_k+2}$ near T_{i_k+2}, as in Subsection 6.2.

We want to make the window $W^+_{i_k}$ on one side of the Birkhoff zone of instability correctly aligned with the window $W^-_{i_k+1}$ on the other side of the Birkhoff zone of instability, under some iterate of F. We will use the existence of Birkhoff connecting orbits that go from near one boundary of the Birkhoff zone of instability to near the other boundary of the zone.

For this reason, we first consider the topological rectangles in $R^+_{i_k}$ corresponding to $W^+_{i_k}$, and $R^-_{i_k+1}$ corresponding to $W^-_{i_k+1}$, both rectangles being in Λ. First we want to make these rectangles correctly aligned. By construction, the rectangle $R^+_{i_k}$ contains a curve segment $\mathcal{S}^+_{b_0(i_k)}$ (corresponding to some curve $w^+_{i_k}(B^1\times\{0\}, y_0)$

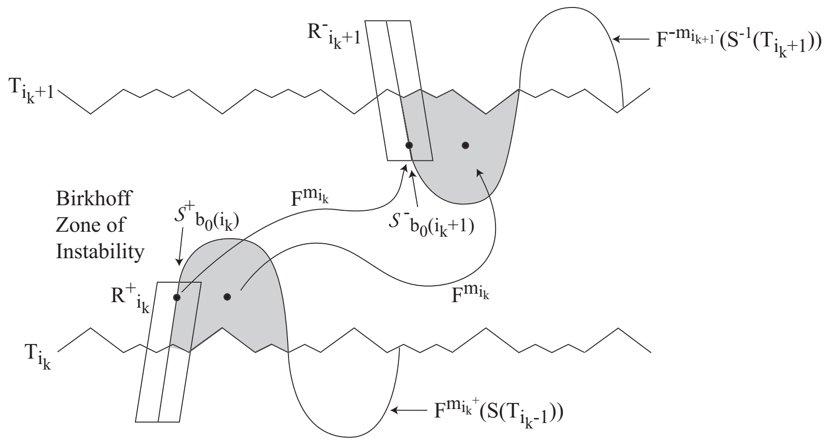

FIGURE 4. Birkhoff connecting orbits.

contained in some unstable-like leaf of $W^+_{i_k}$) that is contained in $F^{m^+_{i_k}}(S(T_{i_k-1}))$. Also, the rectangle $R^-_{i_k+1}$ contains a curve segment $\mathcal{S}^-_{b_0(i_k+1)}$ (corresponding to some curve $w^-_{i_k+1}(x_0, B^1 \times \{0\})$ contained in some stable-like leaf of $W^-_{i_k+1}$) that is contained in $F^{-m^-_{i_k+1}}(S^{-1}(T_{i_k+2}))$.

Condition (A3) implies that the pair of curves $S(T_{i_k-1})$ and T_{i_k} enclose an open neighborhood in Λ of some curve segment of T_{i_k}. It follows that the iterate $F^{m^+_{i_k}}(S(T_{i_k-1}))$ and T_{i_k} also enclose an open neighborhood in Λ of some curve segment of T_{i_k}; let us call this latter neighborhood U. The boundary part of U inside the annulus bounded by T_{i_k} and T_{i_k+1} is entirely contained in $F^{m^+_{i_k}}(S(T_{i_k-1}))$. Similarly, $S^{-1}(T_{i_k+2})$ and T_{i_k+1} enclose an open neighborhood in Λ of some curve segment of T_{i_k+1}. Therefore the iterate $F^{-m^-_{i_k+1}}(S^{-1}(T_{i_k+2}))$ and T_{i_k+1} enclose an open neighborhood V in Λ of some curve segment of T_{i_k+1}. The boundary part of V inside the annulus bounded by T_{i_k+1} and T_{i_k+2} is entirely contained in $F^{-m^-_{i_k+1}}(S^{-1}(T_{i_k+2}))$. By condition (A2) the map F is topologically transitive on both boundary tori T_{i_k+1} and T_{i_k+2}. This implies that there exists a Birkhoff connecting orbit from U to V in Λ, i.e., there exist $x \in U$ and $m_i > 0$ such that $F^{m_i}(x) \in V$. Thus, there exist a point $x_{i_k} \in F^{m^+_{i_k}}(S(T_{i_k-1}))$ whose image $F^{m_{i_k}}(x_{i_k})$ is a point $x_{i_{k+1}} \in F^{-m^-_{i_k+1}}(S^{-1}(T_{i_k+2}))$. Moreover, the point x_{i_k} can be chosen so that the intersection at x_{i_k+1} between $F^{m_{i_k}}(F^{m^+_{i_k}}(S(T_{i_k-1})))$ and $F^{-m^-_{i_k+1}}(S^{-1}(T_{i_k+2}))$ is topologically crossing (otherwise there will be no point in the open set U that goes inside the open set V). See Figure 4.

We now need to perform a series of adjustment to the rectangles $R^+_{i_k}$ and $R^-_{i_k+1}$ in Λ, so that $R^+_{i_k}$ is correctly aligned with $R^-_{i_k+1}$ under $F^{m_{i_k}}$. First, if necessary, we need to extend the original rectangle $R^+_{i_k}$ along the curve $\mathcal{S}^+_{b_0(i_k)}$ such that $R^+_{i_k}$ contains the point $x_{i_k} \in \mathcal{S}^+_{b_0(i_k)}$. We shall similarly extend the rectangle $R^-_{i_k+1}$ along the curve $\mathcal{S}^-_{b_0(i_k+1)}$ such that $R^-_{i_k+1}$ contains the point $x_{i_k+1} \in \mathcal{S}^-_{b_0(i_k+1)}$. Thus the point $x_{i_k} \in R^+_{i_k}$ is taken by $F^{m_{i_k}}$ to the point $x_{i_k+1} \in R^-_{i_k+1}$. We already know, from above, that the image of $\mathcal{S}^+_{b_0(i_k)}$ under $F^{m_{i_k}}$ intersects $\mathcal{S}^-_{b_0(i_k+1)}$ at

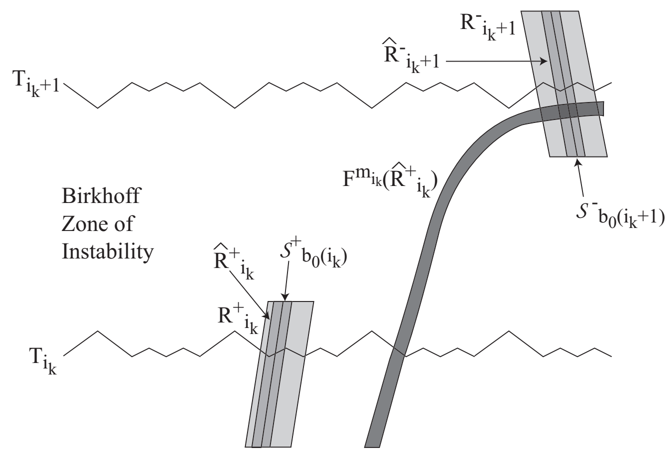

FIGURE 5. Correctly aligned windows across a Birkhoff zone of instability.

x_{i_k+1} in a topologically crossing manner. Then, if necessary, we shrink $R^+_{i_k}$ along its stable-like leaves, and shrink $R^-_{i_k+1}$ along its unstable-like leaves, such that the image of each of the curves $\mathcal{S}^+_{b(i_k)}$ under $F^{m_{i_k}}$ intersects each of the curves $\mathcal{S}^-_{b(i_k+1)}$ in a topologically crossing manner. This is possible due to the stability property of topological crossing. Moreover, we adjust the rectangle $R^+_{i_k}$ so that its stable-like leaves $\mathcal{T}^+_{a(i_k)}$ have their endpoints lying on the images under $F^{m^+_{i_k}} \circ S$ of some invariant tori $T_{j^-(i_k-1)}$ and $T_{j^+(i_k-1)}$ neighboring T_{i_k-1}. Such neighboring tori that are sufficiently close to the torus T_{i_k-1} exist due to condition (A5). In summary, the adjusted rectangle $R^+_{i_k}$ has a pair of edges on opposite sides of T_{i_k} in Λ, and the other pair of edges lying on some iterate of the images under the scattering map of a pair of tori near the previous torus in the sequence. Similarly, the adjusted rectangle $R^-_{i_k+1}$ is constructed such that its unstable-like leaves $\mathcal{T}^-_{a(i_k+1)}$ have their endpoints lying on the images under $F^{-m^-_{i_k+1}} \circ S^{-1}$ of some tori $T_{j^-(i_k+2)}$ and $T_{j^+(i_k+2)}$ neighboring T_{i_k+2}.

Since the unstable-like leaves of $R^+_{i_k}$ are the curves $\mathcal{S}^+_{b(i_k)}$, and the stable-like leaves of $R^-_{i_k+1}$ are the curves $\mathcal{S}^-_{b(i_k+1)}$, then it follows that the adjusted $R^+_{i_k}$ is correctly aligned with the adjusted $R^-_{i_k+1}$ under $F^{m_{i_k}}$. We will denote the rectangles $R^+_{i_k}$ and $R^-_{i_k+1}$ adjusted through the above maneuvers by $\hat{R}^+_{i_k}$ and $\hat{R}^-_{i_k+1}$, respectively. See Figure 5.

The adjustment of the rectangle $R^-_{i_k+1}$ requires an appropriate adjustment of the corresponding window $W^-_{i_k+1}$. Namely, the window $W^-_{i_k+1}$ will be replaced by a window $\hat{W}^-_{i_k+1}$ of the form

$$\hat{W}^-_{i_k+1} = \bigcup_{(I,\phi)\in \hat{R}^-_{i_k+1}} Q^-_{i_k+1}(I,\phi),$$

where the rectangle $\hat{R}^-_{i_k+1}$ is the adjusted rectangle from above, and, for each $(I,\phi) \in \hat{R}^-_{i_k+1}$ the $(n_s + n_u)$-dimensional topological rectangle $Q^-_{i_k+1}(I,\phi)$ is the same rectangle as the one corresponding to the original window $W^-_{i_k+1}$. Consequently, the windows W_{i_k+1,i_k+2} near x_{i_k+1,i_k+2}, and $W^+_{i_k+2}$ near T_{i_k+2} have to be replaced by some corresponding windows $\hat{W}_{i_k+1,i_k+2}$ and $\hat{W}^+_{i_k+2}$ respectively, so that $\hat{W}^-_{i_k+1}$ is correctly aligned with $\hat{W}_{i_k+1,i_k+2}$ under $F^{m^-_{i_k+1}}$, and $\hat{W}_{i_k+1,i_k+2}$ is correctly aligned with $\hat{W}^+_{i_k+2}$ under $F^{m^+_{i_k+2}}$. We stress that the orders of the iterates $F^{m^-_{i_k+1}}$ and $F^{m^+_{i_k+2}}$ for these correct alignments do not change from the ones for the test windows $W^-_{i_k+1}$, W_{i_k+1,i_k+2} and $W^+_{i_k+2}$, since the adjustments do not involve the hyperbolic directions of these windows. At this stage, the construction of windows about the heteroclinic connection between T_{i_k+1} and T_{i_k+2} is complete. Then, beginning with T_{i_k+2}, the construction of correctly aligned windows is continued inductively forward along the transition chain $T_{i_k+1}, T_{i_k+2}, \ldots, T_{i_{k+1}-1}, T_{i_{k+1}}$.

We have also performed an adjustment of the rectangle $R^+_{i_k}$ about the torus T_{i_k} at the other boundary of the Birkhoff zone of instability. This requires an appropriate adjustment of the corresponding window $W^+_{i_k}$. The window $W^+_{i_k}$ will be replaced by a window $\hat{W}^+_{i_k}$ of the form

$$\hat{W}^+_{i_k} = \bigcup_{(I,\phi)\in \hat{R}^+_{i_k}} Q^+_{i_k}(I,\phi),$$

where for each $(I,\phi) \in \hat{R}^+_{i_k}$ the (n_s+n_u)-dimensional topological rectangle $Q^+_{i_k}(I,\phi)$ is the same rectangle as the one corresponding to the original window $W^+_{i_k}$. The previously constructed windows W_{i_k-1,i_k} near x_{i_k-1,i_k}, and $W^-_{i_k-1}$ near T_{i_k-1} will be replaced by corresponding windows $\hat{W}_{i_k-1,i_k}$ and $\hat{W}^-_{i_k-1}$ respectively, so that $\hat{W}^-_{i_k-1}$ is correctly aligned with $\hat{W}_{i_k-1,i_k}$ under $F^{m^-_{i_k-1}}$, and $\hat{W}_{i_k-1,i_k}$ is correctly aligned with $\hat{W}^+_{i_k}$ under $F^{m^+_{i_k}}$. The orders of the iterates $F^{m^-_{i_k-1}}$ and $F^{m^+_{i_k}}$ remain the same as for the original windows. The intersection between $\hat{W}^-_{i_k-1}$ and Λ is a topological rectangle $\hat{R}^-_{i_k-1}$. Since on $\hat{R}^+_{i_k}$ we imposed that the endpoints of its stable-like leaves lie on some iterate of the images under the scattering map of a pair of tori near T_{i_k-1}, in order to ensure the correct alignment of windows, we choose the rectangle $\hat{R}^-_{i_k-1}$ so that its stable-like leaves $\mathcal{S}^-_{b(i_k-1)}$ have their endpoints lying on a pair of invariant tori $T'_{j^-(i_k-1)}$ and $T'_{j^+(i_k-1)}$ neighboring T_{i_k-1}, that are sufficiently close to T_{i_k-1}. This is possible due to condition (A5).

Now we consider the rectangle $R^+_{i_k-1}$ corresponding to the window $W^+_{i_k-1}$. By construction, its unstable-like leaves $\mathcal{S}^+_{b(i_k-1)}$ have their endpoints lying on opposite sides of T_{i_k-1}, on a pair of invariant tori $T_{j^-(i_k-1)}$ and $T_{j^+(i_k-1)}$. If the annulus bounded by $T_{j^-(i_k-1)}$ and $T_{j^+(i_k-1)}$ is contained inside the annulus bounded by $T'_{j^-(i_k-1)}$ and $T'_{j^+(i_k-1)}$, then the rectangle $R^-_{i_k-1}$ can be made correctly aligned with the rectangle $\hat{R}^+_{i_k-1}$ under some sufficiently large iterate $F^{m'_{i_k-1}}$, as in Subsection 6.3. We stress here that the order of the iterate m'_{i_k-1} may be larger than the order m_{i_k-1} of the iterate for the alignment of $R^+_{i_k-1}$ and $R^-_{i_k-1}$, since the window $R^-_{i_k-1}$ has been replaced by a window $\hat{R}^-_{i_k-1}$ whose unstable-like leaves may be bigger than the unstable-like leaves of the original rectangle $R^-_{i_k-1}$. (In order to get the point x_{i_k} of the Birkhoff connecting orbit inside the window $W^+_{i_k}$, we extended, if necessary, the original rectangle $R^+_{i_k}$ along the curve $\mathcal{S}^+_{b_0(i_k)}$ to $\hat{R}^+_{i_k}$ such that $\hat{R}^+_{i_k}$ contains the

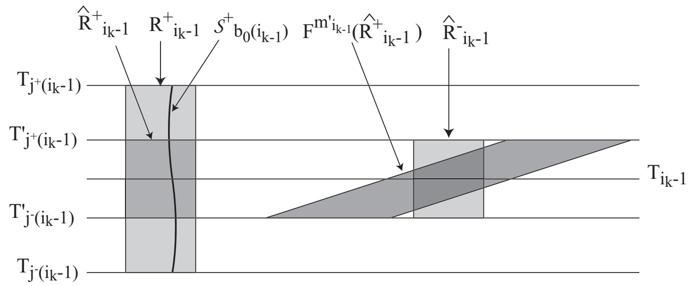

FIGURE 6. Correctly aligned windows across a Birkhoff zone of instability.

point $x_{i_k} \in \mathcal{S}^{+}_{b_0(i_k)}$. The effect of this extension along the unstable-like leaves is that the corresponding window $R^{-}_{i_k-1}$, by the previous torus in the chain, might needed to be extended along its unstable-like leaves as well.) Consequently, the window $W^{-}_{i_k-1}$ is in this case correctly aligned with the window $\hat{W}^{+}_{i_k-1}$ under some power $F^{m'_{i_k-1}}$. Note that here we use the fact that F restricted to T_{i_k-1} is topologically transitive, as specified in (A2). If the annulus bounded by $T_{j^-(i_k-1)}$ and $T_{j^+(i_k-1)}$ is not contained inside the annulus bounded by $T'_{j^-(i_k-1)}$ and $T'_{j^+(i_k-1)}$, then the rectangle $R^{+}_{i_k-1}$ needs to be 'shaved-off' so that its unstable-like leaves have their endpoints lying on $T'_{j^-(i_k-1)}$ and $T'_{j^+(i_k-1)}$. Hence $R^{+}_{i_k-1}$ will be replaced with a rectangle $\hat{R}^{+}_{i_k-1}$ whose unstable-like leaves $\hat{\mathcal{S}}^{+}_{b(i_k-1)}$ have their endpoints lying on $T'_{j^-(i_k-1)}$ and $T'_{j^+(i_k-1)}$. Thus $\hat{R}^{-}_{i_k-1}$ is correctly aligned with the rectangle $\hat{R}^{+}_{i_k-1}$ under some sufficiently large iterate $F^{m'_{i_k-1}}$. An important remark in this case is that the rectangle $\hat{R}^{-}_{i_k-1}$ can be chosen as a vertical sub-rectangle of R_{i_k}, since we only need to shrink the unstable-like leaves of R_{i_k}. See Figure 6. Consequently, the window $W^{-}_{i_k-1}$ will be adjusted to a window $\hat{W}^{-}_{i_k-1}$ that is a vertical sub-window of $W^{-}_{i_k-1}$. The key observation now is that the series of adjustments stops here as we do not need to modify any of the previously constructed windows. Indeed, by Lemma 5.7, since W_{i_k-2,i_k-1} is correctly aligned with $W^{+}_{i_k-1}$, replacing $W^{+}_{i_k-1}$ by a vertical sub-window $\hat{W}^{+}_{i_k-1}$ does not destroy the previous correct alignment. So we have $W^{-}_{i_k-2}$ correctly aligned with W_{i_k-2,i_k-1} under some power of F, and W_{i_k-2,i_k-1} correctly aligned with $\hat{W}^{+}_{i_k-1}$ under some power of F; also $\hat{W}^{-}_{i_k-1}$ is correctly aligned with $\hat{W}_{i_k-1,i_k}$, and $\hat{W}_{i_k-1,i_k}$ is correctly aligned with $\hat{W}^{+}_{i_k}$ under some power of F.

To simplify the notation, we drop the $\widehat{\ }$ symbol from the notation of all the adjusted windows.

In conclusion, at the end of this step, we have obtained the following:

(i) A sequence of correctly aligned windows

$$W^{-}_{i_{k-1}+1}, W_{i_{k-1}+1,i_{k-1}+2}, W^{+}_{i_{k-1}+2}, \ldots, W^{-}_{i_k-1}, W_{i_k-1,i_k}, W^{+}_{i_k},$$

along the transition chain $T_{i_{k-1}+1}, T_{i_{k-1}+2}, \ldots, T_{i_k-1}, T_{i_k}$.

(ii) A sequence of correctly aligned windows

$$W^{-}_{i_k+1}, W_{i_k+1,i_k+2}, W^{+}_{i_k+2}, \ldots, W^{-}_{i_{k+1}-1}, W_{i_{k+1}-1,i_{k+1}}, W^{+}_{i_{k+1}},$$

along the transition chain $T_{i_k+1}, T_{i_k+2}, \ldots, T_{i_{k+1}-1}, T_{i_{k+1}}$.

(iii) The window $W^+_{i_k}$ by one boundary of the Birkhoff zone of instability between T_{i_k} and T_{i_k+1} is correctly aligned with the window $W^-_{i_k+1}$ by the other boundary of the Birkhoff zone of instability.

6.5. **Construction of windows along the transition chains and across the Birkhoff zones of instability.** To summarize, in Subsection 6.2, we described the construction of correctly aligned windows about a topologically crossing heteroclinic connection. In Subsection 6.3 we described the inductive construction along a topological transition chain $T_{i_{k-1}+1}, T_{i_{k-1}+2}, \ldots, T_{i_k-1}, T_{i_k}$, starting at $T_{i_{k-1}+1}$ and moving forward along the transition chain. In Subsection 6.4 we continued this inductive construction across the Birkhoff zone of instability between T_{i_k} and T_{i_k+1} and along the subsequent transition chain $T_{i_k+1}, T_{i_k+2}, \ldots, T_{i_{k+1}-1}, T_{i_{k+1}}$. This process required the revision of the last windows about T_{i_k-1} and T_{i_k}, while the rest of the windows remained unchanged. Thus, starting from some initial torus $T_{i_{k-1}+1}$ and moving forward, we can construct correctly aligned windows along infinitely many topological transition chains interspersed with Birkhoff zones of instability. Such construction does not need to revise the windows constructed at the initial step by $T_{i_{k-1}+1}$. Therefore, a similar construction can be performed backwards in time, along infinitely many topological transition chains interspersed with Birkhoff zones of instability. In conclusion, one obtains a bi-infinite sequence of correctly aligned windows of the type $W^-_i, W^+_i, W_{i,i+1}$, with the windows W^-_i, W^+_i contained in an (ε_i)-neighborhood of T_i. The Shadowing Lemma-type of result Theorem 5.5 implies the existence of an orbit (z_i) that (ε_i)-shadows the tori (T_i).

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